

A symmetry's tale: from the material to the celestial

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Abstract

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Symmetry has played a crucial role in our understanding of physical systems. In this thesis, we review several works based on investigating the symmetry properties of theories. We examine and improve the Noether's theorem and the coset construction, both powerful tools when studying the symmetry aspects of a physical system. We manipulate the intrinsic ambiguities in the derivation of the stress-energy tensor using Noether's theorem to systematically compute, without any guesswork, the necessary "improvement terms" which make the tensor satisfy certain algebraic properties such as symmetry and tracelessness, even off-shell. We then construct a new type of coset construction, which can accommodate relativistic particles with arbitrary spins. This is the first work we know of to incorporate arbitrary spin degrees of freedom into coset construction. We then present two interesting examples of condensed matter systems described by effective field theories that come from spontaneous symmetry breaking. For the so-called framid, we present the peculiar behavior of its stress-energy tensor that it is Lorentz-invariant even though the system breaks Lorentz boosts spontaneously. An analogy is drawn to the cosmological constant problem since the vacuum energy there and the Lorentz-breaking terms here are all surprisingly zero. Lastly, we describe how the inflation of the universe can be driven by a solid. We focus on the icosahedral inflation model, where the isotropies of background evolution and scalar power spectrum are guaranteed although the system is anisotropic. We discuss some observational signatures of this model.

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Dedication

To everyone in my loving and inspiring family.

Chapter 1: Introduction and Background

Symmetries of physical systems are at the core of the studies of the laws of nature. At the dawn of theoretical physics, it was assumed in Newton's laws that physical rules are invariant under Galilean transformations. Later, in the 19th century, Maxwell put together his famous equations for electromagnetism, which possessed the more general Poincaré-invariance. Studies of this symmetry then led to the formulation of special relativity in the early 20th century. As the hero of relativity, Einstein himself is famously obsessed with the idea that a more symmetric, and hence more beautiful, theory should be closer to reality¹. Symmetries have continued to grow more important in physics, all the way up to contemporary research in the 21st century. For example, they are the necessary ingredient in the definition of particles, and therefore vital to the understanding of the standard model of particle physics. Moreover, the principles of effective field theory live on the idea that symmetry is one of the leading criteria for determining the effective Lagrangian. It is then important to develop and upgrade the theoretical toolbox we have for investigating the symmetry aspect of a physical system, and to scrutinize the applications. Following this line of thought, in this thesis, we will explore a few specific tools for and applications of the principle of symmetry, trying to make some extensions and improvements. This thesis is a collection of works done on several specific spots in the vast land administrated by symmetries, and is far from being comprehensive, nor do we try to make it so. We hope that these works contribute to a better understanding of what symmetries are capable of in contemporary theoretical research.

This thesis is mainly based on the work in [1, 2, 3, 4]. Several other works during my graduate study that are in preparation are not included in this thesis. For completeness, they will be briefly

¹As a famous quote from H. Bondi goes, "What I remember most clearly was that when I put down a suggestion that seemed to me cogent and reasonable, Einstein did not in the least contest this, but he only said, 'Oh, how ugly.' As soon as an equation seemed to him to be ugly, he really rather lost interest in it and could not understand why somebody else was willing to spend much time on it. He was quite convinced that beauty was a guiding principle in the search for important results in theoretical physics."

discussed at the end of this chapter.

1.1 Noether's theorem

Noether's theorem is a key manifestation of the importance of the principle of symmetries, relating continuous symmetries to conservation laws. When this theorem is applied to local field theories, one usually calculates the locally conserved current J^μ corresponding to each independent continuous symmetry generator through the Noether's procedure. The procedure for such a calculation is reviewed below.

For the purpose of the discussion of the Noether's theorem, we will be using $\underline{\phi}(x)$ to denote a generic multiplet of (real) fields, not necessarily scalars, and the dot '·' to denote the contraction of indices in field space. Assume that we have an action $S[\phi]$ that is invariant under some continuous symmetries, enumerated by a ,

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) + \epsilon^a \underline{\Delta}_a , \quad (1.1)$$

where ϵ^a are constant infinitesimal parameters, which we will always keep up to first order only, and $\underline{\Delta}_a$ are given local functionals of the fields. When we are concerned only with internal symmetry, the Noether current is calculated to be

$$J_a^\mu = - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \underline{\phi})} \cdot \underline{\Delta}_a , \quad (1.2)$$

where \mathcal{L} denotes the Lagrangian of the system. For the case where the transformation is a spacetime symmetry, the current takes a different form, which we will see in Chapter 2; and for the specific case of spacetime translations, we obtain the famous canonical stress-energy tensor $T_c^\mu{}_\nu$ through Noether's procedure,

$$T_c^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \underline{\phi})} \cdot \partial_\nu \underline{\phi} - \delta_\nu^\mu \mathcal{L} . \quad (1.3)$$

However, this conserved current J^μ is famously ambiguous, and the ambiguities are routinely exploited in the case of spacetime symmetries to “improve” the current associated with spacetime

translations, the canonical stress-energy tensor $T_c^{\mu\nu}$, to some form $T^{\mu\nu}$ with specific properties: if the theory has spatial rotational invariance, T^{ij} can be made symmetric; if the theory also has Lorentz invariance, the full $T^{\mu\nu}$ can be made symmetric [5, 6]; if the theory further has scale-invariance, $T^{\mu\nu}$ can be made traceless up to a total divergence [7, 8, 9]; finally, if the theory has full conformal symmetry, $T^{\mu\nu}$ can be made fully traceless [7, 8, 9]. Notice that these algebraic properties of symmetry and tracelessness are generically only valid on-shell, and frequently require great insights in “guessing” the correct final form. In Chapter 2 of this work, we will present a systematic procedure which calculates the necessary improvement terms for all the different cases mentioned above without any guesswork. Even better, we will derive the additional terms needed to make the stress-energy tensor symmetric and traceless even off-shell.

1.2 Coset construction

Apart from giving rise to conserved quantities, symmetries are also important when they are spontaneously broken. Spontaneous symmetry breaking happens when the ground state solution of a system does not exhibit one or more symmetries that are present in the Lagrangian, or the most symmetric formulation, of the system. Research around spontaneous symmetry breaking has been extremely fruitful in modern theoretical physics, with one of the most prominent examples being the Higgs mechanism. It has since become, in many cases, an important aspect when it comes to understanding a physical system.

This fact becomes particularly evident when we are primarily concerned with low-energy dynamics. When continuous symmetries are spontaneously broken, conventional knowledge states that the spectrum of low-energy excitations include gapless (massless) Goldstone modes. If only internal symmetries are broken and the underlying physics is Poincaré-invariant, Goldstone’s theorem guarantees the existence of one Goldstone boson for each spontaneously broken symmetry. When Poincaré symmetry is broken, however, no such one-to-one correspondence is guaranteed. At least one gapless excitation must exist, but it need not be describable as a scalar particle. In fact, it could have any spin and need not be a particle—or even a quasiparticle—in any conventional

sense. A striking example occurs in fermi liquids at zero temperature, which spontaneously breaks Lorentz boosts. The corresponding gapless excitations that satisfy Goldstone's theorem are the well-known particle-hole pairs, which evidently have no single-particle interpretation [10].

Oftentimes, the only gapless excitations of a given system are the Goldstone modes. Whenever this is the case, as long as we only concern ourselves with the deep infrared (IR), we can integrate out all gapped excitations, yielding a theory consisting of only Goldstone modes. In practice, we often do not explicitly integrate out the gapped modes. Instead, we use the principles of effective field theory (EFT) to construct an action by writing down a linear combination of all symmetry-invariant terms at any given order in a derivative expansion. The coefficients of this linear combination are phenomenological parameters that can be determined experimentally. It is important to note that even when symmetries are spontaneously broken, they remain symmetries of the IR theory of Goldstones; however, their actions on the Goldstones are non-linear and as a result can be quite complicated. Therefore, constructing the full set of symmetry-invariant terms can be rather challenging. To facilitate the formulation of Goldstone EFTs, a procedure known as the coset construction has been devised [11, 12, 13, 14, 15].

For a Poincaré-invariant system that spontaneously breaks internal symmetries while leaving spacetime symmetries intact, there is no ambiguity or choice in how to formulate an action using the coset construction (except for the numerical values of various coefficients that can be fixed experimentally). This uniqueness is a clear reflection of the strict nature of Goldstone's theorem when Poincaré symmetry is preserved; in particular, Goldstones must be massless, scalar particles that exist in a one-to-one correspondence with the spontaneously broken symmetry generators. When Poincaré symmetry is broken, however, Goldstone's theorem is not always so restrictive. In particular, the number or type of Goldstones in the EFT is not determined by the spontaneous symmetry-breaking pattern alone [16, 17]. We must supply additional information regarding the spontaneous symmetry-breaking *mechanism* if we are to uniquely determine the EFT (up to experimentally-determined coefficients) [12]. Ideally, the coset construction for spontaneously broken spacetime symmetries should reflect this extended set of possibilities. While the last few

decades have seen a great effort to understand the coset construction when Poincaré symmetry is spontaneously broken, there remain many kinds of systems that spontaneously break Poincaré symmetry, but whose effective action is difficult to be generated from the traditional method of cosets. In Chapter 3 of this thesis, we will extend the traditional coset construction to allow for a wider range of Goldstones. In particular, we focus on a simple yet illustrative class of systems that have largely evaded descriptions at the level of the coset construction, namely relativistic point particles. This class of EFTs is a useful testing-ground for extensions of the coset construction for two main reasons:

- All particles exhibit identical (or almost identical) spontaneous symmetry breaking (SSB) patterns, yet there are infinitely many distinct kinds of particles, each of which exhibits markedly different behavior. In particular, each kind of particle is classified by its spin $s \in \mathbb{N}/2$ and mass m . Massive and massless particles behave quite differently and particles of different spins have qualitatively very different actions.
- Almost all previous attempts at formulating coset constructions for systems without fermionic symmetries have assumed that the Goldstone excitations involve only bosonic fields;² however, there are Goldstone’s effective actions which involve fermionic degrees of freedom, like that of the fermi liquid [19]. Since point-particles of half-integer spin are fermions, we have a simple testing-ground for fermionic extensions of the coset construction. Thus, an important aim of this work is to extend the coset construction to account for situations in which the Goldstone excitations involve fermions even when the global symmetry group is purely bosonic (i.e. the symmetry algebra is an ordinary algebra as opposed to a graded algebra).

We will review in Chapter 3 the standard coset construction, and outline our new philosophy for the coset construction and comment on the advantages of this novel perspective. We then develop

²Notable successful attempts to include fermions are given in [18, 19], but these fermionic degrees of freedom were included by hand as opposed to being generated by the coset itself.

our method in detail to build up effective actions for relativistic point particles of arbitrary spin through coset construction.

1.3 Condensed matter systems

Perhaps one of the most surprising applications of spontaneous symmetry breaking is to classify condensed matter systems. A detailed introduction to this topic can be found in [20]. Although condensed matter systems are mostly highly non-relativistic, and are associated with a specific rest frame, the fundamental laws from which the collective excitations of condensed matter systems emerge remain Lorentz invariant. One can then think of condensed matter systems as states which break Lorentz invariance spontaneously, and classify them according to their symmetry-breaking patterns.

In [20], condensed matter systems are classified into eight different categories according to their symmetry-breaking patterns. It is beyond the scope of this work to discuss all of them. We will focus on two out of the eight systems, namely the (type I) framid and the solid, and explore some applications of the EFTs of these two systems.

1.3.1 Framid

The framid is a kind of relativistic system that spontaneously breaks Lorentz boosts *only*. It is most intuitively described in terms of a vector field $A_\mu(x)$ with a constant time-like expectation value:

$$\langle A_\mu(x) \rangle = \delta_\mu^0 . \quad (1.4)$$

Such an expectation value breaks Lorentz boosts, and the corresponding Goldstone fields $\vec{\eta}(x)$ can be thought of as parametrizing the fluctuations of $A_\mu(x)$ in the directions of the broken symmetries:

$$A_\mu(x) = \left(e^{i\vec{\eta}(x)\cdot\vec{K}} \right)_\mu^\alpha \langle A_\alpha(x) \rangle , \quad (1.5)$$

where \vec{K} are the boost generators. Notice that, even with Goldstone fields present, $A_\mu A^\mu = -1$.

One can construct the Goldstone effective theory either by writing down in a derivative expansion the most general Poincaré invariant action for A_μ , performing the replacement (1.5), and then expanding to any desired order in the $\vec{\eta}$ fields, or by using the coset construction for spontaneously broken spacetime symmetries. These two approaches have different advantages and disadvantages, but they yield the same result [20]. We will follow the former here.

To the second order in the derivative expansion, the most general effective Lagrangian takes the form [20]

$$\mathcal{L} = -\frac{M_1^2}{2} [(c_L^2 - c_T^2)(\partial_\mu A^\mu)^2 + c_T^2(\partial_\mu A_\nu)^2 + (c_T^2 - 1)(A^\rho \partial_\rho A_\mu)^2] , \quad (1.6)$$

where M_1 is an overall mass scale and c_L, c_T represent the propagation speeds of the longitudinal and transverse Goldstones. We want to eventually work with the Goldstone Lagrangian expanded to quadratic order; hence, with two derivatives on two different fields in each term of the Lagrangian (so that each derivative comes with at least one order of the field), it suffices to expand A_μ to first order in the Goldstones,

$$A_0 = \Lambda_0^0 = \cosh |\vec{\eta}| \simeq 1, \quad A_i = \Lambda_i^0 = \frac{\eta^i}{|\vec{\eta}|} \sinh |\vec{\eta}| \simeq \eta^i . \quad (1.7)$$

It is convenient to separate the Goldstones into their longitudinal and transverse modes, $\vec{\eta} = \vec{\eta}_L + \vec{\eta}_T$, where

$$\vec{\nabla} \times \vec{\eta}_L = 0, \quad \vec{\nabla} \cdot \vec{\eta}_T = 0 , \quad (1.8)$$

and also rescale them,

$$\vec{\eta} \rightarrow \frac{\vec{\eta}}{M_1}, \quad (1.9)$$

to eventually obtain a neat form of the effective Goldstone Lagrangian,

$$\mathcal{L}_2 = \frac{1}{2} \left[\dot{\vec{\eta}}^2 - c_L^2 (\vec{\nabla} \cdot \vec{\eta}_L)^2 - c_T^2 (\partial_i \eta_T^j)^2 \right] . \quad (1.10)$$

Framids remain hypothetical at the moment – there has been no observations of framids in nature, despite the fact that it has the simplest symmetry breaking pattern among condensed matter systems. However, this system has an interesting property that could make it a useful theoretical analog of the cosmological constant problem. More precisely, from the setup above, one can derive the stress energy tensor of the theory, which displays a peculiar property: if evaluated on the $\vec{\eta}(x) = 0$ background, it is Lorentz invariant [20]:

$$T_{\mu\nu}(x) = -\Lambda \eta_{\mu\nu} + \mathcal{O}(\partial\vec{\eta}) . \quad (1.11)$$

This is surprising: the ground state of the system spontaneously breaks boosts, so there is no obvious reason why it should have a Lorentz-invariant stress tensor. Certainly, all familiar condensed matter systems in the lab, such as solids, fluids, superfluids, which also spontaneously break Lorentz, have stress-energy tensors in their ground (or equilibrium) states that are *not* Lorentz invariant. In particular, in $c = 1$ units, they typically have mass densities much bigger than their pressures or internal stresses.

We will first argue and then show in Chapter 4 that the Lorentz invariance of the stress-energy tensor holds at one-loop level. Technically, massive cancellations show up miraculously at 1-loop level to preserve this property. The analogy to cosmological constant problem then goes as follows: for the cosmological constant problem, the vacuum energy expectation value is stupendously small, compared to its “natural” value³; in the case of the stress-energy tensor for the framid, terms that violate Lorentz symmetry are zero, without obvious reasons for being so. The hope is that understanding the analogous naturalness problem for the framid stress-energy tensor could give us insight into the cosmological constant problem.

However, a satisfactory understanding of the Lorentz-preserving phenomenon of framid at the level of symmetry is not available at the moment. One may think of it as a property protected by the spontaneously broken Lorentz symmetry, but this requires great insights if what is given is the

³See, for example, [21] for a comparison between the “natural” value that come from theoretical considerations and the observed value.

low-energy effective action (1.10), instead of the manifestly Lorentz-invariant formulation (1.6). Moreover, there exists a completely different system that features precisely the same symmetry breaking pattern as the framid, but saturates the associated Goldstone theorem in a radically different way, and that most definitely does not have a Lorentz-invariant stress-energy tensor in its ground state. Such a system is the familiar Fermi liquid, and we refer the reader to [10] for details about its relationship with spontaneously broken boosts. We will only carry out the explicit 1-loop calculation in this work, and leave the task of formulating a complete argument for this behavior based on symmetry to future projects.

1.3.2 Solids and inflation

We now turn to a more conventional condensed matter system – solids. From the symmetry breaking point of view, solids break boosts, spatial translations, and rotations spontaneously, and we can write down an effective field theory action accordingly. In this work we will follow closely the formulation of EFT for solids in [22, 23, 24] (see also [25]).

To set up our description, think of “naming” the volume elements of the solid in consideration with a three-dimensional internal label, $\phi^I(\vec{x}, t)$, where I takes the value of 1, 2, or 3. Using this description, we are essentially monitoring the volume elements that go through a certain spatial position \vec{x} over time. Although the choice of the labeling could in principle be arbitrary, an obvious choice that makes our later discussions of the internal symmetries easy and intuitive is the one with equilibrium value

$$\langle \phi^I \rangle = x^I . \tag{1.12}$$

This background breaks translational and rotational symmetry (and of course, boosts symmetry). However, usually a solid system is homogenous at large distances, and is invariant under (a subgroup of) rotations. To restore these properties, we enforce the internal symmetries,

$$\phi^I \rightarrow \phi^I + a^I, \quad \phi^I = O^I_J \phi^J , \tag{1.13}$$

where the a^I 's are constant shifts and O is an element of $SO(3)$ or one of its subgroups, depending on the system. The background configurations are now invariant under a combination of internal and spatial translations/rotations. In other words, the full spatial and internal symmetries are broken to the diagonal combination. Because of this, there is no need to further distinguish between the internal indices I, J and K , and the spatial indices i, j and k . We then write down the Lagrangian in the spirit of effective field theory. To do so, we employ the basic building block

$$B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J, \quad (1.14)$$

which is the only shift-invariant Lorentz scalar quantity to the lowest order in derivative expansion. The most general action we can write down is then

$$S_{\text{solid}} = \int d^4x F(B^{IJ}), \quad (1.15)$$

where F is a generic function invariant under rotations O acting on the I, J indices. Expanding this action around its background values with perturbations π^I ,

$$\phi^I = x^I + \pi^I, \quad (1.16)$$

which are interpreted as the phonons, leads to rich results about the dynamics of phonons in the solid. We will not dive into great detail in describing solids in a lab. Instead, we will focus on the application of this knowledge to primordial cosmology, namely, solid inflation [24]. The idea is to drive inflation with the solid system we just constructed, written in terms of the action

$$S_0 = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R + F(B^{IJ}) \right]. \quad (1.17)$$

As a note on the terminology, from now on, when we talk about “solid inflation”, we choose the O^I in equation (1.13) to be an element of the full $SO(3)$ group. Now, in addition to fluctuations of

the scalar fields, we also expand in the fluctuations of the metric,

$$g_{\mu\nu} = g_{\mu\nu}^{\text{FRW}} + h_{\mu\nu} , \quad (1.18)$$

where $g_{\mu\nu}^{\text{FRW}}$ is the Friedmann-Lemaître-Robertson-Walker metric. We can then carry out the standard program of inflationary calculations to obtain the power spectra and bispectra. However, as we mentioned, the solid system breaks spatial translations and rotations, without ever touching time translation. But the conventional understanding of inflation requires at least a spontaneous breaking of time, since inflation has to be able to keep track of time, and eventually come to an end. In fact, the effective field theory of inflation [26] is worked out exactly based on this understanding. Solid inflation then proves the point that this is not necessary – we can construct inflationary models without spontaneous breaking of time translation symmetry. The role of the clock, in solid inflation, is played by the metric itself – a time-dependent metric gives rise to time-dependent behaviors of the stress-energy tensor of the system, which could then be used to keep track of time.

One can further specialize the solid inflation model and develop the so-called icosahedral inflation [27], which exhibits some additional unique properties. To setup the icosahedral inflation model, one requires that the O^I in equation (1.13) be elements of the icosahedral group. What makes icosahedral inflation special, compared with, say, tetrahedral inflation, is that it is the only model in the solid inflation family that guarantees an isotropic scalar power spectrum, while being intrinsically anisotropic, allowing anisotropies in other power spectra, the bispectrum, and higher-point correlation functions.

Why do we want a model with such properties? Usually we say in cosmology that cosmological observations indicate that the very early universe was approximately homogeneous and isotropic. However, this is based on the only primordial observables we have had access to so far, which are the background cosmology and the two-point function of scalar perturbations. Focusing on the observed isotropy, it is interesting to ponder whether the universe could secretly be anisotropic—in terms of other observables, such as higher-point correlation functions, and the underlying theory—

but featuring an accidental isotropy for the observables we have detected so far. By “accidental” we mean something precise: that isotropy of those observables is a so-called accidental symmetry, akin to baryon number conservation in the standard model of electroweak interactions. That is, an approximate symmetry that is enforced by the fundamental symmetries of the theory to some low order in a perturbative expansion. Based on what we have seen, icosahedral inflation is a concrete implementation of this idea.

In Chapter 5, we will discuss icosahedral inflation in more detail and calculate its unique mixed tensor-scalar power spectrum. Observational implications will also be briefly analyzed.

1.4 Other works

In addition to the works mentioned above, there are several other works that are carried out during my graduate study, currently in preparation.

First, we carried out a more detailed analysis of the observational consequences of icosahedral inflation. Using the Einstein-Boltzmann code CLASS [28], we calculated the cosmic microwave background (CMB) power spectra predicted by icosahedral inflation. As is detailed in Chapter 5, among the different types of correlation functions, we will focus on the so-called T-B and E-B correlations, where T modes are fluctuations in the CMB temperature, and E and B modes are two different CMB polarization modes. This is because such modes are only present when we break either isotropy or parity, and is a hint that the underlying primordial cosmology theory may not be isotropic or parity-even. At the same time, breaking isotropy generates “off-diagonal” correlations, $C_{lm'l'm'}^{XY}$, with X and Y being T (temperature), E (E -mode) or B (B -mode), for $l \neq l'$ and $m \neq m'$, whereas for homogeneous and isotropic models, we are forced to have $l = l'$ and $m = m'$. Further, the value of $C_{lm'l'm'}^{XY}$ is not independent of m or m' . Therefore, contrary to the usual practice of averaging over different $m = m'$ values for a fixed $l = l'$, we did an analysis using bipolar spherical harmonics, first formulated in [29]. We plan to publish our analysis soon. The hope is that some comparisons between our analysis and the *Planck* data (specifically section 6 of [30] from *Planck* 2015) can be carried out in the near future.

More generally for inflationary models, we have been investigating the power law behavior of the scalar power spectrum. The scalar power spectrum for inflationary models are predicted (and also measured, for example, in *Planck* 2018 [31]) to scale with the wavenumber k as k^{-3+n_S-1} , where $n_S - 1$ is a small number ($\ll 1$), constant to the leading order in slow-roll, usually referred to as the “tilt”. Typically, power-law correlation functions come from some scaling symmetry of the theory, and in the case of inflation, a dilatation symmetry would follow from exact de Sitter symmetry. To allow for an exit from the inflationary phase, usually we break exact de Sitter slightly. However, the power law behavior of the power spectrum is preserved, and the tilt can be interpreted as a parametrization of the theory’s deviation from exact de Sitter. The task is to figure out the mechanism behind the preservation of the power law behavior under a slightly broken scaling symmetry. We have found an empirical procedure that could help us read off the tilt from the theory without going through the full calculation of the power spectrum. Starting from this empirical method, we hope to gain a deeper understanding of the situation, and eventually unveil the theoretical construct behind the power law correlation function and our empirical procedure.

Lastly, we carried out an analysis of the combined effect of accretion and superradiance for a scalar field permeating the vicinity of a Kerr black hole. The idea of superradiance is that when a light bosonic field is present around a Kerr black hole, there is an instability in the configuration, through which energy and angular momentum can be extracted from the black hole. At the same time, accretion of ambient matter onto the black hole is always active, and in most cases the rate of accretion is smaller than that of superradiance. However, when the system comes close to the “boundaries” of superradiance – regions in the parameter space of the system where the rate of superradiance is small enough so that accretion can compete efficiently with it, we need to be more careful about our analysis. We carried out analytical and numerical analyses to study the behavior of the system in these regions, showing how the accretion and superradiance cooperate to move the system along the boundaries we mentioned above. We plan to publish the finalized work in the near future, and hope that it could help understand future observations of Kerr black holes.

1.5 Note on Conventions

Throughout this thesis, we will use the “mostly-plus” metric signature, $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. We will work in natural units, $\hbar = c = 1$. Summation over repeated indices is usually implied, unless stated otherwise. In Chapter 2, for Lorentz generators, we use the same normalization as [8, 9], which differs from that of Weinberg [6], $\mathcal{J}_{\text{here}}^{\mu\nu} = -i \mathcal{J}_{\text{Weinberg}}^{\mu\nu}$. Also in Chapter 2, when addressing scale- and conformal-invariance, we keep the spacetime dimensionality D generic. Otherwise, we work in $D = 4$. In Chapter 3, we start with a generic D but restrict to $D = 4$ when we start to construct our effective actions of point particles.

Chapter 2: An Improved Noether's Theorem for Spacetime Symmetries

2.1 Ambiguous currents from an ambiguous theorem

In Chapter 1 we have seen that Noether's theorem relates continuous symmetries to conservation laws, with the conserved current calculated by Noether's procedure given in equations (1.2) and (1.3). We have also mentioned that this current is notoriously ambiguous. In fact, it is ambiguous for two main reasons:

1. First, given a current J^μ that is conserved "on-shell"—that is, on solutions of the equations of motion—one can always add to it a contribution of the form

$$\Delta J^\mu \equiv \partial_\alpha \Sigma^{\alpha\mu} , \quad \Sigma^{\alpha\mu} = -\Sigma^{\mu\alpha} , \quad (2.1)$$

where $\Sigma^{\alpha\mu}$ is any local functional of the fields that is antisymmetric in α and μ . Such an addition is conserved "off-shell"—that is, on any field configuration, regardless of whether this solves the equations of motion or not. Moreover, it does not contribute to the global charge associated with the current, $Q \equiv \int d^3x J^0$, because, using the antisymmetry of $\Sigma^{\alpha\mu}$ and assuming the fields vanish sufficiently fast at spatial infinity, one has

$$\int d^3x \Delta J^0 = \int d^3x \partial_i \Sigma^{i0} = 0 . \quad (2.2)$$

So, the two currents, J^μ and $J'^\mu = J^\mu + \Delta J^\mu$, obey equivalent conservation laws and yield the same global charge.

2. Second, since the current J^μ is only conserved on-shell anyway, one can add to it contributions that vanish on-shell. At the classical level, this does not modify the value of J^μ or of Q on

solutions of the equations of motion, and so it does not modify the associated conservation laws either. At the quantum level, this modifies the Ward identities in the contact terms only, since the equations of motion are obeyed in correlation functions up to contact terms.

In standard textbook treatments, it is exactly the first kind of ambiguities that allow people to derive the improvement terms to treat the stress-energy tensor such that it has the desired properties – symmetric, traceless, or symmetric and traceless. We will see later that the second kind of ambiguities can be utilized to make the stress-energy tensors satisfy these properties even off-shell.

For the purposes of what follows, it is instructive to trace the ambiguities discussed above back to Noether’s theorem: it is the theorem itself that is ambiguous. To see this, let’s review how the theorem usually works. Once again, as in equation (1.1), we assume that our $S[\phi]$ is invariant under some continuous symmetries,

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) + \epsilon^a \underline{\Delta}_a(\underline{\phi}, \partial \underline{\phi}, \dots) . \quad (2.3)$$

To be precise, let’s say that under (2.3) the Lagrangian density changes by a total derivative,

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon^a \partial_\mu F_a^\mu , \quad (2.4)$$

where F_a^μ are some functionals of the fields. Then, as the theorem goes, let’s see how the Lagrangian density changes if we make ϵ^a in (2.3) spacetime-dependent, $\epsilon^a = \epsilon^a(x)$. To first order in ϵ^a , the variation must take the form

$$\delta \mathcal{L} = \epsilon^a(x) \partial_\mu F_a^\mu + \partial_\mu \epsilon^a(x) G_a^\mu + \partial_\mu \partial_\nu \epsilon^a(x) G_a^{\mu\nu} + \dots \quad (2.5)$$

where the G ’s are suitable functionals of the fields, and usually the series in derivatives of ϵ^a truncates at finite order. For example, for a Lagrangian with at most N derivatives on a single field, the series usually truncates at $\partial^N \epsilon^a$ order. Notice that if one sets ϵ^a to a constant, only the first term survives, and one goes back to eq. (2.4). If we now integrate $\delta \mathcal{L}$ over spacetime we get the

variation of the action for generic $\epsilon^a(x)$. Restricting to functions $\epsilon^a(x)$ that go to zero at infinity, we can integrate all derivatives of ϵ^a by parts and end up with

$$\delta S = \int d^4x \delta \mathcal{L} = \int d^4x \epsilon^a(x) \partial_\mu J_a^\mu \quad (\text{off-shell}) , \quad (2.6)$$

where J_a^μ are whatever functionals of the fields emerging from the procedure just described. This defines the Noether currents. The last step is to recognize that eq. (2.3) for generic $\epsilon^a(x)$ vanishing at infinity is a particular field variation that vanishes at infinity, but on-shell the action should be stationary for *all* field variations that vanish at infinity. So, on-shell one must have

$$\partial_\mu J_a^\mu = 0 \quad (\text{on-shell}) . \quad (2.7)$$

Why are we saying that such a procedure is ambiguous? The ambiguity of item 1 above is easy to spot: Adding to J_a^μ identically conserved terms of the form (2.1) does nothing to the integrand in (2.6), precisely because such terms are identically conserved. The ambiguity of item 2 instead is more subtle to unveil, but more relevant for what follows. It has to do with the very first step of the Noether procedure, when we make ϵ^a spacetime dependent: it is usually assumed that that corresponds to replacing (2.3) simply with

$$\underline{\phi} \rightarrow \underline{\phi} + \epsilon^a(x) \underline{\Delta}_a , \quad (2.8)$$

but, in fact, in the theorem as we just described it, nowhere are we using this specific form of the transformation. The only property that we are using is that the x -dependent transformation that we perform should reduce to the symmetry (2.3) in the limit in which ϵ^a are constants. Then, instead of (2.8), we could use [3]

$$\underline{\phi} \rightarrow \underline{\phi} + \epsilon^a(x) \underline{\Delta}_a + \partial_\mu \epsilon^a(x) \underline{\Phi}_a^\mu + \partial_\mu \partial_\nu \epsilon^a(x) \underline{\Phi}_a^{\mu\nu} + \dots , \quad (2.9)$$

where the $\underline{\Phi}$'s are *arbitrary* functionals of the fields. By definition of functional derivatives, these new terms in the transformation of $\underline{\phi}$ modify the variation of the action (2.6) by

$$\delta S \supset \int d^4x \frac{\delta S}{\delta \underline{\phi}} \cdot [\partial_\mu \epsilon^a(x) \underline{\Phi}_a^\mu + \partial_\mu \partial_\nu \epsilon^a(x) \underline{\Phi}_a^{\mu\nu} + \dots] \quad (\text{off-shell}) . \quad (2.10)$$

Integrating by parts all derivatives of ϵ^a and comparing to (2.6), we see that the current gets new contributions of the form

$$J_a^\mu \supset -\frac{\delta S}{\delta \underline{\phi}} \cdot [\underline{\Phi}_a^\mu - \partial_\nu \underline{\Phi}_a^{\mu\nu} + \dots] \quad (\text{off-shell}) , \quad (2.11)$$

which clearly vanish on-shell.

So, in summary, both ambiguities discussed above are inherent features of Noether's theorem itself. The second one—the one we just discussed—is more interesting, in that it ties new terms in the current to a modification of how the fields are declared to transform under the spacetime-modulated version of the symmetry. This will allow us to derive systematically, directly from the theorem, the improvement terms for the stress-energy tensor that are associated with spacetime symmetries beyond translations. As we now explain, the main idea is to tailor, each time, the *translation* Noether theorem to the particular *additional* symmetry one wants to exploit¹.

2.2 The main idea

We want to improve the Noether's theorem for spacetime translations, exploiting the ambiguities discussed above, especially the second one. This, as we saw, is related to modifying the transformation properties of the fields as in (2.9). We then have to ask under what condition we can gain something by considering (2.9), perhaps with some judicious choice of the $\underline{\Phi}$ functionals, instead of the apparently simpler (2.8).

¹Brauner, Torriero, and Yonekura have brought to our attention refs. [17, 32, 33], which also explore ambiguities in Noether's theorem and use them to derive, in a systematic way, a number of desired properties for the conserved currents. The overlap of our work with those papers is, in fact, substantial, especially with ref. [32]. We plan to analyze the connection more closely in the near future.

In fact, for internal symmetries, we see no general benefit of using (2.9) in place of (2.8). On the other hand, for translations, we can make use of the fact that other possible spacetime symmetries, such as Lorentz invariance, scale invariance, and conformal invariance, can in fact be thought of as very specific spacetime-modulated translations. For instance, an infinitesimal Lorentz transformation of constant parameter $\omega_{\mu\nu} = -\omega_{\nu\mu}$, shifts the coordinates by

$$x^\mu \rightarrow x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \quad (\text{Lorentz}) , \quad (2.12)$$

which is formally a spacetime modulated translation

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x) , \quad (2.13)$$

with parameter

$$\epsilon^\mu(x) = \epsilon_L^\mu(x) \equiv \omega^\mu{}_\nu x^\nu \quad (2.14)$$

(‘ L ’ for ‘Lorentz’.) Notice that the index a of the infinitesimal parameters now becomes the spacetime index, $\epsilon^a \rightarrow \epsilon^\mu$.

Our fields transform under a translation of constant parameter ϵ^μ as²,

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \epsilon^\mu \partial_\mu \underline{\phi}(x) \quad (\text{translations}) , \quad (2.15)$$

and usually we would run the translation Noether theorem by generalizing this to

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \epsilon^\mu(x) \partial_\mu \underline{\phi}(x) , \quad (2.16)$$

for generic $\epsilon^\mu(x)$, which yields the so-called canonical stress-energy tensor. But, in alternative, we can notice that since under a Lorentz transformation of constant parameter $\omega_{\mu\nu}$ the fields transform

²We will stick to the active point of view for transformations throughout the work, unless otherwise clarified.

as

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \underline{\phi}(x) - \frac{1}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \cdot \underline{\phi}(x) \quad (\text{Lorentz}) , \quad (2.17)$$

we can run the Noether's theorem for translations with a "Lorentz-friendly" version of (2.9):

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \epsilon^\mu(x) \partial_\mu \underline{\phi}(x) - \frac{1}{2} \partial_\mu \epsilon_\nu(x) \mathcal{J}^{\mu\nu} \cdot \underline{\phi}(x) . \quad (2.18)$$

Such a transformation rule has the property that for a very specific $\epsilon^\mu(x)$ —eq. (2.14)—it corresponds to a symmetry of the action: Lorentz invariance. This implies that the stress-energy tensor one derives running the Noether's theorem in this way will have an additional property besides conservation. In particular, as we will see, the fact that eq. (2.14) corresponds to the most general $\epsilon^\mu(x)$ that has constant, antisymmetric first derivatives, will force the stress-energy tensor to be automatically *symmetric*, off-shell.

This of course matches the standard conclusion—Lorentz symmetry implies a symmetric $T^{\mu\nu}$ —but in our procedure there is no guesswork involved: by making the translation Noether theorem sensitive to Lorentz invariance, we automatically get a symmetric stress-energy tensor. And, contrary to the standard Belinfante procedure, nowhere do we have to use the equations of motion. Of course, this means that the Belinfante stress-energy tensor differs from ours by terms proportional to the equations of motion. As we tried to emphasize in section 2.1, such an ambiguity is to be expected on general grounds.

This is the basic idea that we will try to exploit. As usual, the devil is in the details, so let's see explicitly how these work out in the case of Lorentz, scale, and conformal invariance.

2.3 Passive vs. active, action vs. Lagrangian

As a preliminary step, it is useful to be precise about the symmetries we want to consider. At least for the cases we study here, we can think of a spacetime symmetry as a symmetry of the action

that is associated with a specific change of coordinates,

$$x^\mu \rightarrow x'^\mu(x) = x^\mu - \epsilon^a A_a^\mu(x) \quad (2.19)$$

under which the fields transform in a certain way,

$$\underline{\phi}(x) \rightarrow \underline{\phi}'(x') = \underline{\phi}(x) + \epsilon^a \tilde{\Delta}_a, \quad (2.20)$$

where A_a^μ are functions of the spacetime coordinates only, and $\tilde{\Delta}_a$ are given local functionals of the fields, minus all terms that contains only one factor of the fields $\underline{\phi}$ and with only one derivative. This is the so-called passive viewpoint. According to it, the transformation above is a symmetry if the infinitesimal action element does not change,

$$\mathcal{L}[\underline{\phi}'(x')] d^4x' = \mathcal{L}[\underline{\phi}(x)] d^4x, \quad (2.21)$$

that is, if the Lagrangian density changes as

$$\mathcal{L}[\underline{\phi}'(x')] = \left| \det \frac{\partial x}{\partial x'} \right| \mathcal{L}[\underline{\phi}(x)] \quad (2.22)$$

In principle we could insist on a weaker requirement—that the action change only by boundary terms—but, at least for standard spacetime symmetries, this subtlety happens to be relevant only in the case of conformal transformations and we will discuss it in due time. For the moment, we are going to ignore it.

Now, it so happens that, for an infinitesimal transformation, it is more convenient to adopt the so-called active viewpoint, whereby we transform the fields directly, evaluating all fields and derivatives at the same values of their arguments. In this point of view, the field transforms as

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) + \epsilon^a A_a^\mu(x) \partial_\mu \underline{\phi}(x) + \epsilon^a \tilde{\Delta}_a(\underline{\phi}, \partial \underline{\phi}, \dots). \quad (2.23)$$

We can now carry out the Noether's procedure and write down the most general formula for the Noether's current,

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \underline{\phi})} \cdot \left(A_a^\nu \partial_\nu \underline{\phi} - \tilde{\Delta}_a \right) - A_a^\mu \mathcal{L} . \quad (2.24)$$

Keeping these A_a^μ 's generic functions of spacetime is an over generalization for our purpose in this chapter. We keep equation (2.24) in its current form anyway in case it is useful for future follow-up studies. To study the case of spacetime coordinate translations, simply set $A_a^\mu \rightarrow A_a^\mu = \delta_a^\mu$ and $\tilde{\Delta}_a = 0$. One can easily check that this leads to the canonical stress-energy tensor (1.3).

In deriving Noether's current, as shown in section 2.1, we turn on a spacetime dependence for the ϵ^a . We will do the same here. Writing

$$x'^\mu(x) = x^\mu + \epsilon^\mu(x) \quad (2.25)$$

for the specific $\epsilon^\mu(x)$ that corresponds to the infinitesimal symmetry we want to consider, from (2.22) we get

$$\mathcal{L}[\underline{\phi}'(x)] + \epsilon^\mu \partial_\mu \mathcal{L} = \mathcal{L}[\underline{\phi}(x)] - \partial_\mu \epsilon^\mu \mathcal{L} , \quad (2.26)$$

where we kept terms up to first order in ϵ^μ . Thus, from the active viewpoint, the spacetime transformation under study is a symmetry of the action if the Lagrangian density changes by a *specific* total derivative term [34]:

$$\delta \mathcal{L} \equiv \mathcal{L}[\underline{\phi}'(x)] - \mathcal{L}[\underline{\phi}(x)] = -\partial_\mu (\epsilon^\mu \mathcal{L}) . \quad (2.27)$$

Notice that every term in the infinitesimal transformation $\delta \underline{\phi} = \underline{\phi}'(x) - \underline{\phi}(x)$ that is of the form $f^\mu(x) \partial_\mu \underline{\phi}$, where $f^\mu(x)$ is an infinitesimal function only of spacetime coordinate, can be interpreted as a part of some local spacetime coordinate transformation, taking the form of equation (2.25) in the passive point of view. We are now ready to run our improved translation Noether theorems.

2.4 Improved translation Noether theorem

2.4.1 Generalities

For simplicity, we focus our attention on Poincaré invariant field theories, with the field multiplet $\underline{\phi}$ transforming linearly under Lorentz symmetry, according to a generic representation $\mathcal{J}^{\mu\nu}$, not necessarily irreducible. Moreover, when we consider scale-invariant or conformal-invariant theories, we assume that the fields transform linearly under those as well. In other words, we assume that no spacetime symmetry is spontaneously broken. As we hope it will be clear shortly, these simplifying assumptions are not really needed for our strategy to work, and in principle, our analysis can be straightforwardly extended to non-linear realizations as well.

Likewise, for simplicity we consider Lagrangian densities that depend at most on the first derivatives of the fields,

$$\mathcal{L}[\phi] = \mathcal{L}(\phi, \partial_\mu \phi) . \quad (2.28)$$

In principle we could repeat our analysis with any number of higher derivatives. Even better, in this day and age, given the ongoing proliferation of effective field theories and derivative expansions, it would be nicer to find a more general functional approach that does not require specifying the maximum number of derivatives entering the Lagrangian. We leave this task for future work.

Under rigid translations, our fields transform as in (2.15). We want to run the associated Noether's theorem generalizing that transformation law to

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \epsilon^\mu(x) \partial_\mu \underline{\phi}(x) - \partial_\mu \epsilon_\nu(x) \underline{\Psi}^{\mu\nu}(x) , \quad (2.29)$$

for generic $\epsilon^\mu(x)$, and for a specific (field-dependent) $\underline{\Psi}^{\mu\nu}(x)$, which will change from case to case, depending on the additional symmetries we want to consider.

The variation of the Lagrangian density is

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \underline{\phi}} \cdot \delta \underline{\phi} + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \underline{\phi}} \cdot \partial_\alpha \delta \underline{\phi} , \quad (2.30)$$

which, after some straightforward algebra, can be written as

$$\delta\mathcal{L} = -\partial_\mu(\epsilon^\mu\mathcal{L}) - \partial_\mu\epsilon_\nu\mathcal{T}^{\mu\nu} - \partial_\rho\partial_\mu\epsilon_\nu\mathcal{S}^{\rho\mu\nu} \quad (2.31)$$

with

$$\mathcal{T}^{\mu\nu} = T_c^{\mu\nu} + \frac{\delta S}{\delta\phi} \cdot \underline{\Psi}^{\mu\nu} + \partial_\rho\mathcal{S}^{\rho\mu\nu} \quad (2.32)$$

$$\mathcal{S}^{\rho\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\rho\phi} \cdot \underline{\Psi}^{\mu\nu}, \quad (2.33)$$

where $T_c^{\mu\nu}$ is the canonical energy-momentum tensor as in equation (1.3), and the equations of motion $\frac{\delta S}{\delta\phi}$ are nothing but the Euler-Lagrange equations,

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right). \quad (2.34)$$

Also notice that the quantity $\mathcal{T}^{\mu\nu}$ differs from the canonical stress-energy tensor, $T_c^{\mu\nu}$, by two terms: one is proportional to the equations of motion, the other is a total derivative. As a result, at low-energies $\mathcal{T}^{\mu\nu}$ reduces to $T_c^{\mu\nu}$ on-shell.

According to the general logic of Noether's theorem as reviewed in section 2.1, for *generic* $\epsilon^\mu(x)$ the variation of the Lagrangian density must take the form

$$\delta\mathcal{L} = -\partial_\mu\epsilon_\nu T^{\mu\nu} + \text{total derivatives}, \quad (2.35)$$

and this can be taken as the definition of the stress-energy tensor. To rewrite (2.31) as (2.35), we must integrate by parts the third term. The obvious way to do this would be to write

$$-\partial_\rho\partial_\mu\epsilon_\nu\mathcal{S}^{\rho\mu\nu} = \partial_\mu\epsilon_\nu\partial_\rho\mathcal{S}^{\rho\mu\nu} + \text{total derivatives}, \quad (2.36)$$

but, in fact, we can be more general and keep in mind an ambiguity related to that of item 1 in section

2.1: since $\partial_\rho \partial_\mu \epsilon_\nu$ is symmetric in ρ and μ , we can add to whatever multiplies it any functional of the fields that is antisymmetric in ρ and μ . This will prove useful for what follows. We can thus write that, for our generalization of the translation Noether theorem, the stress-energy tensor is

$$T^{\mu\nu} = \mathcal{T}^{\mu\nu} - \partial_\rho \left(\mathcal{J}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu} \right), \quad \Sigma^{\rho\mu\nu} = -\Sigma^{\mu\rho\nu}, \quad (2.37)$$

where $\Sigma^{\rho\mu\nu}$ is a generic functional of the fields that is antisymmetric in ρ and μ .

Notice that, crucially, to arrive to (2.31) we did *not* drop total-derivative terms. So, we can use the result of the last section: for the choices of $\epsilon^\mu(x)$ and $\underline{\Psi}^{\mu\nu}(x)$ that make (2.29) a *symmetry* transformation, only the first term in (2.31) should survive. As we'll see, this will typically imply some algebraic property for $\mathcal{T}^{\mu\nu}$. One can then try to use the ambiguity associated with $\Sigma^{\rho\mu\nu}$ to extend that property to the full $T^{\mu\nu}$ ³. It may seem that at this point our procedure could use some guesswork, despite our bragging about the opposite. In practice, however, one parametrizes $\Sigma^{\rho\mu\nu}$ as the most general linear combination of the tensors at one's disposal with the right symmetries, and checks whether there is a choice of coefficients that achieves the desired result. Phrased in this way, this step is a linear algebra problem and, as advertised, there is no guesswork involved. We will see this explicitly at work in the three examples that follow.

2.4.2 Lorentz-friendly version

To begin with, consider a Poincaré-invariant field theory, with the fields $\underline{\phi}$ transforming according to some representation $\mathcal{J}^{\mu\nu}$ under Lorentz, as in eq. (2.17). So, if we choose the functionals $\underline{\Psi}^{\mu\nu}$ in (2.29) to be simply

$$\underline{\Psi}^{\mu\nu}(x) = \underline{\Psi}_L^{\mu\nu}(x) \equiv \frac{1}{2} \mathcal{J}^{\mu\nu} \cdot \underline{\phi}(x) \quad (2.38)$$

(‘ L ’ for ‘Lorentz’), we have that for $\epsilon^\mu(x) = \epsilon_L^\mu(x) \equiv \omega^\mu{}_\nu x^\nu$, with constant and antisymmetric $\omega_{\mu\nu}$, eq. (2.29) corresponds to a symmetry transformation, that is, only the first term in (2.31) should survive.

³For this reason, one should resist the temptation to cancel the last term in $\mathcal{T}^{\mu\nu}$ (see eq. (2.32)) against the first term inside the $\partial_\rho(\dots)$ in eq. (2.37) until all the symmetries have been exploited.

On the other hand, for these specific choices, eq. (2.31) reads

$$\delta_L \mathcal{L} = -\partial_\mu (\epsilon_L^\mu \mathcal{L}) - \omega_{\mu\nu} \mathcal{T}^{\mu\nu} . \quad (2.39)$$

This immediately implies that $\mathcal{T}^{\mu\nu}$ is symmetric, since $\omega_{\mu\nu}$ is the most general antisymmetric constant tensor:

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu} . \quad (2.40)$$

Keep in mind that $\mathcal{T}^{\mu\nu}$ is not conserved by itself, and has to be complemented by a correction $\Delta T^{\mu\nu}$ in order to restore conservation. The reason behind this is that (2.39) is obtained using a specific transformation, while in (2.35) we are using an arbitrary ϵ^μ . We should then ask whether we can choose Σ in (2.37) to make the rest of the stress-energy tensor,

$$\Delta T^{\mu\nu} \equiv -\partial_\rho \left(\mathcal{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu} \right) , \quad (2.41)$$

also symmetric. We parametrize Σ as the most general linear-in- \mathcal{S} combination of \mathcal{S} and η tensors with the correct anti-symmetry property. Taking into account that $\mathcal{S}^{\rho\mu\nu}$ is, in this case, antisymmetric in $\mu\nu$ (because of (2.33) and (2.38)), we write

$$\Sigma^{\rho\mu\nu} = \Sigma_L^{\rho\mu\nu} \equiv \alpha \mathcal{S}^{\nu\rho\mu} + \beta (\mathcal{S}^{\rho\mu\nu} - \mathcal{S}^{\mu\rho\nu}) + \gamma (\mathcal{S}_\sigma^{\sigma\rho} \eta^{\mu\nu} - \mathcal{S}_\sigma^{\sigma\mu} \eta^{\rho\nu}) , \quad (2.42)$$

with arbitrary α , β , and γ . The only choice for which eq. (2.41) is symmetric in μ and ν is $\alpha = -\beta = 1$ and $\gamma = 0$.

Thus, for this choice, putting everything together we arrive at our Lorentz-friendly version of the stress-energy tensor:

$$T_L^{\mu\nu} = T_c^{\mu\nu} + \frac{1}{2} \frac{\delta S}{\delta \underline{\phi}} \cdot \mathcal{S}^{\mu\nu} \cdot \underline{\phi} + \frac{1}{2} \partial_\rho \left[\frac{\partial \mathcal{L}}{\partial \partial_\rho \underline{\phi}} \cdot \mathcal{S}^{\mu\nu} \cdot \underline{\phi} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \underline{\phi}} \cdot \mathcal{S}^{\rho\nu} \cdot \underline{\phi} - \frac{\partial \mathcal{L}}{\partial \partial_\nu \underline{\phi}} \cdot \mathcal{S}^{\rho\mu} \cdot \underline{\phi} \right] . \quad (2.43)$$

As we proved, as a consequence of Lorentz-invariance, this is guaranteed to be symmetric, on- and

off-shell:

$$T_L^{\mu\nu} = T_L^{\nu\mu} \quad (\text{off-shell}) . \quad (2.44)$$

It differs from the standard Belinfante expression [5, 6], which in general is symmetric only on-shell, by the second term on the r.h.s. of (2.43), which is manifestly zero on the equations of motion.

2.4.3 Scale-friendly version

The case of scale-invariance proceeds along the same lines. Suppose that we have a scale-invariant theory in D spacetime dimensions, and let's call d the matrix of scaling dimensions of the fields. So, under a scale transformation $x^\mu \rightarrow (1 + \omega) x^\mu$, with infinitesimal, constant ω , the fields transform as

$$\underline{\phi}(x) \rightarrow \underline{\phi}(x) - \omega x^\mu \partial_\mu \underline{\phi}(x) - d \cdot \underline{\phi}(x), \quad (2.45)$$

In order to make use of this symmetry for our purposes, we can choose $\Psi^{\mu\nu}$ in (2.29) to be

$$\underline{\Psi}^{\mu\nu}(x) = \underline{\Psi}_S^{\mu\nu}(x) = \frac{1}{D} \eta^{\mu\nu} d \cdot \underline{\phi} \quad (2.46)$$

(‘ S ’ for ‘scale’), so that we have a symmetry when $\epsilon(x)$ takes the form appropriate for a scale transformation,

$$\epsilon^\mu(x) = \epsilon_S^\mu(x) \equiv \omega x^\mu , \quad (2.47)$$

with constant ω . Plugging this particular choice of $\epsilon(x)$ into (2.31), we get

$$\delta_S \mathcal{L} = -\partial_\mu (\epsilon_S^\mu \mathcal{L}) - \omega \mathcal{T}^\mu{}_\mu , \quad (2.48)$$

while, for scale-invariance to be a symmetry, only the first term should survive. So, scale-invariance guarantees that $\mathcal{T}^{\mu\nu}$ is traceless, off-shell:

$$\mathcal{T}^\mu{}_\mu = 0 . \quad (2.49)$$

Similarly to the case of Lorentz, we now have to ask whether we can choose $\Sigma^{\rho\mu\nu}$ in (2.37) in order to extend this property—tracelessness—to the rest of the stress-energy tensor, eq. (2.41). Since now $\mathcal{S}^{\rho\mu\nu}$ is proportional to $\eta^{\mu\nu}$ (see (2.33) and (2.46)), the most general linear-in- \mathcal{S} combination of \mathcal{S} and η tensors with the right antisymmetry property is simply

$$\Sigma^{\rho\mu\nu} = \Sigma_S^{\rho\mu\nu} \equiv \delta(\mathcal{S}^{\rho\mu\nu} - \mathcal{S}^{\mu\rho\nu}) , \quad (2.50)$$

with generic δ . Eq. (2.41) is traceless only for $\delta = -D/(D - 1)$. With this choice, putting everything together we arrive at our scale-friendly stress-energy tensor:

$$T_S^{\mu\nu} = T_c^{\mu\nu} + \frac{1}{D} \eta^{\mu\nu} \frac{\delta S}{\delta \underline{\phi}} \cdot d \cdot \underline{\phi} + \frac{1}{D-1} \partial_\rho \left(\eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial \partial_\rho \underline{\phi}} \cdot d \cdot \underline{\phi} - \eta^{\rho\nu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \underline{\phi}} \cdot d \cdot \underline{\phi} \right) . \quad (2.51)$$

As a consequence of scale invariance, this is guaranteed to be traceless, on- and off-shell,

$$T_{S\mu}^\mu = 0 \quad (\text{off-shell}) . \quad (2.52)$$

Before proceeding, there is a small puzzle that we need to address. According to standard results [7, 8, 9], scale-invariance is not enough to make the stress-energy tensor traceless. The best one can do, usually, is to make it traceless, on-shell, up to a total divergence:

$$T^\mu{}_\mu = -\partial_\mu V^\mu \quad (\text{standard result}) . \quad (2.53)$$

Here V^μ is a quantity known as the ‘virial current’, which we will encounter and explore further in the next section. Only if the theory enjoys full conformal symmetry can the stress-tensor be further improved to eliminate this total divergence. In our case, there is no sign of this: scale-invariance alone guarantees that eq. (2.51) is completely traceless, off-shell. How is this possible?

The resolution of the puzzle is that eq. (2.51) is not symmetric in general. The standard result (2.53) assumes that one is only considering symmetric stress-energy tensors, like the Belinfante

one [5, 6], which is symmetric on-shell, or our version (2.43), which is symmetric off-shell as well. What we just proved is that if one gives up this requirement, in a scale-invariant theory one can make the stress-energy tensor completely traceless, off-shell. In fact, with hindsight, this is obvious already from eq. (2.53): by adding to $T^{\mu\nu}$ the trivially conserved improvement term

$$\frac{1}{D-1}(\eta^{\mu\nu}\partial_\alpha V^\alpha - \partial^\nu V^\mu), \quad (2.54)$$

one can cancel the trace of $T^{\mu\nu}$ at the expense of giving up its being symmetric.

2.4.4 Conformal-friendly version

Before moving on to the case of full conformal invariance, it is instructive to first combine the two strategies that we adopted in the previous subsections. This will also shed some light on the tension between tracelessness and symmetry of the stress-energy tensor we just alluded to.

Consider then a Lorentz-invariant, scale-invariant theory. To make use of both symmetries, we can combine the $\underline{\Psi}^{\mu\nu}$'s in (2.38) and (2.46), and use for the translation Noether theorem the transformation rule (2.29) with

$$\underline{\Psi}^{\mu\nu}(x) = \underline{\Psi}_S^{\mu\nu}(x) + \underline{\Psi}_L^{\mu\nu}(x). \quad (2.55)$$

In this case, eq. (2.29) reduces to a translation for constant ϵ , to a Lorentz transformation for $\epsilon(x)$ as in (2.14), and to a scale transformation for $\epsilon(x)$ as in (2.47).

Precisely because of the same reasons as in the last two subsections—Lorentz invariance and scale invariance—the $\mathcal{T}^{\mu\nu}$ contribution to the stress-energy tensor is both symmetric *and* traceless, off-shell:

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu}, \quad \mathcal{T}^\mu{}_\mu = 0. \quad (2.56)$$

The question is what to do with the rest, eq. (2.41). We can first notice that, as far as the μ and ν indices are concerned, our $\mathcal{S}^{\rho\mu\nu}$ has the same algebraic properties as our $\underline{\Psi}^{\mu\nu}$ in (2.55): it is made

up of a scale part, which is pure trace, and a Lorentz one, which is antisymmetric:

$$\mathcal{I}^{\rho\mu\nu} = \mathcal{I}_S^{\rho\mu\nu} + \mathcal{I}_L^{\rho\mu\nu}, \quad \mathcal{I}_S^{\rho\mu\nu} \propto \eta^{\mu\nu}, \quad \mathcal{I}_L^{\rho\mu\nu} = -\mathcal{I}_L^{\rho\nu\mu}. \quad (2.57)$$

In turn, the most generic $\Sigma^{\rho\mu\nu}$ we can add is simply the combination of (2.42) and (2.50):

$$\begin{aligned} \Sigma^{\rho\mu\nu} &= \Sigma_L^{\rho\mu\nu} + \Sigma_S^{\rho\mu\nu} \\ &= \alpha \mathcal{I}_L^{\nu\rho\mu} + \beta (\mathcal{I}_L^{\rho\mu\nu} - \mathcal{I}_L^{\mu\rho\nu}) + \gamma (\mathcal{I}_L \sigma^{\sigma\rho} \eta^{\mu\nu} - \mathcal{I}_L \sigma^{\sigma\mu} \eta^{\rho\nu}) + \delta (\mathcal{I}_S^{\rho\mu\nu} - \mathcal{I}_S^{\mu\rho\nu}). \end{aligned} \quad (2.58)$$

Symmetry of $\Delta T^{\mu\nu}$ requires $\alpha = -\beta = 1$, $\gamma = \delta = 0$ in the equation above, while tracelessness requires $\alpha - \beta - \gamma(D - 1) = 0$, $\delta = -D/(D - 1)$. The two solutions of this linear algebra problem are inconsistent, meaning one can make $\Delta T^{\mu\nu}$ symmetric, *or* traceless, but not both. In either case, $\Delta T^{\mu\nu}$ contributes a total derivative to the stress-energy tensor. And so, in particular, if one decides to make it symmetric, its trace will be a total divergence, in agreement with the standard result (2.53).

Now, what happens in the case of full conformal invariance? It so happens that infinitesimal special conformal transformations are precisely a spacetime-modulated specific combination of Lorentz- and scale-transformations of the form (2.55), when $\epsilon(x)$ in (2.29) is taken to be

$$\epsilon^\mu(x) = \epsilon_C^\mu(x) \equiv b^\mu x^2 - 2b \cdot x x^\mu \quad (2.59)$$

(‘ C ’ for ‘conformal’), with constant b^μ . The derivatives of such an $\epsilon(x)$ are in fact a combination of a trace part and an antisymmetric one,

$$\partial_\mu \epsilon_{C\nu} = 2(b_\nu x_\mu - b_\mu x_\nu) - 2b \cdot x \eta_{\mu\nu}, \quad (2.60)$$

as befits a (spacetime-dependent) combination of Lorentz- and scale-transformations. So, if we use this $\epsilon^\mu(x)$ in (2.31), the $\mathcal{I}^{\mu\nu}$ term vanishes because of Lorentz- and scale-invariance, and we

are left with

$$\delta_C \mathcal{L} = -\partial_\mu (\epsilon^\mu \mathcal{L}) + 2b_\mu (\mathcal{S}^{\mu\alpha}{}_\alpha + 2\mathcal{S}_\alpha^{[\mu\alpha]}) . \quad (2.61)$$

Reasoning as before, we would be tempted to say that, if conformal transformations are a symmetry, only the first term should survive. But this is where the subtlety we briefly alluded to in section 2.3 becomes relevant, and so we must finally address it.

In most common cases, even under a *passive* transformation, the action of a (classically) conformally invariant theory is not strictly invariant under conformal transformations, but changes by a boundary term. This is not necessarily related to the existence of Wess-Zumino terms, like for instance that studied in [35]. Rather, it usually happens because one is not really using the most symmetric version of the action. To make this very explicit, consider a free massless real scalar field $\Phi(x)$ in four spacetime dimensions:

$$S[\phi] = - \int d^4x \frac{1}{2} (\partial\Phi)^2 . \quad (2.62)$$

The passive version of a special conformal transformation is

$$x^\mu \rightarrow x'^\mu = x^\mu + b^\mu x^2 - 2b \cdot x x^\mu , \quad \Phi(x) \rightarrow \Phi'(x') = \Phi(x) - 2(b \cdot x)\Phi(x) , \quad (2.63)$$

and it is easy to check that the action above changes by a boundary term:

$$d^4x' \frac{1}{2} (\partial'\Phi')^2 = d^4x \left(\frac{1}{2} (\partial\Phi)^2 - b^\mu \partial_\mu (\Phi^2) \right) . \quad (2.64)$$

However, if one instead starts from the equivalent action

$$\tilde{S}[\phi] = \int d^4x \frac{1}{2} \Phi \square \Phi , \quad (2.65)$$

that boundary term is gone:

$$d^4x' \frac{1}{2} \Phi' \square' \Phi' = d^4x \frac{1}{2} \Phi \square \Phi . \quad (2.66)$$

For simplicity, whenever possible, we tend to rewrite actions in a way that they only involve up to first derivatives of the fields, and, in fact, we have assumed just that in all of our derivations above. So, if we insist on this assumption, in general for conformal transformations we have to allow that the action change by a total derivative *beyond* that of eq. (2.27). This means that, for a conformally invariant theory, the last term in (2.61) must be a total derivative:

$$V^\mu \equiv \mathcal{S}^{\mu\alpha}{}_\alpha + 2 \mathcal{S}_\alpha^{[\mu\alpha]} = \partial_\alpha \sigma^{\alpha\mu} , \quad (2.67)$$

for some local functional $\sigma^{\alpha\mu}$. V^μ is traditionally called the ‘virial current’ [8, 9].

We can now go back to the form of the stress-energy tensor, eq. (2.37). We already saw that its $\mathcal{T}^{\mu\nu}$ part is symmetric and traceless, off-shell. As to the rest, eq. (2.41), we saw in (2.57) that $\mathcal{S}^{\rho\mu\nu}$ is made up of the equivalent scale and Lorentz parts. The virial current (2.67) relates these two parts. And so, for example using (2.67) we can eliminate the scale part,

$$\mathcal{S}^{\rho\mu\nu} = \frac{1}{D} [\eta^{\mu\nu} \partial_\alpha \sigma^{\alpha\rho} + D \mathcal{S}_L^{\rho\mu\nu} - 2 \mathcal{S}_L \alpha^{\rho\alpha} \eta^{\mu\nu}] , \quad (2.68)$$

and rewrite (2.41) as⁴

$$\begin{aligned} \Delta T^{\mu\nu} = & -\partial_\rho \left(\mathcal{S}_L^{\rho\mu\nu} - \frac{2}{D} \mathcal{S}_L \alpha^{\rho\alpha} \eta^{\mu\nu} + \Sigma^{\rho\mu\nu} \right) \\ & - \frac{1}{D} \eta^{\mu\nu} \partial_\alpha \partial_\rho \sigma^{(\alpha\rho)} . \end{aligned} \quad (2.69)$$

This expression, together with the tracelessness and symmetry of $\mathcal{T}^{\mu\nu}$, encodes conformal invariance at the level of the stress-energy tensor. There is no reference anymore to the transformation properties of the fields under scale transformations because, for conformal invariant theories, those are related to the fields’ Lorentz transformation properties through the virial current (2.67).

The question now is whether one can choose an (antisymmetric in ρ and μ) $\Sigma^{\rho\mu\nu}$ in such a way as to make $\Delta T^{\mu\nu}$ also traceless and symmetric. For vanishing $\sigma^{\mu\nu}$, the answer is simply the same

⁴Notice that only the symmetric part of $\sigma^{\mu\nu}$ enters the stress-energy tensor.

as in the Lorentz-friendly case—see section 2.4.2:

$$\Sigma^{\rho\mu\nu} = \Sigma_L^{\rho\mu\nu} \equiv \mathcal{S}_L^{\nu\rho\mu} - (\mathcal{S}_L^{\rho\mu\nu} - \mathcal{S}_L^{\mu\rho\nu}) . \quad (2.70)$$

The reason is that, as we know from that section, this choice makes (2.41) symmetric, and the extra terms in (2.69) are already symmetric. Moreover, as to the trace, we have

$$\Sigma_L^{\rho\mu}{}_{\mu} = 2 \mathcal{S}_L^{\rho\mu}{}^{\rho\mu} , \quad (2.71)$$

which makes the trace of the first line in (2.69) vanish.

For nonvanishing $\sigma^{\mu\nu}$, we can supplement $\Sigma_L^{\rho\mu\nu}$ with total derivative terms, which, in order not to spoil the $\mu\nu$ symmetry just obtained, and recalling that $\Sigma^{\rho\mu\nu}$ must be antisymmetric in ρ and μ and that it is acted on by a ∂_ρ in (2.69), should take the form

$$\Sigma^{\rho\mu\nu} = \Sigma_L^{\rho\mu\nu} + \partial_\alpha \Xi^{[\rho\mu][\alpha\nu]} , \quad (2.72)$$

where Ξ should be symmetric under the $\rho\mu \leftrightarrow \alpha\nu$ pair-exchange, while we are displaying the needed antisymmetries explicitly. The trace of (2.69) then is

$$\Delta T^\mu{}_\mu = -\partial_\alpha \partial_\rho (\eta_{\mu\nu} \Xi^{[\rho\mu][\alpha\nu]} + \sigma^{(\alpha\rho)}) . \quad (2.73)$$

Following the same logic as before, we parametrize Ξ as the most general tensor with the right symmetries and constructed out of σ and η tensors:

$$\begin{aligned} \Xi^{[\rho\mu][\alpha\nu]} = & A(\eta^{\mu\nu}\sigma^{(\alpha\rho)} + \eta^{\alpha\rho}\sigma^{(\mu\nu)} - \eta^{\mu\alpha}\sigma^{(\nu\rho)} - \eta^{\nu\rho}\sigma^{(\mu\alpha)}) \\ & + B(\eta^{\alpha\rho}\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\rho})\sigma^\beta{}_\beta . \end{aligned} \quad (2.74)$$

Demanding that eq. (2.73) vanish, we get $A = -\frac{1}{D-2}$ and $B = \frac{1}{(D-1)(D-2)}$.

Putting everything together, we find that the conformal-friendly version of the stress-energy

tensor is:

$$\begin{aligned}
T_C^{\mu\nu} = & T_c^{\mu\nu} + \frac{\delta S}{\delta \underline{\phi}} \cdot \left(\frac{1}{D} \eta^{\mu\nu} d + \frac{1}{2} \mathcal{J}^{\mu\nu} \right) \cdot \underline{\phi} + \frac{1}{2} \partial_\rho \left[\frac{\partial \mathcal{L}}{\partial (\partial_\rho \underline{\phi})} \mathcal{J}^{\mu\nu} \underline{\phi} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \underline{\phi})} \mathcal{J}^{\rho\nu} \underline{\phi} - \frac{\partial \mathcal{L}}{\partial (\partial_\nu \underline{\phi})} \mathcal{J}^{\rho\mu} \underline{\phi} \right] \\
& + \left[\frac{1}{D-2} \eta^{\mu\nu} \partial_\rho \partial_\sigma \sigma^{(\rho\sigma)} - \frac{1}{D-2} (\partial^\nu \partial_\rho \sigma^{(\rho\mu)} + \partial^\mu \partial_\rho \sigma^{(\rho\nu)}) + \frac{1}{D-2} \square \sigma^{(\mu\nu)} \right. \\
& \left. + \frac{1}{(D-1)(D-2)} \partial^\mu \partial^\nu \sigma^\rho{}_\rho - \frac{1}{(D-1)(D-2)} \eta^{\mu\nu} \square \sigma^\rho{}_\rho \right], \tag{2.75}
\end{aligned}$$

where $\sigma^{\mu\nu}$ is related to the virial current by (2.67), $V^\mu = \partial_\alpha \sigma^{\alpha\mu}$. This stress-energy tensor is guaranteed to be symmetric *and* traceless, off-shell:

$$T_C^{\mu\nu} = T_C^{\nu\mu}, \quad T_C^\mu{}_\mu = 0 \quad (\text{off-shell}). \tag{2.76}$$

Before we conclude, notice that $(D-2)$ is showing up in denominators in many terms. Indeed, our procedure does not work for $D=2$, and this special case needs to be dealt with separately. It is also important that the virial current be a total derivative, as per eq. (2.67). The reason is that in (2.75) there are combinations like $\square \sigma^{(\mu\nu)}$ and $\sigma^\rho{}_\rho$. These cannot be written directly in terms of the virial current V^μ —rather, one needs to extract the σ tensor from $V^\mu = \partial_\alpha \sigma^{\alpha\mu}$.

2.5 The other currents, for free

We have seen how modifying the translation Noether theorem along the lines of section (2.4.1) can make the derivation of “improved” stress-energy tensors systematic. The reason the strategy works is that it makes explicit use of the fact that Lorentz, scale, and conformal transformations can be thought of as suitably modulated translations. They are still defined by certain *constant* parameters, respectively $\omega_{\mu\nu}$, ω , and b_μ . Now, suppose we wanted to run the Noether’s theorem not for translations, but for those extra spacetime symmetries. These parameters would have to be modulated in x in an arbitrary way to find the corresponding currents. But the advantage of how we organized our Lorentz-friendly, scale-friendly, and conformal-friendly *translation* Noether’s theorems is that these can be used directly also as Noether’s theorems for, respectively, Lorentz,

scale, and conformal transformations ⁵.

Concretely, adopting the general transformation law (2.29) for completely generic $\epsilon^\mu(x)$, the Lagrangian changes as in (2.35). This can be taken as the definition of the stress-energy tensor associated with this particular implementation of the Noether's theorem. If we are running the Lorentz-friendly version of the theorem (section 2.4.2), and we perform an x -modulated *Lorentz* transformation,

$$\epsilon^\mu(x) = \omega^\mu{}_\nu(x)x^\nu, \quad \omega_{\mu\nu}(x) = -\omega_{\nu\mu}(x), \quad (2.77)$$

the variation of the Lagrangian density reduces to

$$\delta\mathcal{L} = -\partial_\mu(\omega_{\nu\alpha}x^\alpha)T_L^{\mu\nu} + \text{total derivatives} \quad (2.78)$$

$$= -\partial_\mu\omega_{\nu\alpha}x^\alpha T_L^{\mu\nu} + \text{total derivatives}, \quad (2.79)$$

where we used that, as a consequence of Lorentz invariance, $T_L^{\mu\nu}$ is symmetric, off-shell. Now, by definition, whatever multiplies the derivatives of the $\omega_{\mu\nu}$ parameters in $\delta\mathcal{L}$ is the Noether current $M^{\mu\nu\alpha}$ associated with *Lorentz* transformations,

$$\delta\mathcal{L} = -\partial_\mu\omega_{\nu\alpha}M^{\mu\nu\alpha} + \text{total derivatives}. \quad (2.80)$$

Taking into account that the ω 's are antisymmetric, we thus have

$$M^{\mu\nu\alpha} = \frac{1}{2}(x^\alpha T_L^{\mu\nu} - x^\nu T_L^{\mu\alpha}), \quad \partial_\mu M^{\mu\nu\alpha} = 0 \quad (\text{on-shell}), \quad (2.81)$$

in agreement with the standard result [5, 6].

Likewise, if we are running the scale-friendly version of the theorem (section (2.4.3)), and we perform an x -modulated *scale* transformation,

$$\epsilon^\mu(x) = \omega(x)x^\mu, \quad (2.82)$$

⁵Similar ideas were exploited in [36] in the case of Galilean invariance.

the variation of the Lagrangian reduces to

$$\delta\mathcal{L} = -\partial_\mu(\omega x_\nu) T_S^{\mu\nu} + \text{total derivatives} \quad (2.83)$$

$$= -\partial_\mu\omega x_\nu T_S^{\mu\nu} + \text{total derivatives} , \quad (2.84)$$

where we used that, as a consequence of scale invariance, $T_S^{\mu\nu}$ is traceless, off-shell. By definition, whatever multiplies the derivatives of $\omega(x)$ is the Noether current S^μ associated with scale transformations,

$$\delta\mathcal{L} = -\partial_\mu\omega S^\mu + \text{total derivatives} . \quad (2.85)$$

We thus get

$$S^\mu = x_\nu T_S^{\mu\nu} , \quad \partial_\mu S^\mu = 0 \quad (\text{on-shell}) . \quad (2.86)$$

Related to our comments at the end of section 2.4.3, notice that this differs from the standard expression of the scale current in a non-conformal theory,

$$S^\mu = x_\nu T^{\mu\nu} + V^\mu \quad (\text{standard result}) . \quad (2.87)$$

The V^μ appearing here is precisely the same as in eq. (2.53). In fact, using the conservation of S^μ and of $T^{\mu\nu}$, from (2.87) one derives (2.53), which shows that eq. (2.53) is only valid on-shell. Again, the difference between our result (2.86) and the standard one (2.87) stems from our using a traceless but, in general, non-symmetric stress-energy tensor.

Finally, consider the conformal-friendly case (section 2.4.4). If we choose $\epsilon^\mu(x)$ to be an x -modulated special conformal transformation,

$$\epsilon^\mu(x) = b^\mu(x) x^2 - 2b(x) \cdot x x^\mu , \quad (2.88)$$

the Lagrangian changes by

$$\delta \mathcal{L} = -\partial_\mu (b_\nu x^2 - 2b \cdot x x_\nu) T_C^{\mu\nu} + \text{total derivatives} \quad (2.89)$$

$$= -\partial_\mu b_\nu (x^2 \delta_\alpha^\nu - 2x^\nu x_\alpha) T_C^{\mu\alpha} + \text{total derivatives} , \quad (2.90)$$

where we used that $T_C^{\mu\nu}$ is symmetric and traceless, off-shell. Following the same logic as above, we see that the Noether current associated with special conformal transformations is

$$K^{\mu\nu} = (x^2 \delta_\alpha^\nu - 2x^\nu x_\alpha) T_C^{\mu\alpha} , \quad \partial_\mu K^{\mu\nu} = 0 \quad (\text{on-shell}) , \quad (2.91)$$

in agreement with the standard results [7, 8, 9].

2.6 Examples

Let us work through a few explicit examples of the different improvements we have presented for the translation Noether theorem:

- **Symmetric $T^{\mu\nu}$ from Lorentz invariance**

We can start by taking a look at the non-trivial case of a Dirac spinor $\psi(x)$, with Lagrangian

$$\mathcal{L} = -\bar{\psi}(i\not{\partial} + m)\psi . \quad (2.92)$$

The canonical stress-energy tensor is as usual not symmetric,

$$T_c^{\mu\nu} = -i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L} . \quad (2.93)$$

Applying our formula (2.43) from the Lorentz-friendly procedure, we get:

$$T_L^{\mu\nu} = -i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L} - \frac{i}{2}\bar{\psi}[\mathcal{J}^{\mu\nu}, \gamma^\rho]\partial_\rho\psi + \frac{i}{2}\partial_\rho(\bar{\psi}\gamma^\mu\mathcal{J}^{\rho\nu}\psi + \bar{\psi}\gamma^\nu\mathcal{J}^{\rho\mu}\psi) , \quad (2.94)$$

with the Lorentz generators given by

$$\mathcal{J}^{\mu\nu} = -\frac{1}{4} [\gamma^\mu, \gamma^\nu] . \quad (2.95)$$

The commutator $[\mathcal{J}^{\mu\nu}, \gamma^\rho] = (-\gamma^\mu \eta^{\nu\rho} + \gamma^\nu \eta^{\rho\mu})$ fixes the non-symmetric part coming from the first term in $T_L^{\mu\nu}$, while the rest of the expression is already symmetric. With a bit of γ -matrix algebra, the final expression becomes

$$T_L^{\mu\nu} = -\frac{i}{2} \bar{\psi} \gamma^{(\mu} \partial^{\nu)} \psi + \frac{i}{2} \partial^{(\mu} \bar{\psi} \gamma^{\nu)} \psi - \frac{i}{2} \eta^{\mu\nu} \partial_\rho (\bar{\psi} \gamma^\rho \psi) - \eta^{\mu\nu} \mathcal{L}, \quad (2.96)$$

which is manifestly symmetric, off-shell, as promised.

- **Traceless $T^{\mu\nu}$ from scale invariance**

For our scale-invariance example we may look at scalar theories of the form

$$\mathcal{L} = \phi^4 f \left(\frac{(\partial\phi)^2}{\phi^4} \right), \quad (2.97)$$

which are scale-invariant for any function f in $D = 4$. The canonical stress-energy tensor for such theories is

$$T_c^{\mu\nu} = 2f' \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} f \phi^4, \quad (2.98)$$

where the prime means $f' = \partial_X f$, $X \equiv \frac{(\partial\phi)^2}{\phi^4}$. Notice that $T_c^{\mu\nu}$ happens to be symmetric, since we are dealing with a scalar field, but is not traceless in general.

Our prescription for a traceless stress-energy tensor (2.51) (with $d \rightarrow 1$) gives the following improved expression for such theories:

$$T_S^{\mu\nu} = \frac{4}{3} f' \partial^\mu \phi \partial^\nu \phi - \frac{1}{3} \eta^{\mu\nu} f' (\partial\phi)^2 + \frac{1}{6} \eta^{\mu\nu} \phi \partial_\rho (f' \partial^\rho \phi) - \frac{2}{3} \phi \partial^\nu (f' \partial^\mu \phi), \quad (2.99)$$

for which one can readily check that the trace vanishes off-shell.

However, $T_S^{\mu\nu}$ is not symmetric in general: the first three terms are manifestly symmetric, but the last one is symmetric only if f' is a constant, that is, only if

$$f(X) = \text{const} + \text{const} \times X . \quad (2.100)$$

It is clear from (2.97) that this choice corresponds, in $D = 4$, to a conformally invariant theory.

- **Traceless, symmetric $T^{\mu\nu}$ from conformal invariance**

Up to changing the normalization of ϕ , the conformally invariant theory mentioned above is

$$\mathcal{L} = -\frac{1}{2} (\partial\phi)^2 - \lambda\phi^4 . \quad (2.101)$$

For such a simple theory, the virial term (2.67) takes the form $V^\mu = \phi \partial^\mu \phi$ and hence $\sigma^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu} \phi^2$. Following (2.75), and given that $\mathcal{J}^{\mu\nu} = 0$ for scalar fields, the improved stress-energy tensor in $D = 4$ becomes

$$T_C^{\mu\nu} = -\frac{2}{3} \partial^\mu \phi \partial^\nu \phi + \frac{1}{6} \eta^{\mu\nu} (\partial\phi)^2 - \frac{1}{12} \eta^{\mu\nu} \phi \square\phi + \frac{1}{3} \phi \partial^\mu \partial^\nu \phi , \quad (2.102)$$

which is manifestly symmetric and traceless, off-shell. In fact, recalling that compared to our previous example now we have $f' = \frac{1}{2}$, we see our scale-friendly $T_S^{\mu\nu}$ in (2.99) reduces precisely to our conformal-friendly $T_C^{\mu\nu}$ above.

Notice that, compared to the more standard improved stress-energy tensor associated with (2.101),

$$\begin{aligned} T^{\mu\nu} &= -\partial^\mu \phi \partial^\nu \phi + \frac{1}{2} \eta^{\mu\nu} (\partial\phi)^2 + \lambda \eta^{\mu\nu} \phi^4 + \frac{1}{6} (\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) \phi^2 && \text{(standard result)} \\ &= -\frac{2}{3} \partial^\mu \phi \partial^\nu \phi + \frac{1}{6} \eta^{\mu\nu} (\partial\phi)^2 + \lambda \eta^{\mu\nu} \phi^4 - \frac{1}{3} \eta^{\mu\nu} \phi \square\phi + \frac{1}{3} \phi \partial^\mu \partial^\nu \phi , && (2.103) \end{aligned}$$

ours has a different coefficient for the $\phi \square\phi$ term and, perhaps more surprisingly, has no sign

of the potential. In particular, our $T_C^{\mu\nu}$ does not depend on λ . The reason is that, as we tried to emphasize, our expressions for improved stress-energy tensors differ from the more standard ones by terms proportional to the equations of motion. For solutions of the equations of motion, the value of potential can be related to that of $\phi \square \phi$:

$$\lambda \phi^4 = \phi \cdot \lambda \phi^3 = \frac{1}{4} \phi \square \phi \quad (\text{on-shell}). \quad (2.104)$$

Using this on-shell relationship, the two expressions (2.102) and (2.103) coincide.

- **Electromagnetism**

Finally, we may also look at the electromagnetic field's Lagrangian, which is Lorentz-, scale-, and, in $D = 4$, conformal-invariant. We can then apply and compare all of the three different prescriptions. Consider then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (2.105)$$

The canonical stress-energy tensor is

$$T_c^{\mu\nu} = \partial^\nu A_\lambda F^{\lambda\mu} + \frac{\eta^{\mu\nu}}{4} F^2, \quad (2.106)$$

which is neither symmetric nor traceless.

The Lorentz generators for spin-1 fields are given by $(\mathcal{J}^{\mu\nu})^\rho{}_\sigma = -\eta^{\nu\rho} \delta_\sigma^\mu + \eta^{\mu\rho} \delta_\sigma^\nu$. Our formula for Lorentz-invariance yields the manifestly symmetric stress-energy tensor

$$T_L^{\mu\nu} = \frac{\eta^{\mu\nu}}{4} F^2 + A_\lambda \partial^{(\mu} F^{\nu)\lambda} + \partial_\lambda (A^{(\mu} F^{\nu)\lambda}). \quad (2.107)$$

Our scale-friendly prescription (2.51) (with $d \rightarrow \frac{D-2}{2}$) instead gives

$$T_S^{\mu\nu} = \partial^\nu A_\lambda F^{\lambda\mu} + \frac{\eta^{\mu\nu}}{4} F^2 - \frac{D-2}{2(D-1)} \partial^\nu (A_\lambda F^{\lambda\mu}) + \frac{D-2}{2D(D-1)} \eta^{\mu\nu} (A_\lambda \partial_\rho F^{\lambda\rho} + D \partial_\rho A_\lambda F^{\lambda\rho}). \quad (2.108)$$

Using $\eta^\mu{}_\mu = D$ and $\partial_\mu A_\lambda F^{\lambda\mu} = -F^2/2$, we see that the trace vanishes, in generic D ,

$$T_{S\mu}^\mu = 0 , \quad (2.109)$$

without the use of equations of motion. However, notice that this stress-energy tensor is no longer symmetric.

As well known, in $D = 4$ the theory is also conformally invariant. In fact, the virial term vanishes, and the resulting conformal-friendly stress-energy tensor is

$$T_C^{\mu\nu} = \partial_\rho A^\mu \partial^\rho A^\nu - \partial^\mu A^\rho \partial^\nu A_\rho - A^{(\mu} \partial_\rho F^{\nu)\rho} + \frac{\eta^{\mu\nu}}{4} A_\lambda \partial_\rho F^{\lambda\rho} , \quad (2.110)$$

which is manifestly both symmetric and traceless, without using the equations of motion.

Notice that *none* of these stress-energy tensors for the electromagnetic field is gauge invariant. This is a common issue, and it is usually fixed, on-shell, by adding ad hoc further improvement terms. For a more constructive approach, somewhat similar to ours, see instead [37] and references therein.

2.7 Summary and discussion

Despite the dryness and length of our algebra, our strategy and findings are easy to summarize:

- There are ambiguities in the standard formulation of Noether's theorem. One of these is related to a modification of how the fields are *chosen* to transform in the case of a spacetime-modulated symmetry transformation. Since spacetime symmetries beyond translations can be thought of as suitably modulated translations, such an ambiguity can be used to one's advantage, constructively, to derive directly from the translation Noether theorem the algebraic properties of $T^{\mu\nu}$ associated with said additional spacetime symmetries.
- We formulated this strategy in general, and applied it to the cases of Lorentz invariance, scale invariance, and conformal invariance. We reobtained the standard results, albeit with

some modifications: first, our stress-energy tensors have the standard algebraic properties (symmetry and/or tracelessness) *off-shell*; second in the case of combined Lorentz and scale invariance, we noted a tension between tracelessness and symmetry of the stress-energy tensor. The standard choice corresponds to making the stress-energy tensor symmetric. But we showed that there is an equally valid choice in which the stress-energy tensor is traceless, off-shell, but in general non-symmetric.

- Since the additional spacetime symmetries are incorporated into the structure of the translation Noether theorem, this serves as Noether's theorem for those additional symmetries as well, yielding directly their associated currents in terms of the stress-energy tensor.

Our unified framework shows that the standard improvement terms that make the stress-energy tensor symmetric in the case of Lorentz invariance, and traceless in the case of scale and conformal invariance have the same origin: they are a direct consequence of the fact that all those additional spacetime symmetries are suitably modulated translations.

We have already mentioned two possible extensions of our analysis at the beginning of section 2.4.1: the case of non-linearly realized spacetime symmetries, and the case of Lagrangians with higher-than-first derivatives of the fields. Another possible extension would be to push the starting point of our improved Noether's theorem, eq. (2.29), to higher orders in derivatives of $e^a(x)$. We see no obvious use for this at the moment, but maybe there is one, perhaps related to the question of non-linear realizations alluded to above. Also, as stated in our last sample, electromagnetism, our method does not automatically guarantee gauge-invariant results, and it could be helpful to find an extension to our method to incorporate that. Finally, we wonder whether the viewpoint we have put forward here can prove useful for the ongoing conversation on scale vs. conformal invariance (see for instance the recent [38] and references therein.)

Chapter 3: Coset construction for particles of arbitrary spin

In section 1.2 we briefly discussed the importance of spontaneous symmetry breaking, and the role the coset construction plays in constructing low-energy effective field theories of systems with the corresponding Goldstone modes. In this Chapter we will show the details of our construction of a coset formalism for relativistic particles of arbitrary spin. We begin with a review of the standard coset construction when Poincaré symmetry is spontaneously broken. Next, we outline a new philosophy for the coset construction and comment on the advantages of this novel perspective. As a warm-up we construct the well-known EFTs for massive spin-0 point-particles. It turns out that to construct an effective action for a spin s particle, we must impose an $\mathcal{N} = 2s$ super-reparameterization gauge symmetry on the particle worldline. A brief review of the superspace formalism can be found in Appendix A. We will then impose a gauged $\mathcal{N} = 1$ supersymmetry (SUSY) at the level of the coset construction and use it to formulate actions for spin-1/2 particles. After that, we go to an extended superspace to allow for $\mathcal{N} = 2s$ supercharges and formulate actions for arbitrary-spin particles using the method of cosets. It turns out that for $\mathcal{N} > 1$, the situation becomes complicated and we must supplement the superspace formalism with a multiplet calculus similar to that presented in [39]. To demonstrate that these supersymmetric actions do in fact describe spinning point-particles, we quantize these actions and demonstrate that they have the correct state-spectrum. We find that for massless particles, a new kind of inverse Higgs (IH) constraint, which bears some resemblance to the dynamical IH constraints of [18, 19] are needed to remove Lorentz Goldstones. To further investigate the nature of these novel IH constraints we construct two distinct actions for massless spin-0 particles. Finally, we conclude by commenting on how our new perspective on the coset construction can open up exciting possibilities for future research.

3.1 The coset construction: a review

To keep it concise, we only discuss the key steps in the coset construction. For detailed introductions, see [12, 14] and the references therein. Consider a relativistic quantum field theory with global symmetry group \mathcal{G} that is spontaneously broken to the subgroup \mathcal{H} . We include both internal and spacetime symmetries in \mathcal{G} and \mathcal{H} . Let the symmetry generators be

$$\begin{aligned}
 \bar{P}_{\bar{\mu}} &= \text{unbroken translations,} \\
 T_A &= \text{other unbroken generators,} \\
 P_{\mu'} &= \text{broken translations,} \\
 \tau_\alpha &= \text{other broken generators,}
 \end{aligned}
 \tag{3.1}$$

where $\bar{\mu}$ and μ' run over complementary subsets of $0, \dots, D-1$, where D is the dimension of spacetime. Supposing that $\bar{\mu} = 0, \dots, p$, the above SSB pattern describes a D_p -brane running parallel to translations generated by $P_{\bar{\mu}}$. We permit τ_α and T_A to be some combinations of internal and spacetime generators. Importantly, $\bar{P}_{\bar{\mu}}$ need not be a subset of the spacetime translation generators, P_μ , of the Poincaré group. We insist on the existence of unbroken $\bar{P}_{\bar{\mu}}$ so that states may be classified by some notion of energy and momentum parallel to the brane. For example, solids spontaneously break spatial translations, but phonons can be classified according to lattice momentum, which is conserved (in the vanishing-Umklapp scattering limit) [20, 40, 41].

Although $\bar{P}_{\bar{\mu}}$ and T_A are both unbroken generators, they will play very different roles in the coset construction. As a result, it is helpful to define the subgroup $\mathcal{H}_0 \subset \mathcal{H}$ that is generated exclusively by T_A .

To construct an effective action of only Goldstones, we parameterize the coset $\mathcal{G}/\mathcal{H}_0$ of non-linearly realized symmetries by¹

$$\gamma = e^{ix^{\bar{\mu}}\bar{P}_{\bar{\mu}}} e^{iX^{\mu'}(x)P_{\mu'}} e^{i\pi^\alpha(x)\tau_\alpha}.
 \tag{3.2}$$

¹Although $\bar{P}_{\bar{\mu}}$ are unbroken and hence linearly realized on the fields, they generate shifts (i.e. are non-linearly realized) when acting on the coordinates. As a result, we include these generators in the coset $\mathcal{G}/\mathcal{H}_0$.

The coordinates are $x^{\bar{\mu}}$ and the Goldstone fields are $X^{\mu'}$ and π^α , up to normalization. The symmetry transformations act on γ by left-multiplication. That is for constant $U \in \mathcal{G}$, we have

$$\gamma \rightarrow U \cdot \gamma, \quad (3.3)$$

which can be used to determine the transformation rules for the Goldstones and coordinates [12]. In the world of mathematics, it is a well-known fact that the Maurer-Cartan form, defined by $\gamma^{-1}d\gamma$, is a Lie algebra-valued one-form. As a result, we may express it as a linear combination of the symmetry generators

$$\gamma^{-1}\partial_{\bar{\mu}}\gamma = iE_{\bar{\mu}}^{\bar{\nu}} \left(\bar{P}_{\bar{\nu}} + \nabla_{\bar{\nu}}X^{\mu'}P_{\mu'} + \nabla_{\bar{\nu}}\pi^\alpha\tau_\alpha + \mathcal{B}_{\bar{\nu}}^A T_A \right). \quad (3.4)$$

Under a global symmetry transformation, it can be checked that $E_{\bar{\mu}}^{\bar{\nu}}$ transforms as a vielbein², $\nabla_{\bar{\nu}}\pi^\alpha$ and $\nabla_{\bar{\nu}}X^{\mu'}$ transform covariantly, and $\mathcal{B}_{\bar{\mu}}^A$ transforms as a connection under \mathcal{H}_0 . As a result, we refer to $\nabla_{\bar{\nu}}\pi^\alpha$ ($\nabla_{\bar{\nu}}X^{\mu'}$) as the covariant derivative of π^α ($X^{\mu'}$) and to take further covariant derivatives, we may use

$$\nabla_{\bar{\mu}}^{\mathcal{H}} \equiv [(E^{-1})_{\bar{\mu}}^{\bar{\nu}}\partial_{\bar{\nu}} + i\mathcal{B}_{\bar{\mu}}^A]. \quad (3.5)$$

Notice that we have been completely general about the dimension of the symmetry-breaking object; it can be a brane of any dimension. In the simple case of a 0-brane (i.e. a point-particle), we construct our EFT as a one-dimensional field theory on the time coordinate $t \equiv x^0$.

We mentioned earlier that when only internal symmetries are broken, the number of Goldstones exactly matches the number of broken generators. However, when spacetime symmetries are broken, this need not be the case. The removal of extraneous Goldstones can be implemented at the level of the coset construction by way of imposing inverse Higgs (IH) constraints [11, 13, 15, 42].

²Although we use the same set of notation for the upper and lower indices of the vielbein, the two indices actually have different transformation properties. The upper index transforms linearly under the subgroup \mathcal{H}_0 , while the lower index is covariant under coordinate transformations. The fact that E is a vielbein can also be understood as the fact that the integral over a D_p brane (introduced in the next section) $\int d^{p+1}x \det E$ transforms as a scalar under all symmetries.

Pragmatically, the rules of the game are as follows: Suppose that the commutator between an unbroken translation generator \bar{P} and a broken generator τ' contains another broken generator τ , that is $[\bar{P}, \tau'] \supset \tau$. Suppose further that τ and τ' do not belong to the same irreducible multiplet under \mathcal{H}_0 . Then it turns out that it is consistent with symmetry transformations to set the covariant derivative of the τ -Goldstone in the direction of \bar{P} to zero. This gives a constraint that relates the τ' -Goldstone to derivatives of the τ -Goldstone, allowing the removal of the τ' -Goldstone. The setting of this covariant derivative to zero is known as an inverse Higgs constraint. It turns out that in certain situations, the set of allowed IH constraints is significantly expanded beyond what we have mentioned above [40, 43]. There are two main physical motivations for imposing IH constraints. First, sometimes two Goldstones in the coset will not induce independent fluctuations. Then, we can view IH constraints as convenient gauge-fixing conditions to remove redundant Goldstones. Second, when the conditions for IH conditions are met, we will find that our EFT has gapped Goldstones. As a result, below the energy of this gap, we must integrate out these gapped degrees of freedom. In such a case, IH constraints can be thought of as integrating out gapped Goldstones. For further explanations of the origins of IH constraints, consult [12, 42, 44, 45] and for more on the coset construction, see [12, 14, 40, 43, 46, 47].

3.2 Our philosophy

The coset construction derives its name from the broken symmetry coset $\mathcal{G}/\mathcal{H}_0$. As explained in the previous section, the coset is parameterized by the Goldstone fields. As a result, the number of Goldstones that appear in the coset exactly equals the number of spontaneously broken symmetry generators; however, after imposing IH constraints, extraneous Goldstones are removed, meaning that the total number of Goldstones that exist in the EFT may be less than the number of broken generators. This picture is all well and good for a wide range of systems, but we can begin to see issues as soon as we consider the spin-0 point particle. In particular, the leading-order effective action is

$$S = -m \int dt \sqrt{1 - \dot{X}^i \dot{X}^i}, \quad (3.6)$$

where $X^i(t)$ gives the instantaneous position of the particle at time t and m is the mass [12]. As we will see in the next section, the fields $X^i(t)$ appear in the coset as Goldstones associated with spontaneously broken spatial translations. If we consider a quantum mechanical particle, however, this is cause for alarm. The set of energy eigenstates of a free bosonic point particle are labeled by their three-momentum, $|\vec{p}\rangle$, with corresponding energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. Thus the ground state of this system is $|\vec{0}\rangle$. But notice that such a state spontaneously breaks Lorentz boosts, while preserving spatial rotations and spacetime translations. How is it possible that the field content of our action (3.6) consists of Goldstones corresponding to spatial translations when such translations are not spontaneously broken?

One response is that a classical point-particle has a well-defined position and momentum simultaneously. As a result, the ground state must pick a particular spatial position and hence spontaneously breaks spatial translations. Thus it is natural to suppose that the resulting EFT should have spatial translation Goldstones. But it seems rather odd that in order to describe a quantum particle, we should have to rely so strongly on classical intuition.

It turns out that a similar problem arises when formulating EFTs for finite-temperature systems. For example, the EFT for a fluid consists of Goldstones associated with spacetime translations despite the fact that the equilibrium density matrix spontaneously breaks Lorentz boosts while preserving spatial rotations and spacetime translations (i.e. the same SSB pattern as a *quantum* point particle). The solution proposed in [43] involves a similar appeal to classical intuition: while the equilibrium density matrix of a fluid may not spontaneously break translations, semiclassically the density matrix accounts for a statistical ensemble of highly chaotic micro-states, each of which spontaneously breaks *every* symmetry, including translations. As a result, the coset should not be parametrized by Goldstones associated with the broken generators alone, but should instead be parameterized by Goldstones as if every symmetry of the theory were broken. We will refer to Goldstones associated with broken symmetries as *broken Goldstones* and those associated with unbroken symmetries as *unbroken Goldstones*. It turns out that broken and unbroken Goldstones behaved rather differently; at the level of the coset this difference manifests as gauge redundancies. In

this work, we will borrow this approach and generalize it. We unfortunately have no first-principles understanding of *why* the inclusion of unbroken Goldstone fields is necessary for quantum point particles. The intuition employed to justify their existence in the finite-temperature case does not obviously apply to zero-temperature systems of any variety. What is clear, however, is that the inclusion of unbroken Goldstones is often necessary. It is our goal to outline the most general rules imaginable for including unbroken Goldstones at the level of the coset construction. Then we will apply this generalized coset construction to formulate actions for relativistic particles with spin.

The approach we take to the cosets is as follows. Let the symmetry generators be given by (3.1), where once again $\bar{\mu}$ and μ' run over complementary subsets of the Lorentz index $\mu = 0, \dots, D - 1$, where D is the dimension of the spacetime.³ In general, $\bar{P}_{\bar{\mu}}$ need not be the spacetime translation generators $P_{\bar{\mu}}$; however for all examples considered in this work they will be. The rules of our new coset construction are the following.

- Instead of constructing the effective action on the spacetime coordinates $x^{\bar{\mu}}$, introduce world-volume (or worldline / worldsheet depending on dimension) coordinates σ^M . To keep things completely general, we will allow σ^M to include both bosonic (Grassmann even) and fermionic (Grassmann odd) coordinates. We let $M = 0, \dots, p$ for $p \leq D - 1$ represent bosonic coordinates and $M = p + 1, \dots, \mathcal{N} + p$ be fermionic coordinates. Importantly, the dimension of the worldvolume could be any non-negative number that is no larger than the dimension of the physical spacetime. As such the worldvolume coordinates can be thought of as defining a D_p -brane. In particular, we will construct spinning point-particle actions on manifolds with $p = 0$ and $\mathcal{N} = 2s$, where s is the spin of the particle. Notice that a (classical) D_p -brane cannot have more than $p + 1$ unbroken spacetime translations. To be consistent, the unbroken translation index runs over $\bar{\mu} = 0, \dots, d \leq p$, where d is the number of unbroken spatial translations.

³E.g. we may have $\bar{\mu} = 0$ and $\mu' = 1, \dots, D - 1$, but we may not have $\bar{\mu} = 0, 1$ and $\mu' = 1, \dots, D - 1$.

- Parameterize the coset with Goldstones associated with *every* symmetry generator

$$g(\sigma) = e^{iX^{\bar{\mu}}(\sigma)\bar{P}_{\bar{\mu}}} e^{iX^{\mu'}(\sigma)P_{\mu'}} e^{i\pi^{\alpha}(\sigma)\tau_{\alpha}} e^{ib^A(\sigma)T_A}. \quad (3.7)$$

Notice that at this point, broken and unbroken Goldstones appear on equal footing. Global symmetries act via left-multiplication, so for constant $U \in \mathcal{G}$, we have $g \rightarrow U \cdot g$.

- If spacetime translations parallel to the D_p -brane are unbroken, we may impose some sort of worldvolume-reparameterization symmetry—this could include super-reparameterization—as needed

$$\sigma^M \rightarrow \sigma^M + \alpha^M(\sigma). \quad (3.8)$$

This symmetry can be total reparameterization or partial reparameterization. There is a great deal of freedom here. In many of the examples to come, it will be convenient to introduce a worldvolume vielbein as a gauge-field that transforms under these diffeomorphisms. If the translations parallel to the D_p -brane are broken, (e.g. by a lattice) we may impose a rigid linear symmetry of the coordinates

$$\sigma^M \rightarrow L^M_N \sigma^N + \alpha^M, \quad L^M_N, \alpha^M = \text{const.} \quad (3.9)$$

Such rigid symmetries arise in solids and certain phases of liquid crystals [40], though we will not consider liquid crystals in this thesis.

- To distinguish unbroken Goldstones associated with T_A from broken Goldstones associated with τ_{α} , impose gauge symmetries associated with T_A of the form

$$g(\sigma) \rightarrow g(\sigma) \cdot e^{i\lambda^A(\sigma)T_A}. \quad (3.10)$$

Just like with the reparameterization invariance, there is a great deal of freedom here. We could allow $\lambda^A(\sigma)$ to be a generic function of σ , in which case the Goldstones associated

with τ_α can be gauge-fixed to zero. Alternatively, we can put constraints on the allowed form of $\lambda^\alpha(\sigma)$. Thus, the extent of these gauge symmetries are not strictly determined by the symmetry-breaking pattern. For the broken generators τ_α we may (though are not required to) impose rigid transformations of the form

$$g(\sigma) \rightarrow g(\sigma) \cdot e^{i\kappa^\alpha \tau_\alpha}, \quad \kappa^\alpha = \text{const.} \quad (3.11)$$

Such rigid symmetries arise, for example, in superfluids that spontaneously break spatial rotations [20]. Finally, notice that all gauge symmetry acts via right-multiplication on g and the physical symmetries act via left-multiplication. As a result the gauge symmetries must commute with the physical symmetries.

- The EFT is constructed using invariant building-blocks generated by the Maurer-Cartan form

$$g^{-1}dg. \quad (3.12)$$

All terms generated by the Maurer-Cartan form that are manifestly invariant under the coordinate reparameterization and gauge symmetries can be used as building-blocks of the EFT.

- Inverse Higgs constraints can be imposed whenever there is a set of covariant terms (i.e. they transform linearly under global, reparameterization, and gauge symmetries) that when set to zero, enables the removal of one set of Goldstones from all invariant building-blocks. This includes the rather unusual *dynamical* IH constraints presented in [18, 19] that enable the removal of certain Goldstones at the price of introducing non-trivial operator-constraints on the remaining fields. We will find that similarly unusual IH constraints can be imposed when considering massless particles.

We allow for a great deal of freedom in this coset construction. The reason for doing so is that we are treating the coset as a pragmatic tool to construct symmetry-invariant actions that satisfy

Goldstone's theorem. In particular, since there are no local gauge symmetries associated with any of the broken generators, Goldstone's theorem will be satisfied. The terms of the Maurer-Cartan form that arise from this coset construction are automatically invariant under all internal symmetries.

Finally, to recover the usual coset construction in which only broken Goldstones appear, one need only choose σ^M to be purely bosonic and allow α^M in (3.9) and λ^A in (3.10) to be completely general functions of σ . In this way, we may gauge-fix $\sigma^M = \delta_{\bar{\mu}}^M X^{\bar{\mu}}$ and $b^A = 0$.

3.3 Massive spin-0 point-particles

From now on we consider particles in $D = 4$ flat spacetime. We will construct the effective action for the spin-0 point-particle in two different ways. First, we employ the usual coset construction techniques to formulate a classical action, reminiscent of the Nambu-Goto action. Next, we use our new philosophy of cosets to formulate a point-particle action in the style of the Polyakov action that can easily be quantized. We let P_μ generate spacetime translations and $J_{\mu\nu}$ generate Lorentz boosts, which satisfy the usual Poincaré algebra

$$\begin{aligned} i[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\sigma\mu}J_{\rho\nu} + \eta_{\sigma\nu}J_{\rho\mu}, \\ i[P_\mu, J_{\rho\sigma}] &= \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \\ i[P_\mu, P_\nu] &= 0. \end{aligned} \tag{3.13}$$

We will often find it convenient to use the basis of Lorentz generators

$$K_i = J_{0i}, \quad J_i = \frac{1}{2}\epsilon^{ijk}J_{jk}. \tag{3.14}$$

3.3.1 À la Nambu-Goto

Consider a *classical* scalar point-particle sitting at rest. The SSB pattern is rather straightforward: Lorentz boosts K_i and spatial translations P_i are spontaneously broken, while spatial rotations J_i and temporal translations P_0 are unbroken. As pointed out in the previous section, the

fact that P_i are spontaneously broken is a feature of the classical point-particle only; the ground-state of the quantum point-particle breaks boosts alone.

We begin with the coset of non-linearly realized symmetries

$$g(t) = e^{itP_0} e^{iX^i(t)P_i} e^{i\eta^i(t)K_i}. \quad (3.15)$$

The Maurer-Cartan form is then

$$g^{-1}\partial_t g = iE(P_0 + \nabla X^i P_i + \nabla \eta^i K_i + \Omega^i J_i), \quad (3.16)$$

where the einbein E , covariant derivatives ∇X^i and $\nabla \eta^i$, and spin connection Ω^i are given by

$$\begin{aligned} E &= \Lambda_0^0 + \dot{X}^i \Lambda_i^0, \\ \nabla X^i &= E^{-1}(\Lambda_0^i + \dot{X}^j \Lambda_j^i), \\ \nabla \eta^i &= E^{-1}(\Lambda^{-1} \partial_t \Lambda)^{0i}, \\ \Omega^i &= \frac{E^{-1}}{2} \epsilon^{ijk} (\Lambda^{-1} \partial_t \Lambda)^{jk}, \end{aligned} \quad (3.17)$$

where $\Lambda^\mu{}_\nu \equiv (e^{i\eta^i(t)K_i})^\mu{}_\nu$ and $\dot{X}^i \equiv \partial_t X^i$. We can now impose the IH constraints $\nabla X^i = 0$, which can be solved to give

$$\frac{\eta^i}{\eta} \tanh \eta = \dot{X}^i, \quad (3.18)$$

where $\eta \equiv \sqrt{\eta^2}$. These IH constraints imply that $\nabla \eta^i$ and Ω^i are sub-leading in the derivative expansion, so the only building-block at leading order is E , which becomes $E = \sqrt{1 - \dot{X}^i \dot{X}^i}$.

Hence the action is

$$S = -m \int dt \sqrt{1 - \dot{X}^i \dot{X}^i}, \quad (3.19)$$

where m is a constant parameter that we interpret as the mass. We can see, therefore, that X^i are the only Goldstones that survive the IH constraints.

Finally, it is worth pointing out that the above action can be thought of as a gauge-fixed version

of a manifestly-covariant action. Let τ be the coordinate of the particle's worldline and let $X^\mu(\tau)$ be the fields. Then we can define the action

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}, \quad (3.20)$$

where $\dot{X}^\mu \equiv \partial_\tau X^\mu$. Notice that this action is invariant under reparameterization of the coordinate $\tau \rightarrow \tau'(\tau)$. Thus, we can gauge-fix $\tau = X^0(\tau) \equiv t$, recovering our initial action (3.19).

3.3.2 À la Polyakov

We now throw off the shackles of the ordinary coset construction and proceed with our new philosophy. Begin by parameterizing the full symmetry group

$$g(\tau) = e^{iX^\mu(\tau)P_\mu} e^{i\eta^i(\tau)K_i} e^{i\vartheta^i(\tau)J_i}, \quad (3.21)$$

where τ is the particle's worldline coordinate. With our new philosophy, since P_0 is unbroken, we have free reign to impose any kind of diffeomorphism (i.e. reparameterization) symmetry on the coordinate τ , and since J_i are unbroken we may impose any gauge symmetries of the form $g \rightarrow g \cdot e^{i\lambda^i(\tau)J_i}$. Evidently, there are many possibilities. On the one hand, the excess of possibilities is a draw-back; one of the most appealing features of the coset construction is that it is so constraining that one need barely think in order to use it effectively. On the other hand, Goldstone's theorem for spontaneously broken spacetime symmetries (unlike its internal symmetry counterpart) can be satisfied in all sorts of unusual and unexpected ways. We should therefore not be disappointed that our novel approach to the coset construction now reflects this diversity of possibilities more fully.

With some foreknowledge of the desired point-particle action, we impose the following gauge symmetries.

- Reparameterization invariance: Let $e(\tau)$ be the einbein of the particle's worldline. Then we

have

$$\delta\tau = -\xi(\tau), \quad \delta e = \partial_\tau(e\xi), \quad (3.22)$$

where ξ is an arbitrary infinitesimal function of τ .

- We do not want any rotational Goldstones ϑ^i in the EFT, so we impose total rotational gauge symmetry

$$g \rightarrow g \cdot e^{i\lambda^i(\tau)J_i}, \quad (3.23)$$

where λ^i is an arbitrary function of τ . We can use this gauge symmetry to fix $\vartheta^i = 0$.

With this gauge-fixing condition, our group-element is now

$$g(\tau) = e^{iX^\mu(\tau)P_\mu} e^{i\eta^i(\tau)K_i}. \quad (3.24)$$

We may compute the Maurer-Cartan form, using e as the einbein,

$$g^{-1}\partial_\tau g = ie(\nabla X^\mu P_\mu + \nabla\eta^i K_i + \Omega^i J_i), \quad (3.25)$$

where the covariant derivatives and spin connections are given by

$$\begin{aligned} \nabla X^\mu &= e^{-1}\dot{X}^\nu \Lambda_\nu{}^\mu, \\ \nabla\eta^i &= e^{-1}(\Lambda^{-1}\partial_\tau\Lambda)^{0i}, \\ \Omega^i &= \frac{1}{2e}\epsilon^{ijk}(\Lambda^{-1}\partial_\tau\Lambda)^{jk}, \end{aligned} \quad (3.26)$$

such that $\Lambda^\mu{}_\nu \equiv (e^{i\eta^i(\tau)K_i})^\mu{}_\nu$ and $\dot{X}^\mu \equiv \partial_\tau X^\mu$.

We can impose the IH constraints $\nabla X^i = 0$, which can be solved to give

$$\frac{\eta^i}{\eta} \tanh \eta = \frac{\dot{X}^i}{\dot{X}^0}. \quad (3.27)$$

These IH constraints ensure that $\nabla\eta^i$ and Ω^i do not contribute to the leading order. Thus, the only

leading-order building block is ∇X^0 , which now becomes

$$\nabla X^0 = e^{-1} \sqrt{-\dot{X}^\mu \dot{X}_\mu}. \quad (3.28)$$

Thus, our leading-order action is

$$\begin{aligned} S &= \int d\tau e ((\nabla X^0)^2 + m^2) \\ &= - \int d\tau \left(\frac{1}{e} \dot{X}^\mu \dot{X}_\mu - em^2 \right), \end{aligned} \quad (3.29)$$

where m is again a constant parameter which we interpreted as the mass. We have used the fact that $\int d\tau e$ is the invariant integration measure. We have thus recovered the standard effective action for a spin-0 point-particle in the Polyakov form. Since such an action has an external einbein, it is clear that the ordinary coset construction would not generate it. We therefore see a clear (albeit small) benefit of our new cost construction philosophy. In the following sections, we will see more significant advantages.

3.4 Spin-1/2 point-particles

It was demonstrated in [39, 48, 49, 50, 51, 52, 53] that the effective action describing a spin-1/2 particle could be derived by imposing $\mathcal{N} = 1$ worldline supersymmetry. A quick review of worldline supersymmetry can be found in Appendix A. In this section we will impose the worldline supersymmetry at the level of the coset construction. It turns out that including a mass term requires a little extra machinery, so we begin by considering the massless case.

3.4.1 Massless

A massless, spinning particle must always travel at the speed of light with the direction of the spin parallel (or anti-parallel) to the velocity. Without loss of generality, let us choose the direction

of motion to be parallel to \hat{z} . Then, the unbroken Poincaré symmetry generators are

$$P_u \equiv P_0 - P_3, \quad \mathcal{J}_i \equiv J_i + \epsilon^{3ij} K_j, \quad (3.30)$$

and the broken generators are

$$P_v = P_0 + P_3, \quad P_m, \quad \mathcal{K}_i \equiv K_i + \epsilon^{3ij} J_j, \quad (3.31)$$

where the indices $m, n = 1, 2$. Notice that \mathcal{J}_i are the generators of the (full) little group of massless particles; it is easy to check that their commutation relations are just like those of the generators for the two-dimensional Euclidean group. Additionally, in this basis of translation generators, the non-zero components of the metric are $\eta_{uv} = \eta_{vu} = 1/2$ and $\eta_{mn} = \delta_{mn}$. Lastly, we assume that there are no internal symmetries, so Poincaré is the full symmetry group.

Since spin-1/2 particles enjoy worldline SUSY, we define our coset on the superspace coordinates $\sigma^M = (\tau, \theta)^M$. The most general group element is

$$g(\sigma) = e^{i\mathbb{X}^\alpha(\sigma)P_\alpha + i\mathbb{X}^m(\sigma)P_m} e^{i\eta^i(\sigma)\mathcal{K}_i} e^{i\theta^i(\sigma)\mathcal{J}_i}, \quad (3.32)$$

where α run over u, v , and \mathbb{X}^μ are bosonic fields as a function of $\sigma^M = (\tau, \theta)^M$, as shown in (A.1). Similarly for η and θ . We now impose the following gauge symmetries. First, since P_u is unbroken, we have the freedom to impose super-reparameterization symmetry, acting on σ^M , the worldline einbein e and its super partner χ by (A.12) and (A.13). And since \mathcal{J}_i are unbroken, we may impose the rotational gauge symmetry

$$g \rightarrow g \cdot e^{i\mathbb{X}^i(\sigma)\mathcal{J}_i}, \quad (3.33)$$

where \mathbb{X}^i is a generic function of σ . We may therefore gauge-fix $\theta^i = 0$.

With this gauge-fixing condition, the Maurer-Cartan form is

$$g^{-1}\partial_M g = i\mathbb{E}_M^A (\nabla_A \mathbb{X}^\alpha P_\alpha + \nabla_A \mathbb{X}^m P_m + \nabla_A \eta^i \mathcal{K}_i + \Omega_A^i \mathcal{J}_i), \quad (3.34)$$

where the covariant derivatives of \mathbb{X}^μ are

$$\begin{aligned}\nabla_A \mathbb{X}^\alpha &= \mathbb{E}_A^M \partial_M \mathbb{X}^\mu \mathbb{A}_\mu^\alpha, \\ \nabla_A \mathbb{X}^m &= \mathbb{E}_A^M \partial_M \mathbb{X}^\mu \mathbb{A}_\mu^m,\end{aligned}\tag{3.35}$$

such that $\mathbb{A}^\mu{}_\nu = (e^{i\eta^i(\sigma)\mathcal{K}_i})^\mu{}_\nu$, and \mathbb{E}_M^A is the superzweibein as in (A.10). The precise forms of $\nabla_A \eta^i$ and \mathcal{Q}_A^i will not be important for the construction of the leading-order action.

We may now impose IH constraints. In particular, it is consistent with symmetries to fix

$$0 = \nabla_0 \mathbb{X}^v, \quad 0 = \nabla_0 \mathbb{X}^m.\tag{3.36}$$

These IH constraints can be used to solve for η^i in terms of \mathbb{X}^μ , thereby leaving \mathbb{X}^μ as the only remaining Goldstone. However, these constraints also force $\nabla_0 \mathbb{X}^\mu$ to be null, which we do not wish to impose by hand. Instead, the fact that the particle moves along a null-trajectory should be a result of the equation of motion. The reason these IH constraints are over-constraining is that if we were to include η^i in our effective action, they would serve as Lagrange multipliers. In particular, the constraints they would place on the dynamics of \mathbb{X}^μ would be redundant with the constraints imposed by e and χ . Thus, we could just as well not include η^i in our EFT at all as they would be entirely redundant. We will investigate this further in the following subsection.

We now can construct the leading-order effective action⁴

$$S = \frac{1}{2i} \int d\tau d\theta \mathbb{E} \nabla_0 \mathbb{X}^\mu \nabla_1 \mathbb{X}_\mu,\tag{3.37}$$

which can be immediately simplified to

$$S = \frac{1}{2i} \int d\tau d\theta \mathbb{E} D_0 \mathbb{X}^\mu D_1 \mathbb{X}_\mu.\tag{3.38}$$

Here \mathbb{E} is the superdeterminant $\text{sdet}(\mathbb{E}_M^A)$ of the superzweibein, and its exact form is shown in

⁴We choose to normalize the fields so that the overall coefficient in front of the action is $1/2i$.

(A.10). We can explicitly perform the $d\theta$ integral, which yields an action defined only on the particle worldline τ . Letting $\mathbb{X}^\mu(\tau, \theta) = X^\mu(\tau) + i\theta\psi^\mu(\tau)$, we have

$$S = \int d\tau \left(\frac{1}{2e} \dot{X}^\mu \dot{X}_\mu + \frac{i}{2} \dot{\psi}^\mu \psi_\mu - \frac{i}{e} \chi \psi^\mu \dot{X}_\mu \right). \quad (3.39)$$

Upon quantization (see section 3.6), we will see that this action describes a spin-1/2 point-particle.

3.4.2 On-shell inverse Higgs

We now investigate the nature of the on-shell IH constraint for the case of the massless spin-1/2 particle. To do so, we construct the leading-order action without removing the Lorentz Goldstones η^i . Using the covariant derivatives of (3.35), we may construct our leading-order action. We have

$$S = \frac{1}{2i} \int d\tau d\theta \mathbb{E} (\nabla_0 \mathbb{X}^\mu \nabla_1 \mathbb{X}_\mu + C \nabla_1 \mathbb{X}^u), \quad (3.40)$$

for some constant C . Expanding the covariant derivatives, we have $\nabla_1 \mathbb{X}^u = D_1 \mathbb{X}^\mu \wedge_\mu^u$. At this point, it is convenient to define the superfield $\mathbb{L}_\mu \equiv C \wedge_\mu^u$, which can then be expanded as

$$\mathbb{L}_\mu = L_\mu + i\theta\lambda_\mu. \quad (3.41)$$

Notice that \mathbb{L}_μ is constrained to be a null-vector, meaning that $L^2 = 0$ and $L_\mu \lambda^\mu = 0$. Performing the $d\theta$ integral, our action becomes

$$S = \int d\tau \left(\frac{1}{2e} (\dot{X}^\mu - i\chi\psi^\mu)^2 + \frac{i}{2} \dot{\psi}^\mu \psi_\mu + \frac{1}{2} \dot{X}^\mu L_\mu + \frac{ie}{2} \psi^\mu \lambda_\mu \right). \quad (3.42)$$

Now let us compute the equations of motion. By varying e and χ we find the respective equations

$$e^{-2} (\dot{X}^\mu - i\chi\psi^\mu)^2 = \psi^\mu \lambda_\mu, \quad \psi^\mu \dot{X}_\mu = 0. \quad (3.43)$$

And by varying L_μ and λ_μ subject to the constraints $L^2 = 0$ and $L_\mu \lambda^\mu = 0$, we have

$$\dot{X}^\mu \propto L^\mu, \quad \psi^\mu \propto L^\mu. \quad (3.44)$$

Notice that the second equation of (3.44) implies that $\psi^\mu \lambda_\mu = 0$. Plugging this expression into the first equation of (3.43) gives $(\dot{X} - i\chi\psi)^2 = 0$, which is exactly the constraint obtained by varying e in the action (3.39). Additionally, the second equation of (3.43) is the same constraint found by varying χ in (3.39). We thus have reproduced the constraint equations arising from the standard action (3.39) that does not contain any Lorentz Goldstones. Then, we can interpret (3.44) as equations that specify the Lorentz Goldstones \mathbb{L}_μ .

We therefore conclude that the inclusion of the Lorentz Goldstones does not give us anything new, so they are entirely extraneous degrees of freedom and can simply be omitted from the EFT.

3.4.3 Massive

Notice that in the scalar action (3.29), the term involving the mass is essentially a cosmological constant-term. In the spin-1/2 action (3.39), however, there is no such mass term. The reason is that in a theory with SUSY, an ordinary cosmological constant-term of the form $\int d\tau e$ does not respect the full gauge symmetry group. In this section, we will see how a symmetry-invariant cosmological constant-term can be included in the action.

The intuition is as follows. Suppose that the point-particle existed in a five-dimensional spacetime such that the momentum along the fifth dimension is fixed by $p^5 = m$, for some constant $m > 0$. Since the particle is massless, the on-shell condition is $p^\mu p_\mu + (p^5)^2 = 0$, or equivalently, $p^\mu p_\mu + m^2 = 0$. Thus, if we are only concerned with the motion of the particle along the four spacetime coordinates x^μ for $\mu = 0, 1, 2, 3$, then the particle behaves as if it has mass m .

Going through a procedure almost identical to that of the previous subsection, we find the

effective action for a massless spin-1/2 particle in 5D is

$$S = \int d\tau \left(\frac{1}{2e} \dot{X}^\mu \dot{X}_\mu + \frac{i}{2} \dot{\psi}^\mu \psi_\mu - \frac{i}{e} \chi \psi^\mu \dot{X}_\mu + \frac{1}{2e} \dot{X}^5 \dot{X}_5 + \frac{i}{2} \dot{\psi}^5 \psi_5 - \frac{i}{e} \chi \psi^5 \dot{X}_5 \right), \quad (3.45)$$

where we have defined $\mathbb{X}^5(\tau, \theta) = X^5(\tau) + i\theta\psi^5(\tau)$. At this point, it is helpful to work in the ‘‘Hamiltonian picture’’ in which we include the conjugate momentum p^5 of X^5 . We thus have

$$S = \int d\tau \left(\frac{1}{2e} \dot{X}^\mu \dot{X}_\mu + \frac{i}{2} \dot{\psi}^\mu \psi_\mu - \frac{i}{e} \chi \psi^\mu \dot{X}_\mu + \dot{X}^5 p_5 - \frac{e}{2} (p^5)^2 + \frac{i}{2} \dot{\psi}^5 \psi_5 - i\chi \psi^5 p_5 \right). \quad (3.46)$$

Fixing $p^5 = m$ for constant $m > 0$ we have⁵

$$S = \int d\tau \left(\frac{1}{2e} \dot{X}^\mu \dot{X}_\mu + \frac{i}{2} \dot{\psi}^\mu \psi_\mu - \frac{i}{e} \chi \psi^\mu \dot{X}_\mu - \frac{m^2 e}{2} + \frac{i}{2} \dot{\psi}^5 \psi_5 - im\chi \psi^5 \right), \quad (3.47)$$

which agrees with the massive spin-1/2 action from [39]. In particular, notice that the equation of motion for e is (gauge fixing $\chi = 0$)

$$\frac{1}{e^2} \dot{X}^\mu \dot{X}_\mu + m^2 = 0, \quad (3.48)$$

which is exactly the on-shell condition for a massive particle.

3.5 Higher-spin point-particles

The symmetry-breaking pattern of the Poincaré group for massless particles is the same for all values of spin s , namely the unbroken generators are (3.30) and the broken generators are (3.31). Thus, the only possible distinguishing features among the effective actions for particles of differing spins are the choices of worldvolume and gauge symmetries. It has been demonstrated in [50] that the EFT for a particle of spin s enjoys $\mathcal{N} = 2s$ SUSY. We present an extended worldline SUSY formalism as well as a method of multiplets for calculations in this formalism in Appendix B. At this

⁵Notice that we are not integrating out p^5 ; instead, we are constraining the system to have $p^5 = m$.

stage, we will split the problem into two pieces. The first deals with spin 1 particles, corresponding to $\mathcal{N} = 2$. The effective action will be defined in terms of an integral over superspace and will thus lead to an off-shell SUSY-invariant action. The second will be valid for all $\mathcal{N} \geq 1$, but will instead rely on the multiplet formalism and as a result will yield local on-shell SUSY-invariant actions.

3.5.1 Spin-1 point particles

We define our coset on the superspace coordinates $\sigma^M = (\tau, \vec{\theta})^M$ for $\vec{\theta} = (\theta^1, \theta^2)$. We take the unbroken translation generator P_u as an opportunity to impose the super-reparameterization invariance which acts on σ^M , e and $\vec{\chi}$ by (B.6) and (B.7), while the supervielbein \mathbb{E}_M^A is given by (B.4). Then, as in the spin-1/2 case, we take the unbroken little group generators \mathcal{J}_i as an opportunity to impose a gauge invariance that can be used to fix $\theta^i = 0$.

With this gauge-fixing condition the most general group element is

$$g(\sigma) = e^{i\mathbb{X}^\alpha(\sigma)P_\alpha + i\mathbb{X}^m(\sigma)P_m} e^{i\eta^i(\sigma)\mathcal{K}_i}, \quad (3.49)$$

where $\alpha = u, v$ and $m = 1, 2$. The Maurer-Cartan form is

$$g^{-1}\partial_M g = i\mathbb{E}_M^A (\nabla_A \mathbb{X}^\alpha P_\alpha + \nabla_A \mathbb{X}^m P_m + \nabla_A \eta^i \mathcal{K}_i + \Omega_A^i \mathcal{J}_i), \quad (3.50)$$

where the covariant derivatives of \mathbb{X}^μ are

$$\begin{aligned} \nabla_A \mathbb{X}^\alpha &= \mathbb{E}_A^M \partial_M \mathbb{X}^\alpha \wedge_\mu^\alpha, \\ \nabla_A \mathbb{X}^m &= \mathbb{E}_A^M \partial_M \mathbb{X}^m \wedge_\mu^m, \end{aligned} \quad (3.51)$$

such that $\wedge_\nu^\mu = (e^{i\eta^i(\sigma)\mathcal{K}_i})^\mu{}_\nu$. The precise forms of $\nabla_A \eta^i$ and Ω_A^i will not be important for the construction of the leading-order action.

We impose on-shell IH constraints

$$0 = \nabla_0 \mathbb{X}^v, \quad 0 = \nabla_0 \mathbb{X}^m, \quad (3.52)$$

which allow us to remove η^i from the effective action. The leading-order effective action for $\mathcal{N} = 2$ is therefore

$$S = \frac{1}{4} \int d\tau d^2\theta \mathbb{E} \epsilon^{ab} \nabla_a \mathbb{X}^\mu \nabla_b \mathbb{X}_\mu, \quad (3.53)$$

which can be immediately simplified to

$$S = \frac{1}{4} \int d\tau d^2\theta \mathbb{E} \epsilon^{ab} D_a \mathbb{X}^\mu D_b \mathbb{X}_\mu, \quad (3.54)$$

where ϵ^{ab} is the totally antisymmetric tensor such that $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$. We can now expand the superfield \mathbb{X}^μ in powers of $\vec{\theta}$, yielding

$$\mathbb{X}^\mu = X^\mu + i\theta^a \psi^{a\mu} + \frac{i}{2} \epsilon^{ab} \theta^a \theta^b F^\mu. \quad (3.55)$$

The integral over $d^2\theta$ can now be explicitly evaluated, yielding an action defined on the worldline coordinate τ by

$$S = \int d\tau \left(\frac{1}{2e} (\dot{X} - i\vec{\chi} \cdot \vec{\psi})^2 + \frac{i}{2} \dot{\psi}^\mu \cdot \vec{\psi}_\mu + \frac{e}{2} F^2 - \frac{i}{2} A^{ab} \psi^{\mu a} \psi_\mu^b \right). \quad (3.56)$$

Notice that the field F^μ appears with no derivatives and that its equation of motion is simply $F^\mu = 0$. We can thus integrate it out at no cost, except the resulting action will only be invariant under SUSY on-shell.

3.5.2 Arbitrary-spin point particles

As mentioned earlier, for particles of spin $s > 1$ —which correspond to $\mathcal{N} > 2$ —we must admit defeat, at least in part, if we wish to construct a SUSY-invariant action using the superspace formalism. The reason is that no superspace integral can give rise to the desired invariant action. To see why, we propose the following power-counting argument. We let μ count powers of energy; e.g. $[\partial_\tau] = [d\tau]^{-1} = \mu^1$. Begin by noticing that the actions we have calculated so far, (3.29,3.39,3.56)

all contain a term of the form

$$S \supset \int d\tau \frac{1}{2e} \dot{X}^2 + \dots \quad (3.57)$$

By assuming S scales as μ^0 , we therefore have the scaling of X , and hence \mathbb{X} , namely $[\mathbb{X}] = \mu^{-1/2}$. Further, notice that the commutator of two super-translations give rise to one temporal translation, so $[D_\theta] = [D_0]^{1/2} = \mu^{1/2}$. Moreover, integration and differentiation with respect to Grassmann numbers are equivalent operations, so $[d\theta] = \mu^{1/2}$. Lastly, our action should involve two factors of \mathbb{X} and two covariant derivatives D and D' , which can be either D_0 or D_θ . Then the action scales like

$$[S] = [d\tau][d^{\mathcal{N}}\theta][D][\mathbb{X}][D'][\mathbb{X}] = [D][D']\mu^{s-2}, \quad (3.58)$$

where we have used the fact that $\mathcal{N} = 2s$. But also, the action ought to scale like μ^0 , so we have $[D][D'] = \mu^{2-s}$. But since D and D' must scale as either μ^1 or $\mu^{1/2}$, $[S] = 0$ can only be satisfied if $s \leq 1$. In particular, for $s = 0$, we have $D = D' = D_0$; for $s = 1/2$, we have $D = D_0$ and $D' = D_\theta$; and for $s = 1$, we have $D = D' = D_\theta$.

We now use the multiplet formalism to construct actions invariant under global SUSY for arbitrary \mathcal{N} and hence arbitrary spin. Let $\Sigma^\mu = (X^\mu, \vec{\psi}^\mu)$ and $\eta^i = (\eta^i, \vec{\phi}^i)$ be bosonic multiplets, as defined in (B.10). We take the unbroken generators \mathcal{J}_i as an invitation to impose gauge symmetries that allow us to remove the Goldstones associated with \mathcal{J}_i . Then, we can express the most general group element as the SUSY multiplet⁶

$$g = e^{i\Sigma^\alpha P_\alpha} e^{i\Sigma^m P_m} e^{i\eta^i \mathcal{K}_i}. \quad (3.59)$$

Using the SUSY covariant derivative (not to be confused with the covariant derivative generated

⁶Notice that the addition and multiplication rules (B.14) and (B.15) enable us to define arbitrary functions of the multiplets by employing a Taylor expansion.

by the coset) defined by (B.16), we can compute the Maurer-Cartan form by

$$g^{-1}D_A g = i(\nabla_A \Sigma^\alpha P_\alpha + \nabla_A \Sigma^m P_m + \nabla_A \eta^i \mathcal{K}_i + \Omega^i \mathcal{J}_i), \quad (3.60)$$

where the covariant derivatives of Σ^μ generated by the coset are

$$\begin{aligned} \nabla_A \Sigma^\alpha &= D_A \Sigma^\mu \Lambda_\mu^\alpha, \\ \nabla_A \Sigma^m &= D_A \Sigma^\mu \Lambda_\mu^m, \end{aligned} \quad (3.61)$$

where $\Lambda_\mu^\nu = (e^{i\eta^i \mathcal{K}_i})^\mu_\nu$. The precise form of the other terms in the Maurer-Cartan form will not be important for the construction of the leading-order action. Imposing the on-shell IH constraints⁷ removes the Goldstones η^i ,

$$0 = \nabla_0 \Sigma^v, \quad 0 = \nabla_0 \Sigma^m. \quad (3.62)$$

To construct the effective action, we must build a symmetry-invariant fermionic multiplet (see (B.12) for the definition of the fermionic multiplet), which can then be integrated according to the prescription (B.18). We see that at leading order, the only such option is $\Phi \equiv \nabla_0 \Sigma^\mu \nabla_a \Sigma_\mu$, which can be immediately simplified to $\Phi = D_0 \Sigma^\mu D_a \Sigma_\mu$. We then have the leading-order action

$$S = \int_{\text{SUSY}} \Phi = \int d\tau \left(\frac{1}{2} \dot{X}^2 + \frac{i}{2} \dot{\vec{\psi}}^\mu \cdot \vec{\psi}_\mu \right). \quad (3.63)$$

We must now gauge the $\mathcal{N} = 2s$ extended worldline SUSY. In particular, we want an action that is invariant, up to terms proportional to the equations of motion, under the infinitesimal local transformations (B.7) and the on-shell SUSY transformations

$$\begin{aligned} \delta X^\mu &= \xi \dot{X}^\mu + i \vec{\varepsilon} \cdot \vec{\psi}^\mu, \\ \delta \psi^a &= \xi \dot{\psi}^{a\mu} + \varepsilon^a \frac{1}{e} \left(\dot{X}^\mu - i \vec{\chi} \cdot \vec{\psi}^\mu \right) - \beta^{ab} \psi^{b\mu}, \end{aligned} \quad (3.64)$$

⁷It should be noted that these are only legitimate on-shell IH constraints because we are going to ultimately gauge the worldline SUSY. If we were to keep the SUSY global, then there would be no constraints imposed by gauge fields and the equations of motion arising from boost Goldstones would in fact be extremely important.

where ξ , $\vec{\varepsilon}$, and β^{ab} are now generic infinitesimal functions of τ . In particular, we have

$$S = \int d\tau \left(\frac{1}{2e} z^2 + \frac{i}{2} \vec{\zeta}^\mu \cdot \vec{\psi}_\mu \right), \quad (3.65)$$

where where $z \equiv \dot{X} - i\vec{\chi} \cdot \vec{\psi}$ and $\zeta^a \equiv \dot{\psi}^a + A^{ab}\psi^b$. Equivalently, the expanded form of the action is

$$S = \int d\tau \left(\frac{1}{2e} (\dot{X} - i\vec{\chi} \cdot \vec{\psi})^2 + \frac{i}{2} \dot{\psi}^\mu \cdot \vec{\psi}_\mu - \frac{i}{2} A^{ab} \psi^{\mu a} \psi_\mu^b \right). \quad (3.66)$$

Comparing with [51], we find that we have successfully reproduced the action for on-shell extended local SUSY.

If we wish to include a mass-term in this action, we can follow the steps of section 3.4.3 and arrive at the following additional terms in the action

$$S_{\text{mass}} = \int d\tau \left(-\frac{m^2 e}{2} + \frac{i}{2} \dot{\psi}^5 \cdot \vec{\psi}_5 - im\vec{\chi} \cdot \vec{\psi}^5 - \frac{i}{2} A^{ab} \psi^{5a} \psi_5^b \right). \quad (3.67)$$

Thus, the full action for the massive higher-spin point particle is the sum of (3.66) and (3.67).

3.6 Quantization

After employing our novel coset construction to formulate actions with worldline SUSY, we now wish to demonstrate that they in fact describe point particles with intrinsic quantum spin. By quantizing the action with $\mathcal{N} = 2s \in \mathbb{Z}$ worldline SUSY, we will find that the resulting wave function describes a point-particle of spin s . We will follow closely the procedure shown in [51].

Starting with equation (3.66), we find it convenient to express the action in the ‘‘Hamiltonian form,’’

$$S = \int d\tau \left(p^\mu \dot{X}_\mu - \frac{1}{2} e p^\mu p_\mu - i\vec{\chi} \cdot \vec{\psi}^\mu p_\mu + \frac{i}{2} \dot{\psi}^\mu \cdot \vec{\psi}_\mu - \frac{i}{2} A^{ab} \psi^a \psi^b \right). \quad (3.68)$$

It is then clear from the equations of motion for e , χ^a and A^{ab} that

$$\begin{aligned}
p^\mu p_\mu &= 0, \\
p^\mu \psi_\mu^a &= 0, \\
(\psi^\mu)_{[a} (\psi_\mu)_{b]} &= 0.
\end{aligned} \tag{3.69}$$

We then impose the (anti-)commutation relations

$$[X^\mu, p_\nu] = i\delta_\nu^\mu, \quad \{\psi_a^\mu, \psi_b^\nu\} = \eta^{\mu\nu} \delta_{ab}. \tag{3.70}$$

To facilitate the quantization procedure, it is convenient to work with the usual identification $p_\mu = -i\partial/\partial X^\mu$. To realize the second anti-commutation relation, we work in the representation

$$\psi_a^\mu = \frac{1}{\sqrt{2}} \underbrace{\gamma_5 \otimes \cdots \otimes \gamma_5}_{a-1} \otimes \gamma^\mu \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{\mathcal{N}-1}, \tag{3.71}$$

where we have used the Dirac γ matrices, which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Now, assuming $a < b$, it is straightforward to determine that

$$\begin{aligned}
(\psi^\mu)_{[a} (\psi_\mu)_{b]} &= -\frac{1}{2} \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{a-1} \otimes \gamma_5 \gamma^\mu \otimes \underbrace{\gamma_5 \otimes \cdots \otimes \gamma_5}_{b-a-1} \\
&\quad \otimes \gamma_\mu \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{\mathcal{N}-b}.
\end{aligned} \tag{3.72}$$

The constraint equations coming from equations (3.69) for a spinor $\Psi_{\alpha_1 \cdots \alpha_{\mathcal{N}}}$ are, respectively,

$$\begin{aligned}
\Box \Psi_{\alpha_1 \cdots \alpha_{\mathcal{N}}} &= 0, \\
\not{\partial}_{\alpha_a}^{\beta_a} \Psi_{\alpha_1 \cdots \beta_a \cdots \alpha_{\mathcal{N}}} &= 0, \\
(\gamma^\mu)_{\alpha_a}^{\beta_a} (\gamma_\mu)_{\alpha_b}^{\beta_b} \Psi_{\alpha_1 \cdots \beta_a \cdots \beta_b \cdots \alpha_{\mathcal{N}}} &= 0.
\end{aligned} \tag{3.73}$$

Following the procedures in [51], these constraints will finally give, in the $SL(2, \mathbb{C})$ notation,

$$\partial_{\beta_a}^{\dot{\alpha}_a} \Psi_{\dot{\alpha}_1 \dots \dot{\alpha}_a \dots \dot{\alpha}_N} = 0, \quad \square \Psi_{\alpha_1 \dots \alpha_N} = 0. \quad (3.74)$$

But these are just the conditions that $\Psi_{\alpha_1 \dots \alpha_N}$ be the wave function for a relativistic particle of spin $s = \mathcal{N}/2$. We therefore conclude that our action (3.66) indeed describes a spin- $\mathcal{N}/2$ point particle.

3.7 Massless spin-0 point-particles

When we constructed the EFTs for the massive spin-0 particle, the resulting IH constraints could be solved directly for the Lorentz Goldstones, η^i in terms of the translation Goldstones, X^μ . However, when dealing with massless particles, the IH constraints over-constrained the translation Goldstones and we were forced to interpret them as on-shell IH constraints. In this section we will explore in more detail the relationship between Lorentz Goldstones and massless particles.

3.7.1 Without external einbein

Our aim is to construct an EFT for massless scalars without the introduction of an external einbein. The SSB pattern is identical to that of the spin-1/2 (and higher) massless particles. Explicitly, the unbroken generators are (3.30) and the broken generators are (3.31). The most generic group element is

$$g(\tau) = e^{iX^\alpha(\tau)P_\alpha + iX^m(\tau)P_m} e^{i\eta^i(\tau)\mathcal{K}_i} e^{i\vartheta^i(\tau)\mathcal{J}_i}, \quad (3.75)$$

where $\alpha = u, v$ and $m = 1, 2$. We impose full reparameterization symmetry $\delta\tau = -\xi(\tau)$ as well as a full \mathcal{J}_i gauge symmetry

$$g \rightarrow g \cdot e^{i\lambda^i(\tau)\mathcal{J}_i}, \quad (3.76)$$

which allows us to gauge-fix $\vartheta^i = 0$. With this gauge-fixing condition, the Maurer-Cartan form is

$$g^{-1}\partial_\tau g = iE(P_u + \nabla X^v P_v + \nabla X^m P_m + \nabla \eta^i) + \Omega^i, \quad (3.77)$$

where the einbein is given by $E = \dot{X}^\mu \Lambda_\mu^u$ and $\Lambda_\mu^\nu = (e^{i\eta^i(\tau)\mathcal{K}_i})^\mu{}_\nu$. The covariant derivatives and spin connection will not appear at leading order in our EFT.

One might be tempted to impose the IH constraints $\nabla X^v = 0$ and $\nabla X^m = 0$, but this would imply that \dot{X}^μ is null. Further, since there are no external gauge fields that might constrain \dot{X}^μ to be null, these constraints do not admit an interpretation as on-shell IH constraints. Thus, we are forced to include the Lorentz Goldstones η^i in our EFT.

The leading-order action is then

$$S = \int d\tau E = \int d\tau \dot{X}^\mu L_\mu, \quad (3.78)$$

where $L_\mu \equiv \Lambda_\mu^u$. Importantly, L_μ is constrained to be a null vector, that is $L^2 = 0$.

The equations of motion found by varying L_μ and X^μ are respectively

$$\dot{X}^\mu \propto L^\mu, \quad \dot{L}_\mu = 0. \quad (3.79)$$

Since $L^2 = 0$ the first equation tells us $\dot{X}^2 = 0$, meaning that the particle travels along a null-trajectory. Then, the second equation tells us that $\ddot{X}^\mu = 0$, so the particle does not accelerate. This is exactly what we should expect of a massless particle. Thus, we have an effective action for a massless boson that does not include any external einbein at all; the price we pay is the inclusion of Lorentz Goldstones.

3.7.2 With external einbein

We now consider what happens if we include an external einbein e such that worldline diffeomorphism symmetry acts by

$$\delta\tau = -\xi(\tau), \quad \delta e = \partial_\tau(e\xi). \quad (3.80)$$

As in the previous subsection we gauge-fix $\vartheta^i = 0$. The Maurer-Cartan form is

$$g^{-1}\partial_\tau g = ie(\nabla X^\alpha P_\alpha + \nabla X^m P_m + \nabla\eta^i) + \Omega^i, \quad (3.81)$$

where $\alpha = u, v$ and $m = 1, 2$. Since we have an external einbein, we should expect that on-shell IH constraints are permitted, and this is indeed the case. To be explicit about how these on-shell IH constraints can be used, we will wait to impose any IH constraints and keep η^i in the action.

The leading-order action is

$$S = - \int d\tau e (\nabla X^\mu \nabla X_\mu + C \nabla X^u + m^2), \quad (3.82)$$

for some constants C and m^2 . Define the field $L_\mu \equiv C\Lambda_\mu^u$, which results in $L^2 = 0$. Then our action becomes

$$S = - \int d\tau \left(\frac{1}{e} \dot{X}^\mu \dot{X}_\mu + \dot{X}^\mu L_\mu - m^2 e \right), \quad (3.83)$$

The constraint equation coming from e is

$$\frac{1}{e^2} \dot{X}^2 = -m^2 \quad (3.84)$$

and the constraint equations coming from L_μ are

$$\dot{X}^\mu \propto L_\mu. \quad (3.85)$$

Upon squaring (3.85), we find that $\dot{X}^2 = 0$, which then forces $m^2 = 0$ on self-consistency grounds.

Thus, the effect of the on-shell IH constraint is to fix $m^2 = 0$. After we impose this constraint on m^2 , we can simply drop the Lorentz Goldstones from our action. Thus the leading-order action is

$$S = - \int d\tau \frac{1}{e} \dot{X}^\mu \dot{X}_\mu, \quad (3.86)$$

which is the standard action for a massless spin-0 point-particle.

Finally, it is worth noting that ordinary IH constraints allow one to solve for the extraneous Goldstone in terms of other Goldstones; however, dynamical IH constraints [18, 19] instead serve to impose operator constraints on the terms of the action. Notice that the equations of motion for L_μ in the action (3.78) impose the operator constraint $\dot{X}^2 = 0$. We therefore may conceive of the equations of motion for L_μ as imposing dynamical IH constraints. Similarly when we impose on-shell IH constraints in the action (3.82) (which is equivalent to computing the equations of motion for L_μ) we similarly find a constraint equation that forces $m^2 = 0$. Thus on-shell IH constraints bear a striking resemblance to dynamical IH constraints.

3.8 Summary and discussion

In this chapter, we defined a generalized coset construction for systems with spontaneously broken Poincaré symmetry. The motivation was that when Poincaré symmetry is preserved, Goldstone's theorem is extremely restrictive: all Goldstones must be spin-0 bosonic particles with vanishing mass; however, when Poincaré symmetry is spontaneously broken, many more possibilities exist. In such cases, systems with identical symmetry-breaking patterns may possess inequivalent Goldstone spectra. To illustrate this diversity, we chose to focus on the relativistic point particle. All relativistic point-particles have identical (or nearly identical) SSB patterns; yet they can have any mass $m \geq 0$ and have any spin $s \in \mathbb{N}/2$. With our new-and-improved coset construction, we formulated effective actions for point-particles of arbitrary mass and spin. Along the way we identified a novel kind of inverse Higgs constraint that we termed the *on-shell IH constraint*, which arises when constructing EFTs for massless particles. This IH constraint bears

a striking resemblance to the so-called dynamical IH constraint used to construct EFTs for fermi liquids [18, 19].

With our new coset philosophy, the SSB pattern is no longer the only object of concern. Inspired by [40, 43], we parameterized the full symmetry group with Goldstone fields defined on a worldvolume of our choosing and then imposed Gauge symmetries associated with the unbroken symmetry generators. Thus, the three ingredients that go into this novel coset construction are (1) identifying the SSB pattern, (2) choosing a worldvolume on which to construct the action and (3) picking a particular set of gauge symmetries. We were free to choose whatever gauge symmetries we liked; if we imposed the largest possible set of gauge symmetries, then the Goldstones associated with unbroken generators could be gauge-fixed to zero, thereby recovering the standard coset construction. To allow for a wide range of spins—including both bosonic and fermionic particles—we found that it was necessary to impose an $\mathcal{N} = 2s$ local SUSY on the worldline, where s is the spin of the particle. This gauged SUSY is imposed at the level of the coset, meaning that once the gauge symmetries are specified, constructing an invariant action is just a matter of ‘turning the crank’ and using the coset to read-off symmetry-invariant terms. Thus, this new coset construction provides the same advantages as the usual coset construction.

We expect that this new coset construction will prove useful in a number of areas. In particular, the ordinary coset construction has proved to be a valuable tool when constructing EFTs for condensed matter systems [20, 44, 46]. Since our new coset construction allows for fermionic degrees of freedom, it is our hope and expectation that these new techniques will facilitate useful extensions of the coset construction to account for phases of matter with low-energy fermions. For example EFTs for fermi liquids, non-fermi liquids, and bad metals might now be realizable using this novel method of cosets. We also hope to extend this coset construction to allow for non-equilibrium EFTs defined on the Schwinger-Keldysh contour [43] and to allow for explicitly broken symmetries [54] that exhibit fermionic degrees of freedom. Finally, it would be of great interest to identify the rules that determine which gauge symmetries ought to be imposed. Since we are free to choose from an infinite set of gauge symmetries, it would be extremely useful to

construct a dictionary between physical attributes of the system and emergent gauge symmetries in the coset construction.

Chapter 4: A technical analog of the cosmological constant problem and a solution thereof

4.1 A tale of two zeros

The cosmological constant problem has been a topic of heated debate for decades. In fact, it is difficult to find two theoretical physicists who agree on what exactly the problem is, how to phrase it, how to quantify it, or how many problems there are. Some maintain that there is no problem at all. (We realize that some of our colleagues will take issue with this paragraph as well.)

In this chapter we do not intend to review the extensive literature on the subject¹. Instead, as described in section 1.3, we aim to show in detail how the system of framid [20] presents a technically similar problem, and how that problem is “solved” there. To be more precise, the problem is that the stress-energy tensor of the framid is Lorentz-invariant, despite the fact that the framid is a state that spontaneously breaks Lorentz boosts. The quotes are in order: from a low-energy effective field theory viewpoint, this problem is solved by apparently miraculous cancellations. However, there is a more fundamental viewpoint, according to which those cancellations have to happen. Needless to say, it is a symmetry that ultimately enforces those cancellations, but a symmetry that is spontaneously broken. In particular, that symmetry does not enforce the cancellations through standard selection rules—the ground state is not invariant under the symmetry—but only in a roundabout way, which is utterly obscure in the low-energy effective theory computation we will perform.

We have argued in section 1.3 that a fully satisfactory symmetry argument is not available. Here we argue that it cannot be that the structure in (1.11) is only a tree-level statement, and gets modified

¹It is impossible for us to do justice to all the attempts that have been made at tackling the cosmological constant problem. For an overview, we refer the reader to [21, 55, 56, 57, 58] and references therein. In particular, ref. [58] reviews recent attempts based on relaxation mechanisms.

upon taking into account quantum corrections. Concretely, this would mean that the expectation value of the stress-energy tensor on the framid’s ground state is not Lorentz-invariant,

$$\langle T_{\mu\nu}(x) \rangle \neq -\Lambda \eta_{\mu\nu} , \quad (4.1)$$

for any Λ . This, however, does not seem to be consistent with renormalization theory and, more in general, renormalization group ideas. The reason is that we can think of our framid effective theory as a theory for physics below a certain energy scale M , with all the physics above M having been integrated out. If we integrate out some more physics, say down to a scale $M' < M$, the coefficients in the framid low-energy effective action change, but the set of allowed terms in such an action remains the same. So, what was a “quantum correction” in the energy window between M' and M , now becomes a new contribution to a tree-level coupling in the low-energy theory. But at tree-level the effective theory yields

$$\langle T_{\mu\nu}(x) \rangle = -\Lambda \eta_{\mu\nu} . \quad (4.2)$$

This suggests that the same should hold at the quantum level, up to a renormalization of Λ .

As pointed out in section 1.3, this is our technical analog of the cosmological constant problem. In the cosmological constant problem, one has that the expectation value of the real world’s stress-energy tensor that couples to gravity is zero (or fantastically smaller than the “natural” value it should have), without any manifest symmetry reasons for why it should be so. In our case, we have that the Lorentz-violating components of the expectation value of the framid’s stress-energy tensor are exactly zero, without any manifest symmetry reasons for why it should be so.

Our work is devoted to explicitly verifying eq. (4.2) for the framid to one-loop order. As one might expect, the computation will involve UV-divergent loop integrals, and we will have to pay particular attention to how we regulate such integrals. The reason is that our question has to do with Lorentz invariance, and so we need to make sure that our regulator respects it. However, since Lorentz invariance is spontaneously broken, it is not manifest in the effective theory or, more

importantly, in our loop integrals. And so, for instance, cutting off our integrals in (Euclidean) momentum space in a manifestly Lorentz invariant fashion makes no sense: the framids' Goldstones already have propagation speeds different from unity, and from one another, and so why should their loops be cutoff in a Lorentz-invariant fashion?

To address this problem, we use two regulators that can be made straightforwardly compatible with our spontaneously broken Lorentz invariance. The first is a generalization of Pauli-Villars. The second is dimensional regularization. Notice that, for the latter, since the Goldstone's effective theory has no mass parameters, the relevant integrals for our one-loop computation will all be trivially zero, which would make our check moot. To circumvent this, we couple the framid to a massive scalar particle, in all ways allowed by symmetry, and run our check in that case.

With both regulators, we find that, indeed, eq. (4.2) is obeyed at one-loop order. However, the cancellations involved in making the Lorentz-violating components vanish are absolutely nontrivial, and we were not able to find a clear structure in the actual computation that would ensure, or even suggest, those cancellations. As a further check of our techniques and of the non-triviality of the cancellations, we run the same computation in the case of a superfluid, where we *don't* expect analogous cancellations—a superfluid certainly does not respect (4.2). Indeed, in that case we find deviations from (4.2), and our result matches precisely that of an independent computation carried out elsewhere [59].

4.2 Warm-up: the free relativistic scalar

Before attempting to compute the expectation value of $T^{\mu\nu}$ for our framid, it is instructive to review how things work out for a free relativistic scalar field, even if we perform computations in a way that is *not* manifestly Lorentz covariant.

Consider thus the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2, \quad (4.3)$$

which yields the stress-energy tensor

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L} . \quad (4.4)$$

A quadratic Lagrangian can always be rewritten as a total derivative plus a term proportional to the equations of motion. So, since the vacuum is translationally invariant, the second term in $T^{\mu\nu}$ does not contribute to our expectation value, and we simply have

$$\langle T^{\mu\nu}(x) \rangle = \langle \partial^\mu \phi(x) \partial^\nu \phi(x) \rangle \quad (4.5)$$

$$= - \lim_{y \rightarrow x} \partial_x^\mu \partial_x^\nu G_W(x-y) \quad (4.6)$$

$$= \int \frac{d^4 k}{(2\pi)^4} k^\mu k^\nu \tilde{G}_W(k) , \quad (4.7)$$

where G_W is the Wightman two-point function of ϕ :

$$\tilde{G}_W(k) = \theta(k^0) (2\pi) \delta(k^2 + m^2) . \quad (4.8)$$

At this stage one could use Lorentz-invariance and conclude that

$$\langle T^{\mu\nu}(x) \rangle = \frac{1}{4} \eta^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} k^2 \tilde{G}_W(k) . \quad (4.9)$$

The integral is UV divergent and must be regulated. However, whatever its value, we see that $\langle T^{\mu\nu}(x) \rangle$ is proportional to $\eta^{\mu\nu}$, as expected.

Let's instead go back to eq. (4.7), give up manifest Lorentz invariance, and use the delta function in \tilde{G}_W to perform the integral over k^0 . Using spatial rotational invariance, we get

$$\rho \equiv \langle T^{00} \rangle = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \omega_k , \quad \langle T^{0i} \rangle = 0 , \quad p \equiv \frac{1}{3} \langle T^{ii} \rangle = \frac{1}{6} \int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k}^2}{\omega_k} , \quad (4.10)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. $\langle T^{\mu\nu} \rangle$ is proportional to $\eta^{\mu\nu}$ if and only if $\rho + p$ vanishes, but now this

seems impossible: the integrals entering ρ and p are manifestly positive definite, so, how can there be any cancellations? Then again, such integrals are UV divergent, so one should regulate them properly before jumping to conclusions.

The UV regulator used should preserve Lorentz invariance. In particular, a hard cutoff in momentum space will not do, because we have already performed the integral in k^0 , and so at this point there is no way to introduce a hard cutoff compatible with Lorentz invariance (usually this involves Wick-rotating the d^4k integral to Euclidean space, and then imposing a 4D rotationally invariant cutoff there.)

One possibility is to use dimensional regularization directly for our 3-dimensional integrals. This is compatible with Lorentz invariance because it corresponds to formulating the original theory (4.3) in $d + 1$ -dimensions, going through the manipulations (4.5)-(4.7) in $d + 1$ dimensions, and then performing the k^0 integral explicitly to be left with d -dimensional integrals. We get:²

$$\rho = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \omega_k = -m^{d+1} \frac{\Gamma\left(-\frac{d+1}{2}\right)}{2(4\pi)^{\frac{d+1}{2}}} \quad (4.11)$$

$$p = \frac{1}{2d} \int \frac{d^d k}{(2\pi)^d} \frac{\vec{k}^2}{\omega_k} = m^{d+1} \frac{\Gamma\left(-\frac{d+1}{2}\right)}{2(4\pi)^{\frac{d+1}{2}}} \quad (4.12)$$

(notice the $\frac{1}{3} \rightarrow \frac{1}{d}$ replacement in the definition of p), in agreement with $\rho + p = 0$.

Another possibility is to use a generalization of Pauli-Villars. Recall that, in the simplest case, Pauli-Villars amounts to regulating a log-divergent loop integral by a modification of the Feynman propagator of the form

$$\tilde{G}_F(k) = \frac{i}{-k^2 - m^2 + i\epsilon} \rightarrow \tilde{G}_F^{PV}(k) = \frac{i}{-k^2 - m^2 + i\epsilon} - \frac{i}{-k^2 - M^2 + i\epsilon}, \quad (4.13)$$

with M being a very large mass scale, in particular $M \gg m$. This improves the UV behavior of

²Here and for the rest of this chapter we avoid introducing the MS renormalization scale μ , which would be needed to make our dim-reg formulae dimensionally correct. We do so for notational simplicity, to avoid clutter. If one wants to reinstate μ in our formulae, one can do so just by dimensional analysis, interpreting our formulae as being expressed in $\mu = 1$ units.

the integral without affecting the IR one:

$$\tilde{G}_F^{PV}(k \ll M) \simeq \tilde{G}_F(k) , \quad \tilde{G}_F^{PV}(k \gg M) \simeq -\frac{i(M^2 - m^2)}{k^4} . \quad (4.14)$$

This is manifestly a Lorentz invariant modification of the propagator. More importantly for our applying these ideas to the framid case, such a modification corresponds to adding certain Lorentz-invariant higher derivative terms to the Lagrangian, as clear from the exact rewriting (up to $i\epsilon$'s)

$$\tilde{G}_F^{PV}(k) = \frac{i}{-\frac{M^2+m^2}{M^2-m^2} \cdot k^2 - \frac{1}{M^2-m^2} \cdot k^4 - \frac{M^2}{M^2-m^2} \cdot m^2} , \quad (4.15)$$

or, keeping only the leading order in m/M ,

$$\tilde{G}_F^{PV}(k) \simeq \frac{i}{-k^2 - k^4/M^2 - m^2} . \quad (4.16)$$

The introduction of the Pauli-Villars propagator is enough to regulate log-divergent integrals. In our case, we have quartically divergent integrals, with subleading quadratic and log divergences as well. It turns out that to cancel all these divergences, we need a three-fold modifications of the Feynman propagator. That is, calling $\tilde{G}_F(k; M)$ the Feynman propagator for generic mass M , we want to perform the replacement

$$\tilde{G}_F(k; m) \rightarrow \tilde{G}_F^{PV}(k) = \tilde{G}_F(k; m) + \sum_{a=1}^3 c_a \tilde{G}_F(k; \alpha_a M) , \quad (4.17)$$

where the c 's and α 's are suitable (order-one) coefficients, and M is a common very large mass scale, $M \gg m$.

Since we need to improve the high energy behavior of our loop integrals by four powers of k compared to the standard Pauli-Villars case of (4.14), we want our modified propagator to have the high energy behavior

$$\tilde{G}_F^{PV}(k \gg M) \sim \frac{1}{k^8} , \quad (4.18)$$

which requires the α_a 's to be all different, and the c_a 's to be

$$c_a = \prod_{b \neq a} \frac{\alpha_b^2 - \frac{m^2}{M^2}}{\alpha_b^2 - \alpha_a^2}. \quad (4.19)$$

By combining denominators as done above in (4.15), one can see that this still corresponds to adding suitable higher-derivative Lorentz-invariant terms to the Lagrangian.

Our expressions for ρ and p in eq. (4.10) do not involve directly a Feynman propagator. However, we have to remember that they came from integrating over k^0 an expression involving the Wightman two-point function of ϕ . In our case, we must replace this with

$$\tilde{G}_W(k; m) \rightarrow \tilde{G}_W^{PV}(k) = \tilde{G}_W(k; m) + \sum_{a=1}^3 c_a \tilde{G}_W(k; \alpha_a M). \quad (4.20)$$

Then, integrating again over k^0 , for ρ and p we simply get

$$\rho = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[\omega_{k; m} + \sum_{a=1}^3 c_a \omega_{k; \alpha_a M} \right] \quad (4.21)$$

$$p = \frac{1}{6} \int \frac{d^3 k}{(2\pi)^3} \vec{k}^2 \left[\frac{1}{\omega_{k; m}} + \sum_{a=1}^3 c_a \frac{1}{\omega_{k; \alpha_a M}} \right], \quad \omega_{k; M} \equiv \sqrt{\vec{k}^2 + M^2}. \quad (4.22)$$

As expected, but still surprisingly enough, choosing the c_a 's as in eq. (4.19) makes both of these integrals finite, and, in fact, opposite to each other. Namely:

$$\rho = -p = f(\alpha)M^4 + g(\alpha)m^2M^2 + \frac{1}{32\pi^2}m^4 \log(m/M) + h(\alpha)m^4, \quad (4.23)$$

where f , g , and h are somewhat complicated functions of the α coefficients, whose explicit form we spare the reader. On the other hand, the coefficient of $m^4 \log m$ is finite (for $M \rightarrow \infty$) and α -independent, in agreement with standard renormalization theory: like all non-analyticities in external momenta and mass parameters, it should be finite and calculable, that is, independent of the regulator used. In fact, if we expand the dim-reg result (4.11) for $d \rightarrow 3$, we get exactly the same coefficient for $m^4 \log m$.

4.3 The framid stress-energy tensor

We have briefly laid out the theoretical setup for framids in section 1.3. Our goal now is to check whether the stress-energy tensor resulting from this Lorentz-violating theory remains Lorentz-invariant when including quantum corrections: $\langle T_{\mu\nu}(x) \rangle = -\Lambda \eta_{\mu\nu}$. We know the off-diagonal components of the stress-energy tensor respect this condition due to rotational invariance of the ground state, hence our task reduces to proving that

$$\langle T^{00}(x) \rangle + \frac{1}{3} \langle T^{ii}(x) \rangle = 0, \quad (4.24)$$

where the spatial indices are implicitly summed over.

In order to compute the one-loop correction to $\langle T^{\mu\nu}(x) \rangle$, we start from the full covariant theory (1.6) and compute the canonical (Noether) stress-energy tensor,

$$\begin{aligned} T^{\mu\nu} &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda + g^{\mu\nu} \mathcal{L} \\ &= M_1^2 [(c_L^2 - c_T^2) \partial^\nu A^\mu \partial_\lambda A^\lambda + c_T^2 \partial^\mu A_\lambda \partial^\nu A^\lambda + (c_T^2 - 1) A^\mu \partial^\nu A_\lambda A^\rho \partial_\rho A^\lambda] + g^{\mu\nu} \mathcal{L}. \end{aligned} \quad (4.25)$$

We wish to introduce the framid Goldstones as in (1.5) and expand $T^{\mu\nu}$ up to quadratic order. The diagonal components of (4.25) reduce to

$$T^{00} = -\mathcal{L} + \dot{\vec{\eta}} \cdot \dot{\vec{\eta}}, \quad (4.26)$$

$$T^{ii} = 3\mathcal{L} + c_T^2 \partial_i \vec{\eta} \cdot \partial_i \vec{\eta} + (c_L^2 - c_T^2) (\vec{\nabla} \cdot \vec{\eta})^2. \quad (4.27)$$

Not surprisingly, this is the same result we would have gotten by applying Noether's theorem directly to eq. (1.10): even though boosts are spontaneously broken, spacetime translations are not, and so one can compute the associated Noether current using directly the $\vec{\eta}$ parametrization of the action and having $\vec{\eta}$ transform in the usual way under translations, $\vec{\eta} \rightarrow \vec{\eta} - \epsilon^\mu \partial_\mu \vec{\eta}$.

We now perform manipulations similar to those of section 4.2. Dropping the terms proportional

to the Lagrangian for the same reason as made explicit there, the ground-state expectation values of the expressions above can be written as

$$\rho \equiv \langle T^{00} \rangle = \lim_{y \rightarrow x} \partial_{t_x} \partial_{t_y} \langle \vec{\eta}(x) \cdot \vec{\eta}(y) \rangle \quad (4.28)$$

$$p \equiv \frac{1}{3} \langle T^{ii} \rangle = \frac{1}{3} \lim_{y \rightarrow x} \left[c_T^2 \partial_x^i \partial_y^i \langle \vec{\eta}(x) \cdot \vec{\eta}(y) \rangle + (c_L^2 - c_T^2) \partial_x^i \partial_y^j \langle \eta^i(x) \eta^j(y) \rangle \right] \quad (4.29)$$

Decomposing $\vec{\eta}$ into longitudinal and transverse components, the energy density becomes

$$\begin{aligned} \rho &= - \lim_{y \rightarrow x} \partial_{t_x}^2 \langle \vec{\eta}_L(x) \cdot \vec{\eta}_L(y) + \vec{\eta}_T(x) \cdot \vec{\eta}_T(y) \rangle \\ &= - \lim_{y \rightarrow x} \partial_{t_x}^2 (G_L(x-y) + 2G_T(x-y)) \\ &= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} (\omega_L + 2\omega_T), \end{aligned} \quad (4.30)$$

where G_L and G_T are the (scalar) Wightman two-point functions for longitudinal and transverse modes,

$$\tilde{G}_{L/T}(\omega, \vec{k}) = \theta(\omega) (2\pi) \delta(\omega^2 - \omega_{L/T}^2), \quad (4.31)$$

and ω_L and ω_T are the corresponding energies, $\omega_{L/T} = c_{L/T} |\vec{k}|$. The relative factor of two in (4.30) comes from the fact that $\hat{k} \cdot \hat{k} = 1$ and $\sum_i \delta^{ii} - \hat{k}^i \hat{k}^i = 2$.

Similarly, the pressure can be rewritten as

$$p = \frac{1}{6} \int \frac{d^3 k}{(2\pi)^3} \left(\frac{c_L^2 \vec{k}^2}{\omega_L} + 2 \frac{c_T^2 \vec{k}^2}{\omega_T} \right). \quad (4.32)$$

These expressions for ρ and p are very simple generalizations of the corresponding ones in section 4.2 for a generic massive scalar. However, as we did there, we first need to regularize them before we can check whether $\rho + p$ vanishes. As we emphasized in section 4.1, the regulators used should be consistent with the spontaneously broken Lorentz invariance.

4.4 Pauli-Villars regularization

Let us first consider a suitable generalization of Pauli-Villars regularization. Given the technical similarities between the derivation we just performed and that of section. 4.2, it's clear that *if* we are allowed to introduce independent Pauli-Villars modifications for the longitudinal and transverse phonons' propagators, taking into account their different speeds, then we get $\rho + p = 0$ for the framid as well.

More explicitly, consider the longitudinal phonons' contributions to ρ and p ,

$$\rho_L \equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_L, \quad p_L \equiv \frac{1}{6} \int \frac{d^3k}{(2\pi)^3} \frac{c_L^2 \vec{k}^2}{\omega_L}, \quad (4.33)$$

with $\omega_L = c_L |\vec{k}|$. Apart from the integration measure, \vec{k} always appears here in the combination $c_L |\vec{k}|$. The same is true for the Wightman two-point function (4.31), where these expressions come from, and for the associated Feynman propagator,

$$\tilde{G}_F^L(\omega, \vec{k}) = \frac{i}{\omega^2 - c_L^2 \vec{k}^2 + i\epsilon}. \quad (4.34)$$

So, upon changing the integration variable, $\vec{k}' = c_L \vec{k}$, we have

$$\rho_L = \frac{1}{c_L^3} \rho_{\text{rel}}(0), \quad p_L = \frac{1}{c_L^3} p_{\text{rel}}(0), \quad (4.35)$$

where $\rho_{\text{rel}}(m)$ and $p_{\text{rel}}(m)$ are the relativistic expressions for the energy density and pressure of a massive scalar of mass m , eq. (4.10). Then, the same Pauli-Villars regularization that was applied in section 4.2 can be applied here, yielding

$$\rho_L + p_L = 0. \quad (4.36)$$

Mutatis mutandis, the same considerations can be applied to the transverse phonons' contributions

ρ_T and p_T , yielding

$$\rho_T + p_T = 0 . \quad (4.37)$$

And so, combining the longitudinal and transverse sectors, $\rho + p = 0$.

Notice however that the longitudinal and transverse Goldstones have different speeds in general, and so the Pauli-Villars regularization procedure that we are advocating has to be different for the two sectors. Namely, referring to the explicit analysis of section 4.2, whenever there is a k^2 in a propagator, when we deal with the longitudinal sector we have to replace that with $-\omega^2 + c_L^2 \vec{k}^2$, and when we deal with the transverse sector we have to replace that with $-\omega^2 + c_T^2 \vec{k}^2$.

Recall that in standard relativistic cases, such as that studied in section 4.2, a Pauli-Villars modification of the propagator can be thought of as coming directly from a suitable local higher-derivative modification of the action. So, in our case the question is whether there is a local and Lorentz-invariant higher-derivative modification of the framid's action that corresponds to *independent* Pauli-Villars modifications of the longitudinal and transverse propagators of the desired type. The requirements of locality and Lorentz-invariance are nontrivial—the former because the longitudinal/transverse splitting of $\vec{\eta}$ is non-local, the latter because $\vec{\eta}$ transforms nonlinearly under Lorentz boosts.

Let's start with locality. The modifications of the Feynman propagators we are after, upon combining denominators as explained in section. 4.2, take the form

$$\tilde{G}_F^{L,PV}(\omega, \vec{k}) = \frac{i}{-k_L^2 - \frac{1}{\Lambda_1^2} k_L^4 - \frac{1}{\Lambda_2^4} k_L^6 - \frac{1}{\Lambda_3^6} k_L^8} , \quad k_L^2 \equiv -\omega^2 + c_L^2 \vec{k}^2 , \quad (4.38)$$

and similarly for the transverse propagator. The Λ_a 's are suitable combinations of the Pauli-Villars pole masses. Importantly for what follows, they are the same for the longitudinal and transverse propagators, as long as the pole masses are chosen to be the same for the two sectors. This can be understood easily by thinking about the structure of the propagators at $\vec{k} = 0$, in which case the fact that the longitudinal and transverse propagation speeds are different does not matter.

Using the canonical normalization of (1.10), up to total derivatives these propagators correspond

to the quadratic Lagrangian

$$\mathcal{L}_2^{PV} = \frac{1}{2} \vec{\eta}_L \cdot \left[\square_L - \frac{1}{\Lambda_1^2} \square_L^2 + \frac{1}{\Lambda_2^4} \square_L^3 - \frac{1}{\Lambda_3^6} \square_L^4 \right] \vec{\eta}_L + (L \rightarrow T), \quad (4.39)$$

where \square_L denotes the differential operator

$$\square_L \equiv -\partial_t^2 + c_L^2 \nabla^2, \quad (4.40)$$

and ‘ $(L \rightarrow T)$ ’ stands for a similar structure involving the transverse field $\vec{\eta}_T$ and its propagation speed. Our question of locality is thus reduced to the question of whether this quadratic Lagrangian can be written as a local quadratic Lagrangian for the full $\vec{\eta}$ field.

Notice that as long as at least one spatial laplacian acts on η_L or η_T , one can easily perform the longitudinal/transverse decomposition in a local fashion: defining the local differential operator matrix,

$$D_{ij} = \partial_i \partial_j, \quad (4.41)$$

we simply have

$$\nabla^2 \vec{\eta}_L = D \cdot \vec{\eta}, \quad \nabla^2 \vec{\eta}_T = (\nabla^2 - D) \cdot \vec{\eta}. \quad (4.42)$$

Notice also that, in the quadratic action (4.39), it is enough that the $\vec{\eta}$ ’s on the right—those acted upon by derivatives—be split into longitudinal and transverse. The undifferentiated $\vec{\eta}$ ’s on the left can be replaced with the full $\vec{\eta}$ field, because, as usual, at quadratic order all longitudinal-transverse mixings automatically vanish.

So, the only remaining question concerning locality is whether the terms in (4.39) with time-derivatives *only* can be rewritten in a local fashion. However, the coefficients of such terms are the same for the longitudinal and transverse sectors. This, upon using again the vanishing of longitudinal-transverse mixings, allows us to combine these terms into purely time-derivative terms

for the full $\vec{\eta}$ field:

$$\mathcal{L}_2^{PV} \supset \frac{1}{2} \vec{\eta}_L \cdot \left[-\partial_t^2 - \frac{1}{\Lambda_1^2} \partial_t^4 - \frac{1}{\Lambda_2^4} \partial_t^6 - \frac{1}{\Lambda_3^6} \partial_t^8 \right] \vec{\eta}_L + (L \rightarrow T) \quad (4.43)$$

$$= \frac{1}{2} \vec{\eta} \cdot \left[-\partial_t^2 - \frac{1}{\Lambda_1^2} \partial_t^4 - \frac{1}{\Lambda_2^4} \partial_t^6 - \frac{1}{\Lambda_3^6} \partial_t^8 \right] \vec{\eta} \quad (4.44)$$

So, in summary, a local rewriting of (4.39) involves the following building blocks:

$$\vec{\eta} \cdot \partial_t^{2a} (\nabla^2)^b D^c \cdot \vec{\eta}, \quad (4.45)$$

with different integer non-negative values for a , b , and c , up to $a+b+c = 4$, and suitable coefficients. In particular, we can restrict to $c = 0, 1$, because D is proportional to a projector operator (its only role is to isolate the longitudinal component of $\vec{\eta}$):

$$D \cdot D = \nabla^2 \cdot D. \quad (4.46)$$

We can now ask whether these building blocks are compatible with the spontaneously broken Lorentz invariance. In particular, we can ask whether there exist manifestly Lorentz-invariant combinations of A_μ and ∂_μ that, when expanded to quadratic order in the $\vec{\eta}$ fields, reduce precisely to these building blocks.

It is quite easy to convince oneself that the answer is yes. To this end, it is convenient to integrate by parts half of the derivatives in (4.45). Then, up to a possible sign, the building blocks are

$$\left((\partial_t)^a \partial_{i_1} \dots \partial_{i_b} \eta^k \right)^2 \quad (c = 0) \quad (4.47)$$

and

$$\left((\partial_t)^a \partial_{i_1} \dots \partial_{i_b} (\vec{\nabla} \cdot \vec{\eta}) \right)^2 \quad (c = 1) \quad (4.48)$$

Recalling that, to first order in $\vec{\eta}$, A_μ is simply

$$A_0 \simeq 1, \quad \vec{A} \simeq \vec{\eta}, \quad (4.49)$$

we see that two simple Lorentz-invariant generalizations of (4.47) and (4.48) that reduce to them to quadratic order in $\vec{\eta}$ are

$$((\partial^\parallel)^a \partial_{\mu_1}^\perp \dots \partial_{\mu_b}^\perp A_\nu)^2 \quad (c = 0) \quad (4.50)$$

and

$$((\partial^\parallel)^a \partial_{\mu_1}^\perp \dots \partial_{\mu_b}^\perp (\partial_\nu A^\nu))^2, \quad (c = 1) \quad (4.51)$$

where ∂^\parallel and ∂_μ^\perp are defined as

$$\partial^\parallel \equiv -A^\alpha \partial_\alpha, \quad \partial_\mu^\perp \equiv \partial_\mu + A_\mu A^\alpha \partial_\alpha. \quad (4.52)$$

In conclusion: there exist local Lorentz-invariant higher-derivative corrections to the framid action that modify the $\vec{\eta}$ propagator in a Pauli-Villars fashion, with suitable independent modifications for the longitudinal and transverse modes, such that the Pauli-Villars analysis of section 4.2 can be separately applied to the two sectors, yielding $\rho + p = 0$.

4.5 Dimensional regularization

Let's now consider dimensional regularization. As for the relativistic case considered in section 4.2, dimensional regularization of our spatial momentum integrals (4.30) and (4.32) is consistent with Lorentz invariance. This is because we can think of it as corresponding to formulating the original manifestly Lorentz-invariant theory for A_μ , eq. (1.6), in $d + 1$ spacetime dimensions, and then going through all the subsequent steps that led us to (4.30) and (4.32) keeping d generic. Were we to do so, we would end up with the same integrals (4.30) and (4.32), but in d rather than

3 dimensions, and with a $1/2d$ rather than $1/6$ prefactor for the pressure.

Now, since our effective theory does not feature mass parameters for the Goldstone fields, the integrals (4.30) and (4.32) vanish in dimensional regularization. This is certainly consistent with $\rho + p = 0$, but in a trivial way. To run a nontrivial check, we need to deform our theory. We must do so consistently with Lorentz invariance, of course. And so, in particular, mass parameters for the Goldstones are not allowed.

A particularly physical way to deform the theory is to couple the framid to a massive field, in all ways allowed by symmetry. For simplicity, let's take this to be a scalar field, ϕ . If its mass is below the cutoff of the effective theory (the M_1 of section 4.3, up to suitable powers of $c_{L/T}$), ϕ must be included in our computation. Assuming it has zero expectation value, for our one-loop computation we are interested in all Lorentz-invariant combinations of A_μ and ϕ with up to two derivatives and which, once expanded about the framid's background configuration, yield quadratic terms in the $\vec{\eta}$ and ϕ fields.

We find that there are only three interactions with these properties:

$$\mathcal{L} \supset A_\mu \partial^\mu \phi, \quad A_\mu A_\nu \partial^\mu \partial^\nu \phi, \quad A_\mu A_\nu \partial^\mu \phi \partial^\nu \phi. \quad (4.53)$$

All other possibilities are either related to these through integration by parts or, when expanded in the Goldstone fields to the desired order, yield terms that are total derivatives themselves, and can thus be neglected. Adding the terms above to the framid action, and including also standard kinetic and mass terms for ϕ , which are Lorentz-invariant by themselves, our effective Lagrangian at quadratic order becomes

$$\begin{aligned} \mathcal{L}_2 \rightarrow \frac{1}{2} & \left[\dot{\vec{\eta}}^2 - c_L^2 (\vec{\nabla} \cdot \vec{\eta}_L)^2 - c_T^2 (\partial_i \eta_T^j)^2 + \dot{\phi}^2 - (\vec{\nabla} \phi)^2 - m^2 \phi^2 \right. \\ & \left. + 2b_1 \phi \vec{\nabla} \cdot \vec{\eta}_L + 2b_2 \dot{\phi} \vec{\nabla} \cdot \vec{\eta}_L + b_3 \dot{\phi}^2 \right], \end{aligned} \quad (4.54)$$

where the b_a 's are generic coupling constants. Notice that the transverse components of the Goldstone fields don't mix with the massive scalar, since $\vec{\nabla} \cdot \vec{\eta}_T = 0$.

The relevant components of the stress-energy tensor now read

$$T^{00} = \dot{\vec{\eta}}^2 + b_2 \dot{\phi} \vec{\nabla} \cdot \vec{\eta}_L + (1 + b_3) \dot{\phi}^2, \quad (4.55)$$

$$T^{ii} = c_L^2 (\vec{\nabla} \cdot \vec{\eta}_L)^2 + c_T^2 \partial_i \vec{\eta}_T \cdot \partial_i \vec{\eta}_T - b_1 \phi \vec{\nabla} \cdot \vec{\eta}_L - b_2 \dot{\phi} \vec{\nabla} \cdot \vec{\eta}_L + (\vec{\nabla} \phi)^2, \quad (4.56)$$

where we have omitted terms proportional to the Lagrangian, since, as before, they do not contribute to our expectation values.

Notice that, at this order, the transverse Goldstones $\vec{\eta}_T$ are completely decoupled from ϕ , both at the level of the Lagrangian and as far as their contributions to the stress-energy tensor are concerned. The system is thus divided into two sectors: a transverse one, for which the presence of ϕ is irrelevant, and which thus has vanishing energy density and pressure in dimensional regularization,

$$\rho_T = \int \frac{d^d k}{(2\pi)^d} c_T |\vec{k}| = 0, \quad p_T = \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} c_T |\vec{k}| = 0, \quad (4.57)$$

and a longitudinal one, consisting of $\vec{\eta}_L$ and ϕ , for which the computation of the energy and pressure is, as we will now see, quite involved.

From now on, we will restrict to the longitudinal sector, which amounts to setting $\vec{\eta}_T$ to zero in the formulae above. Also, for notational simplicity we will drop the subscript ‘L’ from $\vec{\eta}_L$ (but we will remember that we are dealing with a longitudinal field.)

4.5.1 Analysis for b_3

Consider first the case in which b_1 and b_2 are set to zero while b_3 is nonzero. In that case, $\vec{\eta}$ and ϕ are completely decoupled (at this order), and their contributions to $\rho + p$ can be analyzed separately. For $\vec{\eta}$, there isn’t much to say: the relevant integrals for ρ and p vanish in dimensional regularization, because they do not involve any mass scale—that is, they are integrals of pure powers.

The situation is more interesting for ϕ . Its action is that of a massive scalar with a propagation

speed different from one,

$$S = \int d^4x \frac{1}{2} [\dot{\phi}^2 - c_\phi^2 (\vec{\nabla}\phi)^2 - M_\phi^2 \phi^2] , \quad c_\phi^2 \equiv \frac{1}{1 + b_3} , \quad M_\phi \equiv c_\phi m , \quad (4.58)$$

where for convenience we redefined the normalization of ϕ by $\phi \rightarrow c_\phi \phi$.

Applying to the purely ϕ parts of eqs. (4.55), (4.56) the same manipulations as in the case of a relativistic scalar (see section. 4.2), we find that the integrals we should compute are

$$\rho_\phi = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \sqrt{c_\phi^2 \vec{k}^2 + M_\phi^2} , \quad p_\phi = c_\phi^2 \times \frac{1}{2d} \int \frac{d^d k}{(2\pi)^d} \frac{\vec{k}^2}{\sqrt{c_\phi^2 \vec{k}^2 + M_\phi^2}} . \quad (4.59)$$

Up to a redefinition of the integration variable, $\vec{k} = \vec{k}'/c_\phi$, these are clearly the same integrals as those of section 4.2. What's perhaps surprising is that, after such a change of variables, the overall powers of c_ϕ we are left with are the same for ρ_ϕ and for p_ϕ :

$$\rho_\phi = \frac{1}{c_\phi^d} \rho_{\text{rel}}(M_\phi) , \quad p_\phi = \frac{1}{c_\phi^d} p_{\text{rel}}(M_\phi) , \quad (4.60)$$

where $\rho_{\text{rel}}(m)$ and $p_{\text{rel}}(m)$ are the energy density and pressure for a relativistic scalar of mass m , eqs. (4.11) and (4.12). So, once again, we get

$$\rho_\phi + p_\phi = 0 . \quad (4.61)$$

This apparent accident is in fact a consequence of a formal (spurionic) invariance of the action (4.58): if we rescale the spatial coordinates but not time, and compensate for this by a rescaling of c_ϕ and of ϕ ,

$$\vec{x} \rightarrow \lambda \vec{x} , \quad c_\phi \rightarrow \lambda c_\phi , \quad \phi \rightarrow \lambda^{-3/2} \phi , \quad (4.62)$$

the action does not change. As usual for spurion analyses, this informs how physical quantities can

depend on c_ϕ . In particular, we are interested in

$$\rho_\phi = \langle T^{00} \rangle, \quad p_\phi = \frac{1}{d} \langle T^{ii} \rangle, \quad (4.63)$$

for a state that is invariant under translations. These expectation values are thus constant in \vec{x} , and so their transformation properties under rescalings of coordinates must be completely taken care of by explicit powers of c_ϕ . T^{00} is a spatial density of energy. Since energy $\sim \text{time}^{-1}$ does not change under a rescaling of spatial coordinates, we must have

$$T^{00} \rightarrow \frac{1}{\lambda^d} T^{00} \quad \Rightarrow \quad \langle T^{00} \rangle \propto \frac{1}{c_\phi^d}. \quad (4.64)$$

T^{ij} is not the density of a conserved quantity. However, it is related to the momentum density T^{0j} by the conservation equation

$$\partial_0 T^{0j} + \partial_i T^{ij} = 0. \quad (4.65)$$

The momentum density rescales as

$$T^{0j} \rightarrow \frac{1}{\lambda^{d+1}} T^{0j}, \quad (4.66)$$

because momentum itself is the inverse of a length, and so it must rescale as $\vec{P} \rightarrow \vec{P}/\lambda$. Using (4.65) and (4.66), we thus have

$$T^{ij} \rightarrow \frac{1}{\lambda^d} T^{ij} \quad \Rightarrow \quad \langle T^{ij} \rangle \propto \frac{1}{c_\phi^d} \quad (4.67)$$

We thus see that both ρ_ϕ and p_ϕ depend on c_ϕ exactly in the same way, in agreement with our explicit result in (4.60).

In more intuitive terms, all of the above stems from the statement that, even if we abandon natural units and we give independent units to mass (m), length (ℓ), and time (t), energy density

and pressure still have the same units³:

$$\left[\frac{\text{energy}}{\text{volume}} \right] = \frac{m}{\ell \cdot t^2}, \quad [\text{pressure}] = \left[\frac{\text{force}}{\text{surface}} \right] = \frac{m}{\ell \cdot t^2}. \quad (4.68)$$

Their ratio is thus dimensionless, and must be the same in all unit systems. In particular if it is -1 in units such that $c_\phi = 1$, then it must be -1 even when $c_\phi \neq 1$.

The arguments above show that, as far as our check is concerned, we can consistently set b_3 to zero, even when we turn b_1 and b_2 back on. The reason is that we can work in units such that $c_\phi = 1$, which corresponds to $b_3 = 0$, and run our check in those units. Going from natural units ($c = 1$) to $c_\phi = 1$ units certainly affects the values of the other parameters of the theory: c_L , c_T , M_1 , b_1 , b_2 , and m . However, since we are leaving these generic anyway, such a change has no repercussions for our check⁴.

4.5.2 Analysis for b_1 and b_2

We may then switch off b_3 and perform our computation for non-zero values of the mixing coefficients b_1 and b_2 . Combining eqs. (4.55) and (4.56), switching to a generic number of spatial dimensions, $3 \rightarrow d$, and dropping again terms proportional to the quadratic Lagrangian, we obtain the following expression for $\rho + p$:

$$\rho + p \equiv \langle T^{00} + \frac{1}{d} T^{ii} \rangle = \frac{d+1}{d} \left(\dot{\vec{\eta}}^2 + \dot{\phi}^2 \right) - \frac{1}{d} m^2 \phi^2 + \frac{1}{d} b_1 \phi \vec{\nabla} \cdot \vec{\eta} + \frac{d+1}{d} b_2 \dot{\phi} \vec{\nabla} \cdot \vec{\eta}. \quad (4.69)$$

³For instance, in SI units, $1 \text{ J/m}^3 = 1 \text{ kg/m s}^2 = 1 \text{ Pa}$.

⁴For more general questions, say computing ρ and p separately rather than just checking $\rho + p = 0$, one can still first work in $c_\phi = 1$ units and then reinstate the dependence on c_ϕ after the fact, by taking into account how the other parameters change when we change units. For instance the parameter c_L in $c_\phi = 1$ units becomes c_L/c_ϕ in any other unit system.

We can now perform the same manipulations as in section 4.2. We find:

$$\begin{aligned}
\rho + p &= \frac{1}{d} \lim_{x \rightarrow y} \left[- (d+1) \partial_t^2 \langle \vec{\eta}(x) \cdot \vec{\eta}(y) \rangle - ((d+1) \partial_t^2 + m^2) \langle \phi(x) \phi(y) \rangle \right. \\
&\quad \left. + (b_1 - (d+1)b_2 \partial_t) \vec{\nabla} \cdot \langle \vec{\eta}(x) \phi(y) \rangle \right] \\
&= \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{d\omega}{2\pi} \left[(d+1) \omega^2 \tilde{G}_W^{m\eta}(\omega, \vec{k}) + ((d+1) \omega^2 - m^2) \tilde{G}_W^{\phi\phi}(\omega, \vec{k}) \right. \\
&\quad \left. + (ib_1 - (d+1)b_2 \omega) |\vec{k}| \tilde{G}_W^{\eta\phi}(\omega, \vec{k}) \right], \tag{4.70}
\end{aligned}$$

where \tilde{G}_W^{ab} is the matrix of Wightman two-point functions for the fields $\psi^a = (\eta, \phi)$ and, thanks to the longitudinality of $\vec{\eta}$, we were able to switch to a purely scalar notation,

$$\tilde{\eta}(\omega, \vec{k}) \equiv \hat{k} \tilde{\eta}(\omega, \vec{k}). \tag{4.71}$$

As usual, from the quadratic Lagrangian we can easily compute the matrix of Feynman propagators, $\tilde{G}_F^{ab}(\omega, \vec{k})$. However, going from these to the Wightman two-point functions requires some work. The general relationship, which we review in Appendix C, is

$$\tilde{G}_W^{ab}(\omega, \vec{k}) = \int \frac{d\omega'}{2\pi} \left[\frac{i}{\omega - \omega' + i\epsilon} \tilde{G}_F^{ab}(\omega', \vec{k}) - \frac{i}{\omega - \omega' - i\epsilon} \tilde{G}_F^{ab*}(-\omega', -\vec{k}) \right] \tag{4.72}$$

$$= \sum_n \left[K^{-1}(\omega, \vec{k})(\omega - \omega_n) \right]^{ab} (2\pi) \delta(\omega - \omega_n), \tag{4.73}$$

where K is the kinetic matrix appearing in the quadratic action,

$$S = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{d\omega}{2\pi} \tilde{\psi}^{*a}(\omega, \vec{k}) K_{ab}(\omega, \vec{k}) \tilde{\psi}^b(\omega, \vec{k}), \tag{4.74}$$

and the $\omega_n = \omega_n(\vec{k})$ are the positive-energy poles of the Feynman propagators.

Using this in eq. (4.70) and performing the ω integral leaves us with

$$\rho + p = \frac{1}{d} \sum_n \int \frac{d^d k}{(2\pi)^d} \left[(d+1) \omega_n^2 R_n^{\eta\eta}(\vec{k}) + ((d+1) \omega_n^2 - m^2) R_n^{\phi\phi}(\vec{k}) + (ib_1 - (d+1)b_2 \omega_n) |\vec{k}| R_n^{\eta\phi}(\vec{k}) \right], \quad (4.75)$$

where the R 's are the residues

$$R_n^{ab}(\vec{k}) = \lim_{\omega \rightarrow \omega_n} \left[K^{-1}(\omega, \vec{k}) (\omega - \omega_n) \right]^{ab}. \quad (4.76)$$

The positions of the poles can be computed for generic b_1 and b_2 , and so can the associated residues. However, the resulting expressions involve somewhat complicated double square root structures, which makes it impossible for us to perform the final integral in \vec{k} explicitly. To circumvent this problem, we can consider the small b_1, b_2 limit, and expand to the first nontrivial order in these couplings.

The kinetic matrix for η and ϕ associated with the quadratic Lagrangian (4.54) is

$$K = \begin{pmatrix} \omega^2 - c_L^2 \vec{k}^2 & -(b_2 \omega + ib_1) |\vec{k}| \\ -(b_2 \omega - ib_1) |\vec{k}| & \omega^2 - \vec{k}^2 - m^2 \end{pmatrix} \quad (4.77)$$

The poles of the Feynman propagators are the zeros of $\det K$, which to quadratic order in b_1, b_2 read

$$\begin{aligned} \omega_1 &\simeq c_L |\vec{k}| - \frac{|\vec{k}| (b_1^2 + b_2^2 c_L^2 \vec{k}^2)}{2c_L ((1 - c_L^2) \vec{k}^2 + m^2)}, \\ \omega_2 &\simeq \sqrt{\vec{k}^2 + m^2} + \frac{\vec{k}^2 (b_1^2 + b_2^2 (\vec{k}^2 + m^2))}{2\sqrt{\vec{k}^2 + m^2} ((1 - c_L^2) \vec{k}^2 + m^2)}. \end{aligned} \quad (4.78)$$

Upon inverting K , expanding the residues (4.76) also to quadratic order, and plugging these

expansions into our expression for $\rho + p$, eq. (4.75), we end up with

$$\rho + p = \frac{1}{d} \sum_A C_A \int \frac{d^d k}{(2\pi)^d} \frac{|\vec{k}|^{\alpha_A}}{((1 - c_L^2) \vec{k}^2 + m^2)^{\gamma_A}} \sqrt{\vec{k}^2 + m^2}^{\beta_A}, \quad (4.79)$$

where the C_A 's are suitable coefficients, and $\alpha_A, \beta_A, \gamma_A$ are powers that vary in combinations, yielding in total eleven structurally distinct terms, as outlined in Table 4.1.

| α | β | γ | C |
|----------|---------|----------|---|
| 1 | 0 | 0 | $\frac{1}{2} c_L (d + 1)$ |
| 2 | -1 | 0 | $\frac{1}{2}$ |
| 0 | 1 | 0 | $\frac{d}{2}$ |
| 6 | -1 | 2 | $\frac{b_2^2}{4} (1 - c_L^2) (d + 1)$ |
| 4 | -1 | 2 | $\frac{b_1^2}{4} (d - 1) + \frac{b_2^2}{4} m^2 (3 + d(2 - c_L^2))$ |
| 2 | -1 | 2 | $\frac{b_1^2}{4} m^2 (d + 2) + \frac{b_2^2}{4} m^4 (d + 2)$ |
| 5 | 0 | 2 | $-\frac{b_2^2}{4} c_L (1 - c_L^2) (d + 1)$ |
| 3 | 0 | 2 | $-\frac{b_1^2}{4c_L} (1 - c_L^2) (d - 1) - \frac{b_2^2}{4} c_L m^2 (d + 3)$ |
| 1 | 0 | 2 | $-\frac{b_1^2}{4c_L} m^2 (d + 1)$ |
| 6 | -3 | 2 | $-\frac{b_1^2}{4} c_L^2 (d - 1)$ |
| 4 | -3 | 2 | $-\frac{b_1^2}{4} c_L^2 m^2 d$ |

Table 4.1: Coefficients C_A for all the values of the powers $\alpha_A, \beta_A, \gamma_A$ that appear in (4.79).

Notice that the first three terms in Table 4.1 are the contributions one gets from the free theories of the Goldstones and the massive scalar. Namely, the first term is

$$\rho + p \supset \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(c_L |\vec{k}| + \frac{1}{d} c_L |\vec{k}| \right) \equiv \rho_L + p_L, \quad (4.80)$$

in agreement with equations (4.30) and (4.32) and, since it involves only pure powers of the momentum, it integrates to zero in dimensional regularization. The second and third terms are the

pressure and energy densities of the free massive scalar,

$$\rho + p \supset \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{d} \frac{\vec{k}^2}{\sqrt{\vec{k}^2 + m^2}} + \sqrt{\vec{k}^2 + m^2} \right) \equiv p_\phi + \rho_\phi, \quad (4.81)$$

whose sum also equals zero, as was shown in section 4.2.

For the rest of the terms, we switch to polar coordinates and perform the integral in the radial k -direction; the closed-form result for this type of integrals, dropping the coefficients C_A , is

$$\begin{aligned} & \frac{1}{2} m^{d+\alpha_j+\beta_j-2\gamma_j} \left[\frac{(1-c_L^2)^{\frac{-d-\alpha_j}{2}} \Gamma\left(\frac{d+\alpha_j}{2}\right) \Gamma\left(\frac{-d-\alpha_j+2\gamma_j}{2}\right) {}_2F_1\left(\frac{d+\alpha_j}{2}, -\frac{\beta_j}{2}; \frac{2+d+\alpha_j-2\gamma_j}{2}; \frac{1}{1-c_L^2}\right)}{\Gamma(\gamma_j)} \right. \\ & \left. + \frac{(1-c_L^2)^{-\gamma_j} \Gamma\left(\frac{d+\alpha_j-2\gamma_j}{2}\right) \Gamma\left(\frac{-d-\alpha_j-\beta_j+2\gamma_j}{2}\right) {}_2F_1\left(\gamma_j, \frac{-d-\alpha_j-\beta_j+2\gamma_j}{2}; \frac{2-d-\alpha_j+2\gamma_j}{2}; \frac{1}{1-c_L^2}\right)}{\Gamma\left(-\frac{\beta_j}{2}\right)} \right], \end{aligned} \quad (4.82)$$

where ${}_2F_1$ is a hypergeometric function. For some structures in the integrand of equation (4.79), the integration result takes much simpler forms; for instance the term with $\alpha = 5, \beta = 0, \gamma = 2$ yields

$$\int \frac{d^d k}{(2\pi)^d} \frac{|\vec{k}|^5}{((1-c_L^2)\vec{k}^2 + m^2)^2} = -\frac{\pi}{4} m^{1+d} (1-c_L^2)^{\frac{-d-5}{2}} (3+d) \sec\left(\frac{d\pi}{2}\right). \quad (4.83)$$

Other structures, particularly when $\beta \neq 0$, integrate to totally nontrivial combinations of hypergeometric, Gamma, and trigonometric functions, as seen in the generalized form in (4.82), and so displaying them here wouldn't provide much intuition.

It is miraculous to see that all these terms, when summed, cancel exactly with each other, despite the highly complicated structures and combinations of coefficients involved. What's also interesting, is that we don't need to specify a number of spatial dimensions to obtain the final answer. That is, for any d , we find:

$$\rho + p = 0. \quad (4.84)$$

4.6 Superfluid check

As a check of our methods and computational tools, we now look at the case of a superfluid, for which we know that the stress-energy tensor does *not* have a Lorentz invariant expectation value. In particular, we want to make sure that the mysterious cancellations that yield $\rho + p = 0$ in the framid case are not a result of potential nuances of the computation. For a superfluid, $\langle T_{\mu\nu}(x) \rangle \neq -\Lambda \eta_{\mu\nu}$ already at tree level, and so an analogous computation should yield a nonzero result.

The simplest implementation of a superfluid EFT involves a single scalar field $\psi(x)$ with a shift symmetry, $\psi \rightarrow \psi + \text{const}$, and a time-dependent vev, $\langle \psi(x) \rangle = \mu t$ [20, 60, 61], where μ is the chemical potential. The superfluid phonon field, $\pi(x)$, parametrizes fluctuations around this background, $\psi(x) \equiv \mu t + \pi(x)$. To lowest order in derivatives, the most general low-energy effective action is

$$S = \int d^4x P(X), \quad X \equiv -\partial_\mu \psi \partial^\mu \psi, \quad (4.85)$$

where $P(X)$ is a generic function, in one-to-one correspondence with the superfluid's equation of state⁵. Upon expanding to quadratic order in the phonon field and choosing canonical normalization, one gets

$$S \rightarrow \frac{1}{2} \int d^4x \left[\dot{\pi}^2 - c_s^2 (\vec{\nabla} \pi)^2 \right], \quad (4.86)$$

with the sound speed given in terms of derivatives of the Lagrangian,

$$c_s^2 = \frac{P'(X)}{P'(X) + 2XP''(X)}. \quad (4.87)$$

Similarly to the framid case, in order to run a nontrivial check in dimensional regularization we couple the superfluid to a massive scalar ϕ . We look for all possible shift-symmetric, Lorentz-invariant couplings that, when expanded in the superfluid phonons $\pi(x)$, yield quadratic terms in π and ϕ , with at most two derivatives acting on them. Since $X = \mu^2 + 2\mu\dot{\pi} + \dot{\pi}^2 - (\vec{\nabla}\pi)^2$, the

⁵Namely, the function P relates the pressure to the chemical potential: $p = P(\mu^2)$.

couplings that produce Lorentz-violating structures are:

$$\begin{aligned}
f_1(X)\phi &\rightarrow \dot{\pi}\phi \\
f_2(X)\partial_\mu\psi\partial^\mu\phi &\rightarrow \dot{\pi}\dot{\phi}, \vec{\nabla}\pi \cdot \vec{\nabla}\phi \\
f_3(X)\partial_\mu\psi\partial_\nu\psi\partial^\mu\phi\partial^\nu\phi &\rightarrow \dot{\phi}^2.
\end{aligned}$$

For simplicity, we focus only on the first type of coupling, which introduces a $\dot{\pi}\phi$ term in the quadratic Lagrangian. Moreover, we choose the simplest possible form for f_1 , that is $f_1(X) = X$. Finally, we work in the $c_s^2 = 1$ limit (which implies $P''(X) = 0$). These choices give us the opportunity to compare our result directly to an independent path-integral calculation [59].

So, in summary, we start from

$$\mathcal{L} = P(X) + X\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \quad (4.88)$$

whose stress-energy tensor is

$$T^{\mu\nu} = 2P'(X)\partial^\mu\psi\partial^\nu\psi + 2\partial^\mu\psi\partial^\nu\psi\phi + \partial^\mu\phi\partial^\nu\phi, \quad (4.89)$$

omitting, as before, terms proportional to the Lagrangian. Expanding to quadratic order and canonically normalizing π , we get

$$\mathcal{L} \simeq \frac{1}{2} \left[\dot{\pi}^2 - (\vec{\nabla}\pi)^2 + 4b\dot{\pi}\phi + \partial_\mu\phi\partial^\mu\phi - m^2\phi^2 \right], \quad (4.90)$$

and

$$T^{00} \simeq \dot{\pi}^2 + \dot{\phi}^2 + 4b\dot{\pi}\phi, \quad T^{ii} \simeq (\vec{\nabla}\pi)^2 + (\vec{\nabla}\phi)^2, \quad (4.91)$$

where the spatial indices are summed over and $b \equiv \mu/\sqrt{2P'(\mu^2)}$ ⁶.

⁶Notice that the expressions (4.91) can be obtained by applying Noether's theorem directly to the quadratic Lagrangian (4.90), but only if one takes into account that π transforms nonlinearly under time-translations, since these are spontaneously broken by the background $\psi = \mu t$. This is related to the statement that the superfluid's ground state

Performing manipulations similar to those of the framid case, we rearrange some terms and obtain

$$T^{00} + \frac{1}{d}T^{ii} = \frac{d+1}{d}(\dot{\pi}^2 + \dot{\phi}^2) - \frac{1}{d}m^2\phi^2 + 4\frac{d+1}{d}b\dot{\pi}\phi - \frac{2}{d}\mathcal{L}, \quad (4.92)$$

whose expectation value is given by

$$\rho + p \equiv \langle T^{00} + \frac{1}{d}T^{ii} \rangle = \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{d\omega}{2\pi} \left[(d+1)\omega^2 \left(\tilde{G}_W^{\pi\pi}(\omega, \vec{k}) + \tilde{G}_W^{\phi\phi}(\omega, \vec{k}) \right) - m^2 \tilde{G}_W^{\phi\phi}(\omega, \vec{k}) - 4i(d+1)\omega b \tilde{G}_W^{\pi\phi}(\omega, \vec{k}) \right]. \quad (4.93)$$

Like before, we expand the Feynman propagators and their poles for small values of the coupling constant b , and then construct the Wightman two-point functions. As expected, all expressions take simpler forms in this case. Namely, the (positive-frequency) poles are

$$\begin{aligned} \omega_1 &\simeq \sqrt{\vec{k}^2 + m^2} + \frac{2b^2}{m^2} \sqrt{\vec{k}^2 + m^2}, \\ \omega_2 &\simeq |\vec{k}| - \frac{2b^2}{m^2} |\vec{k}|, \end{aligned} \quad (4.94)$$

and the corresponding residues are

$$\begin{aligned} R_1^{\pi\pi}(\vec{k}) &\simeq \frac{2b^2}{m^4} \sqrt{\vec{k}^2 + m^2}, & R_2^{\pi\pi}(\vec{k}) &\simeq \left[\frac{1}{2|\vec{k}|} - \frac{b^2(2\vec{k}^2 + m^2)}{m^4|\vec{k}|} \right], \\ R_1^{\phi\phi}(\vec{k}) &\simeq \left[\frac{1}{2\sqrt{\vec{k}^2 + m^2}} - \frac{b^2(2\vec{k}^2 + m^2)}{m^4\sqrt{\vec{k}^2 + m^2}} \right], & R_2^{\phi\phi}(\vec{k}) &\simeq \frac{2b^2}{m^4} |\vec{k}|, \\ R_1^{\pi\phi}(\vec{k}) &\simeq -i \frac{b}{m^2}, & R_2^{\pi\phi}(\vec{k}) &\simeq i \frac{b}{m^2}. \end{aligned} \quad (4.95)$$

is an eigenstate of $H - \mu Q$, but not of H , with $H = \int d^3x T^{00}$ being the Hamiltonian, and $Q = \int d^3x J^0$ the charge.

Putting everything together, the integral in question eventually becomes

$$\rho + p = \frac{b^2}{d m^2} \int \frac{d^d k}{(2\pi)^d} \frac{\vec{k}^2 - d(\vec{k}^2 + m^2)}{\sqrt{\vec{k}^2 + m^2}} \quad (4.96)$$

$$= -\frac{b^2 m^2}{4\pi^2(d+1)} \left(\frac{m^2}{4\pi}\right)^{(d-3)/2} \Gamma\left(\frac{1-d}{2}\right), \quad (4.97)$$

which is different from zero, as expected.

We now come to the result of the independent computation alluded to above. As a possible UV completion of a superfluid EFT, one can consider a massive complex scalar Φ with quartic interactions. Putting the system at finite chemical potential, one ends up with a superfluid, with our field ψ being associated with the phase of Φ . On the other hand, the radial mode of Φ is massive, and can thus be thought of as our massive scalar ϕ . For this system, one can explicitly compute first the associated $P(X)$ at tree level, and then the one-loop quantum corrections to it, in the form of the quantum effective action $\Gamma[\psi]$ [59]. (To lowest order in derivatives, this can be done via functional methods akin to those normally used to compute a Coleman-Weinberg effective potential [14, 62].) Applying Noether's theorem to $\Gamma[\psi]$, one obtains directly the one-loop *expectation value* of $T^{\mu\nu}$, which, for $\rho + p$, matches exactly the result above [59].

4.7 Summary and discussion

We close with a few remarks:

1. Our work is about Lorentz-invariance in a system that spontaneously breaks it—that is, in a system in which such a symmetry is not manifest. So, one must be particularly careful in unveiling possible sources of Lorentz breaking coming from the way one does computations. We already discussed at length how to address UV divergences in a way that is compatible with Lorentz invariance. Another subtlety one needs to address is the Lorentz invariance of the path-integral measure. Even though we did not use path integrals for our computations, canonical quantization of the Goldstones' effective theory is equivalent to path-integral

quantization with the measure

$$D\vec{\eta} \equiv \prod_x d^3\eta(x) . \quad (4.98)$$

This is not Lorentz invariant, because the $\vec{\eta}$ fields transform non-linearly under boosts ⁷. As usual, this should not matter as long as one uses dimensional regularization, but with other regulators it might matter. We investigate the issue in Appendix E. The conclusion is that this subtlety does not matter: (i) for our quantity ($\langle T^{\mu\nu} \rangle$), to the order we are doing computations (one loop), regardless of the regulator used; or (ii) for any quantity, to any order, if one uses dimensional regularization. We were thus justified to neglect it.

2. We have been emphasizing, puzzling over, and checking the fact that our expectation value $\langle T^{\mu\nu} \rangle \propto \eta^{\mu\nu}$ is Lorentz-invariant. However, as we hope the discussion in section 4.5.1 has made clear, such an expectation value is in fact compatible with *any* Lorentz invariance, that is, it is invariant under generalized boosts with an arbitrary speed-of-light parameter c . This is because $\langle T^{00} \rangle$ (energy density) and $\langle T^{ij} \rangle$ (pressure, or stresses) always have the same units, and so a statement like $\langle T^{\mu\nu} \rangle \propto \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is independent of the units used. And so, in particular, it is independent of the value of c . We find this to be an interesting twist. It could have important conceptual implications, or it could just be a technical curiosity.
3. Besides the framid, there are other cases in which the expectation value of $T^{\mu\nu}$ is more symmetric than the ground state ⁸. For instance, for a superfluid time-translations are spontaneously broken, but the expectation value in question is invariant under them (see section 4.6). However, in that case such a symmetry property can be explained by standard selection rules: the ground state spontaneously breaks time translations (H) and a $U(1)$ symmetry (Q) down to a linear combination thereof ($H - \mu Q$) [61]. Expectation values and more in general correlation functions should only be invariant under the unbroken combination. However, for operators that are neutral under the $U(1)$ symmetry, this automatically translates into

⁷Equivalently, one can phrase the problem directly in the canonical formalism [6]. See for instance ref. [63] for an analysis of the same issue for the chiral Lagrangian.

⁸We thank Lam Hui for bringing this up.

invariance under time-translations. $T^{\mu\nu}$ is one such operator, and so its expectation value is invariant under time translations, even though the ground state is not. We do not see any mechanism like this at play in the framid case.

What is the general lesson of our analysis? Are there implications for the cosmological constant problem? In general terms, our analysis exhibits an explicit example of a quantum system in which a certain expectation value is invariant under a symmetry even though there are no selection rules (including those of remark 3 above) enforcing this. In order to find a potential application of this phenomenon to the cosmological constant problem, we think one should start by making progress in two directions. The first is to understand how general this phenomenon is: are there other examples, and what are their common features—for instance, do they all require a spontaneous breaking of Lorentz invariance? The second is to find a structure, a pattern in our one-loop check: the cancellations that lead to $\rho + p = 0$ for the framid, especially those of section 4.5.2, are absolutely nontrivial. It is hard to believe that they are not enforced by a hidden structure in the computation. Perhaps there is a better way of organizing the computation that would make such a structure manifest. We plan to explore these questions in the near future.

Chapter 5: Scalar-tensor mixing from icosahedral inflation

5.1 A discrete but dense rotational group

As promised in section 1.3.2, we will now look into the details of icosahedral inflation. To summarize again the main idea of this chapter, icosahedral inflation can be thought of as a model where inflation is driven by a peculiar solid with icosahedral symmetry – a discrete subgroup of 3D rotations ($SO(3)$). In principle, the background cosmological evolution and all correlation functions for perturbations must be invariant under such discrete rotations, but not necessarily under generic continuous rotations. However, icosahedral rotations are so ‘dense’ (in a colloquial sense) in $SO(3)$, that the background evolution and the scalar two-point function at long distances happen to be accidentally isotropic [27]. Beyond these two observables, full isotropy is lost, and one can check explicitly that already the scalar three-point function and the tensor two-point function are generically anisotropic. In particular, the scalar three-point function can be maximally anisotropic [27], i.e., can have vanishing overlap with all isotropic templates used in data analyses, and the tensor power spectrum can have nonzero mixed correlators between the two helicities [64].

How do we lock down that the symmetry we want is the icosahedral symmetry? The argument is as follows. We would like to find a discrete subgroup of $SO(3)$, so that the theory is intrinsically anisotropic. However, to be consistent with our observations, this symmetry has to guarantee the isotropy of the background evolution (or the stress-energy tensor $T^{\mu\nu}$) and the scalar power spectrum, so that we do not rule out our theory right away by comparing with observations. We therefore first need T^{ij} to be isotropic. But the constraint we have at hand is that T^{ij} is invariant under the discrete symmetry we are imposing onto the system. Second, consider the quadratic

terms in the phonon's Lagrangian,

$$\mathcal{L}_2 = A_{ij} \dot{\pi}^i \dot{\pi}^j + B_{ijklm} \partial^i \pi^j \partial^l \pi^m, \quad (5.1)$$

which determines the scalar power spectrum. An isotropic scalar power spectrum then amounts to isotropic A_{ij} and B_{ijklm} , solely as a result of them being invariant under the discrete subgroup we have in mind. We now have the criterion for the subgroup we are looking for: any two-index or four-index tensor that is invariant under this subgroup has to be isotropic. This condition can be tested against all possible discrete subgroups of $SO(3)$, and one can show that the only group that passes the test is the icosahedral group [27]. However, at the six-index level, tensors can start to be anisotropic, giving rise to terms that give non-zero anisotropic contributions to bispectra and the tensor power spectrum.

In the rest of this chapter, we study the mixed scalar-tensor two-point function in icosahedral inflation. Such a mixed correlator vanishes to the lowest order in the derivative expansion. However, it is generically there once higher derivative corrections are taken into account. This makes it suppressed within the regime of validity of the derivative expansion, which is the relevant perturbative expansion for an effective field theory like ours. As a result, it is much smaller than the scalar spectrum. Still, since the tensor spectrum is also suppressed compared to the scalar one, there is a consistent choice of parameters that makes the scalar-tensor mixing more important than the tensor spectrum itself. Schematically,

$$\frac{\langle \zeta \gamma \rangle}{\langle \zeta \zeta \rangle} \sim \Delta c_{\zeta\gamma}^2, \quad \frac{\langle \gamma \gamma \rangle}{\langle \zeta \zeta \rangle} \sim \epsilon c_L^5, \quad (5.2)$$

where $\Delta c_{\zeta\gamma}^2$ is a small dimensionless mixing parameter, $\epsilon = -\dot{H}/H^2$ is the usual slow-roll parameter, and c_L is the propagation speed of scalar perturbations—which at short distances just reduce to longitudinal phonons, hence the ‘ L ’. One sees immediately that for $\epsilon c_L^5 \ll \Delta c_{\zeta\gamma}^2$, the mixed scalar-tensor correlator is bigger than the tensor spectrum.

5.2 The mixed scalar-tensor two-point function

In solid inflation, cosmological perturbations can be classified in terms of tensors (γ_{ij}), vectors/transverse phonons ($\vec{\pi}_T$), and scalars/longitudinal phonons (π_L) [24]. At the two-derivative level, after solving the constraints one finds the quadratic action

$$S_{(2)} = S_\gamma + S_L + S_T, \quad (5.3)$$

with [24]

$$S_\gamma = \frac{1}{4} M_{\text{Pl}}^2 \int dt d^3x a^3 \left[\frac{1}{2} \dot{\gamma}_{ij}^2 - \frac{1}{2a^2} (\partial_m \gamma_{ij})^2 + 2\dot{H} c_T^2 \gamma_{ij}^2 \right] \quad (5.4)$$

$$S_T = M_{\text{Pl}}^2 \int dt \int_{\vec{k}} a^3 \left[\frac{k^2/4}{1 - k^2/4a^2 \dot{H}} |\dot{\pi}_T^i|^2 + \dot{H} c_T^2 k^2 |\pi_T^i|^2 \right] \quad (5.5)$$

$$S_L = M_{\text{Pl}}^2 \int dt \int_{\vec{k}} a^3 \left[\frac{k^2/3}{1 - k^2/3a^2 \dot{H}} |\dot{\pi}_L - (\dot{H}/H)\pi_L|^2 + \dot{H} c_L^2 k^2 |\pi_L|^2 \right]. \quad (5.6)$$

For icosahedral inflation, since the background does not have full $SO(3)$ symmetry, one expects quadratic mixings among these different polarizations—neither spin nor helicity are good quantum numbers. However, as pointed out already in [27, 64], such an effect is invisible to lowest order in the derivative expansion. On the other hand, if one takes into account higher derivative corrections, it is easy to write down mixing terms that are consistent with icosahedral symmetry. Ref. [64] considered the leading anisotropy effects for the tensor spectrum, which include a mixed correlator for helicities $+2$ and -2 . Here we do the same for the scalar-tensor two-point function.

In a derivative expansion, the first icosahedral-invariant bilinear term we can write down that mixes scalars (and vectors) with tensors is

$$S_{\text{mix}} = -M_{\text{Pl}}^2 \int dt d^3x a \Delta c_{\gamma\zeta}^2 T_6^{ijklmn} \partial_i \pi_j \partial_k \partial_l \gamma_{mn}. \quad (5.7)$$

Here, $\Delta c_{\gamma\zeta}^2$ is a free dimensionless parameter—which we expect to depend slowly on time, but which we can take as constant to zeroth-order in the slow-roll expansion—and T_6 is the unique

(up to normalization) spin-6 icosahedral invariant tensor [64]. As we show in Appendix F, the single power of $a(t)$ is consistent with the near scale-invariance of the solid driving inflation, which is ultimately related to the slow-roll expansion [24]. There, we also show that, to this order in derivatives, associated with (5.7) there are no extra scalar-tensor mixings involving N or N^i . Finally, in the spirit of the derivative expansion and according to standard EFT logic, we need higher-derivative corrections to yield small effects at the scales of interest, that is, for typical frequencies of order H . This requires $\Delta c_{\zeta\gamma}^2$ to be generically ‘small’; how small will be made clear in section 5.4. The mixing term above can come from non-minimal couplings between our solid and the Riemann tensor, e.g. of the form $(R^{\mu\nu\rho\sigma} \partial_\mu \phi^I \partial_\nu \phi^J \partial_\rho \phi^K \partial_\sigma \phi^L)^3$ with suitable index contractions. We show this in Appendix F.

The fact that, within the regime of validity of the EFT, the term above can only yield small effects allows us to treat it in perturbation theory. Decomposing the phonon field into its longitudinal and transverse parts,

$$\pi_j = \frac{\partial_j}{\sqrt{-\nabla^2}} \pi_L + \pi_T^j, \quad \vec{\nabla} \cdot \vec{\pi}_T = 0, \quad (5.8)$$

and keeping only the longitudinal one, the mixing term above becomes

$$S_{\text{mix}} = -M_{\text{Pl}}^2 \int \frac{d\tau d^3k}{(2\pi)^3} a^2 \Delta c_{\zeta\gamma}^2 T_6^{ijklmn} k_i \hat{k}_j k_k k_l \pi_L(\vec{k}, \tau) \gamma_{mn}(-\vec{k}, \tau), \quad (5.9)$$

where we switched to Fourier space and conformal time. It is customary to parametrize scalar perturbations in terms of the variable ζ , which for solid inflation is related to π_L by $\zeta = -k \pi_L/3$ [24]. Following standard cosmological perturbation theory [65], to first order in $\Delta c_{\zeta\gamma}^2$ the mixed two-point function we are after thus is

$$\langle \zeta(\vec{k}, \tau) \gamma^s(\vec{q}, \tau) \rangle = -i \int_{-\infty}^{\tau} d\tau' \langle \Omega(-\infty) | [\zeta(\vec{k}, \tau) \gamma^s(\vec{q}, \tau), H_{\text{int}}(\tau')] | \Omega(-\infty) \rangle, \quad (5.10)$$

where $s = \pm$ is either of the two tensor polarizations,

$$\gamma_{ij}(\vec{k}, \tau) = \sum_{s=\pm} \gamma^s(\vec{k}, \tau) \epsilon_{ij}^s(\vec{k}), \quad (\epsilon_{ii}^s = k_i \epsilon_{ij}^s = 0, \epsilon_{ij}^s \epsilon_{ij}^{s'*} = 2\delta^{ss'}) , \quad (5.11)$$

and the interaction Hamiltonian is

$$H_{\text{int}}(\tau') = -3M_{\text{Pl}}^2 \int \frac{d\tau' d^3k'}{(2\pi)^3} a^2 \Delta c_{\zeta\gamma}^2 T_6^{ijklmn} \hat{k}'_i \hat{k}'_j \hat{k}'_k \hat{k}'_l k'^2 \zeta(\vec{k}', \tau') \gamma_{mn}(-\vec{k}', \tau') \quad (5.12)$$

Writing our fields as usual as

$$\gamma^s(\vec{k}, \tau) = \gamma_{cl}(k, \tau) a^s(\vec{k}) + \gamma_{cl}^*(k, \tau) a^{s\dagger}(-\vec{k}) \quad (5.13)$$

$$\zeta(\vec{k}, t) = \zeta_{cl}(\vec{k}, t) b(\vec{k}) + \zeta_{cl}^*(\vec{k}, t) b^\dagger(-\vec{k}), \quad (5.14)$$

and using the relevant mode functions to lowest order in slow roll [24],

$$\gamma_{cl}(k, \tau) = \frac{1}{M_{\text{Pl}} a} \frac{e^{-ik\tau}}{\sqrt{k}} \left(1 - \frac{i}{k\tau}\right) \quad (5.15)$$

$$\zeta_{cl}(k, \tau) = -\frac{1}{M_{\text{Pl}} a} \sqrt{\frac{c_L}{4\epsilon k}} e^{-ikc_L\tau} \left(\frac{i}{c_L^2} + \frac{1}{c_L^3 k\tau} + \frac{k}{3aHc_L}\right), \quad (5.16)$$

after some straightforward (but tedious) algebra we get

$$\langle \zeta \gamma^s \rangle' \equiv \frac{\langle \zeta(\vec{k}, \tau) \gamma^s(\vec{q}, \tau) \rangle}{(2\pi)^3 \delta^3(\vec{k} + \vec{q})} = \frac{\Delta c_{\zeta\gamma}^2}{\epsilon M_{\text{Pl}}^2} T_6^{ijklmn} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \epsilon_{mn}^s(\vec{k}) \times I(\tau), \quad (5.17)$$

where

$$I(\tau) \equiv \frac{3}{2} \frac{c_L}{a^2} \left[\left(\frac{1}{c_L^2} - \frac{1}{k^2 c_L^3 \tau^2} + \frac{1}{3c_L} \right) \frac{c_L^2 + 5c_L + 3}{3k c_L^2 (1 + c_L)^2} + \left(\frac{1}{k c_L^2 \tau} + \frac{1}{k c_L^3 \tau} + \frac{k}{3aHc_L} \right) \left(\frac{1}{k^2 c_L^3 \tau} - \frac{\tau}{3c_L(1 + c_L)} \right) \right]. \quad (5.18)$$

For late times, $k\tau \rightarrow 0^-$, our two-point function becomes time independent and scale invariant,

and reduces to

$$\langle \zeta \gamma^s \rangle' = \frac{3}{2} \frac{2c_L^3 + 4c_L^2 + 6c_L + 3}{(1 + c_L)^2} \cdot T_6^{ijklmn} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \epsilon_{mn}^s(\vec{k}) \cdot \frac{\Delta c_{\zeta\gamma}^2}{\epsilon c_L^5} \frac{H^2}{M_{\text{Pl}}^2 k^3} \quad (5.19)$$

Dropping order-one numerical factors,

$$\langle \zeta \gamma \rangle' \sim \frac{\Delta c_{\zeta\gamma}^2}{\epsilon c_L^5} \frac{H^2}{M_{\text{Pl}}^2 k^3} \sim \frac{\Delta c_{\zeta\gamma}^2}{\epsilon c_L^5} \langle \gamma \gamma \rangle', \quad (5.20)$$

consistently with the estimate in [64], which was derived for $c_L \sim 1$. Recalling that for solid inflation models the tensor-to-scalar ratio is roughly $r \sim \epsilon c_L^5$ [24], we see that for $\Delta c_{\zeta\gamma}^2 \gg r$ the mixed correlator we computed is much bigger than the tensor spectrum itself. As we will see in section 5.4, such a possibility is still within the regime of validity of the effective theory and of a perturbative expansion in $\Delta c_{\zeta\gamma}^2$.

5.3 Visualizing the two-point function

The two-point function (5.19) depends on the orientation of \vec{k} relative to the underlying icosahedral geometry, through the factor

$$M^{\zeta s}(\vec{k}) \equiv T_6^{ijklmn} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \epsilon_{mn}^s(\vec{k}) \quad (5.21)$$

(we are using a notation consistent with that of [64], to facilitate comparison.) As discussed at length in [64], the phase of the polarization tensor $\epsilon_{mn}^s(\vec{k})$ is arbitrary, and, as a function of the direction of \vec{k} , necessarily involves singularities. This makes decomposing $M^{\zeta s}(\vec{k})$ in spherical harmonics or plotting its angular dependence not particularly informative.

One possible way out is to consider the squared absolute value of $M^{\zeta s}(\vec{k})$, so that the ambiguous

and singular phases cancel. Using the results of [64],

$$\begin{aligned}
|M^{\zeta+}|^2 &= |M^{\zeta-}|^2 = \frac{1}{2} \sum_{s=\pm 1} |M^{s\zeta}|^2 \\
&= \frac{1}{2} T_6^{ijklmnpqr} T_6^{opqrst} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \hat{k}_o \hat{k}_p \hat{k}_q \hat{k}_r \sum_{s=\pm 1} \epsilon_{mn}^s(\vec{k}) \epsilon_{st}^{s*}(\vec{k}) \\
&= \frac{1}{2} T_6^{ijklmnpqr} T_6^{opqrst} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \hat{k}_o \hat{k}_p \hat{k}_q \hat{k}_r (P_{ms} P_{nt} + P_{mt} P_{ns} - P_{mn} P_{st}) , \quad (5.22)
\end{aligned}$$

where P_{ij} is the transverse projector,

$$P_{ij}(\hat{k}) \equiv \delta_{ij} - \hat{k}_i \hat{k}_j . \quad (5.23)$$

Following [64], we expect $|M^{\zeta s}|^2$ to contain spherical harmonics with $\ell = 0, 6, 10, 12$ only. Indeed, with the help of Mathematica we find

$$|M^{\zeta s}|^2 = \frac{1}{2} \sum_{\ell m} C_{\ell m} Y_{\ell}^m(\theta, \phi) , \quad (5.24)$$

with the only nonzero $C_{\ell m}$ being

$$\ell = 0: \quad C_{0,0} = \frac{1024\sqrt{\pi}}{3003}(\gamma + 1) \quad (5.25)$$

$$\begin{aligned}
\ell = 6: \quad C_{6,\pm 6} &= -\sqrt{\frac{5}{11}} C_{6,\pm 2} = \frac{32}{323} \sqrt{\frac{11\pi}{273}} (\gamma + 2) \\
C_{6,\pm 4} &= -\sqrt{\frac{7}{2}} C_{6,0} = -\frac{352}{969} \sqrt{\frac{2\pi}{91}} (\gamma + 1) \quad (5.26)
\end{aligned}$$

$$\begin{aligned}
\ell = 10: \quad C_{10,\pm 10} &= -\sqrt{\frac{255}{19}} C_{10,\pm 6} = -\sqrt{\frac{255}{494}} C_{10,\pm 2} = -\frac{20}{23} \sqrt{\frac{21\pi}{46189}} (3\gamma + 1) \\
C_{10,\pm 8} &= \frac{1}{2} \sqrt{\frac{17}{3}} C_{10,\pm 4} = -\sqrt{\frac{187}{130}} C_{10,0} = -\frac{20}{23} \sqrt{\frac{70\pi}{7293}} (\gamma + 1) \quad (5.27)
\end{aligned}$$

$$\begin{aligned}
\ell = 12: \quad C_{12,\pm 12} &= 5\sqrt{\frac{69}{154}} C_{12,\pm 8} = \frac{15}{17} \sqrt{\frac{437}{187}} C_{12,\pm 4} = \frac{5}{58} \sqrt{\frac{5681}{119}} C_{12,0} = \frac{45}{2} \sqrt{\frac{\pi}{676039}} (\gamma + 1) \\
C_{12,\pm 10} &= -\frac{1}{5} \sqrt{\frac{209}{21}} C_{12,\pm 6} = \sqrt{\frac{209}{34}} C_{12,\pm 2} = \frac{33}{23} \sqrt{\frac{3\pi}{29393}} (3\gamma + 1) , \quad (5.28)
\end{aligned}$$

where γ is the golden ratio.

In Figure 5.1 we plot the angular dependence of $|M_{\zeta\gamma}|$, alongside the underlying icosahedral

structure. Clearly, the signal is concentrated around directions pointing towards the edges of the icosahedron.

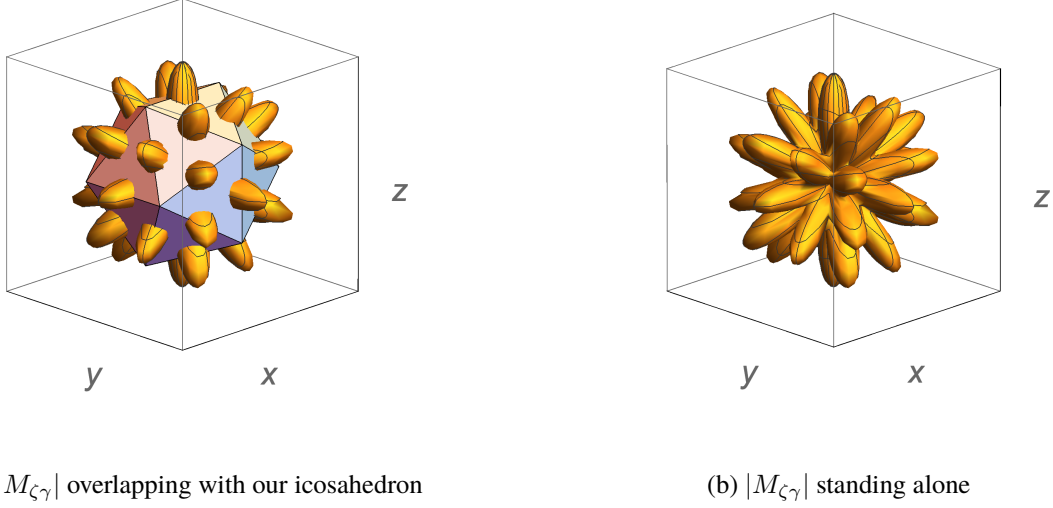


Figure 5.1: Angular plot of $|M_{\zeta\gamma}|$.

Another way to get rid of the ambiguous phases is to consider directly the two-point function $\langle \zeta \gamma_{ij} \rangle$, because the full γ_{ij} field—eq. (5.11)—is unambiguous. Using the results above and the tracelessness of T_6 , we get

$$\begin{aligned}
\langle \zeta \gamma_{ij} \rangle &\propto \sum_{s=\pm 1} M^{\zeta s} \epsilon_{ij}^s(-\vec{k}) \\
&= T_6^{klmnop} \hat{k}_k \hat{k}_l \hat{k}_m \hat{k}_n \sum_{s=\pm 1} \epsilon_{op}^s(\vec{k}) \epsilon_{ij}^{s*}(\vec{k}) \\
&= (P_{oi} P_{pj} + P_{oj} P_{pi} - P_{op} P_{ij}) T_6^{klmnop} \hat{k}_k \hat{k}_l \hat{k}_m \hat{k}_n \\
&= [2T_6^{klmni j} - 2(T_6^{klmni p} \hat{k}_p \hat{k}_j + T_6^{klmni p} \hat{k}_i \hat{k}_p) + T_6^{klmnop} \hat{k}_o \hat{k}_p (\delta_{ij} + \hat{k}_i \hat{k}_j)] \hat{k}_k \hat{k}_l \hat{k}_m \hat{k}_n .
\end{aligned} \tag{5.29}$$

This however is a transverse traceless two-index tensor (because γ_{ij} is), and so it is difficult to visualize its angular dependence: we cannot trace it or contract it with \hat{k} 's to construct a scalar angular function.

5.4 Non-perturbative check

As a check of our results of section 5.2, we now try to calculate the same two-point function in a non-perturbative way. We will be able to do so only in a specific kinematical regime, which however will still allow us to perform a nontrivial check.

Eventually we will still expand our result to linear order in $\Delta c_{\zeta\gamma}^2$, so, we can focus from the start on a two-field system made up of the scalar perturbations and either polarization of the tensor ones, because at linear order there cannot be interference among different sources of mixing. In particular, we can safely neglect the tensor-tensor mixing of ref. [64]. Moreover, it turns out that, because of the time-dependence of $a(\tau)$, even this simple two-field system cannot be diagonalized for generic momenta (or generic times). We show this in Appendix G. So, here we focus on modes well inside the sound horizon, $c_L k/aH \gg 1$, for which the time-dependence of $a(\tau)$ can be neglected.

With these qualifications in mind, to lowest order in slow-roll the quadratic action we need is (see eqs. (5.4), (5.6), (5.9))

$$\begin{aligned}
S_\gamma + S_L + S_{\text{mix}} \rightarrow \frac{1}{2} M_{\text{Pl}}^2 a^2 \int \frac{d\tau d^3k}{(2\pi)^3} & \left[\frac{1}{2} (|\gamma'_s|^2 - k^2 |\gamma_s|^2) + 2\epsilon a^2 H^2 (|\pi'_L|^2 - c_L^2 k^2 |\pi_L|^2) \right. \\
& \left. - \Delta c_{\zeta\gamma}^2 k^3 (M^{\zeta s} \pi_L^* \gamma^s + \text{c.c.}) \right] \quad (c_L k/aH \gg 1), \quad (5.30)
\end{aligned}$$

where s is either $+$ or $-$, $M^{\zeta s}$ is defined in (5.21), and all the fields and coefficients are evaluated at (\vec{k}, τ) . Neglecting the time-dependence of all the coefficients—including a —we can go to frequency space and rewrite this conveniently in a compact form as

$$\int \frac{d\omega d^3k}{(2\pi)^4} \psi^\dagger \cdot K \cdot \psi, \quad (5.31)$$

where

$$\psi \equiv \begin{pmatrix} \gamma_s \\ \pi_L \end{pmatrix}, \quad K \equiv \frac{1}{2} M_{\text{Pl}}^2 a^2 \begin{pmatrix} \frac{1}{2}(\omega^2 - k^2) & -\frac{1}{2} \Delta c_{\zeta\gamma}^2 k^3 M^{\zeta s*} \\ -\frac{1}{2} \Delta c_{\zeta\gamma}^2 k^3 M^{\zeta s} & 2\epsilon a^2 H^2 (\omega^2 - c_L^2 k^2) \end{pmatrix}, \quad (5.32)$$

and all the fields are now evaluated at (\vec{k}, ω) .

To compute the equal-time two-point function we are interested in well inside the sound horizon, we can now simply invert the matrix K , insert the $i\epsilon$'s appropriate for the Feynman prescription for the poles, and take the integral over ω through standard residue methods. The reason this procedure is correct in our limit is that in general it gives the ground state's T -ordered correlation functions for a quantum system with a time-independent Hamiltonian; in our case, T -ordering does not matter, because our fields commute at equal time; moreover, in our inside-the-sound-horizon limit the time-dependence of the perturbations' Hamiltonian is negligible, and the Bunch-Davies ground state is equivalent to the flat-space one.

Then, the Fourier-space Feynman propagator of ψ is schematically

$$\langle \psi \psi^\dagger \rangle_{\omega, \vec{k}} = i(K + i\epsilon)^{-1} (2\pi)^4 \delta^4, \quad (5.33)$$

and so the equal-time two-point function we are interested in is

$$\begin{aligned} \langle \pi_L \gamma^s \rangle'_{\vec{k}, \tau} &= \int \frac{d\omega}{(2\pi)} i(K + i\epsilon)^{-1}_{12} \\ &= \frac{\Delta c_{\zeta\gamma}^2}{a^2 M_{\text{Pl}}^2} k^3 M^{\zeta s} \int \frac{d\omega}{(2\pi)} \frac{i}{\epsilon a^2 H^2 (\omega^2 - k^2 + i\epsilon) (\omega^2 - c_L^2 k^2 + i\epsilon) - \frac{1}{4} \Delta c_{\zeta\gamma}^2 |M^{\zeta s}|^2 k^6} \\ &\simeq -\frac{\Delta c_{\zeta\gamma}^2}{2\epsilon H^2 a^4 M_{\text{Pl}}^2} M^{\zeta s} \frac{1}{c_L(1 + c_L)}, \end{aligned} \quad (5.34)$$

where in the last step we restricted to the first order in $\Delta c_{\zeta\gamma}^2$. Recalling that ζ is related to π_L by $\zeta = -k\pi_L/3$, we see that this result matches precisely our previous one, eq. (5.17), in the high k /early times limit, $c_L k |\tau| \gg 1$.

This computation also makes it clear how small $\Delta c_{\zeta\gamma}^2$ should be for a perturbative analysis to

be applicable: the ω integral above is dominated by poles with $\omega \simeq \pm k$ and $\omega \simeq \pm c_L k$. The scalar-tensor mixing shifts these, respectively, by

$$\frac{\Delta\omega}{\omega} \simeq \frac{(\Delta c_{\zeta\gamma}^2)^2 |M^{\zeta s}|^2}{8(1 - c_L^2)} \frac{k^2}{\epsilon a^2 H^2}, \quad \frac{\Delta\omega}{\omega} \simeq -\frac{(\Delta c_{\zeta\gamma}^2)^2 |M^{\zeta s}|^2}{8c_L^2(1 - c_L^2)} \frac{k^2}{\epsilon a^2 H^2}. \quad (5.35)$$

For these relative shifts to be small up to physical momenta k/a much bigger than H , we need

$$\Delta c_{\zeta\gamma}^2 \ll c_L \sqrt{\epsilon}, \quad (5.36)$$

where we used that $M^{\zeta s}$ and $(1 - c_L^2)$ are both of order one (in solid inflation models, c_L^2 has to be smaller than $1/3$ [24]).

5.5 Imprints on CMB Anisotropies

We now turn our attention to the effects of a scalar-tensor mixing on CMB anisotropies. In more standard cases, where the inflationary theory has both rotational and parity symmetry, a mixing between the so-called E and B modes is forbidden due to symmetry arguments. More precisely, following the convention in [66],

$$\begin{aligned} \langle a_{T,lm}^* a_{T,l'm'} \rangle &= C_{TT,l} \delta_{l,l'} \delta_{m,m'} \\ \langle a_{T,lm}^* a_{E,l'm'} \rangle &= C_{TE,l} \delta_{l,l'} \delta_{m,m'} \\ \langle a_{E,lm}^* a_{E,l'm'} \rangle &= C_{EE,l} \delta_{l,l'} \delta_{m,m'} \\ \langle a_{B,lm}^* a_{B,l'm'} \rangle &= C_{BB,l} \delta_{l,l'} \delta_{m,m'} \\ \langle a_{T,lm}^* a_{B,l'm'} \rangle &= 0 \\ \langle a_{E,lm}^* a_{B,l'm'} \rangle &= 0. \end{aligned}$$

In particular, the T - B and E - B correlators vanish because under a parity transformation one has

$$a_{T,lm} \rightarrow (-1)^l a_{T,lm} , \quad a_{E,lm} \rightarrow (-1)^l a_{E,lm} , \quad a_{B,lm} \rightarrow -(-1)^l a_{B,lm} . \quad (5.37)$$

Therefore, when $l = l'$, such correlators are forbidden because of parity, while for $l \neq l'$, they are forbidden because of rotations.

However, in our case there is no full rotational symmetry, hence modes of different l 's can mix. As a result, one can generically expect nonzero $\langle a_{T,lm}^* a_{B,l'm'} \rangle$ and $\langle a_{E,lm}^* a_{B,l'm'} \rangle$ correlators when $l = l' \pm n$ for odd n (even n 's are still forbidden by parity, which is a symmetry of our theory). A similar argument has been presented in [67] for pseudoscalar inflation. Adapting the notation of [67] and [68],

$$a_{T/E,lm}^{(s)} = 4\pi (-i)^l \int \frac{d^3k}{(2\pi)^3} \mathcal{F}_{T/E,l}^{(s)}(k) \zeta_{\vec{k}} Y_{lm}^*(\hat{k}) \quad (5.38)$$

$$a_{T/E,lm}^{(t)} = 4\pi (-i)^l \int \frac{d^3k}{(2\pi)^3} \mathcal{F}_{T/E,l}^{(t)}(k) \left[\gamma_{\vec{k}}^{(+2)} {}_{-2}Y_{lm}^*(\hat{k}) + \gamma_{\vec{k}}^{(-2)} {}_2Y_{lm}^*(\hat{k}) \right] \quad (5.39)$$

$$a_{B,lm}^{(t)} = 4\pi (-i)^l \int \frac{d^3k}{(2\pi)^3} \mathcal{F}_{B,l}^{(t)}(k) \left[\gamma_{\vec{k}}^{(+2)} {}_{-2}Y_{lm}^*(\hat{k}) - \gamma_{\vec{k}}^{(-2)} {}_2Y_{lm}^*(\hat{k}) \right] \quad (5.40)$$

Here we use s and t to label contributions from scalar and tensor modes, $\mathcal{F}_{T/E/B,l}^{(s/t)}(k)$ is the corresponding radiation transfer function (see, e.g., [68] for their explicit forms), and ${}_{\pm 2}Y_{lm}$ is the spin-weighted spherical harmonics of spin ± 2 . The transfer functions depend only on the cosmology after inflation, and are thus independent of our inflationary model. Inflation enters the correlation functions above only through ζ and γ , evaluated at the end of inflation. (See [66, 67, 68, 69, 70] for details). We thus have

$$\begin{aligned} & \langle a_{T/E,lm}^{(s)*} a_{B,l'm'}^{(t)} \rangle \\ &= (4\pi)^2 i^{l-l'} (-1)^{l'} \times C \times A_{(l,m),(l',m')} \times \int \frac{dk}{k} \mathcal{F}_{T/E,l}^{(s)}(k) \mathcal{F}_{B,l'}^{(t)}(k) \end{aligned} \quad (5.41)$$

where C is a k -independent factor, given by our previous calculation as

$$C = \frac{3}{2} \frac{2c_L^3 + 4c_L^2 + 6c_L + 3}{(1 + c_L)^2} \frac{\Delta c_{\zeta\gamma}^2 H^2}{\epsilon c_L^5 M_{Pl}^2},$$

and $A_{(l,m),(l',m')}$ is a purely geometric factor, defined as

$$A_{(l,m),(l',m')} \equiv \int d\Omega_{\hat{k}} Y_{lm}(\hat{k}) \left[M^{\zeta^+}_{-2} Y_{l'm'}^*(\hat{k}) - M^{\zeta^-}_{-2} Y_{l'm'}^*(\hat{k}) \right]. \quad (5.42)$$

For icosahedral inflation, the parity selection rules spelled out above allow non-vanishing $A_{(l,m),(l',m')}$ only for $l = l' \pm n$, with n odd. In fact, further investigation with Mathematica shows that there is no obvious selection rule based on the value of $m - m'$.

Notice that the arbitrary and singular phase introduced in $M^{\zeta\pm}$ by the polarization tensors is still there—it does not cancel out in the combination entering $A_{(l,m),(l',m')}$. So, in order to evaluate these expressions, one should make an explicit choice of polarization tensors. For instance, the choice of ref. [69] is,

$$\epsilon_{mn}^{\pm 2}(\hat{k}) = \sqrt{2} \epsilon_m^{\pm 1}(\hat{k}) \epsilon_n^{\pm 1}(\hat{k}), \quad \epsilon_m^{\pm 1}(\hat{k}) = \frac{1}{\sqrt{2}} (\hat{\theta}(\hat{k}) \pm i \hat{\phi}(\hat{k})), \quad (5.43)$$

where θ and ϕ are the polar and azimuthal angles of \hat{k} , and $\hat{\theta}$ and $\hat{\phi}$ are the corresponding unit vectors. With this choice, as an example, for $l = 3$, $l' = 2$, $m' = 2$, and arbitrary m , we find

$$\begin{aligned} A_{(3,-2),(2,2)} &= \frac{2 + \gamma}{3\sqrt{21}}, \\ A_{(3,0),(2,2)} &= -\frac{\gamma}{6\sqrt{70}}, \\ A_{(3,2),(2,2)} &= A_{(3,1),(2,2)} = A_{(3,-1),(2,2)} = 0, \end{aligned} \quad (5.44)$$

where γ is, as before, the golden ratio (we computed the relevant integrals with Mathematica.)

Notice that icosahedral inflation also gives rise to nonzero tensor-tensor T - B and E - B correlators, $\langle a_{T/E,lm}^{(t)*} a_{B,l'm'}^{(t)} \rangle$, in addition to the scalar-tensor ones, $\langle a_{T/E,lm}^{(s)*} a_{B,l'm'}^{(t)} \rangle$. However, T - B and

E - B correlators are dominated by the latter contributions, since the anisotropies in the tensor-tensor spectrum are of order $\epsilon c_L^5 \ll 1$ compared to the tensor-scalar mixing [64]¹.

Current CMB observations are able to put constraints on T - B and E - B correlations. In the CMB literature, such correlations are usually assumed to be coming from cosmic birefringence. For example, recent constraints on cosmic birefringence effect coming from ACT can be found in [71] and [72], and similar constraints from Planck can be found in [73] and [74]. However, it is not straightforward to translate constraints on cosmic birefringence into constraints on the parameters of our model. We leave performing this analysis for future work.

5.6 Summary and discussion

We have computed the scalar-tensor correlation function in icosahedral inflation [27], and discussed its possible imprints on CMB anisotropies, in the form of non-vanishing T - E and T - B spectra. Such correlations are allowed because the inflationary model at hand breaks (spontaneously) rotational invariance. Within the regime of validity of the effective field theory, the mixed scalar-tensor correlator can be parametrically larger than the tensor spectrum itself.

It is useful to compare our results and framework to other models of inflation featuring anisotropic effects, such as the model studied in [75, 76, 77]. There, the intrinsic anisotropy of the background evolution enters all correlators of perturbations, including the scalar spectrum. In our case instead, the model is *designed* in such a way as to guarantee that the scalar spectrum is automatically isotropic, while leaving open the possibility of detectable anisotropies in other correlation functions, such as the scalar three-point function [27], the tensor spectrum [64], and the scalar-tensor two-point function (the case considered here). The reason behind this choice is spelled out in section 1.3.2: the scalar spectrum is the only primordial correlation function we have detected, and it appears to be consistent with statistical isotropy.

¹As shown in Appendix F, the parameter Δc_γ^2 that corrects the tensor modes' propagation speed in an anisotropic fashion in ref. [64] is generically of the same order as our mixing parameter $\Delta c_{\zeta\gamma}^2$, since the two effects can arise from the same non-linear combinations of matter fields and curvature tensors.

Conclusion

Symmetry is a core aspect of the theoretical study of physical systems. We have seen in this thesis a (far from complete) set of tools and applications rooted in symmetry considerations. Developing these tools and applications help us refine our understanding of physical systems from condensed matter states to the primordial universe.

We reviewed and improved Noether's theorem for spacetime symmetries. Based on the intrinsic ambiguities in Noether's theorem, we worked out systematically the improvement terms to the stress-energy tensor without any guesswork, according to the symmetry properties of the theories in consideration. For Lorentz invariant theories, our prescription produces the symmetric stress-energy tensor, even off-shell, while agreeing with Belinfante's result on-shell. Similarly, for theories with scale-invariant symmetry, we derived the traceless stress-energy tensor, and for those with conformal symmetry a stress-energy tensor that is both symmetric and traceless, even off-shell. Further, our method unifies the translation Noether theorem with those of the additional spacetime symmetries, yielding at the same time both the improved stress-energy tensor and the additional Noether currents. We hope that the improvements can at least help make computations of stress-energy tensors seem less arbitrary, and it might also shed light on the scale vs. conformal invariance debate, discussed in [38] and the references therein. Possible extensions include turning on higher derivative terms in the Lagrangian, considering non-linearly realized symmetries, and incorporating gauge-invariance.

We then emphasize that spontaneous symmetry breaking leads to a vast variety of gapless excitations. Since the broken symmetries are realized non-linearly, the task of writing down all

terms that are compatible with these symmetries in an effective field theory of the Goldstone modes could be laborious. The coset construction is developed to help with this exact situation. However, there has been little attempt to extend the coset construction to include particles with non-zero spins. We therefore extended the coset construction to the cases of relativistic point particles with arbitrary spins. To do so, we introduced a new philosophy for coset construction, one that reflects the diverse possibilities with broken Poincaré symmetries by associating with every symmetry generator, broken or unbroken, a Goldstone field, and selecting a proper subset of them when constructing the EFT by choosing a particular set of gauge symmetries. We also emphasized that to describe a particle with spin s , one needs to implement an $\mathcal{N} = 2s$ supersymmetric worldline reparameterization gauge symmetry. Since we will be breaking spacetime symmetry, the number of Goldstone modes does not necessarily match the number of broken symmetries, and one needs to go through the inverse Higgs constraints, which we extended under our philosophy. The detailed procedures were then shown with specific examples – massless and massive particles with spin-0, spin-1/2, or higher spin. We hope that this extended coset construction can help us better understand more varieties of physical systems, especially those with Fermionic degrees of freedom.

One interesting application of spontaneous symmetry breaking is that we can classify condensed matter systems by the Poincaré symmetries they break. Among all the possibilities, the framid is the one that breaks the least amount of symmetries – they only break boosts. Although the framid system breaks boosts spontaneously, its stress-energy tensor has the peculiar behavior of still being Lorentz-invariant. We checked through explicit calculations that this property holds up to the one-loop level. One may think of this as a technical analog of the cosmological constant problem, where the observed vacuum energy is stupendously smaller than a “natural” value; for the framid stress-energy tensor, on the other hand, the Lorentz-breaking terms are forced to vanish, seemingly by the broken symmetry. It is unclear whether this phenomenon has a deeper explanation rooted in some symmetry reasons. We hope to better understand it in the near future so that we might have a chance to also improve our understanding of the cosmological constant problem.

Lastly, one may use the tools that we have developed to write down an effective field action

for solids, and furthermore, to use it to drive inflation in the primordial universe. Such solid inflation models do not break time translation spontaneously. Depending on the kind of internal symmetry the system preserves, we can build different kinds of solid inflationary models. In particular, we can construct icosahedral inflation, when the system has an icosahedral symmetry. Invariance under the icosahedral group will guarantee isotropic background evolution and scalar power spectrum. However, since the model is intrinsically anisotropic, anisotropies show up in higher point correlation functions, as well as in the tensor power spectrum. At the same time, a non-zero mixed tensor-scalar correlation is allowed. We calculated the power spectrum of this mixed correlation, and mentioned some observational implications. A more detailed analysis of the observational consequences is close to being completed and will be published in the near future.

To conclude, in this thesis we have discussed and extended the Noether's theorem and the coset construction, and presented two applications when spontaneous symmetry breaking is combined with EFT principles in the cases of framids and solids, relating condensed matter systems to cosmology. We have pushed our understandings of these specific topics further, which are among the countless tools and applications that have their roots in symmetries of theories.

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Appendix A: Worldline SUSY

In order to construct an effective action for spin-1/2 particles, we will need to endow the particle's worldline with local SUSY. In this section, we review the basics of both global and local SUSY in one dimension. For an in-depth discussion of $\mathcal{N} = 1$ SUSY, consult [78].

We begin by considering global worldline SUSY. There are many possible starting-points for a discussion of SUSY, but for our purposes, it is most convenient to work in superspace. Superspace is essentially a mathematical trick to make SUSY manifest. Suppose that our coordinates are $\sigma^M = (\tau, \theta)^M$, where τ is a standard, bosonic (i.e. commuting, or Grassmann even) coordinate and θ is a fermionic (i.e. anti-commuting, or Grassmann odd) coordinate. In other words, $\theta^2 = 0$. It is sometimes convenient to use the notation $\sigma^0 = \tau$ and $\sigma^1 = \theta$. Suppose we have a Grassmann-even field defined on these coordinates, $\mathbb{X}(\tau, \theta)$. Then, because $\theta^2 = 0$, Taylor expanding to linear order in θ gives an exact result, so we may write

$$\mathbb{X}(\tau, \theta) = X(\tau) + i\theta\psi(\tau), \quad (\text{A.1})$$

where X is a real-valued field and ψ is a Grassman-odd field. Thus, \mathbb{X} is Grassmann-even.

We define integration and differentiation with respect to θ as equivalent operations, given by

$$\int d\theta \mathbb{X} = \partial_\theta \mathbb{X} = i\psi. \quad (\text{A.2})$$

The global SUSY and worldline translation transformations act on the coordinates by

$$(\tau, \theta) \rightarrow (\tau', \theta') = (\tau - \xi - i\theta\varepsilon, \theta - \varepsilon), \quad (\text{A.3})$$

where ξ is a constant real number and ε is a Grassmann-odd constant. It is straightforward to check

that there are two linear combinations of the partial derivatives ∂_τ and ∂_θ that do not transform under SUSY, namely

$$D_0 \equiv \partial_\tau, \quad D_1 \equiv \partial_\theta + i\theta\partial_\tau. \quad (\text{A.4})$$

Additionally, it can be checked that $D_1^2 = D_0$. Often D_1 is referred to as the *covariant derivative* (not to be confused with the covariant derivatives in the coset construction). We will refer to both of them as the SUSY covariant derivative.

We now define the (flat) SUSY zweibein \mathbb{E}_M^A for $A, M = 1, 2$ as the 2×2 matrix¹

$$\mathbb{E}_M^A = \begin{pmatrix} 1 & 0 \\ -i\theta & 1 \end{pmatrix}_M^A. \quad (\text{A.5})$$

With this definition, we can succinctly express both SUSY covariant derivatives as the components of $D_A \equiv \mathbb{E}_A^M \partial_M$, where $\partial_M \equiv (\partial_\tau, \partial_\theta)_M$ and \mathbb{E}_A^M is the inverse of \mathbb{E}_M^A .

We now promote SUSY to the full reparameterization invariance of the superspace, namely

$$\sigma^M \rightarrow \sigma^M - \alpha^M(\sigma), \quad (\text{A.6})$$

where α^M is a generic infinitesimal function of σ . Now the superzweibein becomes dynamical and transforms under this symmetry by

$$\delta \mathbb{E}_M^A = \partial_M \alpha^N \mathbb{E}_N^A + \alpha^N \partial_N \mathbb{E}_M^A. \quad (\text{A.7})$$

The scalar field $\mathbb{X}(\sigma)$ transforms by

$$\delta \mathbb{X} = \alpha^M \partial_M \mathbb{X}. \quad (\text{A.8})$$

Now the SUSY covariant derivatives are defined by $D_A \equiv \mathbb{E}_A^M \partial_M$. With this definition, $D_A \mathbb{X}$ transforms as a scalar under (A.6).

¹We use M, N as superspace coordinates indices and A, B as super tangent-space indices.

But now that we have introduced a dynamical superzweibein, we have an additional gauge symmetry, namely local transformations of the tangent space. In particular, we have the symmetries

$$\delta\mathbb{E}_M^0 = 0, \quad \delta\mathbb{E}_M^1 = \mathbb{E}_M^0\varphi, \quad (\text{A.9})$$

where φ is an infinitesimal Grassmann-odd function of σ . We would like to reduce the number of gauge symmetries to simplify matters. We choose a gauge-fixing condition that may appear somewhat strange, but is useful in the sense that the residual gauge symmetries are the standard gauged SUSY transformations. In particular, we postulate that the superzweibein take the form

$$\mathbb{E}_M^A = \begin{pmatrix} \mathbb{E} + i\theta\chi & \chi \\ -i\theta & 1 \end{pmatrix}_{M}^A, \quad \mathbb{E}(\tau, \theta) \equiv e(\tau) + i\theta\chi(\tau), \quad (\text{A.10})$$

and thus the inverse superzweibein is

$$\mathbb{E}_A^M = \frac{1}{\mathbb{E}} \begin{pmatrix} 1 & -\chi \\ i\theta & e \end{pmatrix}_A^M, \quad (\text{A.11})$$

where $e(\tau)$ is a real-valued field and $\chi(\tau)$ is a Grassmann-odd field. We also require that the residual super-reparameterization symmetry have a restricted α^M , namely

$$\alpha^M(\sigma) = \left(\xi(\tau) + \frac{i}{e}\theta\varepsilon(\tau), \varepsilon(\tau) - \frac{i}{e}\theta\varepsilon(\tau)\chi \right)^M, \quad (\text{A.12})$$

where ξ is a real-valued infinitesimal function of τ and ε is a Grassmann-odd infinitesimal function of τ . This restricted form of α^M allows the inverse superzweibein to remain in the desired form (A.11) if we endow $e(\tau)$ and $\chi(\tau)$ with the transformation properties

$$\delta e = \partial_\tau(\xi e) + 2i\varepsilon\chi, \quad \delta\chi = \partial_\tau(\xi\chi) + \dot{\varepsilon}. \quad (\text{A.13})$$

We can therefore identify e as the worldline einbein and χ as its superpartner, i.e. the gravitino. In this gauge, the covariant derivatives are

$$D_0 = \frac{1}{\mathbb{E}}(\partial_\tau - \chi\partial_\theta), \quad D_1 = \frac{1}{\mathbb{E}}(i\theta\partial_\tau + e\partial_\theta), \quad (\text{A.14})$$

and the invariant integration measure is

$$\int d^2\sigma \text{sdet}(\mathbb{E}_M^A) = \int d\tau d\theta \mathbb{E}, \quad (\text{A.15})$$

where sdet is the superdeterminant defined as follows. Let A, D be Grassman-even and B, C be Grassmann-odd matrices. Then define

$$\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - C \cdot A^{-1} \cdot B)^{-1}. \quad (\text{A.16})$$

It is straightforward to check that $\text{sdet}(\mathbb{E}_M^A) = \mathbb{E}$.

Appendix B: Extended worldline SUSY

To construct an effective action for a particle of arbitrary spin $s \in \mathbb{N}/2$, we will need to endow the particle's worldline with a local extended $\mathcal{N} = 2s$ SUSY. In this section, we will extend the superspace formalism of Appendix A to allow for multiple supercharges. To our knowledge no such superspace formalism exists in the current literature; however many common elements can be found in [50]. It will turn out that for $\mathcal{N} > 2$, the superspace formalism will have to be supplemented by a multiplet calculus, which is a straight-forward extension of the one presented in [39].

B.1 Superspace revisited

As before, let us begin by considering global SUSY. We define the coordinates of superspace by

$$\sigma^M = (\tau, \theta^1, \dots, \theta^{\mathcal{N}})^M, \quad (\text{B.1})$$

where $\sigma^0 \equiv \tau \in \mathbb{R}$ and $\sigma^a \equiv \theta^a$ for $a = 1, \dots, \mathcal{N}$ are Grassmann odd. Suppose we have a field defined on these coordinates, $\mathbb{X}(\tau, \vec{\theta})$. Because $\theta^a \theta^b = -\theta^b \theta^a$, the Taylor expansion in θ^a will terminate after $\mathcal{N} + 1$ -terms. It turns out, however, that only the first two terms are dynamical. We have

$$\mathbb{X}(\tau, \vec{\theta}) = X + i\vec{\theta} \cdot \vec{\psi} + \dots, \quad (\text{B.2})$$

where $X(\tau)$ is a real-valued field and $\vec{\psi}(\tau)$ is a vector of Grassmann odd fields; thus \mathbb{X} is Grassmann-even. The terms subsumed by \dots in the above equation end up being non-dynamical fields whose only purpose is to allow SUSY to close off-shell. See an explicit example in section 3.5.1.

We define integration and differentiation with respect to the Grassmann coordinates as equiva-

lent operations, given by

$$\int d\theta^a \mathbb{X} = \partial_a \mathbb{X} = i\psi^a, \quad (\text{B.3})$$

where $\partial_a \equiv \partial/\partial\theta^a$.

Now we promote the global SUSY transformation to a local, gauge symmetry. Instead of introducing full superspace reparameterization invariance as we did in Appendix A, we will cut to the chase. We postulate that the supervielbein takes the block-matrix form

$$\mathbb{E}_M^A = \begin{pmatrix} \mathbb{E} + i\vec{\theta} \cdot \vec{\Gamma} & \Gamma^b \\ -i\theta^a & \delta^{ab} \end{pmatrix}_M^A, \quad (\text{B.4})$$

where $\mathbb{E} = e + i\vec{\theta} \cdot \vec{\chi}$ and $\Gamma^a = \chi^a + A^{ab}\theta^b$. Here, e , χ^a and A^{ab} are fields defined on the coordinate τ . Then the inverse supervielbein is

$$\mathbb{E}_A^M = \frac{1}{\mathbb{E}} \begin{pmatrix} 1 & -\Gamma^b \\ i\theta^a & \mathbb{E}\delta^{ab} - i\theta^a\Gamma^b \end{pmatrix}_A^M. \quad (\text{B.5})$$

In order for the supervielbein to retain the desired form, we take our reparameterization symmetries to be a restricted subset of coordinate transformations given by $\sigma^M \rightarrow \sigma^M - \alpha^M(\sigma)$, such that

$$\begin{aligned} \alpha^0(\sigma) &= \xi(\tau) + \frac{i}{\mathbb{E}} \vec{\theta} \cdot \vec{\varepsilon}(\tau), \\ \alpha^a(\sigma) &= \varepsilon^a(\tau) - \frac{i}{\mathbb{E}} \vec{\theta} \cdot \vec{\varepsilon}(\tau) \Gamma^a + \theta^b \beta^{ab}(\tau), \end{aligned} \quad (\text{B.6})$$

where $\xi(\tau)$ and $\beta^{ab}(\tau) = -\beta^{ba}(\tau)$ are infinitesimal and real-valued, while $\varepsilon^a(\tau)$ are infinitesimal and Grassmann-odd. Under this coordinate transformation, the component fields of the superviel-

bein transform as

$$\begin{aligned}
\delta e &= \partial_\tau(\xi e) + 2i\vec{\varepsilon} \cdot \vec{\chi}, \\
\delta\chi^a &= \partial_\tau(\xi\chi^a) + \varepsilon^a + A^{ab}\varepsilon^b - \beta^{ab}\chi^b, \\
\delta A^{ab} &= \partial_\tau(\xi A^{ab}) + \dot{\beta}^{ab} + \beta^{ac}A^{cb} - \beta^{bc}A^{ca}.
\end{aligned} \tag{B.7}$$

We can therefore interpret e as the worldline einbein, χ^a as the gravitinos and A^{ab} as an $O(\mathcal{N})$ gauge field. Notice that in the case $\mathcal{N} = 1$, A^{ab} vanishes, which is why we did not encounter it in the spin-1/2 case.

The covariant derivatives are given by $D_A \equiv \mathbb{E}_A^M \partial_M$. Explicitly, we have

$$D_0 = \frac{1}{\mathbb{E}}(\partial_\tau - \vec{\Gamma} \cdot \vec{\partial}), \quad D_a \equiv i\theta^a D_0 + \partial_a. \tag{B.8}$$

When acting on the scalar field \mathbb{X} , we have

$$\begin{aligned}
D_0\mathbb{X} &= \frac{1}{\mathbb{E}}(z + i\vec{\theta} \cdot \vec{\zeta}) \dots, \\
D_a\mathbb{X} &= \frac{i}{\mathbb{E}}\theta^a(z + i\vec{\theta} \cdot \vec{\zeta}) + i\psi^a + \dots,
\end{aligned} \tag{B.9}$$

where $z \equiv \dot{X} - i\vec{\chi} \cdot \vec{\psi}$ and $\zeta^a \equiv \dot{\psi}^a + A^{ab}\psi^b$ and \dots denote terms with non-dynamical fields. Finally, the invariant integration measure is $\int d\tau d^{\mathcal{N}}\theta \mathbb{E}$.

B.2 Method of multiplets

The construction of a SUSY-invariant action using the superspace formalism by integrating a Lagrangian over the whole of superspace for $\mathcal{N} > 2$ is still an unsolved problem. In fact, there is good reason to believe that it is impossible, which we address in section 3.5.2. Further, even if we were to accomplish such a feat, the resulting action would contain a potentially very large number of auxiliary fields that are entirely non-dynamical and would only serve to ensure that the SUSY transformations close off-shell. We could then integrate out such non-dynamical fields to obtain a

simpler action that would enjoy SUSY only on-shell. It is the aim of this subsection to explain how to directly construct the on-shell SUSY action without worrying about the non-dynamical fields. We term this approach the method of multiplets.

Our multiplet approach will enable us to construct off-shell SUSY-invariant actions for global SUSY. Only at the end will we gauge this symmetry to obtain the desired local on-shell SUSY-invariant action. We begin by defining bosonic and fermionic multiplets. A bosonic multiplet is an ordered pair

$$\Sigma = (X, \vec{\psi}), \quad (\text{B.10})$$

where X is Grassmann even and $\vec{\psi}$ is a Grassmann-odd \mathcal{N} -component vector. Under an infinitesimal global SUSY transformation, we have

$$\delta X = \xi \dot{X} + i\vec{\varepsilon} \cdot \vec{\psi}, \quad \delta \psi^a = \xi \dot{\psi}^a + \varepsilon^a \dot{X} - \beta^{ab} \psi^b, \quad (\text{B.11})$$

where ξ is an infinitesimal real constant and $\vec{\varepsilon}$ and $\beta^{ab} = -\beta^{ba}$ are Grassmann-odd constants. Next, we define a fermionic multiplet as an ordered pair

$$\Phi = (\vec{f}, b), \quad (\text{B.12})$$

where \vec{f} is a Grassmann-odd \mathcal{N} -component vector and b Grassmann even. The components transform by

$$\delta f^a = \xi \dot{f}^a + i\varepsilon^a b - \beta^{ab} f^b, \quad \delta b = \xi \dot{b} + \vec{\varepsilon} \cdot \vec{f}. \quad (\text{B.13})$$

We can add and multiply these multiplets together. The rule for addition is simply component-wise addition,

$$\begin{aligned} (X_1, \vec{\psi}_1) + (X_2, \vec{\psi}_2) &= (X_1 + X_2, \vec{\psi}_1 + \vec{\psi}_2), \\ (\vec{f}_1, b_1) + (\vec{f}_2, b_2) &= (\vec{f}_1 + \vec{f}_2, b_1 + b_2). \end{aligned} \quad (\text{B.14})$$

We are not permitted to add a bosonic multiplet to a fermionic multiplet. Next, we have the

multiplication rules

$$\begin{aligned}
(X_1, \vec{\psi}_1) \times (X_2, \vec{\psi}_2) &= (X_1 X_2, X_1 \vec{\psi}_2 + X_2 \vec{\psi}_1), \\
(\vec{f}_1, b_1) \times (\vec{f}_2, b_2) &= (\vec{f}_1 \cdot \vec{f}_2, b_1 \vec{f}_2 - b_2 \vec{f}_1), \\
(X, \vec{\psi}) \times (\vec{f}, b) &= (X \vec{f}, X b + \vec{\psi} \cdot \vec{f}).
\end{aligned} \tag{B.15}$$

Notice that the first two products yield bosonic multiplets and the last product yields a fermionic multiplet.

Additionally, we can take derivatives of these multiplets. Acting on a bosonic multiplet we have

$$D_0(X, \vec{\psi}) = (\dot{X}, \dot{\vec{\psi}}), \quad D_a(X, \vec{\psi}) = (i\psi^a, \dot{X}). \tag{B.16}$$

It can be checked that $D_0(X, \vec{\psi})$ and $D_a(X, \vec{\psi})$ are, respectively, bosonic and fermionic multiplets.

Acting on a fermionic multiplet, we have

$$D_0(\vec{f}, b) = (\dot{\vec{f}}, \dot{b}), \quad D_a(\vec{f}, b) = (ib, \dot{\vec{f}}). \tag{B.17}$$

It can be checked that $D_0(\vec{f}, b)$ and $D_a(\vec{f}, b)$ are, respectively, fermionic and bosonic multiplets.

Oftentimes, we will use the compact notation D_A for $A = 0, \dots, \mathcal{N}$. Lastly, given the fermionic multiplet $\Phi = (\vec{f}, b)$, the global SUSY-invariant integral is

$$\int_{\text{SUSY}} \Phi \equiv \int d\tau b. \tag{B.18}$$

We therefore see that the aim of the coset construction will be to construct a symmetry-invariant fermionic multiplet that will then be integrated according to the above rule. After the action is constructed in this manner, we can include the worldline gauge fields to promote the global SUSY to a local, gauge symmetry.

Appendix C: Wightman and Feynman

As we have seen, it is particularly helpful to rewrite the expectation value of the stress-energy tensor in terms of derivatives acting on Wightman two-point functions. Formally, given a set of real fields ψ^a governed by a quadratic Lagrangian, in general one has

$$\langle T^{\mu\nu}(0) \rangle = \lim_{x \rightarrow 0} \sum_{ab} D_{ab}^{\mu\nu}(\partial_x) G_W^{ab}(x) , \quad (\text{C.1})$$

where $D_{ab}^{\mu\nu}(\partial_x)$ is a function of derivatives, usually up to second order, and G_W represents the matrix of Wightman two-point functions,

$$G_W^{ab}(x) \equiv \langle \psi^a(x) \psi^b(0) \rangle . \quad (\text{C.2})$$

Notice that, for real fields, one has

$$G_W^{ba}(x) = G_W^{ab*}(-x) . \quad (\text{C.3})$$

However, we are more familiar with the calculation of Feynman propagators given a certain theory, not the Wightman version. How can we relate G_W to G_F , the Feynman propagator, in a way that is helpful to our calculations?

To begin with, notice that, by definition,

$$G_F^{ab} = \langle \psi^a(x) \psi^b(0) \rangle \theta(t) + \langle \psi^b(0) \psi^a(x) \rangle \theta(-t) \quad (\text{C.4})$$

$$= G_W^{ab}(x) \theta(t) + G_W^{ba}(-x) \theta(-t) \quad (\text{C.5})$$

$$= G_W^{ab}(x) \theta(t) + G_W^{ab*}(x) \theta(-t) . \quad (\text{C.6})$$

As usual, $\theta(t)$ is the step function. This relation can be inverted to give

$$G_W^{ab}(x) = G_F^{ab}(x)\theta(t) + G_F^{ab*}(x)\theta(-t) . \quad (\text{C.7})$$

Computations are usually easier in Fourier transform. Using the Fourier representation of the step function,

$$\theta(t) = \int \frac{d\omega}{2\pi} \frac{i}{\omega + i\epsilon} e^{-i\omega t} , \quad (\text{C.8})$$

we get

$$\tilde{G}_W^{ab}(\omega, \vec{k}) = \int \frac{d\omega'}{2\pi} \left[\frac{i}{\omega - \omega' + i\epsilon} \tilde{G}_F^{ab}(\omega', \vec{k}) - \frac{i}{\omega - \omega' - i\epsilon} \tilde{G}_F^{ab*}(-\omega', -\vec{k}) \right] \quad (\text{C.9})$$

$$= \int \frac{d\omega'}{2\pi} \left[\frac{i}{\omega - \omega' + i\epsilon} \tilde{G}_F^{ab}(\omega', \vec{k}) + \text{h.c.} \right] , \quad (\text{C.10})$$

where the last equality follows from $\tilde{G}_F^{ab}(\omega, \vec{k}) = \tilde{G}_F^{ba}(-\omega, -\vec{k})$ —a direct consequence of the definition (C.4). We thus see that \tilde{G}_W^{ab} is a hermitian matrix, in agreement with (C.3).

The matrix of Feynman propagators is easily computed starting from the quadratic action written in Fourier space,

$$S = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \tilde{\psi}^{*a}(\omega, \vec{k}) K_{ab}(\omega, \vec{k}) \tilde{\psi}^b(\omega, \vec{k}) , \quad (\text{C.11})$$

where the kinetic matrix K is hermitian. In matrix notation, one simply has

$$\tilde{G}_F(\omega, \vec{k}) = i(K(\omega, \vec{k}) + i\epsilon)^{-1} . \quad (\text{C.12})$$

Focusing on the first term in the integral (C.10), and assuming that, as usual, the Feynman propagators decay at infinity and have (simple) poles slightly away from the real axis, we can close the ω' contour in the lower half plane. We thus only pick up the poles of \tilde{G}_F that lie under the real axis—the positive frequency ones, for a stable theory. We get

$$\tilde{G}_W^{ab}(\omega, \vec{k}) = \sum_n \frac{i}{\omega - \omega_n + i\epsilon} \left[K^{-1}(\omega', \vec{k})(\omega' - \omega_n) \right]_{\omega'=\omega_n}^{ab} + \text{h.c.} , \quad (\text{C.13})$$

where the sum is extended over the positive frequency poles, $\omega_n = \omega_n(\vec{k})$.

Using the distributional identity

$$\frac{1}{x + i\epsilon} = P\frac{1}{x} - i\pi\delta(x) \quad (\text{C.14})$$

and the hermiticity of K , we finally get

$$\tilde{G}_W^{ab}(\omega, \vec{k}) = \sum_n \left[K^{-1}(\omega', \vec{k})(\omega' - \omega_n) \right]_{\omega'=\omega_n}^{ab} (2\pi)\delta(\omega - \omega_n) . \quad (\text{C.15})$$

As a check, for a single relativistic massive scalar the kinetic “matrix” is simply

$$K = \omega^2 - \vec{k}^2 - m^2 , \quad (\text{C.16})$$

the positive frequency pole is

$$\omega_k = \sqrt{\vec{k}^2 + m^2} , \quad (\text{C.17})$$

and the Wightman two-point function thus reduces to

$$\tilde{G}_W(\omega, \vec{k}) = \frac{1}{2\omega_k} (2\pi)\delta(\omega - \omega_k) \quad (\text{C.18})$$

$$= (2\pi)\theta(\omega)\delta(k^2 + m^2) , \quad (\text{C.19})$$

which is the correct expression. Notice however that, in the general derivation above, we have never used Lorentz invariance.

Appendix D: Symmetric stress-energy tensors

One may consider performing our calculations in Chapter 4 starting from the more trusted symmetric versions of the stress-energy tensor, i.e. the Hilbert and Belinfante tensors. When deriving the Hilbert tensor for a framid, we have to keep in mind the unit-norm constraint on A^μ ,

$$g^{\mu\nu} A_\mu A_\nu = -1, \tag{D.1}$$

which forbids varying the metric $g^{\mu\nu}$ independently of A^μ . One can introduce a vierbein and vary the action with respect to it instead, yielding [20]

$$T_H^{\mu\nu} = \frac{1}{\sqrt{-g}} \left(2 \frac{\delta S}{\delta g_{\mu\nu}} + \frac{\delta S}{\delta A_\mu} A^\nu \right), \tag{D.2}$$

where now the functional derivatives are unconstrained.

The tensor above is evidently not symmetric in general. In fact, it is symmetric only on-shell, i.e. upon using the equations of motion, which, taking into account the unit-norm constraint once more, are simply

$$(\eta_{\mu\nu} + A_\mu A_\nu) \frac{\delta S}{\delta A_\nu} = 0. \tag{D.3}$$

The end result for the (symmetric) Hilbert tensor is

$$\begin{aligned}
T_H^{\mu\nu} &= \mathcal{L}g^{\mu\nu} \\
&+ 2c_1 \left[A^{(\mu} \partial_\alpha \partial^{\nu)} A^\alpha - A_\alpha \partial^\alpha \partial^{(\mu} A^{\nu)} - \partial^\alpha A_\alpha \partial^{(\mu} A^{\nu)} - \partial^{(\mu} A^\alpha \partial^{\nu)} A_\alpha + \partial_\alpha A^{(\mu} \partial^{\nu)} A^\alpha \right] \\
&+ 2c_2 \left[A^{(\mu} \partial_\alpha \partial^{\nu)} A^\alpha - g^{\mu\nu} A_\alpha \partial_\beta \partial^\alpha A^\beta - g^{\mu\nu} \partial_\alpha A^\alpha \partial^\beta A_\beta \right] \\
&+ 2c_3 \left[A^{(\mu} \partial_\alpha \partial^\alpha A^{\nu)} + \partial_\alpha A^{(\mu} \partial^\alpha A^{\nu)} - A_\alpha \partial^\alpha \partial^{(\mu} A^{\nu)} - \partial^\alpha A_\alpha \partial^{(\mu} A^{\nu)} - \partial_\alpha A^{(\mu} \partial^{\nu)} A^\alpha \right] \\
&+ 2c_4 \left[A^\mu A^\nu \partial_\alpha A^\beta \partial_\beta A^\alpha + A^\alpha A^\mu A^\nu \partial_\beta \partial_\alpha A^\beta - A^\alpha A^{(\mu} \partial_\alpha A^{\nu)} \partial^\beta A_\beta - A^\alpha A_\beta \partial_\alpha A^{(\mu} \partial^\beta A^{\nu)} \right. \\
&\quad \left. - A^\alpha A_\beta A^{(\mu} \partial^\beta \partial_\alpha A^{\nu)} + A^\alpha A^{(\mu} \partial_\alpha A^\beta \partial_\beta A^{\nu)} - A^\alpha A^{(\mu} \partial_\alpha A_\beta \partial^{\nu)} A^\beta \right], \tag{D.4}
\end{aligned}$$

where the c_a 's are coefficients related to the M_1^2 , c_L^2 , and c_T^2 coefficients of sect. 4.3 [20]. Manipulating this tensor to compute our quantum corrections clearly requires considerably more effort compared to the Noether one, eq. (4.25). The same is true for the Belinfante tensor, which turns out to be exactly the same as the Hilbert one.

Namely, the Belinfante stress-energy tensor in general is defined as [6]

$$T_B^{\mu\nu} = T_N^{\mu\nu} - \frac{i}{2} \partial_\kappa \left[\frac{\partial \mathcal{L}}{\partial (\partial_\kappa \Phi^a)} (\mathcal{J}^{\mu\nu})^a_b \Phi^b - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^a)} (\mathcal{J}^{\kappa\nu})^a_b \Phi^b - \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi^a)} (\mathcal{J}^{\kappa\mu})^a_b \Phi^b \right], \tag{D.5}$$

where $T_N^{\mu\nu}$ is the Noether stress-energy tensor, and the $\mathcal{J}^{\mu\nu}$'s are the Lorentz generators in the representation appropriate for the fields Φ^a . For 4-vector fields,

$$(\mathcal{J}^{\rho\sigma})^\kappa_\lambda = i(\eta^{\sigma\kappa} \delta_\lambda^\rho - \eta^{\rho\kappa} \delta_\lambda^\sigma). \tag{D.6}$$

Writing down all terms in (D.5), the result is equal to equation (D.4) plus terms that are proportional to the equations of motion¹. We checked that upon expanding the Belinfante and Hilbert stress-energy tensors to quadratic order in our $\vec{\eta}$ fields, we get exactly the same expressions for $\langle T^{00} \rangle$ and $\langle T^{ij} \rangle$ as those derived from the Noether stress-energy tensor, eqs. (4.28), (4.29).

¹Note that the Belinfante tensor is also non-symmetric unless one uses the equations of motion [6].

Appendix E: The path-integral measure

In order to construct a Lorentz-invariant measure, we can start from the obvious invariant measure for A_μ ,

$$DA_\mu \equiv \prod_x d^4 A(x) , \quad (\text{E.1})$$

and impose an invariant constraint that removes its norm, e.g.

$$\delta(A_\mu A^\mu + 1) . \quad (\text{E.2})$$

We can then parametrize A_μ in terms of our Goldstone fields $\vec{\eta}(x)$ and of a radial mode $\rho(x)$,

$$A_0 = \rho \cosh |\vec{\eta}| , \quad \vec{A} = \rho \frac{\vec{\eta}}{|\vec{\eta}|} \sinh |\vec{\eta}| . \quad (\text{E.3})$$

The path integral then reads

$$\int DA_\mu \delta(A_\mu A^\mu + 1) e^{iS} \dots = \int D\rho D\vec{\eta} \text{Det} J \delta(\rho^2 - 1) e^{iS} \dots , \quad (\text{E.4})$$

where the dots denote insertions of operators, and the functional Jacobian J is

$$J(x, x') \equiv \frac{\delta(A_0(x), \vec{A}(x))}{\delta(\rho(x'), \vec{\eta}(x'))} = \frac{\partial(A_0, \vec{A})}{\partial(\rho, \vec{\eta})} \delta(x - x') . \quad (\text{E.5})$$

Using standard functional methods [14], its determinant can be written in exponential form as

$$\text{Det} J = e^{i\Delta S} , \quad \Delta S \equiv -i \left(\int \frac{d^4 k}{(2\pi)^4} \right) \int d^4 x \log \frac{\sinh^2 |\vec{\eta}|}{|\vec{\eta}|^2} , \quad (\text{E.6})$$

where we used that, thanks to the delta-function in (E.4), $\rho = 1$. The integral over ρ can now be performed explicitly, upon which we are left with the path integral

$$\int D\vec{\eta} e^{i(S+\Delta S)} \dots \quad (\text{E.7})$$

We thus reach the conclusion that, to preserve Lorentz invariance in our computations, we should supplement the $\vec{\eta}$ effective action with ΔS .

If we use dimensional regularization, ΔS vanishes, because its overall coefficient does. This is one of the many reasons why dimensional regularization is convenient, and why we usually don't track functional determinants coming from field redefinitions in the path integral.

If, on the other hand, we use other UV regulators, we should keep ΔS around. Notice that, like all effects coming from functional determinants, ΔS is formally of one-loop order. We should then use it consistently in perturbation theory. For instance, for one-loop computations, we should use ΔS at tree-level.

ΔS is a (UV divergent) potential for our Goldstone fields. In particular, it includes a mass term for them. This is inconsistent with the Goldstone theorem for spontaneously broken boosts [10]. This means that, at one-loop, there must be other contributions that cancel at least the effects of such a mass term. Or, conversely, if we don't keep ΔS around, at one-loop we must find nontrivial contributions to the mass of the Goldstones, in violation of the Goldstone theorem.

We can check this explicitly. For simplicity, let's consider the $c_L = c_T = 1$ case, which is particularly symmetric [20]. The two-derivative Goldstone action takes the form of a relativistic non-linear sigma model,

$$S = -\frac{M_1^2}{2} \int d^4x f_{ij}(\vec{\eta}) \partial_\mu \eta^i \partial^\mu \eta^j, \quad (\text{E.8})$$

with f_{ij} given by

$$f_{ij}(\vec{\eta}) = P_{ij}^\parallel(\vec{\eta}) + \frac{\sinh^2 |\vec{\eta}|}{|\vec{\eta}|^2} P_{ij}^\perp(\vec{\eta}), \quad (\text{E.9})$$

where P^\parallel and P^\perp are the parallel and perpendicular projectors in $\vec{\eta}$ -space. We can compute at once all one-loop contributions to the mass and to non-derivative interactions of the Goldstone fields by

computing the one-loop Coleman-Weinberg potential [62]. Following again standard functional methods [14], we get

$$\Delta\Gamma_{\text{CW}} = \frac{i}{2} \int d^4x \frac{d^4k}{(2\pi)^4} \text{tr} \log(k^2 f_{ij}(\vec{\eta})) \quad (\text{E.10})$$

where the trace is a simple finite-dimensional (3×3) matrix trace. We can split the matrix inside the trace as

$$\log(k^2 f_{ij}(\vec{\eta})) = \log(k^2) \delta_{ij} + \log(f_{ij}(\vec{\eta})) . \quad (\text{E.11})$$

The first term is field independent, and we can discard it. As to the second term, we can evaluate its trace in a basis in which it is diagonal, such as a basis in which $\vec{\eta} \propto (1, 0, 0)$. We thus get

$$\Delta\Gamma_{\text{CW}} = i \left(\int \frac{d^4k}{(2\pi)^4} \right) \int d^4x \log \frac{\sinh^2 |\vec{\eta}|}{|\vec{\eta}|^2} = -\Delta S . \quad (\text{E.12})$$

Regardless of the UV regulator used, this cancels exactly all effects of ΔS at this order, thus recovering agreement with the boost Goldstone theorem.

In conclusion, when using UV regulators other than dim-reg, the correction ΔS coming from the path-integral measure should be kept, and used consistently in perturbation theory. In practice, for our purposes in this work, this ends up not mattering. This is because we computed the one-loop expectation value of the stress-energy on the framid's ground state. Since ΔS is formally already of one-loop order, its contributions to such an expectation value should be considered only at tree level. That is,

$$\langle T^{\mu\nu} \rangle_{1\text{-loop}} = \langle T^{\mu\nu} \rangle_{1\text{-loop}}^0 + \langle \Delta T^{\mu\nu} \rangle_{\text{tree}} , \quad (\text{E.13})$$

where the l.h.s. stands for all one-loop contributions in the full theory (with action $S + \Delta S$), the first term on the r.h.s. stands for the one-loop contributions in the theory without ΔS , and the second term on the r.h.s. stands for the correction to the stress-energy tensor operator coming from ΔS , evaluated on the ground state at tree level only. But at tree-level the ground state simply corresponds to $\vec{\eta} = 0$, so ΔS vanishes there, and so does its contribution to our expectation value.

Appendix F: Origin of the mixing term and powers of a

As emphasized in [27, 64], for icosahedral inflation anisotropies in two-point functions can only arise from higher derivative corrections to the solid's action. A possible candidate is a term schematically of the form $T_6 \cdot (R^{\mu\nu\rho\sigma} \partial_\mu \phi^I \partial_\nu \phi^J \partial_\rho \phi^K \partial_\sigma \phi^L)^3$, with suitable I -type index contractions. In fact, for such a term to be compatible with the approximate internal scale invariance associated with slow-roll [24],

$$\phi^I \rightarrow \lambda \phi^I , \tag{F.1}$$

we need to multiply it by a factor scaling like $(B^{IJ})^{-6}$ (to lowest order in slow roll), again with suitable index contractions.

So, let's consider a higher-derivative action term schematically of the form

$$\Delta S \sim \frac{1}{M^2} \int d\tau d^3x a^4 (B^{IJ})^{-6} T_6 \cdot (R^{\mu\nu\rho\sigma} \partial_\mu \phi^I \partial_\nu \phi^J \partial_\rho \phi^K \partial_\sigma \phi^L)^3 , \tag{F.2}$$

where we introduced an arbitrary dimensionful coupling constant. Notice that we are using directly conformal time, since it makes the analysis that follows simpler: all components of the unperturbed metric scale like a^2 , and so we don't need to differentiate between time and space.

Consider now expanding such a term in spatially flat slicing gauge about our inflationary background. We are only interested in bilinear scalar-mixing terms. We expand our fields in perturbations,

$$\phi \rightarrow x + \pi , \quad g \rightarrow a^2(\tau)(\eta + h) , \tag{F.3}$$

where g is shorthand for the metric (with lower indices), and keep in mind that $h_{\mu\nu}$ contains tensors (γ_{ij}) as well as scalars (h_{00} and $h_{0i} = \partial_i \psi$). We thus need to keep π - h and h - h bilinear terms. However, to this order in derivatives, the latter cannot contribute to our scalar-tensor mixing. The

reason is that for the contraction with T_6 in (F.2) to yield something nonzero, we need six free spatial indices on the fields and their derivatives. If we neglect π , the factors of $\partial\phi$ and B^{IJ} yield terms with no derivatives on the fields, whereas the factors of $R^{\mu\nu\rho\sigma}$ yield terms with zero, one, or two-derivative per field. So, among the h - h terms, the only ones that have a chance of giving a contribution to scalar-tensor mixings are of the form

$$\partial_i\partial_j\gamma_{kl} \times (\partial_m\partial_n h_{00}, \partial_m h_{0n}, \text{ or } \partial_m\partial_0 h_{0n}) \quad (\text{F.4})$$

$$(\partial_0\partial_i\gamma_{jk}, \text{ or } \partial_i\gamma_{jk}) \times \partial_l\partial_m h_{0n}, \quad (\text{F.5})$$

and all come from two factors of $R^{\mu\nu\rho\sigma}$ each expanded to linear order in $h_{\mu\nu}$. But then these terms cannot be there: the spacetime indices of the Riemann tensor are contracted with factors of $\partial\phi$; if we neglect π , $\partial_\mu\phi^I = \delta_\mu^I$, which projects all the indices of the Riemann tensor onto spatial directions; to linear order in the metric fluctuations, this can only yield spatial derivatives of γ_{ij} and, in particular, no term involving h_{00} or h_{0i} .

So, in summary, to figure out our scalar-mixing terms in spatially flat-slicing gauge, we only need to look for π - γ terms with three derivatives overall, and we can neglect everything else. From the expansion of $\partial\phi$ and B^{IJ} we get first derivatives of π ,

$$\partial\phi \rightarrow 1 + \partial\pi, \quad B^{IJ} \sim g^{-1}\partial\phi\partial\phi \rightarrow \frac{1}{a^2}(1 + \partial\pi), \quad (\text{F.6})$$

and from the Riemann tensor we get second derivatives of γ ,

$$R^{\mu\nu\rho\sigma} \sim (g^{-1})^3 R^\mu{}_{\nu\rho\sigma} \quad (\text{F.7})$$

$$\sim (g^{-1})^3 (\partial\Gamma + \Gamma\Gamma) \quad (\text{F.8})$$

$$\sim (g^{-1})^5 (\partial g \partial g + g \partial^2 g) \rightarrow \frac{1}{a^6} (a^2 H^2 + \partial^2 \gamma), \quad (\text{F.9})$$

where we used that the Christoffel symbols are schematically $\Gamma \sim g^{-1}\partial g$ and that $\partial_\tau a = a^2 H$.

Plugging all this into (F.2), and keeping only the bilinear π - γ terms, we get

$$\Delta S \sim \frac{1}{M^2} \int d\tau d^3x a^4 a^{12} (1 + \partial\pi)^{-6} T_6 a^{-18} (a^2 H^2 + \partial^2 \gamma)^3 (1 + \partial\pi)^{12} \quad (\text{F.10})$$

$$\rightarrow \frac{H^4}{M^2} \int d\tau d^3x a^2 T_6 \partial\pi \partial^2 \gamma, \quad (\text{F.11})$$

which, if written in proper time t , is precisely of the form (5.7), with

$$\Delta c_{\zeta\gamma}^2 \sim \frac{H^4}{M^2 M_{\text{Pl}}^2}, \quad (\text{F.12})$$

which is of the same order as the tensor-tensor mixing parameter Δc_γ^2 of ref. [64]. Notice that, given the tracelessness and total symmetry of T_6 , the index contraction displayed in (5.7) is the only non-vanishing one.

More in general, we can prove that the mixing term (5.7) is the only scalar-tensor bilinear mixing allowed by gauge invariance. To minimize the number of extra terms coming from covariant derivatives, here we use cosmic time t instead of conformal time τ , and h now stands for metric perturbations about the FRW metric,

$$g = g_{\text{FRW}} + h. \quad (\text{F.13})$$

First, let's set the fluctuations of ϕ^I to zero by fixing the spatial diffs (unitary gauge),

$$\phi^I = x^I \quad (\text{F.14})$$

In terms of gauge transformations $x^\mu \rightarrow x^\mu + \xi^\mu$, we have fixed the spatial components $\xi^i(x)$.

Under the residual $\xi^0(x)$ transformations, h transforms as

$$h_{00} \rightarrow h_{00} + \partial_0 \xi_0, \quad h_{0i} \rightarrow h_{0i} + \partial_i \xi_0, \quad h_{ij} \rightarrow h_{ij} - 2\delta_{ij} \dot{a} a \xi_0. \quad (\text{F.15})$$

The gauge-invariant building blocks for a possible mixing term thus are

$$\partial_0 h_{0i} - \partial_i h_{00} \quad \text{and} \quad \partial_k h_{ij} + \delta_{ij} \partial_0 (a^2 h_{0k}) - a^2 \delta_{ij} \partial_k h_{00} . \quad (\text{F.16})$$

We can now play the Stueckelberg trick and restore gauge invariance under the broken diffeomorphism $x^i \rightarrow x^i + \xi^i$ by promoting the parameters ξ^i to Goldstone fields:

$$\xi^i(x) = \pi^i(x) . \quad (\text{F.17})$$

Our building blocks, which are now invariant under all gauge transformations, now read

$$\Pi_i = \partial_0 h_{0i} - \partial_i h_{00} + \partial_0 (a^2 \partial_0 \pi_i) \quad (\text{F.18})$$

$$\Xi_{ijk} = \partial_k h_{ij} + \delta_{ij} \partial_0 (a^2 h_{0k}) - a^2 \delta_{ij} \partial_k h_{00} + a^2 \partial_k (\partial_i \pi_j + \partial_j \pi_i) + \delta_{ij} \partial_0 (a^4 \partial_0 \pi_k) . \quad (\text{F.19})$$

Up to integration by parts, the combinations of these building blocks that may give rise to scalar-tensor mixing terms, with the least number of derivatives and correct number of free indices (six), are

$$\partial_i \Pi_j \partial_k \Xi_{mnl} \quad \text{and} \quad \Xi_{ijk} \Xi_{mnl} \quad (\text{F.20})$$

The first term is suppressed in general because it has more derivatives than the second term. The second term produces our $\partial\pi\partial^2\gamma$ mixing, the one appearing in (5.7). Notice that the terms proportional to Kronecker deltas will not contribute to the final result since our icosahedral spin-6 tensor T_6 is traceless.

Appendix G: Diagonalizability

Consider the scalar-tensor sector of our theory, defined by the quadratic action terms (5.4), (5.6), and (5.9). For simplicity, let's focus on a two-dimensional field space, made up of the scalars and a single polarization of the tensors (cf. sect. 5.4). We want to see under what conditions such a system is diagonalizable. It is easier to work directly with the equations of motion rather than the action, since for the former there are no integration-by-parts ambiguities. Certainly, if the equations of motion cannot be made diagonal, neither can the action.

At fixed \vec{k} , upon changing the normalization of π_L in a suitable time-dependent fashion, our equation of motion can be written compactly as

$$\alpha\phi'' + \alpha'\phi' + \alpha M\phi = 0, \quad (\text{G.1})$$

where ϕ is a doublet of fields, α is a scalar function of time, and M is a 2×2 matrix, also time-dependent.

Let us assume that we can diagonalize such an eom through some time-dependent invertible matrix $R(\tau)$. By plugging $\phi = R(\tau)\varphi$ into the equation above, we get

$$\varphi'' + (2R^{-1}R' + \frac{\alpha'}{\alpha})\varphi' + (R^{-1}R'' + \frac{\alpha'}{\alpha}R^{-1}R' + R^{-1}MR)\varphi = 0. \quad (\text{G.2})$$

For this equation to be diagonal, we need:

1. $R^{-1}R'$ to be diagonal, because of the φ' term. However, this implies that $R^{-1}R''$ is also diagonal. This follows from

$$R^{-1}R'' = (R^{-1}R')' - (R^{-1})'R' = (R^{-1}R')' + (R^{-1}R')^2, \quad (\text{G.3})$$

where the last equality can be obtained from taking the time derivative of $R^{-1}R = 1$.

2. $R^{-1}MR$ to be diagonal, because all the other terms multiplying φ already are.

But for $R^{-1}MR$ and $R^{-1}R'$ to be diagonal at the same time, we need them to commute. This in turn implies that M commutes with M' . The reason is that if $R^{-1}MR$ is diagonal at all times, so is its time derivative, which is simply

$$(R^{-1}MR)' = R^{-1}M'R + [R^{-1}MR, R^{-1}R'] = R^{-1}M'R . \quad (\text{G.4})$$

So, $R^{-1}M'R$ is diagonal, which means that it commutes with $R^{-1}MR$. Or, equivalently,

$$[M', M] = 0 . \quad (\text{G.5})$$

This is a nontrivial condition on the time-dependence of M . It is easy to check that such a condition is not satisfied in our case. Therefore, our equation of motion is not diagonalizable.