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Values and coalition configurations

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Abstract

In this paper we consider coalition configurations (Albizuri et al., 2006), that is, families of coalitions not necessarily disjoint whose union is the grand coalition, and give a generalization of the Shapley value (1953) and the Owen value (1977) when coalition configurations form. This will be an alternative definition to the one given by Albizuri et al. (2006).

Keywords: Shapley value, Owen value, coalition configurations.

1 Introduction

There are negotiation situations in which some agents prefer to cooperate together than with others. There is a tool which has been employed to study these kind of negotiations: that of a coalition structure, that is, a partition of the set of agents into disjoint coalitions. Aumann and Dreze (1974) propose and study a value when agents form a coalition structure, and later on, Owen (1977) proposes and characterizes another modification of the Shapley (1953) value also when coalition structures are formed (see also Hart and Kurz, 1983). Even though other coalitional values have been studied (see Albizuri and Aurrekoetxea, 2006, Casajus, 2009, Gómez-Rúa and Vidal-Puga 2010, and references

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herein), the most widely used (see Gómez-Rúa and Vidal-Puga, 2013, and references herein) is the Owen (1977) value.

In Owen's (1977) approach, each member of the coalition structure bargains against the others to allocate the worth available to the grand coalition. Albizuri et al. (2006) consider the more general concept of coalition configuration to model negotiations in which players form coalitions not necessarily disjoint. A coalition configuration is defined as a family of coalitions not necessarily disjoint, whose union is the grand coalition. They generalize the Owen value (1977) (and therefore the Shapley value, 1953) with reference to coalition configurations. In fact they obtain two generalizations of the Owen value. The configuration value and the dual configuration value. Both values are dual each of the other. Let us present the second one by means of an example. The first one could be presented similarly.

Let $N = \{1, 2, 3\}$ be the set of players and consider the transferable utility game v on N which satisfies $v(1) = 0 = v(2) = v(3)$, $v(12) = 3$, $v(13) = 0$, $v(23) = 1$ and $v(123) = 5$. The dual configuration value ϕ associates a vector of outcomes with each coalition configuration. These outcomes can be calculated by means of orderings.

Suppose for example that players form coalition configuration $\mathcal{B}_1 = \{\{1, 2\}, \{2, 3\}\}$. Then we have to consider all the orderings of the elements which form the two coalitions of $\mathcal{B}_1$, in such a way that the elements of the same coalition keep together. So we consider 1223, 1232, 2123, 2132, 2312, 2321, 3212 and 3221. We interpret these orderings as follows. Suppose that each player in each coalition of $\mathcal{B}_1$ has a representative associated with that coalition, and that these four representatives form a queue outside a room in such a way that all the representatives associated with a coalition are together. Then, these representatives enter in the room and form an ordering. We have in this way the orderings above. When a representative of a player enters in the room a coalition forms if all the representatives of that player are in the room, being that coalition the one formed by the players whose representatives are all in the room. For example, given 1232, player 1 is given $v(1) = 0$ for when 1 enters coalition $\{1\}$ forms and 1 is given her contribution to the singleton coalition. When the first representative of player 2 enters in the room neither coalition forms for all the representatives of 2 are not yet in the room. Therefore, 2 is not given anything. Then 3 comes and coalition $\{1, 3\}$ forms, and 3 is given her marginal contribution to this coalition: $v(13) - v(1) = 0$. When the second representative of player 2 enters $\{1, 2, 3\}$ forms and 2 is given $v(123) - v(13) = 5$. If we suppose that all the orderings are equally likely, the expected marginal contribution of a player is her dual configuration value associated with $\mathcal{B}_1$. The value is $\phi(v, \mathcal{B}_1) = \left(1, 3\frac{1}{2}, \frac{1}{2}\right)$.

Consider now the unanimity game u_N on N given by $u_N(N) = 1$ and $u_N(S) = 0$ otherwise. Then, an analogous reasoning as before leads to $\phi(u_N, \mathcal{B}_1) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

If $\mathcal{B}_2 = \{\{1,2\}, \{2,3\}, \{2\}\}$ is formed we can calculate the dual configuration value as before. Now we have to consider the orderings 12232, 12322, 23122, ... In this case $\phi(v, \mathcal{B}_2) = \left(\frac{2}{3}, 4, \frac{1}{3}\right)$ and $\phi(u_N, \mathcal{B}_2) = \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$.

Notice that in both cases player 2 has three representatives in $\mathcal{B}_2$ since she belongs to three coalitions. If player 2 were in four coalitions she would have four representatives and so on. So the more coalitions a player belongs to the more effort or weight she has.

In this work we do not allow any player (as player 2 here) to increase her weight by belonging to more and more coalitions. Each player will have a fix weight and this weight will be spread among the coalitions she belongs to. Think for example of agents that have a certain amount of time they share among the coalitions they belong to, the worth of these coalitions depending on the time spent by the agents inside the coalitions.

The weight of a player will be represented by a positive real number and we will consider all the possible weights for a player. So we have a family of alternative values. They will be called weighted bounded configuration values. The definition will be made also by means of orderings, as in Albizuri et al. (2006).

The paper is structured as follows. Section 2 is a preliminary one, in Section 3 we define the weighted bounded configuration values and in Section 4 we give an axiomatic characterization which gives all the weighted bounded configuration values. In Section 5 we focus on a specific weighted bounded configuration value. As a first approach we can say that we suppose that the weights of all the players in a fixed set U are equal to one and that these players spread equally their weight among the coalitions of the coalition configuration they belong to. Finally, in Section 6 we explain how to obtain the dual family corresponding to the weighted bounded configuration values.

2 Preliminaries

Given a finite set of players N, we denote by $\mathbf{G}^N$ the set formed by the cooperative transferable utility games with player set N, that is by the mappings $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. A *coalition* is a subset $S \subseteq N$, $S \neq \emptyset$. Given $i \in S \subseteq N$ and $T \subseteq N \backslash S$, let $S^{-i,T}$ denote the set $(S \backslash \{i\}) \cup T$. Given $v \in \mathbf{G}^N$ and $T \subseteq N$, $T \neq \emptyset$, we denote by $v_T \in \mathbf{G}^T$ the game on T such that $v_T(S) = v(S)$ for all $S \subset T$. A game $v \in \mathbf{G}^N$ is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T$. A coalition T is a *partnership* in $v \in \mathbf{G}^N$ if $v(C \cup S) = v(S)$ whenever $C \subsetneqq T$ and $S \subseteq N \backslash T$. For every coalition T we denote by

u_T the unanimity game defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

A *coalition structure* of N is a family $\mathcal{B} = \{B_1, ..., B_m\}$ of coalitions of N such that $\bigcup_{q=1}^{m} B_q = N$, and $B_p \cap B_q = \emptyset$ if $B_p, B_q \in \mathcal{B}$ with $p \neq q$. We denote by $\mathbf{B}_0^N$ the set of coalition structures of N and $\mathbf{BG}_0^N = \mathbf{B}_0^N \times \mathbf{G}^N$.

If we fix $\mathcal{B} \in \mathbf{B}_0^N$, a *solution* ψ on $\mathbf{G}^N$ is a function from $\mathbf{G}^N$ into $\mathbb{R}^N$. Vector $\psi(v)$ represents the expectations of players in v when they form coalition structure $\mathcal{B}$.

Let us present the weighted coalition structure values with intercoalitional symmetry defined by Levy and McLean (1989). We present a formalization which fits with the definition for the weighted bounded configuration values that we provide in Section 3.

First we need some notation. Let $\mathcal{B} = \{B_1, ..., B_m\}$ be a coalition structure of N. For $B_q \in \mathcal{B}$, denote by $\Sigma(B_q)$ the set of permutations of B_q and let $\mathcal{R}_{\mathcal{B}}(N)$ be the set of tuples $(\sigma_1, ..., \sigma_m)$ such that for every $l = 1, ..., m$ it holds that

(1) $\sigma_l \in \Sigma(B_q)$ for some $q = 1, ..., m$, and

(2) $l' \neq l$ and $\sigma_{l'} \in \Sigma(B_q)$ implies $\sigma_l \notin \Sigma(B_q)$.

Each element $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ naturally induces a permutation R_α of N in which players in every $B_q \in \mathcal{B}$ appear successively. We denote by $R_\alpha^{B_q}$ the permutation $\sigma_l \in \Sigma(B_q)$.

Let $(v, \mathcal{B}) \in \mathbf{BG}_0^N$ and $\omega \in \mathbb{R}_+^N$.[1] Assume that players in N form a queue outside a room according to the following procedure. Players in every B_q are together, all the orderings of coalitions $B_q \in \mathcal{B}$ are equally likely and players in every B_q are ordered as follows. One player in B_q is picked up at a time and she is placed in the front of the queue which is partially formed. Once a player is picked up she is not picked up any more and the probability of picking up player i is given by her weight ω_i divided by the weights of the players in B_q who are not yet in the queue. Then players proceed to enter in the room. When player i enters in the room a coalition S forms and player i is given her marginal contribution to this coalition, that is, $v(S) - v(S \backslash \{i\})$. The expected marginal contribution of player i is by definition the weighted coalition structure value with intercoalitional symmetry of player i in v associated with ω and $\mathcal{B}$.

This definition is due to Levy and McLean (1989). Let us formalize it. For every $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ and $i \in N$, let $R_\alpha[i]$ denote the set of players which are before player i in order R_α, including player i.

Let $(v, \mathcal{B}) \in \mathbf{BG}_0^N$ and $\omega \in \mathbb{R}_+^N$. For every $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ and $i \in N$ the *marginal*

[1] Given $N \subseteq U$ we denote by $\mathbb{R}_+^N$ the set of $|N|$ -tuples with strictly positive components.

contribution of player i in order R_α is

$$C_i(v, R_\alpha) = v(R_\alpha[i]) - v(R_\alpha[i] \setminus \{i\}).$$

Consider the probability distribution $Q^{(\omega,\mathcal{B})}$ on $\mathcal{R}_{\mathcal{B}}(N)$ such that

$$Q^{(\omega,\mathcal{B})}(\alpha) = \frac{1}{|\mathcal{B}|!} \cdot \prod_{B_q \in \mathcal{B}} Q^{\omega}_{B_q}\left(R^{B_q}_\alpha\right),$$

where $Q^{\omega}_{B_q}$ is the probability distribution on $\Sigma(B_q)$ such that $Q^{\omega}_{B_q}(\sigma_l) = \prod_{t=1}^{|B_q|} \frac{\omega_{i_t}}{\sum_{s=1}^{t} \omega_{i_s}}$ for $\sigma_l = \left(i_1 ... i_{|B_q|}\right) \in \Sigma(B_q)$.

The *weighted coalition structure value with intercoalitional symmetry* of player i in v associated with ω and $\mathcal{B}$ is defined by

$$\eta^{(\omega,\mathcal{B})}_i(v) = \sum_{\alpha \in \mathcal{R}_{\mathcal{B}}(N)} Q^{(\omega,\mathcal{B})}(\alpha) \cdot C_i(v, R_\alpha).$$

These solutions are characterized by Levy and McLean (1989) by means of the following axioms. Let $\mathcal{B} \in \mathbf{B}^N_0$ be fixed and let ψ denote a solution on $\mathbf{G}^N$.

Linearity. For every $v_1, v_2 \in \mathbf{G}^N$ and $\lambda, \mu \in \mathbb{R}$ it holds that

$$\psi(\lambda v_1 + \mu v_2) = \lambda \psi(v_1) + \mu \psi(v_2).$$

Efficiency. For every $v \in \mathbf{G}^N$ it holds that

$$\sum_{i \in N} \psi_i(v) = v(N).$$

$\mathcal{B}$-Positivity. If $v \in \mathbf{G}^N$ is monotonic and $C_i(v, R_\alpha) > 0$ for some $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$, then $\psi_i(v) > 0$.

Intercoalitional Symmetry. Let $v \in \mathbf{G}^N$. If $B_p, B_q \in \mathcal{B}$ are such that for every $\mathcal{C} \subseteq \mathcal{B} \setminus \{B_p, B_q\}$ it holds $v\left(B_p \cup \bigcup_{B_r \in \mathcal{C}} B_r\right) = v\left(B_q \cup \bigcup_{B_r \in \mathcal{C}} B_r\right)$, then

$$\sum_{i \in B_p} \psi_i(v) = \sum_{i \in B_q} \psi_i(v).$$

Intracoalitional Partnership. If $T \subseteq N$ is a partnership in v, then for every B_q such that $B_q \cap T \neq \emptyset$ and every $i \in B_q \cap T$ it holds that

$$\psi_i(u_T) \sum_{j \in B_q \cap T} \psi_j(v) = \psi_i(v) \sum_{j \in B_q \cap T} \psi_j(u_T).$$

Null Player Axiom. Let $v \in \mathbf{G}^N$. If $i \in N$ is a null player in v (i.e., if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$), then

$$\psi_i(v) = 0.$$

Theorem 2.1 *(Levy and McLean, 1989) Let $\mathcal{B} \in \mathbf{B}_0^N$. A function $\psi : \mathbf{G}^N \to \mathbb{R}^N$ satisfies Linearity, Efficiency, the Null Player Axiom, $\mathcal{B}$-Positivity, Intercoalitional Symmetry and Intracoalitional Partnership if and only if there exists a vector $\omega \in \mathbb{R}_+^N$ such that $\psi = \eta^{(\omega,\mathcal{B})}$.*

And finally, Levy and McLean (1989) proved that

$$\eta_i^{(\omega,\mathcal{B})}(u_T) = \frac{1}{|B_p \in \mathcal{B} : B_p \cap T \neq \emptyset|} \cdot \frac{\omega_i}{\sum_{j \in B_q \cap T} \omega_j} \quad \text{if} \quad i \in B_q \cap T. \tag{1}$$

3 The weighted bounded configuration values

In this section we propose an alternative solution to the ones given in Albizuri et al. (2006). Again, players form coalitions and can be in more than one. In addition, we suppose that when a player is in more than one coalition, several players can represent her in the coalitions she belongs to.

To formalize this, we consider two disjoint sets of players, U and U', where players in U' represent players in U. We suppose that these auxiliary players do not belong at once to more than one coalition.

Let U and U' be, respectively, a finite and an infinite set of players with $|U| \geq 2$ and $U \cap U' = \emptyset$. We can always find a function $F : U \times \{2, ..., 2^{|U|-1}\} \to 2^{U'}$ such that $|F(i,k)| = k$ for all $k \in \{2, ..., 2^{|U|-1}\}$, and $F(i,k) \cap F(j,k') = \emptyset$ if $i \neq j$ or $k \neq k'$. The interpretation of $F(i,k)$ is as follows: $F(i,k)$ is the set of representatives of player i when player i belongs to exactly k different coalitions.

Let $A = U \cup F\left(U \times \{2, ..., 2^{|U|-1}\}\right) \subset U \cup U'$ be the set of all possible players and representatives. A *coalition configuration* of $N \subseteq A$ is a family $\mathcal{B} = \{B_1, ..., B_m\}$ of different coalitions of N such that $\bigcup_{q=1}^m B_q = N$, $|\mathcal{B}^i| = 1$ when $i \in N \cap U'$ and it satisfies

$$\left|\mathcal{B}^i\right| > 1 \Longrightarrow F\left(i, \left|\mathcal{B}^i\right|\right) \cap N = \emptyset$$

for all $i \in N$, where $\mathcal{B}^i = \{B_q \in \mathcal{B} : i \in B_q\}$.

The interpretation is as follows. There are two types of players, players in U and players in $F\left(U \times \{2, ..., 2^{|U|-1}\}\right)$, where players in $F\left(U \times \{2, ..., 2^{|U|-1}\}\right)$ can represent players in U. The admissible set of players are the subsets of A. Each player $i \in U$ can be represented by the players in $F(i,k)$ when player i joins k different coalitions. Players in $F(i,k)$ can be seen as split players of player i, and they cannot split (this is the meaning of $|\mathcal{B}^i| = 1$ when $i \in N \cap U'$). So, players in $F(i,k)$ cannot be in more than one coalition. Furthermore, player $i \in U$ cannot be with players that represent her (this is the last requirement).

The set of coalition configurations of N is denoted by $\mathbf{B}^N$. Moreover, we write $\mathbf{BG}^N = \mathbf{B}^N \times \mathbf{G}^N$ and $\mathbf{BG} = \bigcup_{N \subseteq A} \mathbf{BG}^N$.

Given $\mathcal{B} \in \mathbf{B}^N$, for every $B_q \in \mathcal{B}$ we denote by $\Sigma(B_q)$, as in the previous section, the set of permutations of B_q and by $\mathcal{R}_{\mathcal{B}}(N)$ the set of tuples $(\sigma_1, ..., \sigma_m)$ such that for every $l = 1, ..., m$ it holds that

(1) $\sigma_l \in \Sigma(B_q)$ for some $B_q \in \mathcal{B}$, and

(2) $l' \neq l$ and $\sigma_{l'} \in \Sigma(B_q)$ implies $\sigma_l \notin \Sigma(B_q)$.

Now, every element $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ induces a permutation with repetition of order $\sum_{B_q \in \mathcal{B}} |B_q|$ of N in which players in every $B_q \in \mathcal{B}$ appear successively. We denote this permutation with repetition by R_α and by $R_\alpha^{B_q}$ the permutation $\sigma_l \in \Sigma(B_q)$.

A *solution* on $\mathbf{BG}$ is a function ψ from $\mathbf{BG}$ into $\bigcup_{N \subseteq A} \mathbb{R}^N$ such that $\psi(v, \mathcal{B}) \in \mathbb{R}^N$ whenever $(v, \mathcal{B}) \in \mathbf{BG}^N$. Vector $\psi(v, \mathcal{B})$ can be interpreted as the expected value of players in v when players in the game form coalition configuration $\mathcal{B}$.

We define our solution by means of orderings. Suppose that there exists a vector $\omega \in \mathbb{R}_+^A$ such that for each $i \in U$ and $k \in \left\{2, ..., 2^{|U|-1}\right\}$

$$\sum_{j \in F(i,k)} \omega_j = \omega_i. \tag{2}$$

That is, every player belonging to an admissible set of players has a weight ω_i and the weight of every player $i \in U$ is the sum of the weights of the split players associated with her.

Example 3.1 *Take $\omega_i = 1$ for all $i \in U$ and $\omega_i = \frac{1}{k}$ for all $i \in F(j,k)$ with $j \in U$ and $k \in \left\{2, ..., 2^{|U|-1}\right\}$. It is straightforward to check that such ω is well-defined and it satisfies (2).*

Example 3.2 *Let $\mu \in \mathbb{R}_+^A$. Take $\omega_i = \mu_i$ for all $i \in U$ and $\omega_i = \frac{\mu_i}{\sum_{l \in F(j,k)} \mu_l} \mu_j$ for all $i \in F(j,k)$ with $j \in U$ and $k \in \left\{2, ..., 2^{|U|-1}\right\}$. Again, it is straightforward to check that such ω is well-defined and it satisfies (2).*

For each $N \subseteq A$, $i \in N \cap U$ and $\mathcal{B} \in \mathbf{B}^N$ such that $|\mathcal{B}^i| > 1$ there exists some one-to-one function $\pi_{\mathcal{B}}^i : \mathcal{B}^i \to F(i, |\mathcal{B}^i|)$. This function tells us which player in $F(i, |\mathcal{B}^i|)$ can represent player i in each coalition of $\mathcal{B}^i$. If $|\mathcal{B}^i| = 1$ then player i belongs to a unique coalition and therefore player i can be represented by herself. So, we also define $\pi_{\mathcal{B}}^i(B) = i$ for all $i \in N$ such that $|\mathcal{B}^i| = 1$.

Moreover, these mappings should be consistent in the sense that the representatives of some player j do not change if some other player i is replaced by her own representatives.

Namely, let $N \subseteq A$, $i \in N \cap U$ and $\mathcal{B} \in \mathbf{B}^N$ such that $|\mathcal{B}^i| > 1$, and consider

$$\mathcal{B}^{-i,F\left(i,|\mathcal{B}^i|\right)} = \left(\mathcal{B} \backslash \mathcal{B}^i\right) \cup \left\{ B_q^{-i,\left\{\pi_{\mathcal{B}}^i(B_q)\right\}} \right\}_{B_q \in \mathcal{B}^i}.$$

If $i, j \in B_q \in \mathcal{B}$, then we require

$$\pi^j_{\mathcal{B}^{-i,F(i,|\mathcal{B}^i|)}} \left((B_q \backslash \{i\}) \cup \left\{ \pi^i_{\mathcal{B}} (B_q) \right\} \right) = \pi^j_{\mathcal{B}} (B_q). \tag{3}$$

We denote by π such a family of mappings $\{\pi^i_{\mathcal{B}}\}_{i,\mathcal{B}}$.

Let us fix $\mathcal{B} \in \mathbf{B}^N$ and π. Every player $i \in N$ has a representative in each coalition $B_q \in \mathcal{B}^i$, given by π. Suppose that these representatives form a queue outside a room in such a way that all representatives associated with every B_q are together. These queues can be represented by the members $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$. We suppose that all the orderings of coalitions $B_q \in \mathcal{B}$ are equally likely and that players in every B_q are ordered as follows. One player in B_q is picked up at a time and she is placed in the front of the queue which is partially formed. Once a player is picked up she is not picked up any more and the probability of picking up a player $i \in B_q$ is given by the split weight $\omega_{\pi^i_{\mathcal{B}}(B_q)}$ divided by the weights $\omega_{\pi^k_{\mathcal{B}}(B_q)}$ associated with the players $k \in B_q$ who are not yet in the queue. After forming the queue the representatives proceed to enter in the room. When a representative of a player enters in the room a coalition forms if all the representatives of that player have entered in the room. Moreover this coalition, say S, is formed by the players whose representatives are all in the room. When the last representative of player i enters in the room she will be given her marginal contribution to the coalition S, that is, $v(S) - v(S \backslash \{i\})$. The expected marginal contribution of player i will be by definition her *weighted bounded configuration value* associated with (ω,π). It will be denoted by $\phi^{\omega,\pi}_i(v, \mathcal{B})$.

Let us formalize this definition.

Consider the probability distribution $Q^{(\omega,\mathcal{B},\pi)}$ on $\mathcal{R}_{\mathcal{B}}(N)$ such that

$$Q^{(\omega,\mathcal{B},\pi)}(\alpha) = \frac{1}{|\mathcal{B}|!} \cdot \prod_{B_q \in \mathcal{B}} Q^{(\omega,\pi)}_{B_q}\left(R^{B_q}_{\alpha}\right),$$

where $Q^{(\omega,\pi)}_{B_q}$ is the probability distribution on $\Sigma(B_q)$ such that $Q^{(\omega,\pi)}_{B_q}(\sigma_l) = \prod\limits_{t=1}^{|B_q|} \frac{\omega_{\pi^{i_t}_{\mathcal{B}}(B_q)}}{\sum\limits_{s=1}^{t} \omega_{\pi^{i_s}_{\mathcal{B}}(B_q)}}$ for $\sigma_l = \left(i_1 ... i_{|B_q|}\right) \in \Sigma(B_q)$.

Given $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ and $i \in N$ denote by $R_\alpha[i]$ the set of players whose last position in R_α is before the last position of player i, including player i. Notice that this definition generalizes that of $\mathcal{R}_{\mathcal{B}}(N)$ defined in the previous section, so it share the same name.

Given $(v,\mathcal{B}) \in \mathbf{BG}^N$, for every $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ and $i \in N$, the *marginal contribution* of player i in order R_α is defined by $C_i(v,R_\alpha) = v(R_\alpha[i]) - v(R_\alpha[i] \setminus \{i\})$.

The *weighted bounded configuration value* of player i in $(v,\mathcal{B}) \in \mathbf{BG}^N$ associated with ω is by definition

$$\phi_i^{\omega,\pi}(v,\mathcal{B}) = \sum_{\alpha \in \mathcal{R}_{\mathcal{B}}(N)} Q^{(\omega,\mathcal{B})}(\alpha) \cdot C_i(v,R_\alpha).$$

The weighted bounded configuration value ϕ^ω is the corresponding solution on $\mathbf{BG}$.

Next we give a relation between the weighted bounded configuration values and the weighted coalition structure values with intercoalitional symmetry defined by Levy and McLean (1989). It follows that the weighted bounded configuration value of a player is the sum of the Levy and McLean values of the split players associated with that player in an associated game with the natural associated coalition structure.

In this Proposition and in the following, given $\omega \in \mathbb{R}_+^A$ satisfying (2) and $S \subseteq U$, we denote by ω_S the vector ω restricted to S.

Proposition 3.1 *Let ϕ^ω be a weighted bounded configuration value. Then for every $v \in \mathbf{G}^N$ and $i \in N \subseteq A$ it holds that*

$$\phi_i^\omega(v,\mathcal{B}) = \sum_{B_q \in \mathcal{B}^i} \eta_{\pi_{\mathcal{B}}^i(B_q)}^{\left(\omega_{\widehat{N}},\widehat{\mathcal{B}}\right)}(\widehat{v}),$$

where $\widehat{N} = \bigcup_{B_q \in \mathcal{B}} \bigcup_{j \in B_q} \left\{\pi_{\mathcal{B}}^j(B_q)\right\}$, $\widehat{v} \in \mathbf{G}^{\widehat{N}}$ is such that

$$\widehat{v}(T) = v\left(j \in N : \bigcup\nolimits_{B_q \in \mathcal{B}^j} \left\{\pi_{\mathcal{B}}^j(B_q)\right\} \subseteq T\right),$$

and $\widehat{\mathcal{B}} = \left\{\widehat{B_q}\right\}_{B_q \in \mathcal{B}}$, where $\widehat{B_q} = \bigcup_{j \in B_q} \left\{\pi_{\mathcal{B}}^j(B_q)\right\}$.

Proof. Let ϕ^ω and $(v,\mathcal{B}) \in \mathbf{BG}^N$. There is a natural bijection between $\mathcal{R}_{\mathcal{B}}(N)$ and $\mathcal{R}_{\widehat{\mathcal{B}}}\left(\widehat{N}\right)$ that associates each $\alpha = (\sigma_1,...,\sigma_m) \in \mathcal{R}_{\mathcal{B}}(N)$ with $\widehat{\alpha} \in \mathcal{R}_{\widehat{\mathcal{B}}}\left(\widehat{N}\right)$ just replacing player $i \in \sigma_l \in \Sigma(B_q)$ by $\pi_{\mathcal{B}}^i(B_q)$. Furthermore, for every $i \in N$ it holds that $C_i(v,R_\alpha) = \sum_{B_q \in \mathcal{B}^i} C_{\pi_{\mathcal{B}}^i(B_q)}(\widehat{v},R_{\widehat{\alpha}})$ and $Q^{(\omega,\mathcal{B})}(\alpha) = Q^{\left(\omega_{\widehat{N}},\widehat{\mathcal{B}}\right)}(\widehat{\alpha})$. Therefore, we obtain the required result. ■

Notice that all the sets $\bigcup_{B_q \in \mathcal{B}^i} \{\pi_{\mathcal{B}}^i(B_q)\}$ are partnerships in $\widehat{v}$ and that the worth of $\bigcup_{B_q \in \mathcal{B}^i} \{\pi_{\mathcal{B}}^i(B_q)\}$ is just the worth of i in v. Observe also that $\widehat{\mathcal{B}}$ is the natural coalition structure induced by $\mathcal{B}$ on $\widehat{N}$, when every $i \in N$ is replaced in each $B_q \in \mathcal{B}^i$ by her associated split player $\pi_{\mathcal{B}}^i(B_q)$.

Remark 3.1 *When the weights are natural numbers then the weighted bounded configuration value can also be calculated as follows. The weights can be seen as the number* *of representatives that players have in the coalitions they belong to. Suppose for example that* $\{1,2,3\} = N \subset A$, $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = 1$, $F(2,2) = \{4,5\}$, *and* $\omega_4 = 2$, $\omega_5 = 1$. *Let* $\mathcal{B}_1 = \{\{1,2\},\{2,3\}\}$, *and suppose that* 4 *and* 5 *represent player* 2, *respectively, in* $\{1,2\}$ *and in* $\{2,3\}$, *i.e.* $\pi^2_{\mathcal{B}_1}(\{1,2\}) = 4$ *and* $\pi^2_{\mathcal{B}_1}(\{2,3\}) = 5$. *So players* 1, 2 *and* 3 *have, respectively,* ω_1, ω_2 *and* ω_3 *representatives in the coalitions they belong to, and* ω_4 *and* ω_5 *tell us how* ω_2 *is divided in the two coalitions player* 2 *belongs to. Therefore, player* 1 *has one representative in coalition* $\{1,2\}$, *player* 3 *has one representative in* $\{2,3\}$, *and player* 2 *has two representatives in* $\{1,2\}$ *and one in* $\{2,3\}$. *We denote by* $2, 2'$ *the two representatives of player* 2 *in* $\{1,2\}$ *and maintain the denomination of the player for the other representatives. If we consider the orderings of all the representatives in such a way that those associated with the same coalition are together, that is,* $122'23$, $12'223$, $122'32$, ..., *if we suppose that these orderings are equally likely and that the sets form as explained in the Introduction for the dual configuration value, the expected marginal contribution of a player* $i \in \{1,2,3\}$ *is her value according to* $\phi^{\omega,\pi}$. *For the first game* v *presented in the Introduction,* $\phi^{\omega}(v,\mathcal{B}_1) = \left(\frac{2}{3}, 3\frac{5}{6}, \frac{1}{2}\right)$. *For the unanimity game* u_N, $\phi^{\omega}(u_N,\mathcal{B}_1) = \left(\frac{1}{6}, \frac{7}{12}, \frac{1}{4}\right)$.

These results can be proved as in Kalai and Samet's Theorem 9 (1987), taking into account Proposition 3.1.

4 Characterization of the weighted bounded configuration values

In this section we characterize the family formed by the values ϕ^{ω} by means of the following axioms. We denote by ψ a solution on $\mathbf{BG}$ and $N \subseteq A$.

The first five axioms are adaptations to this framework of the axioms that characterize the weighted coalition values with intercoalitional symmetry, except the Null Player Property.

Linearity. For every $(v_1,\mathcal{B}), (v_2,\mathcal{B}) \in \mathbf{BG}^N$ and $\lambda, \mu \in \mathbb{R}$ it holds that

$$\psi(\lambda v_1 + \mu v_2, \mathcal{B}) = \lambda \psi(v_1,\mathcal{B}) + \mu \psi(v_2,\mathcal{B}).$$

Efficiency. For every $(v,\mathcal{B}) \in \mathbf{BG}^N$ it holds that

$$\sum_{i \in N} \psi_i(v,\mathcal{B}) = v(N).$$

$\mathcal{B}$-Positivity. If $v \in \mathbf{G}^N$ is monotonic and $C_i(v, R_\alpha) > 0$ for some $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ then $\psi_i(v,\mathcal{B}) > 0$.

Intercoalitional Symmetry. Let $(v,\mathcal{B}) \in \mathbf{BG}_0^N$. If $B_p, B_q \in \mathcal{B}$ are such that for every $\mathcal{C} \subseteq \mathcal{B}\setminus\{B_p, B_q\}$ it holds $v\left(B_p \cup \bigcup_{B_r\in\mathcal{C}} B_r\right) = v\left(B_q \cup \bigcup_{B_r\in\mathcal{C}} B_r\right)$, then

$$\sum_{i\in B_p} \psi_i(v,\mathcal{B}) = \sum_{i\in B_q} \psi_i(v,\mathcal{B}).$$

Intracoalitional Partnership. Let $(v,\mathcal{B}) \in \mathbf{BG}^N$. If $T \subseteq N$ is a partnership in v and $B_q \in \mathcal{B}$ is such that $B_q \cap T \neq \emptyset$ and $|\mathcal{B}^i| = 1$ for every $i \in B_q \cap T$, then for every $i \in B_q \cap T$ it holds that

$$\psi_i(u_T,\mathcal{B}) \sum_{j\in B_q\cap T} \psi_j(v,\mathcal{B}) = \psi_i(v,\mathcal{B}) \sum_{j\in B_q\cap T} \psi_j(u_T,\mathcal{B}).$$

The following axiom requires the solution to be independent of the null players with no relevant role in a coalition structure.

Null Players Out. Let $(v,\mathcal{B}) \in \mathbf{BG}_0^N$. If $i \in N$ is a null player in v, then

$$\psi_j\left(v_{N\setminus\{i\}}, \mathcal{B}_{-i}\right) = \psi_j(v,\mathcal{B})$$

for every $j \in N\setminus\{i\}$.

For the next axiom we need two notations.

Given $v \in \mathbf{G}^N$, $i \in N\cap U$ and $k \in \left\{2,...,2^{|N|-1}\right\}$, the game $v^{-i,F(i,k)} \in \mathbf{G}^{N^{-i,F(i,k)}}$ is defined by

$$v^{-i,F(i,k)}(T) = \begin{cases} v(T\cap N) & \text{if } F(i,k) \nsubseteq T \\ v((T\cap N)\cup\{i\}) & \text{if } F(i,k) \subseteq T. \end{cases}$$

In this game player i has been substituted by $F(i,k)$, being the proper subsets of $F(i,k)$ powerless, that is, being $F(i,k)$ a partnership.

Given $\mathcal{B} \in \mathbf{B}^N$ and $i \in N$, we write

$$\mathcal{B}^{-i,F\left(i,|\mathcal{B}^i|\right)} = \left(\mathcal{B}\setminus\mathcal{B}^i\right) \cup \left\{B_q^{-i,\left\{\pi^i_{\mathcal{B}}(B_q)\right\}}\right\}_{B_q\in\mathcal{B}^i}.$$

That is, player i is substituted by her representatives in the coalitions (of the coalition configuration) she belongs to.

The next axiom states that if a player belongs to several coalitions of a coalition configuration, then she can be substituted by her representatives associated with this coalition configuration without changing the value of the other players.

Merger. Let $\mathcal{B} \in \mathbf{B}^N$ and $i \in N$ such that $|\mathcal{B}^i| > 1$. Then,

$$\psi_j(v,\mathcal{B}) = \psi_j\left(v^{-i,F\left(i,|\mathcal{B}^i|\right)}, \mathcal{B}^{-i,F\left(i,|\mathcal{B}^i|\right)}\right)$$

for every $j \in N\setminus\{i\}$ and $v \in \mathbf{G}^N$.

The dual configuration value (Albizuri et al., 2006) satisfies all the above properties[2]. Moreover, the weighted bounded configuration value also satisfies them (see Proposition 4.1 below). Hence, we need an additional axiom to decide which of these two values is more suitable in a given situation.

Example 4.1 *Let $N = \{1,2,3,4,5\}$ with $2 \in U$ and $\{4,5\} = F(2,2) \subset U'$. So, players 4 and 5 are the representatives of player 2 when player 2 belongs to exactly two coalitions, as for example in $\mathcal{B}' = \{\{1,2\},\{2,3\}\} \in \mathbf{B}^{\{1,2,3\}}$. Let u_N be the unanimity game on N, so that $\{2,4,5\}$ is a partnership in (N, u_N). Let $\mathcal{B} = \{\{1\},\{2,3,4,5\}\}$. The dual configuration value (Albizuri et al., 2006) assigns:*

$$\phi_2(u_N, \mathcal{B}) = \frac{1}{8} \neq \frac{1}{4} = \phi_4(u_N, \mathcal{B}) + \phi_5(u_N, \mathcal{B}).$$

On the other hand, a weighted bounded configuration value assigns

$$\begin{aligned} \phi_2^{\omega}(u_N, \mathcal{B}) &= \frac{1}{2}\frac{\omega_2}{\omega_2+\omega_3+\omega_4+\omega_5} \\ &= \frac{1}{2}\frac{\omega_4+\omega_5}{\omega_2+\omega_3+\omega_4+\omega_5} \\ &= \phi_4^{\omega}(u_N, \mathcal{B}) + \phi_5^{\omega}(u_N, \mathcal{B}). \end{aligned}$$

We consider that $\phi_2^{\omega}(u_N, \mathcal{B}) = \phi_4^{\omega}(u_N, \mathcal{B}) + \phi_5^{\omega}(u_N, \mathcal{B})$ is a natural requirement for a coalition configuration value, since players 4 and 5 are representatives of player 2, and they belong to a common partnership, being therefore indistinguishable in this game.

In the following axiom we formalize the general situation described in Example 4.1. Consider a player and the players who represent her (with respect to some other coalition configuration) all of them forming a partnership in a game, and therefore being indistinguishable in this game. If we take a coalition configuration in which such a player and her representatives belong to the same coalition, the axiom requires the solution to give the same value to the player and such representatives.

F-Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{B}' \in \mathbf{B}^{N'}$, and $i \in N \cap N' \cap U$ such that $\left|(\mathcal{B}')^i\right| > 1$, $F\left(i, \left|(\mathcal{B}')^i\right|\right) \subset N$, $\mathcal{B}^i = \mathcal{B}^j$ for all $j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)$, and $v \in \mathbf{G}^N$ such that $F\left(i, \left|(\mathcal{B}')^i\right|\right) \cup \{i\}$ is a partnership in v. Then

$$\psi_i(v, \mathcal{B}) = \sum_{j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)} \psi_j(v, \mathcal{B}).$$

[2]See Theorem 5.1 in Albizuri et al. (2006) for Efficiency, Linearity and Intercoalitional Symmetry (called Coalitional Symmetry). $\mathcal{B}$-Positivity, Intracoalitional Parnership, Null Players Out and Merger follow easily from the definition.

Recall that $|\mathcal{B}^j| = 1$ for all $j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)$ since $j \in U'$. Hence, players i and j belong to a unique coalition in $\mathcal{B}$, that is, there exists $B_q \subset N$ such that $\mathcal{B}^i = \mathcal{B}^j = \{B_q\}$.

Proposition 4.1 *The weighted bounded configuration value ϕ^ω satisfies Linearity, Efficiency, $\mathcal{B}$-Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger and F-Partnership Additivity.*

Proof. By definition, it is clear that ϕ^ω satisfies Linearity, Efficiency and $\mathcal{B}$-Positivity. If $\mathcal{B}$ is a coalition structure then the mapping $\psi(v) = \phi^\omega(v, \mathcal{B})$ is a weighted coalition structure value and therefore ψ satisfies Intercoalitional Symmetry and Null Players Out.

Let us prove that ϕ^ω satisfies Intracoalitional Partnership. Let $(v, \mathcal{B}) \in \mathbf{BG}^N$, $T \subseteq N$ be a partnership in v, $B_q \in \mathcal{B}$ such that $B_q \cap T \neq \emptyset$ and $|\mathcal{B}^i| = 1$ for every $i \in B_q \cap T$. Let $i, j \in B_q \cap T$. By Proposition 2 it holds that

$$\phi_j^\omega(v, \mathcal{B}) = \phi_j^\omega\left(\widehat{v}, \widehat{\mathcal{B}}\right), \tag{4}$$

where $\left(\widehat{v}, \widehat{\mathcal{B}}\right) \in \mathbf{BG}^{\widehat{N}}$ is defined as in Proposition 2. Since $\widehat{\mathcal{B}}$ is a coalition structure, the solution $\widehat{\psi}$ on $\mathbf{G}^{\widehat{N}}$ defined by $\widehat{\psi}(w) = \phi^\omega\left(w, \widehat{\mathcal{B}}\right)$ is a weighted coalition structure value with intercoalitional symmetry (the one associated with $\omega^{\widehat{N}}$ and $\widehat{\mathcal{B}}$), and therefore $\widehat{\psi}$ satisfies Intracoalitional Partnership. Notice also that the set $\widehat{T} = \bigcup_{t \in T} \bigcup_{B_p \in \mathcal{B}^t} \{\pi_{\mathcal{B}}^t(B_p)\}$ is a partnership in $\widehat{v}$. Therefore,

$$\widehat{\psi}_i\left(u_{\widehat{T}}\right) \sum_{j \in \widehat{B_q} \cap \widehat{T}} \widehat{\psi}_j(\widehat{v}) = \widehat{\psi}_i(\widehat{v}) \sum_{j \in \widehat{B_q} \cap \widehat{T}} \widehat{\psi}_j\left(u_{\widehat{T}}\right).$$

And taking into account that $\widehat{B_q} \cap \widehat{T} = B_q \cap T$, $u_{\widehat{T}} = \widehat{u_T}$ and the equality (4), we have that

$$\phi_i^\omega(u_T, \mathcal{B}) \sum_{j \in B_q \cap T} \phi_j^\omega(v, \mathcal{B}) = \phi_i^\omega(v, \mathcal{B}) \sum_{j \in B_q \cap T} \phi_j^\omega(u_T, \mathcal{B}),$$

that is, ϕ^ω satisfies Intracoalitional Partnership.

We prove now that ϕ^ω satisfies Merger. Let $\mathcal{B} \in \mathbf{B}^N$ and $i \in N$ such that $|\mathcal{B}^i| > 1$, $j \in N \backslash \{i\}$ and $v \in \mathbf{G}^N$. It is clear that

$$\widehat{\mathcal{B}^{-i,F(i,|\mathcal{B}^i|)}} = \hat{\mathcal{B}}$$

and

$$\widehat{v^{-i,F(i,|\mathcal{B}^i|)}} = \hat{v}.$$

Moreover, by (3),

$$\pi^j_{\mathcal{B}^{-i,F(i,|\mathcal{B}^i|)}}\left((B_q \backslash \{i\}) \cup \left\{\pi_{\mathcal{B}}^i(B_q)\right\}\right) = \pi_{\mathcal{B}}^j(B_q)$$

for all $B_q \in \mathcal{B}^j$.

By Proposition 3.1,

$$\phi_j^\omega (v, \mathcal{B}) = \sum_{B_q \in \mathcal{B}^j} \eta_{\pi_{\mathcal{B}}^j (B_q)}^{(\omega_N, \hat{\mathcal{B}})} (\hat{v})$$

and

$$\phi_j^\omega \left(v^{-i, F\left(i, \left|\mathcal{B}^i\right|\right)}, \mathcal{B}^{-i, F\left(i, \left|\mathcal{B}^i\right|\right)} \right) = \sum_{B'_q \in \left(\mathcal{B}^{-i, F\left(i, \left|\mathcal{B}^i\right|\right)}\right)^j} \eta_{\pi_{\mathcal{B}}^j \left(B'_q\right)}^{(\omega_N, \hat{\mathcal{B}})} (\hat{v}) .$$

It is clear that there exists a one-to-one correspondence between $\mathcal{B}^j$ and $\left(\mathcal{B}^{-i, F\left(i, \left|\mathcal{B}^i\right|\right)}\right)^j$, given by $B'_q = (B_q \backslash \{i\}) \cup \{\pi_{\mathcal{B}}^i (B_q)\}$, so that the two above expressions coincide.

Finally, we prove that ϕ^ω satisfies F-Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{B}' \in \mathbf{B}^{N'}$, $i \in N \cap N' \cap U$ such that $\left|(\mathcal{B}')^i\right| > 1$, $F\left(i, \left|(\mathcal{B}')^i\right|\right) \subset N$, $\mathcal{B}^i = \mathcal{B}^j = \{B_q\}$ for all $j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)$, and $v \in \mathbf{G}^N$ such that $T = F\left(i, \left|(\mathcal{B}')^i\right|\right) \cup \{i\}$ is a partnership in v. By the Intracoalitional Partnership Axiom it holds that

$$\phi_i^\omega (v, \mathcal{B}) = \frac{\phi_i^\omega (u_T, \mathcal{B})}{\sum\limits_{j \in B_q \cap T} \phi_j^\omega (u_T, \mathcal{B})} \sum_{j \in B_q \cap T} \phi_j^\omega (v, \mathcal{B}) ,$$

where the denominator is not zero by $\mathcal{B}$-Positivity. Moreover, $B_q \cap T = T$ and hence

$$\phi_i^\omega (v, \mathcal{B}) = \frac{\phi_i^\omega (u_T, \mathcal{B})}{\sum\limits_{j \in T} \phi_j^\omega (u_T, \mathcal{B})} \sum_{j \in T} \phi_j^\omega (v, \mathcal{B}) .$$

Taking into account Proposition 3.1 and (1) we obtain

$$\phi_i^\omega (v, \mathcal{B}) = \frac{\omega_i}{\sum\limits_{j \in T} \omega_j} \sum_{j \in T} \phi_j^\omega (v, \mathcal{B}) . \tag{5}$$

Analogously, for every $k \in F\left(i, \left|\mathcal{B}^i\right|\right)$ it holds that

$$\phi_k^\omega (v, \mathcal{B}) = \frac{\omega_k}{\sum\limits_{j \in T} \omega_j} \sum_{j \in T} \phi_j^\omega (v, \mathcal{B}) . \tag{6}$$

By (2) we know that $\omega_i = \sum_{k \in T \backslash \{i\}} \omega_k$, which, joint with (5) and (6), implies

$$\phi_i^\omega (v, \mathcal{B}) = \sum_{k \in T \backslash \{i\}} \phi_k^\omega (v, \mathcal{B}) = \sum_{k \in F\left(i, \left|(\mathcal{B}')^i\right|\right)} \phi_k^\omega (v, \mathcal{B}) ,$$

that is, ϕ^ω satisfies F-Partnership Additivity. ■

Theorem 4.1 *A solution ψ on* **BG** *satisfies Linearity, Efficiency, $\mathcal{B}$-Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger and F-Partnership Additivity if and only if there exist ω such that $\psi = \phi^\omega$.*

Proof. Proposition 4.1 is the if part of this Theorem. So it remains to prove that if a solution ψ on $\mathbf{BG}$ satisfies the above axioms then there exist ω and π such that $\psi = \phi^{\omega}$.

Let $\mathcal{B} \in \mathbf{B}_0^N$ be a coalition structure on $N \subseteq A$. The mapping $\psi^R(v) = \psi(v, \mathcal{B})$ satisfies Linearity, Efficiency, the Null Player Axiom, $\mathcal{B}$-Positivity, Intercoalitional Symmetry and Intracoalitional Partnership. Therefore, by Theorem 2.1 there exists $\lambda^N(\mathcal{B}) \in \mathbb{R}_+^N$ such that

$$\psi^R(v) = \psi(v, \mathcal{B}) = \eta^{\left(\lambda^N(\mathcal{B}), \mathcal{B}\right)}(v). \tag{7}$$

Moreover, Levy and McLean (1989) prove that $\lambda(\mathcal{B})_i^N = \psi_i(u_N, \mathcal{B})$ for every $i \in N$.

Let $B_q \in \mathcal{B}$ and $i, j \in B_q \cap N$. By Intracoalitional Partnership,

$$\psi_i\left(u_{B_q \cap N}, \mathcal{B}\right) \sum_{k \in B_q \cap N} \psi_k(u_N, \mathcal{B}) = \psi_i(u_N, \mathcal{B}) \sum_{k \in B_q \cap N} \psi_k\left(u_{B_q \cap N}, \mathcal{B}\right).$$

Under $\mathcal{B}$-Positivity, we can rewrite the above equality as

$$\frac{\psi_i\left(u_{B_q \cap N}, \mathcal{B}\right)}{\psi_i(u_N, \mathcal{B})} = \frac{\sum\limits_{k \in B_q \cap N} \psi_k\left(u_{B_q \cap N}, \mathcal{B}\right)}{\sum\limits_{k \in B_q \cap N} \psi_k(u_N, \mathcal{B})}.$$

Since we have the same equality for player j, it follows that

$$\frac{\psi_i\left(u_{B_q \cap N}, \mathcal{B}\right)}{\psi_i(u_N, \mathcal{B})} = \frac{\psi_j\left(u_{B_q \cap N}, \mathcal{B}\right)}{\psi_j(u_N, \mathcal{B})},$$

that is,

$$\frac{\psi_i(u_N, \mathcal{B})}{\psi_j(u_N, \mathcal{B})} = \frac{\psi_i\left(u_{B_q \cap N}, \mathcal{B}\right)}{\psi_j\left(u_{B_q \cap N}, \mathcal{B}\right)} = \frac{\psi_i\left(u_{B_q \cap N}, \{B_q \cap N\}\right)}{\psi_j\left(u_{B_q \cap N}, \{B_q \cap N\}\right)}, \tag{8}$$

where we have taken into account Null Players Out in the second equality.

Now, we take into account again the Intracoalitional Partnership axiom, but now with $(u_A, \{A\}) \in \mathbf{BG}^A$, coalition $A \in \{A\}$ and $T = B_q \cap N$. Then

$$\psi_i\left(u_{B_q \cap N}, \{A\}\right) \sum_{k \in B_q \cap N} \psi_k(u_A, \{A\}) = \psi_i(u_A, \{A\}) \sum_{k \in B_q \cap N} \psi_k\left(u_{B_q \cap N}, \{A\}\right).$$

Since we have the same equality for player j, reasoning as above we have that

$$\frac{\psi_i(u_A, \{A\})}{\psi_j(u_A, \{A\})} = \frac{\psi_i\left(u_{B_q \cap N}, \{A\}\right)}{\psi_j\left(u_{B_q \cap N}, \{A\}\right)} = \frac{\psi_i\left(u_{B_q \cap N}, \{B_q \cap N\}\right)}{\psi_j\left(u_{B_q \cap N}, \{B_q \cap N\}\right)}. \tag{9}$$

Equalities (8) and (9) imply that

$$\frac{\psi_i(u_N, \mathcal{B})}{\psi_i(u_A, \{A\})} = \frac{\psi_j(u_N, \mathcal{B})}{\psi_j(u_A, \{A\})}.$$

If we denote by $c_{B_q,\mathcal{B}}$ this ratio, we have that $\psi_i(u_N,\mathcal{B}) = c_{B_q,\mathcal{B}} \cdot \psi_i(u_A,\{A\})$ for every $i \in B_q$. Let $\omega_k = \psi_k(u_A,\{A\})$ for every $k \in A$. Hence, $\lambda^N(\mathcal{B})_i = c_{B_q,\mathcal{B}} \cdot \omega_i$ for every $i \in B_q$. Taking into account (1) and (7) we obtain

$$\psi(v,\mathcal{B}) = \eta^{(\omega_N,\mathcal{B})}(v) \tag{10}$$

for every $v \in \mathbf{G}^N$.

Now let $(v,\mathcal{B}) \in \mathbf{BG}^N$. By Merger and Efficiency, for every $i \in N$ with $|\mathcal{B}^i| > 1$,

$$\psi_i(v,\mathcal{B}) = \sum_{B_q \in \mathcal{B}^i} \psi_{\pi^i_{\mathcal{B}}(B_q)}\left(\widehat{v},\widehat{\mathcal{B}}\right),$$

where $\widehat{N}$, $\widehat{\mathcal{B}} = \left\{\widehat{B_q}\right\}_{B_q \in \mathcal{B}^i}$ and $\widehat{v} \in \mathbf{G}^{\widehat{N}}$ are defined as in Proposition 3.1. For $i \in N$ with $|\mathcal{B}^i| = 1$, we have $\pi^i_{\mathcal{B}}(B_q) = i$ and, under Merger and Efficiency, the above equality also holds. By (10) we have that $\psi\left(\widehat{v},\widehat{\mathcal{B}}\right) = \eta^{\left(\omega_{\widehat{N}},\widehat{\mathcal{B}}\right)}(\widehat{v})$, and therefore

$$\psi_i(v,\mathcal{B}) = \sum_{B_q \in \mathcal{B}^i} \eta^{\left(\omega_{\widehat{N}},\widehat{\mathcal{B}}\right)}_{\pi^i_{\mathcal{B}}(B_q)}(\widehat{v}).$$

We still need to prove that ω satisfies (2) so that the above equality implies $\psi = \phi^{\omega}$. Equality (2) is satisfied since by F-Partnership Additivity it follows that

$$\omega_i = \psi_i(u_A,\{A\}) = \sum_{j \in F(i,k)} \psi_j(u_A,\{A\}) = \sum_{j \in F(i,k)} \omega_j.$$

for all $k \in \left\{2,...,2^{|U|-1}\right\}$. ■

The eight axioms used in Theorem 4.1 are independent. We describe eight reasonable values. Each of them satisfies all the axioms but one. Fix ω as in the definition of some ϕ^{ω}.

• For any $v \in \mathbf{G}^N$ let $C(v)$ denote the *carrier* of v, i.e. the set of non-null players. Then, the value $p^{0,\omega}$ defined as $p^{0,\omega}_i(v,\mathcal{B}) = 0$ for all $i \in N \backslash C(v)$ and

$$p^{0,\omega}_i(v,\mathcal{B}) = \frac{v(N)}{|\{B_r \in \mathcal{B} : B_r \cap C(v) \neq \emptyset\}|} \sum_{B_q \in \mathcal{B}^i : B_q \cap C(v) \neq \emptyset} \frac{\omega_{\pi^i_{\mathcal{B}}(B_q)}}{\sum_{j \in B_q} \omega_{\pi^j_{\mathcal{B}}(B_q)}}$$

for all $i \in C(v)$, satisfies all the axioms but Linearity.

• For any $\delta \in (0,1)$, the value $\phi^{\delta,\omega}$ defined as $\phi^{\delta,\omega}(v,\mathcal{B}) = \delta\phi^{\omega}(v,\mathcal{B})$ satisfies all the axioms but Efficiency. Notice that these values are reasonable expected payoff measures when there is a fixed discounting factor δ due to some unavoidable process of bargaining.

• Let $\omega(\mu) \in \mathbb{R}^A_+$ be defined as in Example 3.2 for some $\mu \in \mathbb{R}^A_+$. For each $\alpha \in \mathbb{R}$, let μ^{α} be defined as $\mu^{\alpha}_i = (\mu_i)^{\alpha}$ (μ_i raised to the power of α) for all $i \in A$. When $\mu_i \neq \mu_j$ for some $i,j \in U$, the value $\phi^{\infty,\mu}$ defined as

$$\phi^{\infty,\mu}(v,\mathcal{B}) = \lim_{\alpha \to \infty} \phi^{\omega(\mu^{\alpha})}(v,\mathcal{B})$$

satisfies all the axioms but $\mathcal{B}$-Positivity. These values are priority rules where the highest priority goes to those players with the highest coefficient μ.

• Given $\mathcal{B} \in \mathbf{B}^N$, let $\omega^{\mathcal{B}} \in \mathbb{R}^N_+$ be defined as $\omega^{\mathcal{B}}_i = |\mathcal{B}^i|$ for all $i \in N \cap U$ and $\omega^{\mathcal{B}}_i = \frac{1}{k}$ for all $i \in F(j,k)$. The value $Sh^e(v,\mathcal{B}) = Sh^{\omega^{\mathcal{B}}}(v)$, where $Sh^{\omega^{\mathcal{B}}}$ is the weighted Shapley value (Kalai and Samet, 1987), satisfies all the axioms but Intercoalitional Symmetry.

• For each $(v,\mathcal{B}) \in \mathbf{BG}^N$, let $\mu^{(v,\mathcal{B})} \in \mathbb{R}^A_+$ satisfying $\mu^{(v,\mathcal{B})}_i = \sum_{j \in F(i,k)} \mu^{(v,\mathcal{B})}_j$ for all $(i,k) \in U \times \{2, ..., 2^{|U|-1}\}$, $\mu^{(v,\mathcal{B})}_i = \mu^{(v,\mathcal{B}')}_i$ when $i \in \mathcal{B}'$ and $\mathcal{B}$ is finer[3] than $\mathcal{B}'$, and $\mu^{(v,\mathcal{B})} = \mu^{\left(v^{-i,F\left(i,\left|\mathcal{B}^i\right|\right)},\mathcal{B}^{-i,F\left(i,\left|\mathcal{B}^i\right|\right)}\right)}$ for all $(v,\mathcal{B}) \in \mathbf{BG}^N$ and $i \in N \cap U$. When $\mu^{(v,\mathcal{B})} \neq \mu^{(u_T,\mathcal{B})}$ for some $(v,\mathcal{B}) \in \mathbf{BG}^N$ and T satisfying the conditions stated in the definition of Intracoalitional Partnership, the value f^μ defined as

$$f^\mu_i(v,\mathcal{B}) = \sum_{B_q \in \mathcal{B}^i} \eta^{\left(\mu^{(v,\mathcal{B})}_{\hat{N}},\hat{\mathcal{B}}\right)}_{\pi^i_{\mathcal{B}}(B_q)}(\hat{v})$$

for all $i \in N$, satisfies all the axioms but Intracoalitional Partnership.

• The value p defined as

$$p_i(v,\mathcal{B}) = \frac{v(N)}{|\mathcal{B}|} \sum_{B_q \in \mathcal{B}^i} \frac{\omega_{\pi^i_{\mathcal{B}}(B_q)}}{\sum_{j \in B_q} \omega_{\pi^j_{\mathcal{B}}(B_q)}}$$

for all $i \in N$, satisfies all the axioms but Null Players Out.

• For each $\mathcal{B} \in \mathbf{B}$, let $\mu^{\mathcal{B}} \in \mathbb{R}^A_+$ satisfying $\mu^{\mathcal{B}}_i = \sum_{j \in F(i,k)} \mu^{\mathcal{B}}_j$ for all $(i,k) \in U \times \{2, ..., 2^{|U|-1}\}$ and $\mu^{\mathcal{B}}_i = \mu^{\mathcal{B}'}_i$ when $i \in \mathcal{B}'$ and $\mathcal{B}$ is finer than $\mathcal{B}'$. When $\mu^{\mathcal{B}} \neq \mu^{\mathcal{B}^{-i,F\left(i,\left|\mathcal{B}^i\right|\right)}_{\pi^i_{\mathcal{B}}}}$ for some $\mathcal{B} \in \mathbf{B}^N$ and $i \in N$, the value g^μ defined as

$$g^\mu_i(v,\mathcal{B}) = \sum_{B_q \in \mathcal{B}^i} \eta^{\left(\mu^{\mathcal{B}}_{\hat{N}},\hat{\mathcal{B}}\right)}_{\pi^i_{\mathcal{B}}(B_q)}(\hat{v})$$

for all $i \in N$, satisfies all the axioms but Merger.

• The dual configuration value (Albizuri et al., 2006) satisfies all the axioms but F-Partnership Additivity.

5 The symmetric bounded equally split value

In this section we focus on a specific weighted bounded configuration value. It is the value obtained when the weights are given as in Example 3.1, i.e. all the players in U have

[3]We say that $\mathcal{B} \in \mathbf{B}^N$ is *finer* than $\mathcal{B}' \in \mathbf{B}^{N'}$ when $N \supseteq N'$ and there exists a one-to-one function f from $\mathcal{B}$ to $\mathcal{B}'$ such that $f(B_q) \subset B_q$ for all $B_q \in \mathcal{B}$.

weight 1 and they spread equally their weight over the coalitions they belong to. That is, $\omega_i = 1$ for all $i \in U$ and $\omega_i = \frac{1}{k}$ for all $i \in F(j,k)$ with $j \in U$ and $k \in \left\{2, ..., 2^{|U|-1}\right\}$.

We denote this value by $\phi^{\mathbf{e}}$ and we call it the *symmetric bounded equally split value.*

This value satisfies intracoalitional anonymity when a coalition structure with sets of players contained in U is formed. Formally this property is stated as follows.

U-**Intracoalitional Anonymity.** Let $(v, \mathcal{B}) \in \mathbf{BG}_0^N$ with $N \subseteq U$. If π is a permutation of N such that $\pi(B_q) = B_q$ for every $B_q \in \mathcal{B}$, then for every $i \in N$ it holds that

$$\psi_i(\pi v, \mathcal{B}) = \psi_{\pi i}(v, \mathcal{B}),$$

where $\pi v \in \mathbf{G}^N$ is defined as $(\pi v)(S) = v(\pi S)$ for all $S \subseteq N$.

The symmetric bounded equally split value also satisfies the following variation of F-Partnership Additivity.

F-Equally Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{B}' \in \mathbf{B}^{N'}$ and $i \in N \cap N' \cap U$ such that $\left|(\mathcal{B}')^i\right| > 1$, $F\left(i, \left|(\mathcal{B}')^i\right|\right) \subset N$, $\mathcal{B}^i = \mathcal{B}^j$ for all $j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)$, and $v \in \mathbf{G}^N$ such that $F\left(i, \left|(\mathcal{B}')^i\right|\right) \cup \{i\}$ is a partnership in v. Then

$$\psi_i(v, \mathcal{B}) = \sum_{j \in F\left(i, \left|(\mathcal{B}')^i\right|\right)} \psi_j(v, \mathcal{B})$$

and

$$\psi_j(v, \mathcal{B}) = \psi_k(v, \mathcal{B}) \text{ if } j, k \in F\left(i, \left|(\mathcal{B}')^i\right|\right).$$

This axiom not only requires i and $F\left(i, \left|(\mathcal{B}')^i\right|\right)$ to obtain the same value according to ψ, but also all players in $F\left(i, \left|(\mathcal{B}')^i\right|\right)$ to obtain the same value. Observe that F-Equally Partnership Additivity implies F-Partnership Additivity.

If we add U-Intracoalitional Anonymity in the axiom system considered in the previous Section and substitute F-Partnership Additivity by F-Equally Partnership Additivity, we obtain a characterization of $\phi^{\mathbf{e}}$.

Theorem 5.1 *A solution ψ on* $\mathbf{BG}$ *satisfies Linearity, Efficiency, $\mathcal{B}$-Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger, F-Equally Partnership Additivity, and U-Intracoalitional Anonymity if and only if ψ is the symmetric bounded equally split value.*

Proof. It is straightforward to prove that if ψ is the symmetric bounded equally split value then it satisfies the above axioms.

Conversely, if a solution ψ on $\mathbf{BG}$ satisfies Linearity, Efficiency, $\mathcal{B}$-Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger, and F-Equally Partnership Additivity, have that $\psi = \phi^{\omega}$ with $\omega_i = \psi_i(u_A, \{A\})$ for every $i \in A$. Since ψ satisfies U-Intracoalitional Anonymity there exists $\omega_0 \in \mathbb{R}_+$ such that $\omega_0 = \omega_i = \omega_j$ for every $i, j \in U$. Fix $i \in U$, $k \in \left\{2, ..., 2^{|U|-1}\right\}$ and $j_0 \in F\left(U \times \left\{2, ..., 2^{|U|-1}\right\}\right)$. F-Equally Partnership Additivity implies

$$\omega_0 = \sum_{j \in F(i,k)} \psi_j(u_A, \{A\}) = k\psi_{j_0}(u_A, \{A\}).$$

Therefore

$$\psi_{j_0}(u_A, \{A\}) = \frac{\omega_0}{k},$$

and $\psi = \phi^{\mathbf{e}}$. ■

6 The dual case

Throughout this work we could have defined and studied the dual values of the weighted bounded configuration values. The dual value $\phi^{*\omega}$ can be defined by $\phi^{*\omega}(v, \mathcal{B}) = \phi^{\omega}\left(v^d, \mathcal{B}\right)$, $(v, \mathcal{B}) \in \mathbf{BG}^N$, where $v^d \in \mathbf{G}^N$ is the dual game defined by

$$v^d(S) = v(N) - v(N \backslash S)$$

for all $S \subset N$.

For the dual case we would have to make the following changes. In the Introduction, when considering the orderings of the examples, a set would be formed when the *first* representative of the player entered in the room (instead of the *last* representative) and the player would be given then the corresponding marginal contribution.

In Section 3 the same would happen when the first representative of the player entered in the room. And the probability of $\alpha \in \mathcal{R}_{\mathcal{B}}(N)$ would be

$$(Q^*)^{(\omega,\mathcal{B})}(\alpha) = Q^{(\omega,\mathcal{B})}(\alpha^*),$$

where α^* is the reverse tuple of α. As for Proposition 3.1, though Levy and McLean did not define the dual value $(\eta^*)^{(\omega,\mathcal{B})}$ of $\eta^{(\omega,\mathcal{B})}$, it can be defined as

$$(\eta^*)^{(\omega,\mathcal{B})}(u_T^*) = \eta^{(\omega,\mathcal{B})}(u_T),$$

where u_T^* is defined by

$$u_T^*(S) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

and $\widehat{v}$ would be defined by $\widehat{v}(T)=v\left(j\in N:\bigcup_{B_q\in\mathcal{B}^j}\left\{\pi^j_{\mathcal{B}}(B_q)\right\}\cap T\neq\emptyset\right)$. In this case, $\bigcup_{B_q\in\mathcal{B}^j}\left\{\pi^j_{\mathcal{B}}(B_q)\right\}$ would be a p^*-type coalition in $\widehat{v}$. A coalition S is a p^*-type coalition (Kalai and Samet, 1987) if for each $R\supseteq S$ and $T\subsetneq S$ then $v(R\backslash T)=v(R)$. That is, any proper subset of S has the same effect as S.

In Section 4 we would consider p^*-type coalitions instead of partnership sets in all the axioms, and in Intercoalitional Partnership we would take u^*_T instead of u_T. Moreover, in Merger we would define

$$v^{-i,F(i,k)}(T)=\begin{cases} v(T\cap N) & \text{if } F(i,k)\cap T=\emptyset\\ v((T\cap N)\cup\{i\}) & \text{if } F(i,k)\cap T\neq\emptyset.\end{cases}$$

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