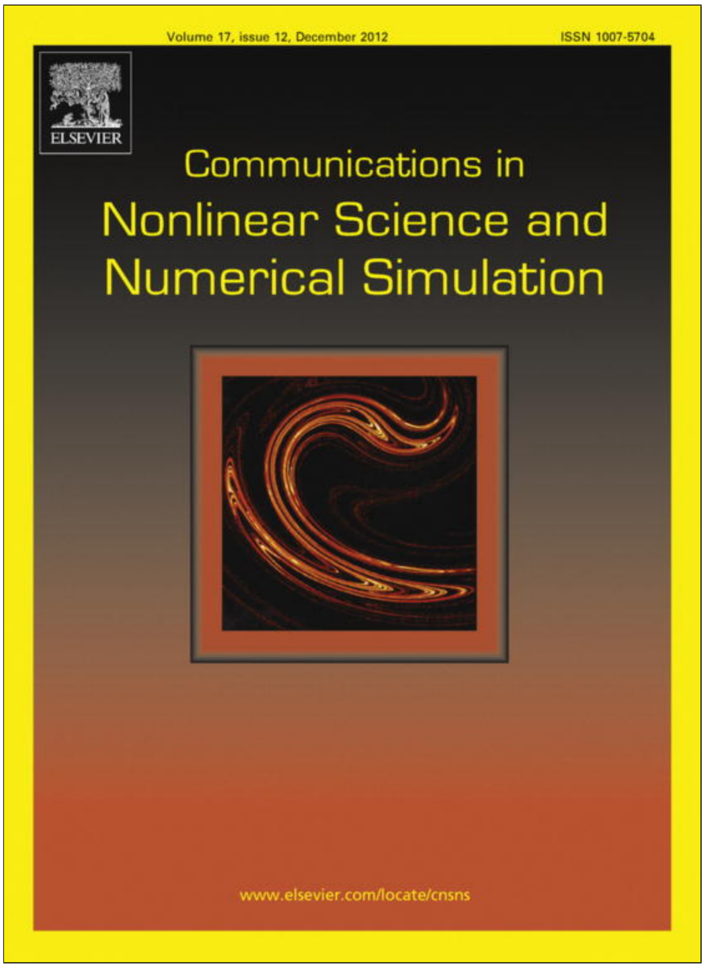
Volume 17, issue 12, December 2012
ISSN 1007-5704
ELSEVIER
Communications in
Nonlinear Science and
Numerical Simulation
www.elsevier.com/locate/cnsns

Commun Nonlinear Sci Numer Simulat 17 (2012) 5131–5142

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

# On the stochastic response of a fractionally-damped Duffing oscillator

Giuseppe Failla [a], Antonina Pirrotta [b,*]

[a] Dipartimento di Meccanica e Materiali (MECMAT), Università di Reggio Calabria, Via Graziella, Località Feo di Vito, 89122 Reggio Calabria, Italy
[b] Dipartimento di Ingegneria Strutturale, Aerospaziale e Geotecnica (DISAG), Università di Palermo, Viale delle Scienze, 90128 Palermo, Italy

ARTICLE INFO

*Article history:*
Received 28 December 2011
Received in revised form 15 February 2012
Accepted 24 March 2012
Available online 2 April 2012

*Keywords:*
Viscoelasticity
Fractional derivative damping
Stochastic response

ABSTRACT

A numerical method is presented to compute the response of a viscoelastic Duffing oscillator with fractional derivative damping, subjected to a stochastic input. The key idea involves an appropriate discretization of the fractional derivative, based on a preliminary change of variable, that allows to approximate the original system by an equivalent system with additional degrees of freedom, the number of which depends on the discretization of the fractional derivative. Unlike the original system that, due to the presence of the fractional derivative, is governed by non-ordinary differential equations, the equivalent system is governed by ordinary differential equations that can be readily handled by standard integration methods such as the Runge–Kutta method. In this manner, a significant reduction of computational effort is achieved with respect to the classical solution methods, where the fractional derivative is reverted to a Grunwald–Letnikov series expansion and numerical integration methods are applied in incremental form. The method applies for fractional damping of arbitrary order $\alpha$ ($0 < \alpha < 1$) and yields very satisfactory results. With respect to its applications, it is worth remarking that the method may be considered for evaluating the dynamic response of a structural system under stochastic excitations such as earthquake and wind, or of a motorcycle equipped with viscoelastic devices on a stochastic road ground profile.



## 1. Introduction

Fractional derivative modeling of linear viscoelastic behavior is a crucial topic in materials science. The first contributions in this field trace back to Nutting [1], who observed that the stress–strain data sets of many complex materials do exhibit a power-law relaxation, and to Gemant [2] and Bosworth [3], the first to propose a fractional derivative model for the constitutive behavior of viscoelastic media. The use of fractional derivatives to fit experimental data was later pursued by Scott-Blair and Gaffyn [4] and Caputo [5]. However, a first attempt to provide a theoretical basis for a fractional derivative modeling of viscoelasticity was due, at the beginning of the 80s, to Bagley and Torvik, who framed their model in the context of molecular theory [6]. They also showed how, in order to capture the frequency-dependence of damping properties in some viscoelastic materials, fractional derivatives are more appropriate than classical linear models such as the Kelvin–Voigt model [7,8].

In the last two decades, fractional derivative modeling of viscoelasticity has been applied in numerous studies and, for its capability of describing complex material behaviors at a macroscopic level, in form of equations involving a small number of parameters, it is now a well-established approach to viscoelastic media [9–12]. Also, a significant effort has been devoted to develop corresponding, suitable mechanical interpretations of the fractional derivative models of viscoelasticity: for

* Corresponding author.
*E-mail addresses:* giuseppe.failla@unirc.it (G. Failla), antonina.pirrotta@unipa.it (A. Pirrotta).

1007-5704/$ - see front matter © 2012 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.cnsns.2012.03.033

instance, Glöckle and Nonnenmacher [13] have interpreted the fractional relaxation equation in the context of non-Markovian process theory, while Heymans [14] has shown that fractional constitutive equations can be obtained from a simple rheological model. Based on the relation between fractional derivatives and fractals theory described by Nigmatullin [15], attempts to derive fractional derivative viscoelastic models from corresponding fractals mechanical model have been pursued by Schiessel and Blumen [16], Heymans and Bauwens [17].

In structural engineering analysis, fractional derivatives have been widely used to model the damping force in systems equipped with viscoelastic dampers, for vibration or seismic isolation purposes [18–21]. Analytical results and experimental data have been found in a very good agreement by Makris and Constantinou [22,23]. To compute the deterministic response of single- or multi-degree of freedom linear systems with a fractional derivative damping, various strategies have been proposed. They involve the Laplace transform [7,8], the Fourier Transform [24], numerical methods [18,25,26] or an eigenvector expansion [27,28]. On the other hand, due to the relevance of stochastic excitations in structural engineering, several methods compute the response statistics have been developed. In this context, a frequency domain approach has been pursued by Spanos and Zeldin [29] and by Rudinger [30]. Alternatively, based on the Laplace transform of the motion equation the system response has been given a time-domain Duhamel integral expression [31] involving pertinent Green's functions, the latter being available in a closed form for certain values of the fractional derivative order $\alpha$. A similar approach, where the Duhamel integral is derived instead based on the Fourier transform of the motion equation, has been later developed by Kun et al. [32]. Further, the response of a system involving two fractional derivatives has been recently addressed by Huang et al. [33], who have derived a Duhamel integral expression by using the Laplace transform in conjunction with the weighted generalized Mittag–Leffler function.

As further, relevant developments in the stochastic analysis of systems with a fractional derivative damping, also a few recent studies concerning nonlinear systems are worth mentioning. Contributions are due to Huang and Jin [34], to Spanos and Evangelatos [35] and to Chen and Zhu [36]. The first have used a classical stochastic averaging procedure for single-degree-of-freedom (SDOF) systems with strongly nonlinear restoring forces and a fractional derivative light damping, subjected to a Gaussian white noise [34]. The second have proposed a general frequency domain solution based on statistical linearization; results have been presented for a Duffing oscillator with a fractional derivative damping, subjected to a Gaussian white noise [35]. The third have studied the stochastic jump and bifurcation of a Duffing oscillator with fractional derivative damping using the stochastic averaging method [36].

The purpose of this paper is to propose an efficient time domain simulation to compute the stochastic response of a Duffing oscillator with fractional damping. In the authors' opinion, this issue may be of particular interest since, on one hand, the time domain simulation still provides a necessary benchmark solution for any approximate method based on stochastic calculus (such as, for instance, the stochastic averaging already used by some authors [34,36]). On the other hand, existing time domain simulations based on standard numerical approximations of the fractional derivative such as, for instance, the Grunwald–Letnikov (GL) or the Riemann–Liouville (RL) series expansions, do involve a high computational effort. This shall be considered indeed as an inevitable consequence of using either the GL or the RL series expansions, which reflect the nature of the fractional derivative operator, that is an operator with memory and, as such, introduce in the motion equation at time $t$ the full displacement response until that time.

With the aim to overcome these shortcomings, in this paper an appropriate discretization of the fractional derivative operator, already and successfully used by the authors for linear systems [37], will be generalized to the Duffing oscillator. It will be shown that the proposed discretization leads to an equivalent system of first order differential equations in terms of state variables, that can be easily solved with a significant reduction of computational effort as compared to the classical solution methods, where the fractional derivative is discretized by the GL or RL series expansion and standard Newmark methods are applied in an incremental form.

With respect to the applications, it is worth remarking that the proposed method may be considered in many fields of engineering. For instance, in the structural field it can be a valid tool to compute the response of a structure equipped with viscoelastic devices under stochastic excitations as earthquake or wind. Further, in mechanical engineering the off-road and racing motorcycles or cars require advanced sophisticated setup of suspensions to improve the comfort and the safety of the rider. Since, due to the ground roughness, suspensions usually experience extreme and stochastic excursions (suspension stroke) to perform their function, it is required to solve nonlinear differential equations governing the motion of the motorcycle or the car equipped with viscoelastic dampers and excited by the stochastic road ground profile. The proposed method lends itself to this purpose.

## 2. Fractionally-damped duffing oscillator driven by stochastic input

Very recently, studies dealing with viscoelastic media have revealed their fundamental interpolating feature between the pure elastic and the pure viscous behavior. In fact, the viscoelastic materials experience both creep and relaxation phases that are well described by a constitutive law ruled by a fractional differential equation [38]. In this context, the structural SDOF system depicted in Fig. 1 and governed by the motion equation

$$m\ddot{x}(t) + f_V(t) + f_E(t) = f(t) \tag{1}$$

is considered. Specifically, in Eq. (1) $m$ is the mass and $f(t)$ is the forcing function; $f_E(t) = kx(t) + \varepsilon x^3(t)$ is the restoring force of the nonlinear spring, where $k$ is the linear stiffness and $\varepsilon$ controls the amount of nonlinearity; $f_V(t) = C_\alpha(\mathcal{D}^\alpha x)(t)$ is the attenuation force of the viscoelastic damper, being $C_\alpha$ the fractional damping coefficient and $(\mathcal{D}^\alpha x)(t)$ a $\alpha$-order fractional

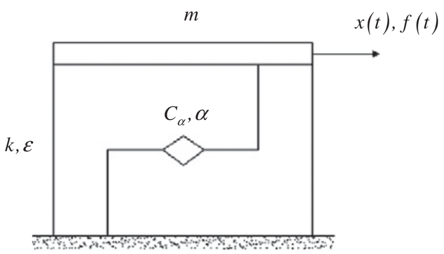


**Fig. 1.** Duffing SDOF system with a fractional damping.

derivative that interpolates the purely elastic behavior ($\alpha = 0$) and the purely viscous behavior ($\alpha = 1$). The motion equation can be obviously reverted to the ordinary form

$$\ddot{x}(t) + c_\alpha(\mathcal{D}^\alpha x)(t) + \omega_0^2 x(t) + \varepsilon_0 x^3(t) = f_0(t) \tag{2}$$

where $c_\alpha = C_\alpha/m$, $\omega_0 = \sqrt{k/m}$ is the natural frequency, $\varepsilon_0 = \varepsilon/m$ and $f_0(t) = f(t)/m$.

It is worth pointing out that many definitions exist for fractional derivative [39], and the most commonly used is the Riemann–Liouville (RL) definition,

$$(\mathcal{D}^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-\infty}^{t}\frac{x(\tau)}{(t-\tau)^\alpha}d\tau; \quad 0 < \alpha < 1, \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function.

In Ref. [38] it has been demonstrated that the power-law decay can be considered as the best function for fitting relaxation tests on viscoelastic materials. Further, it can be shown that upon introducing a power-law decay function as a relaxation function into the kernel of the Boltzmann integral for viscoelastic materials, the fractional constitutive law involving the Caputo's derivative is retrieved. On this solid ground, in this paper we will refer to the Caputo's definition for fractional derivative, i.e.

$$(\mathcal{D}_C^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau; \quad 0 < \alpha < 1 \tag{4}$$

On the other hand, it is known that the Caputo's fractional derivative for a quiescent system at $t = 0$ or for systems that operate from $t = -\infty$ coincides with the RL fractional derivative [39].

Particular attention will be devoted to the response of system (1) to a stochastic input. That is, the solution to the following equation will be sought

$$\ddot{x}(t) + c_\alpha(\mathcal{D}_C^\alpha x)(t) + \omega_0^2 x(t) + \varepsilon_0 x^3(t) = w(t), \tag{5}$$

where $w(t)$ is an arbitrary sample of a zero mean Gaussian white noise $W(t)$ of power spectral density $S_0$, i.e. $E[W(t_1)W(t_2)] = 2\pi S_0\delta(t_1 - t_2)$ (as customary, stochastic processes are denoted by capital letters while an arbitrary sample is denoted by the corresponding lower case).

In a recent study [35] it has been shown that, upon discretizing the fractional derivative by the GL series expansion, Eq. (5) can be recast in incremental form and the time domain simulation can be pursued by classical Newmark methods.

In this paper, an original time domain simulation will be proposed, based on an alternative discretization of the fractional derivative. The results are validated by a comparison with those obtained by using the Newmark method in conjunction with the GL series expansion, as shown in the next sections.

### 2.1. Time domain simulation via Grunwald–Letnikov approximation

The implementation of the Newmark method in conjunction with the GL series expansion can be briefly described as follows.

Consider a quiescent system at $t = 0$. As explained in Ref. [35], if the time interval of interest $[0, t_f]$ is discretized into equally-spaced steps $\Delta t$, for the fractional derivative (4) at time $t_i = i\Delta t$ the following GL series expansion can be adopted:

$$(D_C^\alpha x)(t) = \lim_{\Delta t \to 0}\Delta t^{-\alpha}\sum_{k=0}^{i} GL_k x(t_i - k\Delta t), \tag{6}$$

where $GL_k$ are coefficients to be computed in the recursive form

$$GL_k = \frac{k-\alpha-1}{k}GL_{k-1}, \quad GL_0 = 1.0 \tag{7}$$

Based on Eq. (7), the motion equation (5) at a time instant $t_i = i\Delta t$ can be written as

$$\ddot{x}(t_i) + c_\alpha\Delta t^{-\alpha}\sum_{k=0}^{i} GL_k x(t_i - k\Delta t) + q(t_i) = f(t_i), \tag{8}$$

where, for conciseness, symbol $q(t)$ has been introduced, i.e. $q(t)=\omega_0^2 x(t)+\varepsilon_0 x^3(t)$. Then computing the difference between the two motion equations written, respectively, at the time instant $t_{i+1}=(i+1)\Delta t$ and at the time instant $t_i$, yields the following equation in incremental form

$$\Delta\ddot{x}_{i+1}+c_\alpha \Delta t^{-\alpha} GL_0 \Delta x_{i+1}+\Delta q_{i+1}=\Delta f_{i+1}-c_\alpha \Delta t^{-\alpha} P_i, \tag{9}$$

In Eq. (9) $\Delta\ddot{x}_{i+1}=\ddot{x}(t_{i+1})-\ddot{x}(t_i)$, $\Delta q_{i+1}=q(t_{i+1})-q(t_i)$, $\Delta f_{i+1}=f(t_{i+1})-f(t_i)$ and

$$P_i=\sum_{k=1}^{i} GL_k[x(t_{i+1}-k\Delta t)-x(t_i-k\Delta t)]+GL_{i+1}x(0), \tag{10}$$

where $x(0)$ is the initial condition for $x(t)$. Recognize that $P_i$ in Eq. (10) is a *pseudo*-force, depending on the displacement response history until the time instant $t_i$. At this stage, in order to avoid time-consuming iterations to compute the nonlinear increment $\Delta q_{i+1}$, the following simplifying assumption can be generally made [40]

$$\Delta q_{i+1}=\hat{k}_i \Delta x_{i+1}, \tag{11}$$

where $\hat{k}_i=\omega_0^2+3\varepsilon_0 x^2(t_i)$ is the initial tangent slope at time instant $t_i$ and $\Delta x_{i+1}=x(t_{i+1})-x(t_i)$ is the displacement increment. In this manner, a typical Newmark method can be applied to the motion equation (8) written in incremental form [40]. Specifically, if the linear acceleration method is adopted, the displacement increment is given as

$$\Delta x_{i+1}=\Delta p_{i+1}/(\hat{k}_i+c_\alpha \Delta t^{-\alpha} GL_0+6\Delta t^{-2}), \tag{12}$$

where $\Delta p_{i+1}$ is the effective loading increment

$$\Delta p_{i+1}=\Delta f_{i+1}-c_\alpha \Delta t^{-\alpha} P_i+6\dot{x}(t_i)/\Delta t+3\ddot{x}(t_i) \tag{13}$$

Upon computing the displacement increment (12), the velocity increment can be also derived as

$$\Delta\dot{x}_{i+1}=3\Delta x_{i+1}/\Delta t-3\dot{x}(t_i)-\ddot{x}(t_i)\Delta t/2, \tag{14}$$

where the acceleration $\ddot{x}(t_i)$ is given by the motion equation set at time $t_i$. It is then apparent that, when using a Newmark method in conjunction with the GL approximation (6), at any time instant $t_i$ a summation over the displacement response until $t=t_i$ is generally required to compute both the pseudo-force (10) and the acceleration $\ddot{x}(t_i)$. These summations do lead, in general, to a significant computational effort.

### 2.2. Time domain simulation via proposed method

In this section the proposed method will be introduced, in attempt to overcome the shortcomings of the approach discussed in the previous section. Let us consider the following variable transformation [41]:

$$z=y^2(t-\tau);\quad dz=2(t-\tau)y\,dy. \tag{15}$$

that leads to rewrite the Gamma function as

$$\Gamma(\alpha)=\int_0^\infty e^{-z}z^{\alpha-1}dz=2\int_0^\infty e^{-y^2(t-\tau)}y^{2\alpha-1}(t-\tau)^\alpha dy \tag{16}$$

Being $1/\Gamma(1-\alpha)=\Gamma(\alpha)\sin(\alpha\pi)/\pi$, Eq. (16) allows the coefficient $1/\Gamma(1-\alpha)$ involved in the Caputo's fractional derivative (see Eq. (4)) to be written as

$$\frac{1}{\Gamma(1-\alpha)}=\frac{2\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-y^2(t-\tau)}y^{2\alpha-1}(t-\tau)^\alpha dy. \tag{17}$$

In this manner, the fractional derivative in the motion equation (5) can be recast in the form (the system is quiescent at $t=0$)

$$(\mathcal{D}_C^\alpha x)(t)=\mu_\alpha\int_0^\infty\int_0^t e^{-y^2(t-\tau)}\dot{x}(\tau)d\tau\, y^{2\alpha-1}dy. \tag{18}$$

where $\mu_\alpha=2\sin(\alpha\pi)/\pi$.

Next, denote by $u_y(t)$ the following integral in Eq. (18):

$$u_y(t)=\int_0^t e^{-y^2(t-\tau)}\dot{x}(\tau)d\tau \tag{19}$$

The latter can be considered as a Duhamel integral giving the response of the Maxwell half oscillator in Fig. 2, assumed to be quiescent at $t=0$:

$$\dot{u}_y(t)+y^2u_y(t)=\dot{x}(t);\quad u_y(0)=0. \tag{20}$$

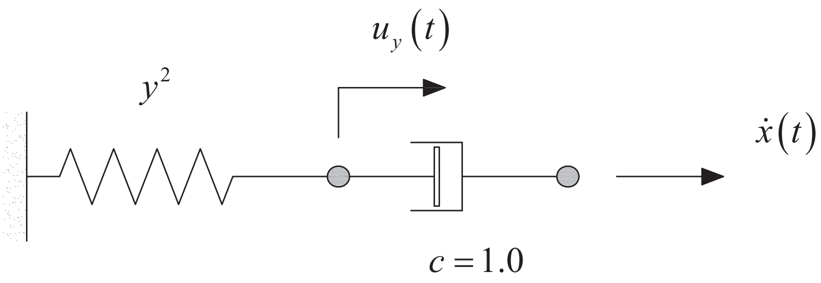


**Fig. 2.** Maxwell oscillator, Eq. (20).

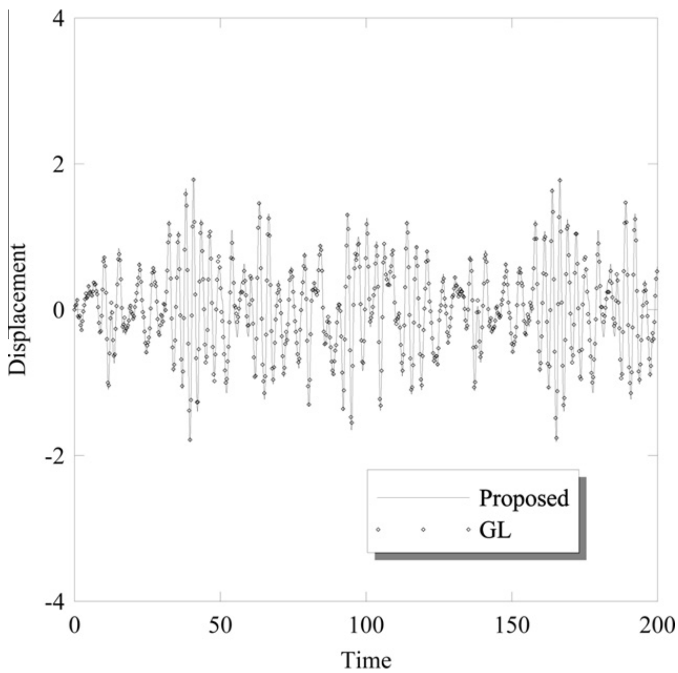


**Fig. 3.** Displacement sample for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

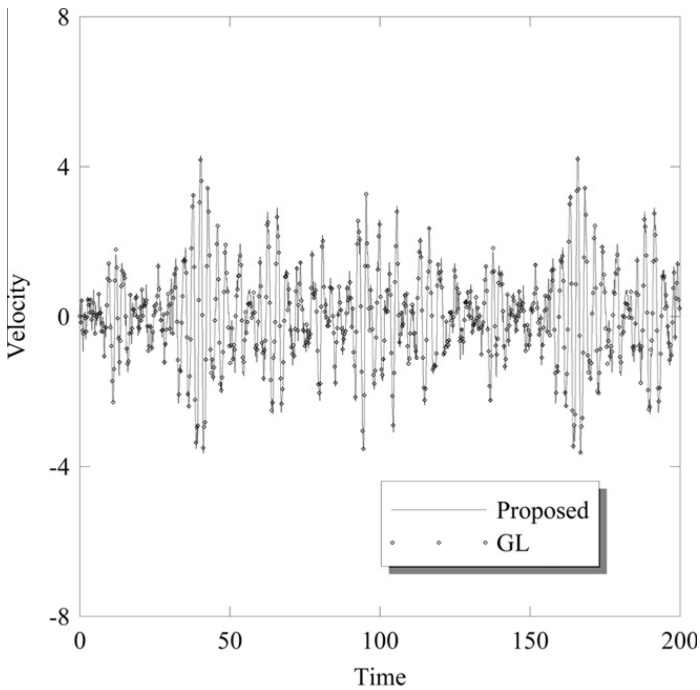


**Fig. 4.** Velocity sample for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

Recognize that the system has a stiffness coefficient equal to $y^2$, a unit damping coefficient and is forced by the velocity response $\dot{x}(t)$. At this stage, it follows that the fractional derivative of $x(t)$, given in Eq. (4), may be rewritten as

$$(\mathcal{D}_C^{\alpha}x)(t) = \mu_{\alpha}\int_0^{\infty} u_y(t)y^{2\alpha-1}dy \tag{21}$$

and may be approximated in discrete form as

$$(\mathcal{D}_C^{\alpha}x)(t) \approx \mu_{\alpha}\sum_{j=1}^{\infty} u_{y_j}(t)y_j^{2\alpha-1}\Delta y \tag{22}$$

where $y_j = j\Delta y$ and, due to Eq. (20), $u_{y_j}(t)$ is the response of the first order differential equation

$$\dot{u}_{y_j}(t) + y_j^2 u_{y_j}(t) = \dot{x}(t); \quad j = 1, 2 \cdots \infty \tag{23}$$

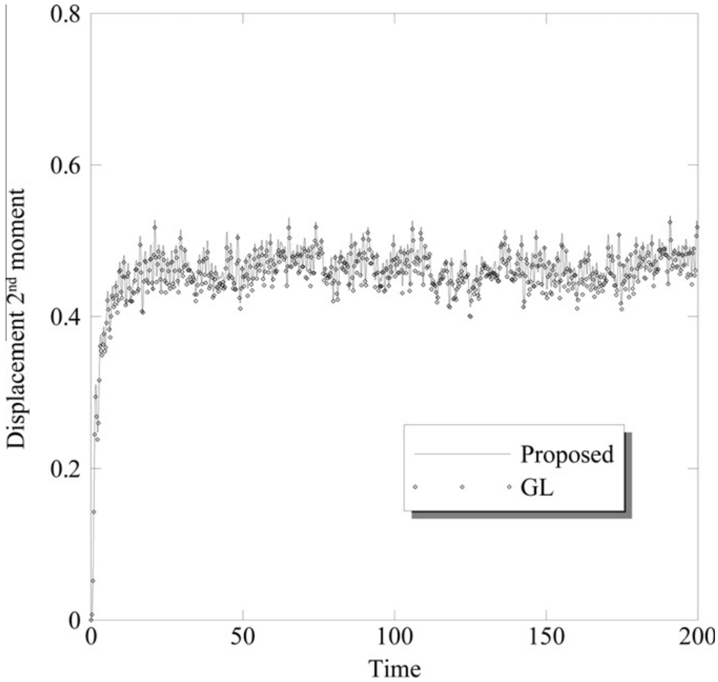


**Fig. 5.** Displacement 2nd moment for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

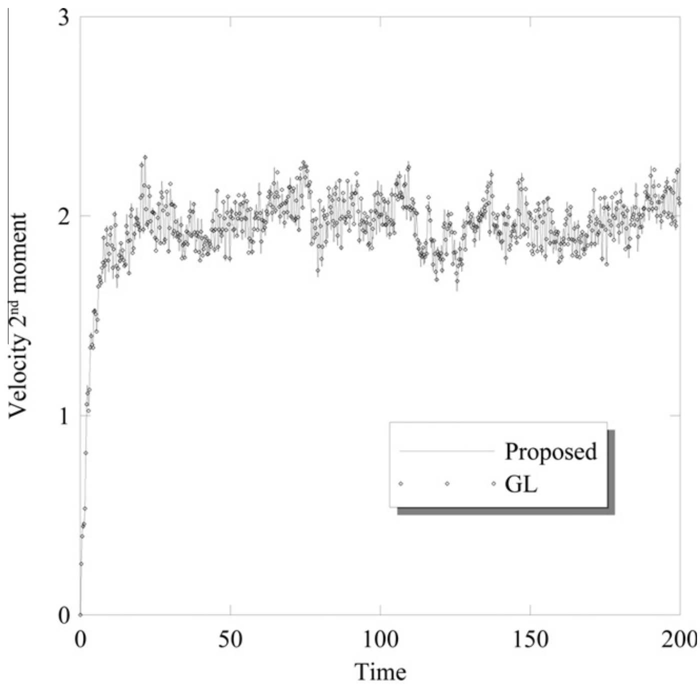


**Fig. 6.** Velocity 2nd moment for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

Based on Eq. (22), the response of the Duffing oscillator with fractional damping (5) can be obtained solving the following system of ordinary differential equations:

$$\begin{aligned} &\ddot{x}(t) + c_\alpha \mu_\alpha \sum_{j=1}^{\infty} u_{y_j}(t) y_j^{2\alpha-1} \Delta y + \omega_0^2 x(t) + \varepsilon_0 x^3(t) = w(t); \\ &\dot{u}_{y_j}(t) + y_j^2 u_{y_j}(t) = \dot{x}(t) \end{aligned} \tag{24}$$

Eq. (24) show that, based on the proposed discretization of the fractional derivative (22), the original system (5), that is governed by a non-ordinary differential equation, can be reverted to the equivalent system (24) of ordinary differential equations, that can be readily handled by standard numerical integration methods, getting rid of the all displacement response history until the time instant $t_i$. Obviously, for numerical purposes a finite number $n$ of terms shall be retained in the first equation of system (24). This leads to the finite degree of freedom linear system

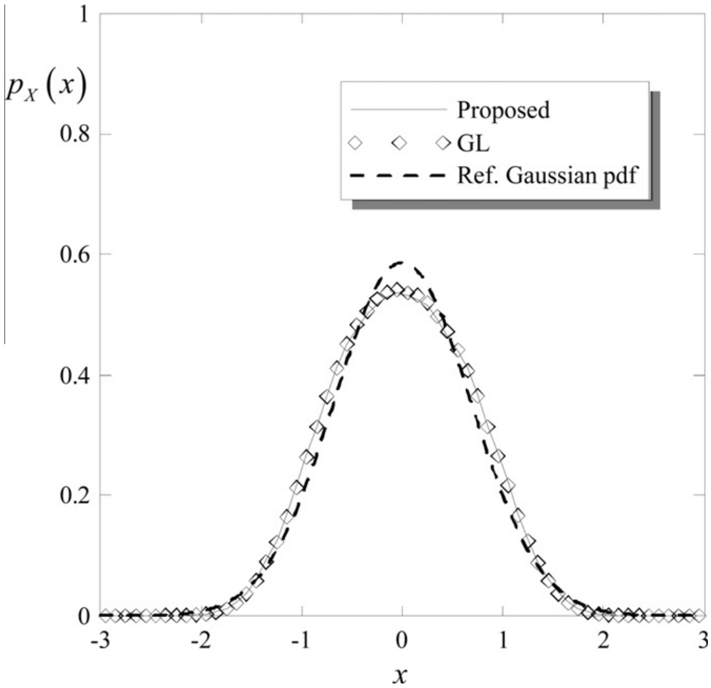


**Fig. 7.** Displacement pdf for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

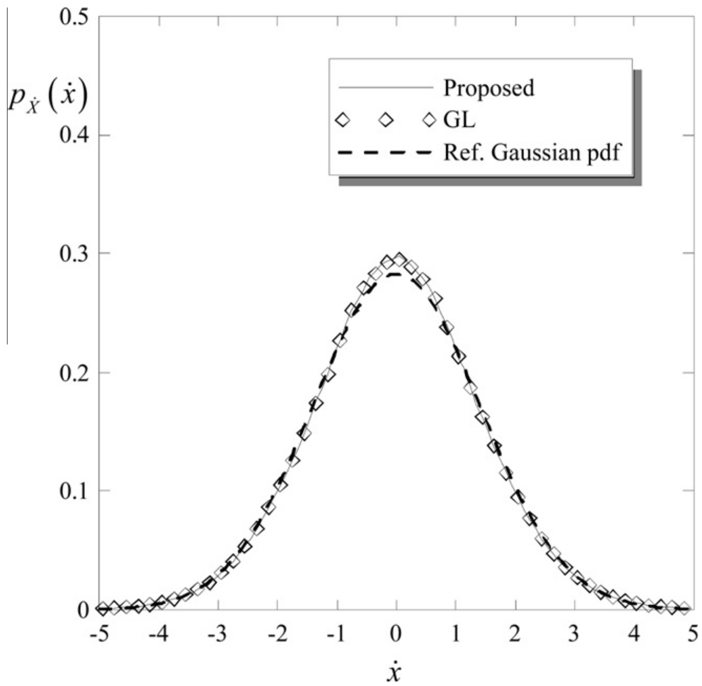


**Fig. 8.** Velocity pdf for $\varepsilon_0 = 2.0$ and $\alpha = 0.3$.

$$\dot{\mathbf{z}} = \mathbf{D}\mathbf{z} + \mathbf{v}w(t), \tag{25}$$

where

$$\mathbf{z}^T = [\,x \quad \dot{x} \quad u_{y_1} \quad u_{y_2} \quad \cdots \quad u_{y_n}\,] \tag{26}$$

is the vector of state variables of order $(n+2)$,

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -\omega_0^2 - \varepsilon_0 x^2 & 0 & -c_\alpha \mu_\alpha y_1^{2\alpha-1}\Delta y & -c_\alpha \mu_\alpha y_2^{2\alpha-1}\Delta y & \cdots & -c_\alpha \mu_\alpha y_n^{2\alpha-1}\Delta y \\ 0 & 1 & -y_1^2 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -y_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & -y_n^2 \end{bmatrix} \tag{27}$$

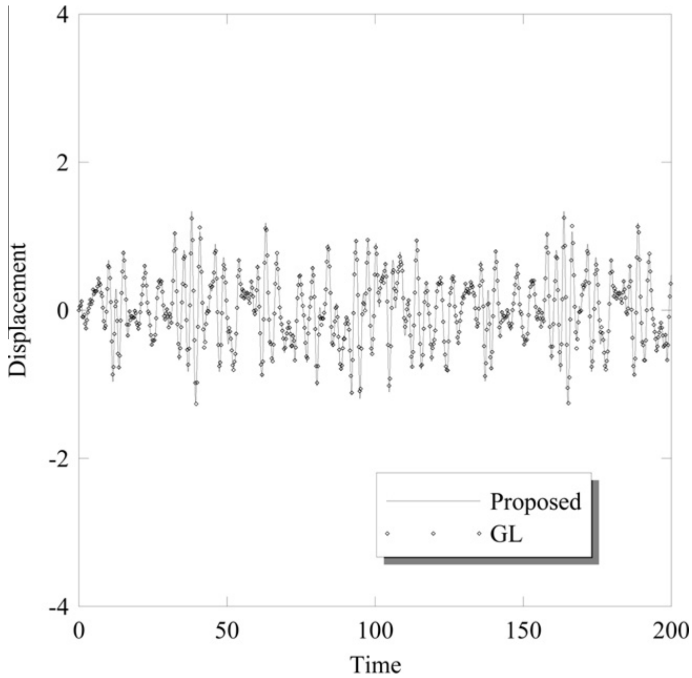


**Fig. 9.** Displacement sample for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

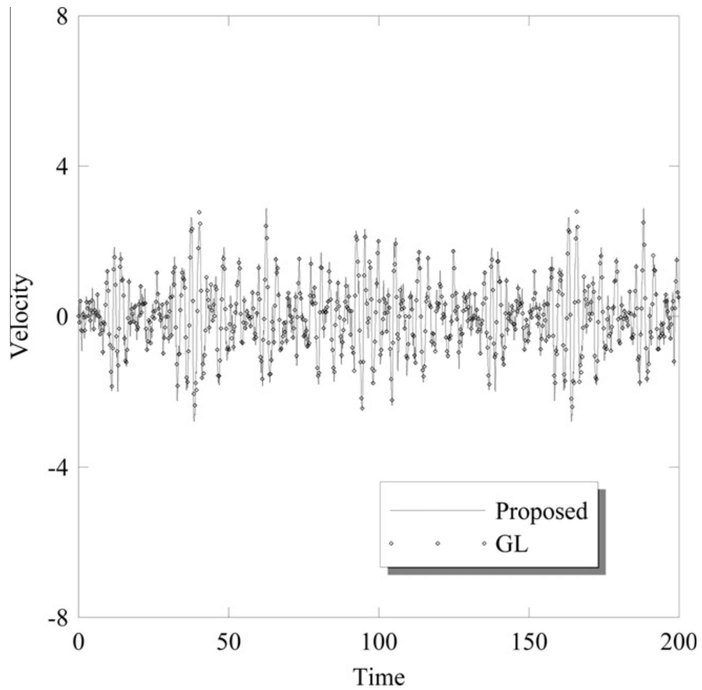


**Fig. 10.** Velocity sample for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

is the system matrix and

$$\mathbf{v}^T = [0 \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0] \tag{28}$$

is the forcing column vector.

At this point, it is apparent that, any time-domain numerical integration scheme can be applied to compute the response of the nonlinear system (25). In this paper the Runge–Kutta method will be adopted.

## 3. Numerical applications

The accuracy of the proposed method is assessed by computing the solution of Eq. (5) that, as already stated in the introduction, may be considered the motion equation of either a structural system or a motorcycle equipped with viscoelastic

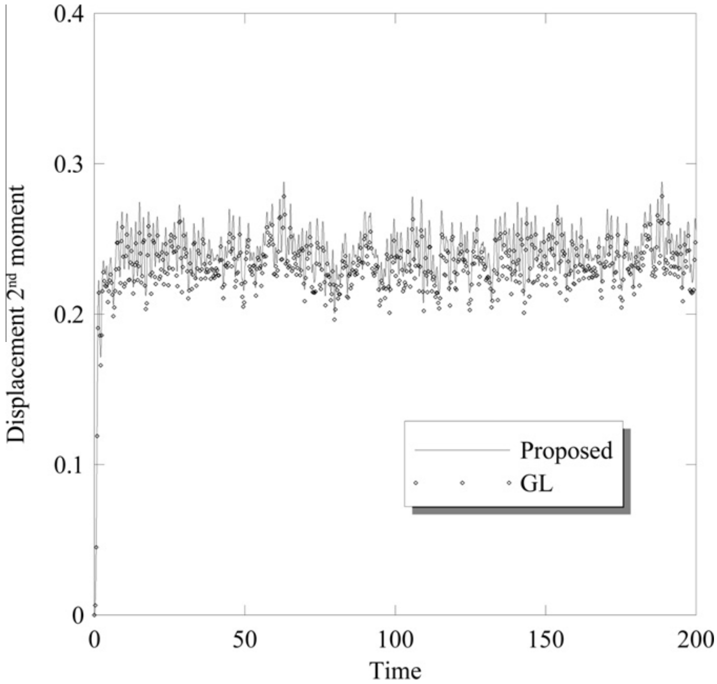


**Fig. 11.** Displacement 2nd moment for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

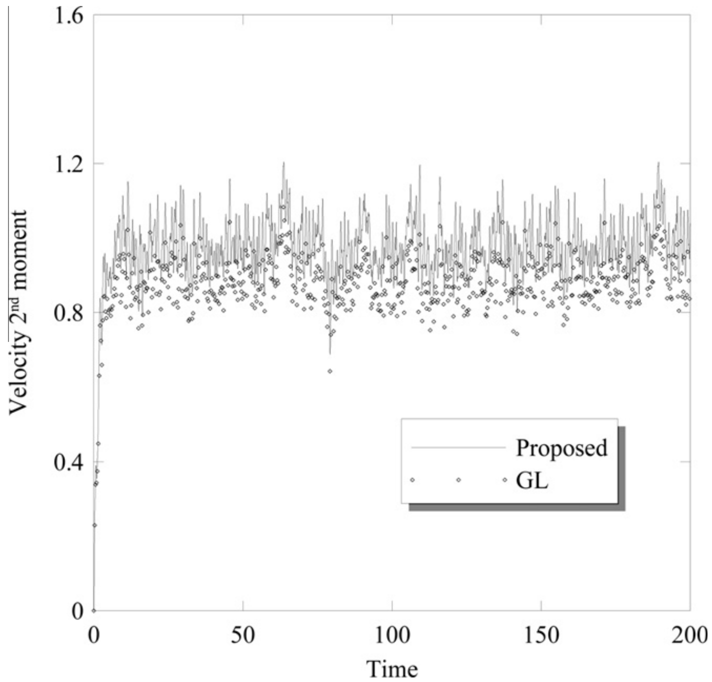


**Fig. 12.** Velocity 2nd moment for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

devices, to a Gaussian white noise, for two sets of system parameters. The first set is given by $\omega_0 = 1.0$, $c_\alpha = 1.0$, $\varepsilon_0 = 2.0$ and $\alpha = 0.3$; the second set by $\omega_0 = 1.0$, $c_\alpha = 1.0$, $\varepsilon_0 = 4.0$ and $\alpha = 0.6$. In both cases the power spectral density of the Gaussian white noise is $S_0 = 1/2\pi$. An arbitrary sample $w(t)$ is built based on the well-known spectral representation in Ref. [42], according to which $w(t)$ is given by

$$w(t) = \sum_{k=1}^{M} \sqrt{4S_0 \Delta\omega} \cos(\omega_k t + \varphi_k). \tag{29}$$

where $\Delta\omega$ is a constant step on the frequency axis, $\omega_k = k\Delta\omega$ are $M$ equally-spaced frequencies and $\varphi_k$ are $M$ random phases uniformly distributed in the interval $[0, 2\pi]$. Specifically, $\Delta\omega = 0.05$ rad/s and $M = 500$ are selected. The time-domain simulation is carried out generating 1000 samples.

The proposed method is implemented by discretizing the $y$-axis (see Eq. (22)) in two subintervals: the first is $[0, 0.05]$ and is divided into 200 equal steps; the second is $[0.05, 20.0]$ and is divided into 100 steps. Therefore, a total number of $n = 300$ additional Maxwell oscillators is considered to build the equivalent system (25). The discretization parameters of the $y$-axis are selected to provide an accurate description of the integrand function $e^{-z} z^{\alpha-1}$ in Eq. (16), taking into account that $z = y^2(t-\tau)$.

The response of the equivalent system (25) is obtained by using the Runge–Kutta method and then compared to the response of the original system (5) as obtained based on Eq. (9). Specifically, the latter is implemented by using the linear acceleration method, discussed in detail in Section 2.1. A time step $\Delta t = 0.002$ is adopted for both the Runge–Kutta method and the linear acceleration method.

For $\varepsilon_0 = 2.0$ and $\alpha = 0.3$, Fig. 3 through Fig. 8 show the results for both the displacement and the velocity process. In particular, Figs. 3 and 4 show an arbitrary sample over the time interval $0 \div 200$ s; the variances over the same interval are shown in Figs. 5 and 6, while Figs. 7 and 8 show the corresponding probability density functions (pdfs). In Figs. 7 and 8 the Gaussian pdfs that feature the same variance as the displacement process (Fig. 7) and the same variance as the velocity process (Fig. 8) are also shown. The same results for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$ are then reported in Fig. 9 through 14 through Fig. 14.

All the results show that the proposed solution and the standard solution of the original system (5), built by using the linear acceleration method in conjunction with the GL approximation of the fractional derivative, are in a very good agreement. It can be then stated that the proposed solution appears capable of reproducing the system behavior with a significant accuracy, in terms of time response to individual samples of the input process as well as in terms of response statistics. In this respect, recognize that the displacement and the velocity pdfs appear both very accurate over the whole domain, i.e. around the peak value and the tails (Figs. 7–8 and Figs. 13–14).

It shall be remarked, however, that for the two cases under study the proposed solution allows a significant reduction of computational effort, to an extent up to 35%, with respect to the standard solution of the original system (5). This can be explained by recognizing that, to obtain the response increment at each time step by the standard solution of the original system (5), time-consuming summations are required over the terms of the GL approximation of the fractional derivative, to compute the pseudo-force term $P_i$, Eq. (10), and the acceleration $\ddot{x}(t_i)$ at the beginning of each time step.

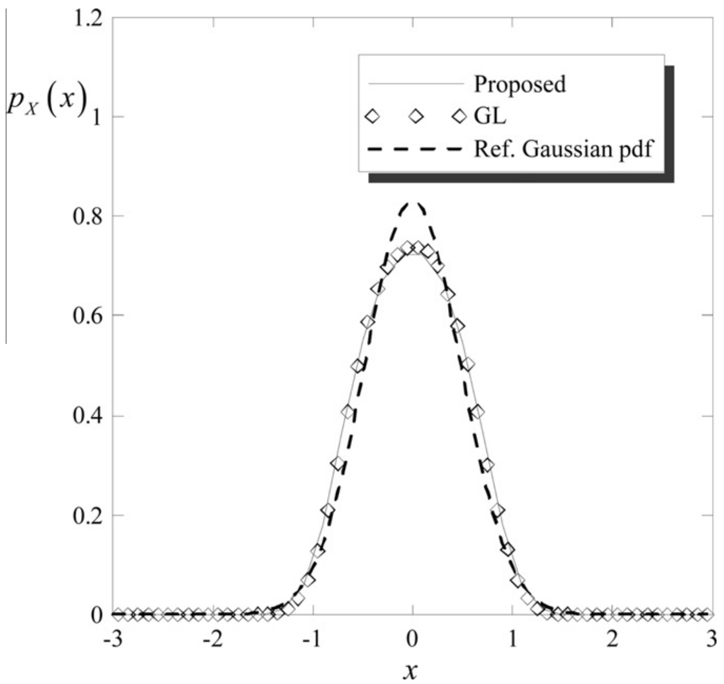


**Fig. 13.** Displacement pdf for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

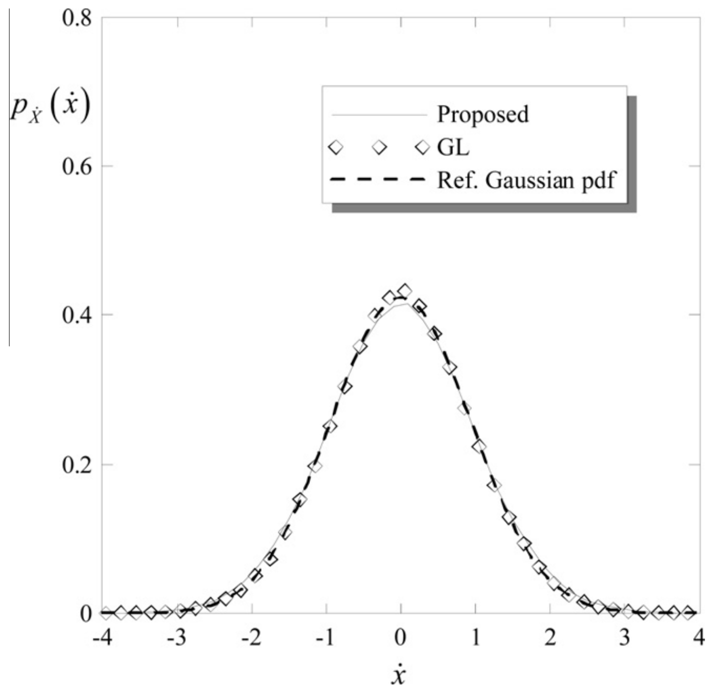


**Fig. 14.** Velocity pdf for $\varepsilon_0 = 4.0$ and $\alpha = 0.6$.

## 4. Conclusions

The stochastic response of a viscoelastic Duffing oscillator with fractional damping has been computed via a new, efficient time domain simulation. The key-idea is to discretize the fractional operator based on an appropriate change of variable. In this manner, the original motion equation has been reverted to a system involving additional oscillators, the number of which depends on the adopted discretization, that can be readily handled by a standard Runge–Kutta method. Since numerical applications show that, in general, a limited number of additional oscillators is requested to achieve accurate results, the method proves to be computationally efficient as compared to existing time domain simulations involving the classical GL discretization of the fractional derivative in conjunction with a standard Newmark method, due to the time-consuming summations that the latter requests over the displacement response history. Similar conclusion could be drawn for alternative classical discretizations of the fractional derivative, such as the RL discretization.

## References


[1] Nutting PG. A new general law deformation. J Franklin Inst 1921;191:678–85.
[2] Gemant A. On fractional differentials. Philos Mag Ser 1938;25:540–9.
[3] Bosworth RCL. A definition of plasticity. Nature 1946;157:447.
[4] Scott-Blair GW, Gaffyn JE. An application of the theory of quasi-properties to the treatment of anomalous strain–stress relations. Philos Mag 1949;40:80–94.
[5] Caputo M. Vibrations on an infinite viscoelastic layer with a dissipative memory. J Acoust Soc Am 1974;56(3):897–904.
[6] Bagley RL, Torvik PJ. A theoretical basis for the application of fractional calculus to viscoelasticity. J Rheol 1983;27:201–10.
[7] Bagley RL, Torvik PJ. Fractional calculus – a different approach to the analysis of viscoelastically damped structures. Amer Inst Aeronaut Astronaut J 1983;21:741–8.
[8] Bagley RL, Torvik PJ. Fractional calculus in the transient analysis of viscoelastically damped structures. Amer Inst Aeronaut Astronaut J 1985;23:918–25.
[9] Rogers L. Operators and fractional derivatives for viscoelastic constitutive equations. J Rheol 1983;27(4):351–72.
[10] Koeller RC. Application of fractional calculus to the theory of viscoelasticity. J Appl Mech 1984;51:299–307.
[11] Pritz T. Analysis of four-parameter fractional derivative model of real solid materials. J Sound Vib 1996;195:103–15.
[12] Adolfsson K, Enelund M, Olson P. On the fractional order model of viscoelasticity. Mech Time-Dependent Mater 2005;9:15–24.
[13] Glöckle WG, Nonnenmacher TF. Fox function representation of non-Debye relaxation processes. J Stat Phys 1993;71:741–57.
[14] Heymans N. Hierarchical models for viscoelasticity: dynamic behavior in the linear range. Rheol Acta 1996;35:508–19.
[15] Nigmatullin RR. Fractional integral and its physical interpretation. Theor Math Phys 1992;90(3):242–51.
[16] Schiessel H, Blumen A. Hierarchical analogues to fractional relaxation equations. J Phys A: Math Gen 1993;26:5057–69.
[17] Heymans N, Bauwens JC. Fractal rheological models and fractional differential equations for viscoelastic behavior. Rheol Acta 1994;33:210–9.
[18] Koh CG, Kelly LM. Application of fractional derivatives to seismic analysis of base isolated models. Earthquake Eng Struct Dynam 1990;19(2):229–41.
[19] Lee HH, Tsai CS. Analytical model for viscoelastic dampers for seismic mitigation of structures. Comput Struct 1994;50(1):111–21.
[20] Shen KL, Soong TT. Modeling of viscoelastic dampers for structural applications. J Eng Mech 1995;121:694–701.
[21] Papoulia KD, Kelly JM. Visco-hyperelastic model for filled rubbers used in vibration isolation. J Eng Mater Technol 1997;119:292–7.
[22] Makris N, Constantinou MC. Fractional-derivative Maxwell model for viscous dampers. J Struct Eng 1991;117:2708–24.
[23] Makris N, Constantinou MC. Spring-viscous damper systems for combined seismic and vibration isolation. Earthquake Eng Struct Dynam 1992;21(8):649–64.

[24] Gaul L, Klein P, Kemple S. Impulse response function of an oscillator with fractional derivative in damping description. Mech Res Commun 1989;16:297–305.
[25] Lixia Y, Agrawal OP. A numerical scheme for dynamic systems containing fractional derivatives. In Proceedings of 1998 ASME design engineering technical conferences, September 13–16, Atlanta, Georgia, 1998.
[26] Shokooh A, Suarez L. A comparison of numerical methods applied to a fractional model of damping materials. J Vib Control 1999;5:331–54.
[27] Suarez L, Shokooh A. An eigenvector expansion method for the solution of motion containing fractional derivatives. J Appl Mech 1997;64:629–35.
[28] Chang TS, Singh MP. Seismic analysis of structures with a fractional derivative model of viscoelastic dampers. Earthquake Eng Eng Vib 2002;1(2):251–60.
[29] Spanos PD, Zeldin BA. Random vibration of systems with frequency-dependent parameters or fractional derivatives. J Eng Mech 1997;123:290–2.
[30] Rudinger F. Tuned mass damper with fractional derivative damping. Eng Struct 2006;28:1774–9.
[31] Agrawal OP. Stochastic analysis of dynamic systems containing fractional derivatives. J Sound Vib 2001;247(5):927–38.
[32] Kun Y, Li L, Jiaxiang T. Stochastic seismic response of structures with added viscoelastic dampers modeled by fractional derivative. Earthquake Eng Eng Vib 2003;2(1):133–9.
[33] Huang ZL, Jin XL, Lim CW, Wang Y. Statistical analysis for stochastic systems including fractional derivatives. Nonlinear Dyn 2010;59:339–49.
[34] Huang ZL, Jin XL. Response and stability of a SDOF strongly nonlinear stochastic system with light damping modeled by a fractional derivative. J Sound Vib 2009;319:1121–35.
[35] Spanos PD, Evangelatos GI. Response of a non-linear system with restoring forces governed by fractional derivatives-time domain simulation and statistical linearization solution. Soil Dynam Earthquake Eng 2010;30:811–21.
[36] Chen L, Zhu W. Stochastic jump and bifurcation of Duffing oscillator with fractional derivative damping under combined harmonic and white noise excitations. Int J Non-Linear Mech 2011;46:1324–9.
[37] Di Paola M, Failla G, Pirrotta A. Stationary and non-stationary stochastic response of linear fractional viscoelastic systems. Probab Eng Mech 2011;28:85–90.
[38] Di Paola M, Pirrotta A, Valenza A. Visco-elastic behavior through fractional calculus: an easier method for best fitting experimental results. Mech Mater 2011;43(12):799–806.
[39] Podlubny I. Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, some methods of their solution and some of their applications. New York: Academic Press; 1999.
[40] Clough RW, Penzien J. Dynamics of structures. New York: Mc-Graw Hill; 1993.
[41] Schmidt A, Gaul L. On a critique of a numerical scheme for the calculation of fractionally damped dynamical systems. Mech Res Commun 2006;33:99–107.
[42] Shinozuka M, Deodatis G. Stochastic process models for earthquake ground motion. Probab Eng Mech 1988;3(3):114–23.