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A discrete mechanical model of fractional hereditary materials

Mario Di Paola · Francesco Paolo Pinnola · Massimiliano Zingales

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Abstract Fractional hereditary materials are characterized for the presence, in the stress-strain relations, of fractional-order operators with order $\beta \in [0, 1]$. In Di Paola and Zingales (J. Rheol. 56(5):983–1004, 2012) exact mechanical models of such materials have been extensively discussed obtaining two intervals for β: (i) Elasto-Viscous (EV) materials for $0 \le \beta \le 1/2$; (ii) Visco-Elastic (VE) materials for $1/2 \le \beta \le 1$. These two ranges correspond to different continuous mechanical models.

In this paper a discretization scheme based upon the continuous models proposed in Di Paola and Zingales (J. Rheol. 56(5):983–1004, 2012) useful to obtain a mechanical description of fractional derivative is presented. It is shown that the discretized models are ruled by a set of coupled first order differential equations involving symmetric and positive definite matrices. Modal analysis shows that fractional order operators have a mechanical counterpart that is ruled by a set of Kelvin-Voigt units and each of them provides a proper contribution to the overall response. The robustness of the proposed discretization scheme is assessed in the paper for different classes of external loads and for different values of $\beta \in [0, 1]$.

M. Di Paola · F.P. Pinnola · M. Zingales (✉)
Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale e dei Materiali (DICAM), Università degli Studi di Palermo, Viale delle Scienze, Ed. 8, 90128 Palermo, Italy
e-mail: massimiliano.zingales@unipa.it

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1 Introduction

In recent years complex materials, often used in engineering applications, have been obtained with the aid of sophisticated industrial processes aiming to enhance stiffness and strength of materials. These features may be easily measured by simple experimental tests (tensile test) that do not account for the past stress/strain histories of materials.

As the hereditary properties of complex materials are investigated with introducing time dependent measures of stress (relaxation test) and/or strain (creep test), it may be observed that accurate description of the results of experimental tests are well-fitted by power-laws with real order exponent [2–4]. This observation yields a description of material properties with the aid of fractional-order integro-differential operators [5–7]. In this context *fractional hereditary materials* (FHM) have been introduced since the beginning of the last century. Based on this observation in the second part of the last century several research works have been carried out to enforce knowledge of fractional description of rheological stress-strain behavior in time domain [8–11] and in frequency domain [12–14].

Among the integration schemes of fractional operators to describe viscoelastic model several important contributions have been provided by various authors. In more details in [15, 16] the fractional operators have been discretized by using the definition of Grünwald-Letnikov (see Appendix). Furthermore, by using the Euler Gamma function properties other method of discretization may be found in [17–19].

In this paper we propose a discretized mechanical model for description of fractional order hereditary materials based upon the exact mechanical models described in [1]. FHM have been classified in two different categories, according to its predominant behavior: the Elasto-Viscous class and the Visco-Elastic class.

Materials belonging to Elasto-Viscous (EV) class are characterized by a marked elastic behavior as compared with viscous properties. This condition leads to stress-strain relations involving fractional operators with index $\beta \in [0, 1/2]$ ($\beta = 0$ represents a perfectly elastic behavior).

Contrariwise Visco-Elastic (VE) materials are characterized by a predominant viscous phase and this condition leads to stress-strain relations with fractional operator order $\beta \in [1/2, 1]$ ($\beta = 1$ represents the condition of pure Newtonian fluid).

These two similar but different features possesses different mechanical analogues. In particular EV material ($0 \leq \beta \leq 1/2$) is described, mechanically, by an indefinite column of fluid resting on a bed of independent springs. The corresponding mechanical analogues to VE material ($1/2 \leq \beta \leq 1$) is represented by an indefinite shear type column resting on a bed of independent dashpots [1]. In all cases both stiffness and coefficients of viscosity decay with a power-law as the distance from the top of the columns increases.

In this paper a discretization of the aforementioned continuous models is presented. To this aim the discretized mechanical models representing EV and VE materials are investigated. In more details the mechanical models of EV materials may be represented as crushproof massless laminae interconnected by dashpots resting on a bed of independent springs. By contrast, in case of VE materials the crushproof laminae are interconnected by springs (like a shear type frame) resting on a bed of external viscous dashpots.

Since in both cases all mechanical characteristics decay with power-law from the top floor the equilibrium equations matrices of the discretized model have the same form for both EV and VE cases. It follows that the eigenvectors of the discretized model do not change. Decoupling of the differential equations, as in classical modal analysis, leads to a set of differential equations correspondent to Kelvin-Voigt units with different time scales.

Concepts exploited for the simpler case of fractional constitutive law may be extended to the case in which such a stress-strain relation is suited by a linear combination of fractional operators like for biological soft tissues and/or compact bones. Moreover cases with order of fractional operators β does not belong to interval [0, 1] may be dealt with the same concepts although without immediate mechanical analogues for $\beta > 3/2$. These issues will be addressed in future studies.

2 The mechanical description of FHM

In this section we introduce preliminarily some fundamental concepts about FHM.

The time-dependent behavior of FHM may be introduced starting from the so-called relaxation function $G(t)$ that represents the stress $\sigma(t)$ for assigned strain history $\gamma(t) = U(t)$, being $U(t)$ the unit step function. In virtue of Boltzmann superposition principle the stress-strain constitutive law is given as:

$$\sigma(t) = \int_0^t G(t-\tau)\, d\gamma(\tau) = \int_0^t G(t-\tau)\dot{\gamma}(\tau)\, d\tau \tag{1}$$

Equation (1) is valid for $\gamma(0) = 0$. If $\gamma(0) = \gamma_0 \neq 0$ then the additional contribution given as $G(t)\gamma_0$ has to be added in Eq. (1). The stress-strain relation described in Eq. (1) involves a convolution integral with kernel $G(t)$. In the context of FHM the functional class of $G(t)$ is of power-law type that reads:

$$G(t) = \frac{C(\beta)}{\Gamma(1-\beta)} t^{-\beta} \tag{2}$$

where $\Gamma(\cdot)$ is the Euler Gamma function, $C(\beta)/\Gamma(1-\beta)$ and β are parameters that depend of the materials at hand and may be evaluated by a proper fit of experimental results. Introducing Eq. (2) in Eq. (1) the stress-strain relation is obtained as:

$$\sigma(t) = C(\beta)\big({}_{\mathrm{C}}\mathrm{D}_{0^+}^{\beta}\gamma\big)(t) \tag{3}$$

where $\left({}_{\mathrm{C}}\mathrm{D}_{0+}^{\beta}\gamma\right)(t)$ is the Caputo's fractional derivative of order β (see Appendix). Advanced engineering materials such as biological polymer, foams and gels show $\beta \in]0,1[$ whereas $\beta = 0$ and $\beta = 1$ corresponds to pure solid and pure fluid materials, respectively.

As a dual consideration the reciprocal stress-strain relation may be obtained starting from the creep function $J(t)$, that represents the strain $\gamma(t)$ for the assigned stress history $\sigma(t) = U(t)$. The use of Boltzmann superposition principle leads to the stress-strain relation in the form:

$$\gamma(t) = \int_0^t J(t-\tau)\, d\sigma(\tau) = \int_0^t J(t-\tau)\dot{\sigma}(\tau)\, d\tau \tag{4}$$

that holds as the initial stress $\sigma(0) = 0$. If $\sigma(0) = \sigma_0 \neq 0$ then the additional contribution $J(t)\sigma_0$ has to be added in Eq. (4). Equation (4) is a convolution integral with kernel $J(t)$ and in Laplace domain an algebraic relation among the Laplace transform of relaxation $\hat{G}(s)$ and the Laplace transform of creep $\hat{J}(s)$ function exists as $\hat{G}(s)\hat{J}(s) = 1/s^2$ [20, 21]. It follows that as $G(t)$ is assigned as in Eq. (2), the corresponding creep function $J(t)$ may be easily obtained in the form:

$$J(t) = \frac{t^{\beta}}{C(\beta)\Gamma(1+\beta)}. \tag{5}$$

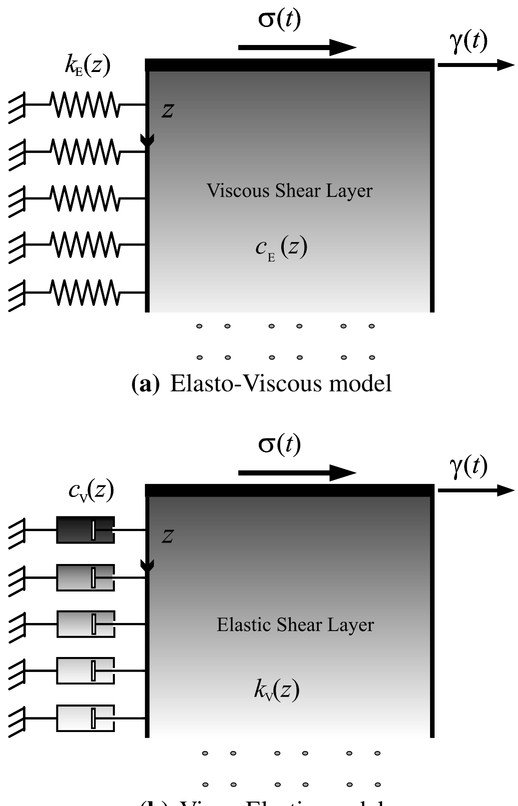
Substitution of such an expression in Eq. (4) yields:

$$\gamma(t) = \frac{1}{C(\beta)}\left(\mathrm{I}_{0+}^{\beta}\sigma\right)(t) \tag{6}$$

where $\left(\mathrm{I}_{0+}^{\beta}\sigma\right)(t)$ (see Appendix) is the Riemann-Liouville fractional integral of order β.

Inspection of Eqs. (3) and (6) reveals that, as soon as we assume that $J(t)$ (or $G(t)$) is of power-law type, then the constitutive law of the materials is ruled by fractional operators, so the name fractional hereditary materials.

In a previous study [1] it has been shown that, from a mechanical prospective, it must be distinguished among values of order $\beta = \beta_E \in [0, 1/2]$ and values of order $\beta = \beta_V \in [1/2, 1]$. Such a difference is reflected into the different mechanical models beyond β_E and β_V. In more details there are two different mechanical models that exactly restitute the stress-strain relation expressed in Eq. (3) or in Eq. (6). As



(a) Elasto-Viscous model

(b) Visco-Elastic model

Fig. 1 Continuous fractional models

$0 \le \beta = \beta_E \le 1/2$ the mechanical model is a massless indefinite fluid column resting on a bed of independent springs as shown in Fig. 1(a) and in this case is referred so elasto-viscous material. If, instead, $1/2 \le \beta = \beta_V \le 1$ the exact mechanical model is represented by indefinite massless shear-type column resting on a bed of independent dashpots as shown in Fig. 1(b), this model is referred to visco-elastic material.

The correspondence of these mechanical models and fractional order operators has been proved by introducing a z vertical axis as shown in Fig. 1 and denoting $\sigma(z,t)$ the shear stress (in the fluid or in the cantilever beam) and $\gamma(z,t)$ the normalized displacement field [1]. Moreover let $\sigma(0,t) = \sigma(t)$ and $\gamma(0,t) = \gamma(t)$ the stress applied on the top of the model and the corresponding strain, respectively. The stress-strain relation in Eq. (3) is captured by the stress $\sigma(t)$ on the upper lamina and its correspondent transverse displacement $\gamma(t)$ (normalized displacement at the top). All the mechanical characteristics, viscosity of fluid $c_E(z)$ and external stiffness $k_E(z)$ for the model in Fig. 1(a) ($0 \le \beta \le 1/2$) as well as shear modulus $k_V(z)$ and external viscous coefficient of external dashpots $c_V(z)$ for the model in Fig. 1(b) ($1/2 \le \beta \le 1$), vary along the z axis with power-law.

In more details we define G_0 and η_0 the reference values of the shear modulus and viscosity coefficient. For the EV materials ($\beta \in [0, 1/2]$) the stiffness and the viscous coefficients decay as:

$$k_E(z) = \frac{G_0}{\Gamma(1+\alpha)} z^{-\alpha}; \qquad c_E(z) = \frac{\eta_0}{\Gamma(1-\alpha)} z^{-\alpha} \tag{7}$$

with $0 \le \alpha \le 1$, whereas the VE materials ($\beta \in [1/2, 1]$) the mechanical characteristics of the model in Fig. 1(b) reads:

$$k_V(z) = \frac{G_0}{\Gamma(1-\alpha)} z^{-\alpha}; \qquad c_V(z) = \frac{\eta_0}{\Gamma(1+\alpha)} z^{-\alpha}. \tag{8}$$

The governing equation of the continuous model depicted in Fig. 1(a) is written as:

$$\frac{\partial}{\partial z}\left[c_E(z)\frac{\partial \dot{\gamma}(z,t)}{\partial z}\right] = k_E(z)\gamma(z,t) \tag{9}$$

while the equilibrium equation of the continuous model depicted in Fig. 1(b) is written as:

$$\frac{\partial}{\partial z}\left[k_V(z)\frac{\partial \gamma(z,t)}{\partial z}\right] = c_V(z)\dot{\gamma}(z,t). \tag{10}$$

The solution of the differential equations in (9) and (10) can be solved by Laplace transform. In this way the solution $\hat{\gamma}(z,s)$ in Laplace domain involves the modified first and second kind Bessel functions, denoted respectively with $Y_\beta(\cdot)$ and $K_\beta(\cdot)$, in particular we obtain for EV case:

$$\hat{\gamma}(z,s) = z^\beta\left[B_{E1}Y_\beta\left(\tfrac{z}{\sqrt{\tau_E(\alpha)s}}\right) + B_{E2}K_\beta\left(\tfrac{z}{\sqrt{\tau_E(\alpha)s}}\right)\right] \tag{11}$$

with $\tau_E(\alpha) = -\eta_0\Gamma(\alpha)/(\Gamma(-\alpha)G_0)$ and $\beta = (1-\alpha)/2$; while for VE case we have:

$$\hat{\gamma}(z,s) = z^\beta\left[B_{V1}Y_\beta\left(z\sqrt{\tau_E(\alpha)s}\right) + B_{V2}K_\beta\left(z\sqrt{\tau_E(\alpha)s}\right)\right] \tag{12}$$

with $\tau_V(\alpha) = -\eta_0\Gamma(-\alpha)/(\Gamma(\alpha)G_0)$ and $\beta = (1+\alpha)/2$. The constants of integration B_{Ei} and B_{Vi} with $i = 1, 2$ are obtained by imposing the following pairs of boundary conditions, for the EV and VE case respectively:

$$\text{(EV)} \quad \begin{cases} \lim_{z\to 0} c_E(z)\dfrac{\partial \dot{\gamma}(z,t)}{\partial z} = \sigma(0,t) = \sigma(t), \\ \lim_{z\to\infty} \gamma(z,t) = 0 \end{cases} \tag{13a}$$

$$\text{(VE)} \quad \begin{cases} \lim_{z\to 0} k_V(z)\dfrac{\partial \gamma(z,t)}{\partial z} = \sigma(0,t) = \sigma(t), \\ \lim_{z\to\infty} \gamma(z,t) = 0 \end{cases} \tag{13b}$$

and by making the inverse Laplace transform we obtain the fractional stress-strain relation in Eq. (6), that is to say:

$$\gamma(t) = \frac{1}{C_E(\beta)}\left(\mathrm{I}^\beta_{0^+}\sigma\right)(t) \quad \text{(EV)} \tag{14a}$$

$$\gamma(t) = \frac{1}{C_V(\beta)}\left(\mathrm{I}^\beta_{0^+}\sigma\right)(t) \quad \text{(VE)} \tag{14b}$$

where the coefficients $C_E(\beta)$ and $C_V(\beta)$ are defined as:

$$C_E(\beta) = \frac{G_0\,\Gamma(\beta)2^{2\beta-1}}{\Gamma(2-2\beta)\Gamma(1-\beta)}\left(\tau_E(\alpha)\right)^\beta, \; 0 \le \beta \le 1/2 \tag{15a}$$

$$C_V(\beta) = \frac{G_0\,\Gamma(1-\beta)\,2^{1-2\beta}}{\Gamma(2-2\beta)\Gamma(\beta)}\left(\tau_V(\alpha)\right)^\beta, \; 1/2 \le \beta \le 1. \tag{15b}$$

At this point we observe that, if the boundary condition applied to the top layer of the model (see Eqs. (13a) and (13b)), respectively, for EV and VE materials, involves Dirichlet specifics, as:

$$\begin{cases} \lim_{z\to 0} \gamma(z,t) = \gamma(t), \\ \lim_{z\to\infty} \gamma(z,t) = 0. \end{cases} \tag{16}$$

The evaluation of the stress at the top lamina in terms of the transverse displacement field yields:

$$\begin{aligned} \sigma(t) &= C_E(\beta)\left({}^C D^\beta_{0^+}\gamma\right)(t) \quad \text{(EV)} \\ \sigma(t) &= C_V(\beta)\left({}^C D^\beta_{0^+}\gamma\right)(t) \quad \text{(VE)} \end{aligned} \tag{17}$$

as reported in [1].

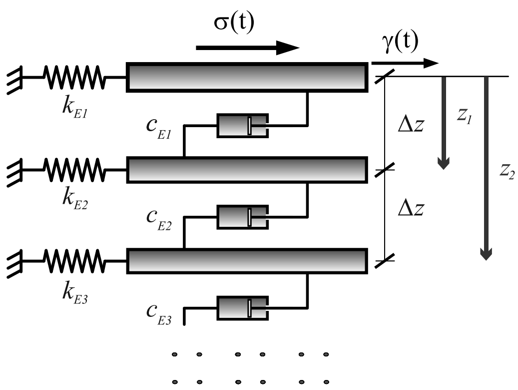


Fig. 2 Discretized counterpart of the continuous model Fig. 1(a) EV case

3 Discretization of FHM

The mechanical representation of fractional order operators discussed in previous section may be used to introduce a discretization scheme that corresponds to evaluate fractional derivative. The two cases corresponding to $\beta \in [0, 1/2]$ and $\beta \in [1/2, 1]$ will be analyzed in this section.

3.1 The discretized model of EV material

By introducing a discretization of the z-axis as $z_j = j\triangle z$ into to the governing equation of the EV material in Eq. (9) yields a finite difference equation of the form:

$$\frac{\triangle}{\triangle z}\left[c_E(z_j)\frac{\triangle\dot{\gamma}(z_j,t)}{\triangle z}\right] = k_E(z_j)\gamma(z_j,t) \tag{18}$$

so that, denoting $k_{Ej} = k_E(z_j)\triangle z$ and $c_{Ej} = c_E(z_j)/\triangle z$ the continuous model is discretized into a dynamical model constituted by massless shear layers, with horizontal degrees of freedom $\gamma(z_j,t) = \gamma_j(t)$, that are mutually interconnected by linear dashpots with viscosity coefficients c_{Ej} and resting on a bed of independent linear springs k_{Ej}.

The stiffness coefficient k_{Ej} and the viscosity coefficient c_{Ej} reads:

$$k_{Ej} = \frac{G_0}{\Gamma(1+\alpha)} z_j^{-\alpha}\triangle z; \qquad c_{Ej} = \frac{\eta_0}{\Gamma(1-\alpha)}\frac{z_j^{-\alpha}}{\triangle z} \tag{19}$$

with $\alpha = 1 - 2\beta$.

The equilibrium equations of the generic shear layer of the model read:

$$\begin{cases} k_{E0}\gamma_1(t) - c_{E0}\triangle\dot{\gamma}_1(t) = \sigma(t), \\ k_{Ej}\gamma_j(t) + c_{Ej-1}\triangle\dot{\gamma}_{j-1}(t) - c_{Ej}\triangle\dot{\gamma}_j(t) = 0, \\ \quad j = 1, 2, \ldots, \infty \end{cases} \tag{20}$$

where $\gamma_1(t) = \gamma(t)$ and $\triangle\dot{\gamma}_j(t) = \dot{\gamma}_{j+1}(t) - \dot{\gamma}_j(t)$. By inserting Eqs. (19) in Eqs. (20), at the limit as $\triangle z \to 0$, the discrete model reverts to Eq. (9). That is the discretized model presented in Fig. 2 represents a proper discretization of the continuous EV counterpart.

As soon as z increase $\gamma(z,t)$ decay and $\lim_{z\to\infty}\gamma(z,t) = 0$ it follows that only a certain number, say n, of equilibrium equation may be accounted for the analysis. It follows that the system in Eqs. (20) may be rewritten in the following compact form:

$$p_E\mathbf{A}\dot{\boldsymbol{\gamma}} + q_E\mathbf{B}\boldsymbol{\gamma} = \mathbf{v}\sigma(t) \tag{21}$$

where:

$$p_E = \frac{\eta_0}{\Gamma(1-\alpha)}\triangle z^{-(1+\alpha)}; \qquad q_E = \frac{G_0}{\Gamma(1+\alpha)}\triangle z^{1-\alpha}. \tag{22}$$

In Eq. (21):

$$\boldsymbol{\gamma}^T = [\gamma_1(t)\ \gamma_2(t)\ \ldots\ \gamma_n(t)]; \quad \mathbf{v}^T = [1\ 0\ 0\ \ldots\ 0] \tag{23}$$

where the apex T means transpose. The coefficient matrices **A** and **B** are given as:

$$\mathbf{A} = \begin{bmatrix} 1^{-\alpha} & -1^{-\alpha} & \ldots & 0 \\ -1^{-\alpha} & 1^{-\alpha}+2^{-\alpha} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (n-1)^{-\alpha}+n^{-\alpha} \end{bmatrix} \tag{24}$$

$$\mathbf{B} = \begin{bmatrix} 1^{-\alpha} & 0 & 0 & \ldots & 0 \\ 0 & 2^{-\alpha} & 0 & \ldots & 0 \\ 0 & 0 & 3^{-\alpha} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & n^{-\alpha} \end{bmatrix}. \tag{25}$$

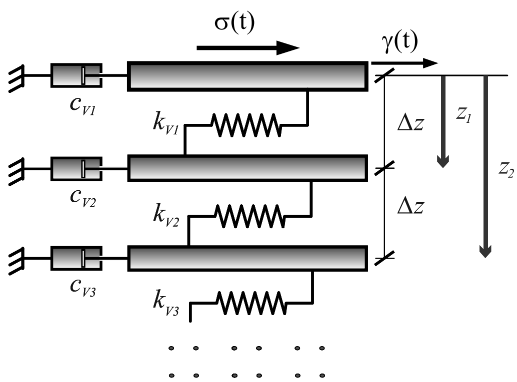


Fig. 3 Discretized counterpart of the continuous model Fig. 1(b) VE case

The matrices **A** and **B** are symmetric and positive definite (in particular **B** is diagonal) and they may be easily constructed for an assigned value of α (depending of the derivative order β) and for a fixed truncation order n. Moreover Eq. (21) may now be easily integrated by using standard tools of dynamic analysis how it will be shown later on.

3.2 The discretized model of VE material

As the fractional order derivative is $\beta = \beta_V \in [1/2, 1]$ the mechanical description of the material is the represented by the continuous model depicted in Fig. 3 and ruled by Eq. (10).

By introducing a discretization of the z-axis in intervals $\triangle z$ in governing equation of the VE materials in Eq. (10) yields a finite difference equation of the form:

$$\frac{\triangle}{\triangle z}\left[k_V(z_j)\frac{\triangle\gamma(z_j,t)}{\triangle z}\right] = c_V(z_j)\dot{\gamma}(z_j,t) \tag{26}$$

that corresponds to a discretized mechanical representation of fractional derivatives. The mechanical model is represented by a set of massless shear layers with state variables $\gamma(z_j,t) = \gamma_j(t)$ that are mutually interconnected by linear springs with stiffness $k_{Vj} = k_V(z_j,t)/\triangle z$ resting on a bed of independent linear dashpots with viscosity coefficient $c_{Vj} = c_V(z_j,t)\triangle z$. Springs and dashpots are given as:

$$k_{Vj} = \frac{G_0}{\Gamma(1-\alpha)}\frac{z_j^{-\alpha}}{\triangle z}; \qquad c_{Vj} = \frac{\eta_0}{\Gamma(1+\alpha)}z_j^{-\alpha}\triangle z \tag{27}$$

with $\alpha = 2\beta - 1$.

The set of equilibrium equations reads:

$$\begin{cases} c_{V0}\dot{\gamma}_1 - k_{V0}\triangle\gamma_1 = \sigma(t), \\ c_{Vj}\dot{\gamma}_j + k_{Vj-1}\triangle\gamma_{j-1} - k_{Vj}\triangle\gamma_j = 0, \\ \quad j = 1, 2, \ldots, \infty. \end{cases} \tag{28}$$

So that, accounting for the contribution of the first n shear layers the differential equation system may be written as:

$$p_V\mathbf{B}\dot{\boldsymbol{\gamma}} + q_V\mathbf{A}\boldsymbol{\gamma} = \mathbf{v}\sigma(t) \tag{29}$$

where:

$$p_V = \frac{\eta_0}{\Gamma(1+\alpha)}\triangle z^{1-\alpha}; \qquad q_V = \frac{G_0}{\Gamma(1-\alpha)}\triangle z^{-(1+\alpha)} \tag{30}$$

while $\boldsymbol{\gamma}$, $\mathbf{v}$ and the matrices **A** and **B** have already been defined in Sect. 3.1.

Up to now we considered a shear stress and a subsequent shear deformation as in the Couette problem, but exact governing equations, and then exact mechanical models, for the axial stress and axial deformation are the same how is depicted in Fig. 4 for both EV and VE case. Also in this case as $\Delta z \to 0$ (continuous problem) both the fractional EV and VE continuous are restored.

In the next section the modal analysis of dynamical system ruled by Eq. (21) and Eq. (29) will be performed leading to a set of decoupled system of first order differential equations.

4 Modal analysis of the discretized models

The observations reported in previous section lead to conclude that, whatever class of FHM material is considered, the time-evolution of the material system may be obtained by the introduction of a proper set of inner state variables, collected in the vector $\boldsymbol{\gamma}(t)$ and ruled by a set first-order linear differential equations. In this perspective the mechanical response of the FHM may be obtained in terms of the vector $\boldsymbol{\gamma}(t)$ by means of the decoupling set of eigenmodes of the differential equations system reported in Eq. (20) for EV materials or in Eq. (28) for VE materials. Since the different mechanical models correspond to Elasto-Viscous

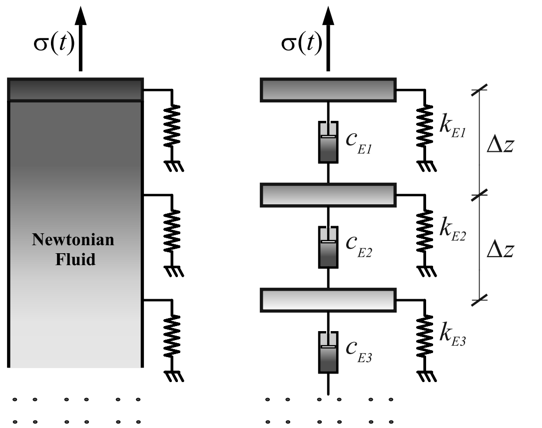


Fig. 4 Continuous and discretized fractional axial models

and Visco-Elastic materials, as discussed in previous section, these two cases will be dealt separately in the following.

4.1 The case of Elasto-Viscous (EV) materials

In this section the governing equation is reported in Eq. (21). As customary we first solve the homogeneous case, that is as $\sigma(t) = 0$. We introduce a coordinate transformation in the form:

$$\mathbf{B}^{1/2}\boldsymbol{\gamma} = \mathbf{x} \tag{31}$$

and premultiplying by $\mathbf{B}^{-1/2}$ a differential equation for the unknown vector $\mathbf{x}$ is obtained as:

$$p_E \mathbf{D}\dot{\mathbf{x}} + q_E \mathbf{x} = \tilde{\mathbf{v}}\sigma(t) \tag{32}$$

where $\tilde{\mathbf{v}} = \mathbf{B}^{-1/2}\mathbf{v}$ and $\mathbf{D}$ is the dynamical matrix $\mathbf{D} = \mathbf{B}^{-1/2}\mathbf{A}\,\mathbf{B}^{-1/2}$ given as:

$$\mathbf{D} = \begin{bmatrix} 1 & -(\frac{2}{1})^{\frac{\alpha}{2}} & 0 & \dots & 0 \\ -(\frac{2}{1})^{\frac{\alpha}{2}} & 1+(\frac{2}{1})^{\alpha} & -(\frac{3}{2})^{\frac{\alpha}{2}} & \dots & 0 \\ 0 & -(\frac{3}{2})^{\frac{\alpha}{2}} & 1+(\frac{3}{2})^{\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1+(\frac{n}{n-1})^{\alpha} \end{bmatrix} \tag{33}$$

that is $\mathbf{D}$ is symmetric and positive definite and it may be obtained straightforwardly once n and α are fixed. Let $\boldsymbol{\Phi}$ be the modal matrix whose columns are the orthonormal eigenvectors of $\mathbf{D}$ that is:

$$\boldsymbol{\Phi}^T \mathbf{D}\boldsymbol{\Phi} = \boldsymbol{\Lambda}; \qquad \boldsymbol{\Phi}^T\boldsymbol{\Phi} = \mathbf{I} \tag{34}$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\Lambda}$ is the diagonal matrix collecting the eigenvalues $\lambda_j > 0$ of $\mathbf{D}$.

In the following we order λ_j in such a way that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. As we indicate $\mathbf{y}(t)$ the modal coordinate vector, defined as:

$$\mathbf{x}(t) = \boldsymbol{\Phi}\mathbf{y}(t); \qquad \mathbf{y}(t) = \boldsymbol{\Phi}^T\mathbf{x}(t) \tag{35}$$

and we substitute in Eq. (32) a decoupled set of differential equation is obtained in the form:

$$p_E \boldsymbol{\Lambda}\,\dot{\mathbf{y}} + q_E \mathbf{y} = \bar{\mathbf{v}}\sigma(t) \tag{36}$$

where $\bar{\mathbf{v}} = \boldsymbol{\Phi}^T\tilde{\mathbf{v}} = \boldsymbol{\Phi}^T\mathbf{B}^{-1/2}\mathbf{v} = \boldsymbol{\Phi}^T\mathbf{v}$. The jth equation of Eq. (36) reads:

$$\dot{y}_j + \rho_j y_j = \frac{\phi_{1,j}}{p_E\lambda_j}\sigma(t); \qquad j = 1, 2, 3, \dots, n \tag{37}$$

where $\rho_j = q_E/p_E\lambda_j > 0$ and $\phi_{1,j}$ is the jth element of the first row of the matrix $\boldsymbol{\Phi}$. Equations (37) represent a decoupled set of Kelvin-Voigt units, as is shown in Fig. 5, and the solution of Eq. (37) is provided in the form:

$$y_j(t) = y_j(0)\,e^{-\rho_j t} + \frac{\phi_{1,j}}{p_E\lambda_j}\int_0^t e^{-\rho_j(t-\tau)}\sigma(\tau)\,d\tau \tag{38}$$

where $y_j(0)$ is the jth component of the vector $\mathbf{y}(0)$ related to the vector of initial conditions $\boldsymbol{\gamma}(0)$ as:

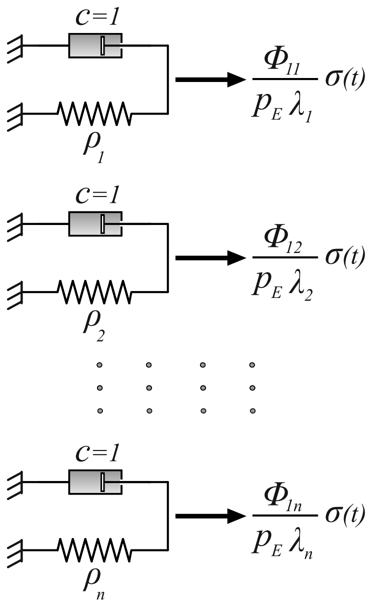


Fig. 5 Kelvin-Voigt elements in modal space (EV case)

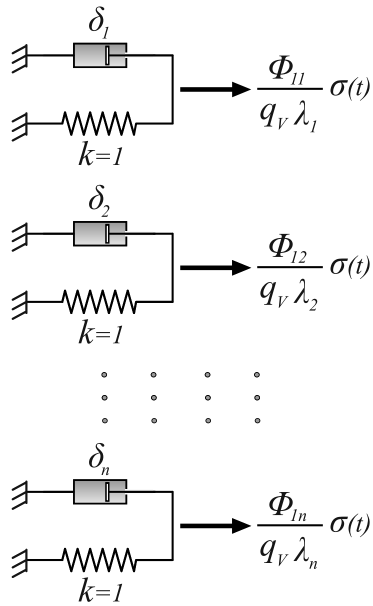


Fig. 6 Kelvin-Voigt elements in modal space (VE case)

$$\mathbf{y}(0) = \boldsymbol{\Phi}^T \mathbf{B}^{1/2} \boldsymbol{\gamma}(0). \tag{39}$$

Solution of the differential equation system in Eq. (21) may be obtained as the modal vector $\mathbf{y}(t)$ has been evaluated by solving Eq. (38) with the aid of Eqs. (31) and (35) as:

$$\boldsymbol{\gamma}(t) = \mathbf{B}^{-1/2} \boldsymbol{\Phi} \mathbf{y}(t). \tag{40}$$

As we are interested to a relation among the shear stress and the normalized transverse displacement of the upper lamina we must evaluate the first element of vector $\boldsymbol{\gamma}(t)$ that is obtained as:

$$\gamma(t) = \mathbf{v}^T \boldsymbol{\gamma}(t). \tag{41}$$

4.2 The case of Visco-Elastic (VE) materials

Modal analysis of the differential equations system representing the behavior of VE material is quite similar to previous section. In this case we substitute Eq. (31) in Eq. (29) and we perform left premultiplication by $\mathbf{B}^{-1/2}$ that reads:

$$p_V \dot{\mathbf{x}} + q_V \mathbf{D} \mathbf{x} = \tilde{\mathbf{v}} \sigma(t) \tag{42}$$

where $\mathbf{D}$ is the dynamical matrix defined in previous section. The dynamical equilibrium equation in modal coordinate reads:

$$p_V \dot{\mathbf{y}} + q_V \boldsymbol{\Lambda} \mathbf{y} = \bar{\mathbf{v}} \sigma(t) \tag{43}$$

so that equilibrium of jth Kelvin-Voigt represented by Eq. (43) is given as:

$$\delta_j \dot{y}_j + y_j = \frac{\phi_{1,j}}{q_V \lambda_j} \sigma(t); \quad j = 1, 2, 3, \ldots n \tag{44}$$

where $\delta_j = p_v / q_V \lambda_j > 0$. In this case the problem in the modal coordinates is decomposed in a set of Kelvin-Voigt units as shown in Fig. 6.

The solution in terms of modal coordinates are obtained in integral form as:

$$y_j(t) = y_j(0) e^{-t/\delta_j} + \frac{\phi_{1,j}}{\delta_j q_V \lambda_j} \int_0^t e^{-(t-\tau)/\delta_j} \sigma(\tau) \, d\tau. \tag{45}$$

The stress-strain relations between shear stress $\sigma(t)$ and normalized displacement $\gamma(t)$ may be obtained as in previous section (see Eqs. (38) and (41)).

The case of $\beta = 1/2$, that is common to both EV and VE mechanical analogues, is a critical value and some additional considerations may be withdrawn from its analysis as it will be shown in the next section.

4.3 The critical value of $\beta : \beta = 1/2$

The case of $\beta = 1/2$ is of particular interest since eigenvalues and eigenvectors are given in closed form.

It follows that the role played by the truncation depth of z and the number of laminae may be evidenced.

The case $\beta = 1/2$ may be treated in two different ways, or by assuming $\beta = 1/2$ starting from the EV case, or by assuming $\beta = 1/2$ starting from the VE case. Starting from the EV-model we get:

$$k_{Ej} = G_0 \triangle z; \qquad c_{Ej} = \frac{\eta_0}{\triangle z} \tag{46}$$

consequently the Eqs. (22) take the following form:

$$p_E = \frac{\eta_0}{\triangle z}; \qquad q_E = G_0 \triangle z \tag{47}$$

and the equilibrium equation system in compact form, similarly to the Eq. (21), reads:

$$\frac{\eta_0}{\triangle z} \mathbf{A} \dot{\boldsymbol{\gamma}} + G_0 \triangle z \, \mathbf{B} \boldsymbol{\gamma} = \mathbf{v} \sigma(t) \tag{48}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ assume the following particular form:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix} \tag{49}$$

and

$$\mathbf{B} = \mathbf{I}. \tag{50}$$

The eigenvalues λ_j and the normalized eigenvectors $\boldsymbol{\phi}_j$ of particular tridiagonal matrix $\mathbf{A}$ may be found in [22, 23] and they are reported below:

$$\lambda_j = 2 - 2\cos\left(\frac{2j-1}{2n+1}\pi\right), \qquad j = 1, 2, \dots, n, \tag{51}$$

$$\phi_{k,j} = \sqrt{\frac{4}{2n+1}} \cos\left[\frac{(2j-1)(2k-1)}{2(2n+1)}\pi\right], \quad j,k = 1, 2, \dots, n. \tag{52}$$

Using the Eq. (52) can be easily calculate the modal matrix $\boldsymbol{\Phi}$, obtaining the following equation in the modal space:

$$\frac{\eta_0}{\triangle z} \boldsymbol{\Lambda} \dot{\mathbf{y}} + G_0 \triangle z \, \mathbf{y} = \bar{\mathbf{v}} \sigma(t) \tag{53}$$

where $\bar{\mathbf{v}} = \boldsymbol{\Phi}^T \mathbf{v}$ and the jth-equation of the system (53), corresponding to the equilibrium equation of the jth-Kelvin-Voigt unit in the modal space, reads:

$$\dot{y}_j + \frac{G_0 \triangle z^2}{\eta_0 \lambda_j} y_j = \frac{\phi_{1,j} \triangle z}{\eta_0 \lambda_j} \sigma(t), \qquad j = 1, 2, \dots, n. \tag{54}$$

The solution of jth-equation in modal space of **EV**-model is given as:

$$y_j(t) = y_j(0)\, e^{-\frac{G_0 \triangle z^2}{\eta_0 \lambda_j} t} + \frac{\phi_{1,j} \triangle z}{\eta_0 \lambda_j} \int_0^t e^{-\frac{G_0 \triangle z^2}{\eta_0 \lambda_j}(t-\tau)} \sigma(\tau)\, d\tau \tag{55}$$

and the normalized transverse displacement of the upper lamina is obtained as:

$$\gamma(t) = \mathbf{v}^T \boldsymbol{\Phi}\, \mathbf{y}(t) = \sum_{j=1}^{n} \left[\phi_{1,j} y_j(0)\, e^{-\frac{G_0 \triangle z^2}{\eta_0 \lambda_j} t} + \frac{\phi_{1,j}^2 \triangle z}{\eta_0 \lambda_j} \int_0^t e^{-\frac{G_0 \triangle z^2}{\eta_0 \lambda_j}(t-\tau)} \sigma(\tau)\, d\tau \right]. \tag{56}$$

Exactly the same result is achieved as we work out on the governing equation of the VE material behavior reported in Sect. 3.2.

5 Numerical examples

In this section two numerical applications will be presented for different load histories and different values of β: (i) A constant stress case $\sigma(t) = \sigma_0 U(t)$; (ii) A sinusoidal variation $\sigma(t) = \sigma_0 \sin(\omega t) U(t)$. For both cases fractional integral in Eq. (6) may be evaluated in closed form and, for case (i), it returns the creep function given in Eq. (5); for case (ii) the result of fractional integral is:

$$\gamma(t) = \frac{\sigma_0 \omega t^{1+\beta}}{C(\beta)\Gamma(2+\beta)} {}_1F_2 \times \left[1; \frac{2+\beta}{2}, \frac{3+\beta}{2}; -\left(\frac{\omega t}{2}\right)^2\right] \tag{57}$$

where ${}_1F_2(\cdot)$ is the hypergeometric function.

In Fig. 7 the results for $\sigma(t) = \sigma_0 U(t)$ and different values of $\beta \in [0, 1/2]$ (EV case) are contrasted with

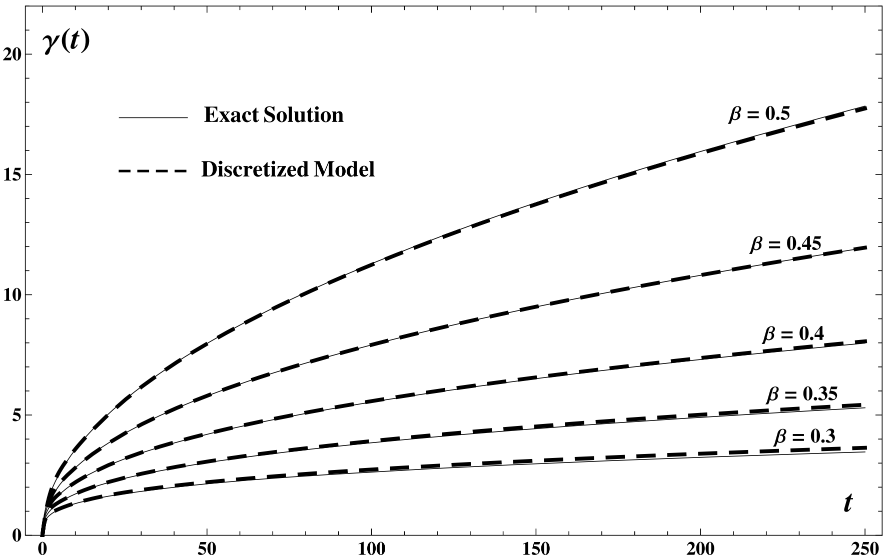


Fig. 7 Creep test of EV model with $\sigma_0 = 1$: comparison between the exact and approximate solution

exact solution reported in Eq. (6). For this case we select $\triangle z = 0.001$ and $n = 1500$, where $\triangle z$ is the discretization step and n is the number of layers considered (total truncation depth is $\bar{h} = n\triangle z = 1.5$), moreover it is assumed $G_0 = 1$ and $\eta_0 = 1$.

In Fig. 8 the results for $\sigma(t) = \sigma_0 U(t)$ and different values of $\beta \in [1/2, 1]$ (VE case) are contrasted with exact solution in Eq. (6), the discretization step selected is $\triangle z = 0.02$, $n = 1500$ (total depth $\bar{h} = 3$). It may be observed that exact solutions are matched with that obtained by the discretization procedure for a large time, moreover for $\beta = 1/2$ the solution obtained with EV case exactly coalesces with that obtained for VE case.

In order to investigate further the role played by truncation depth $\bar{h} = n\triangle z$ and by the number of laminae n, the critical case $\beta = 1/2$ is addressed. The creep function obtained with the discretized model reads:

$$\gamma(t) = \sum_{j=1}^{n}\left[\frac{\phi_{1,j}^2 \triangle z \sigma_0}{\eta_0 \lambda_j}\int_0^t e^{-\frac{G_0 \triangle z^2}{\eta_0 \lambda_j}(t-\tau)} U(\tau) d\tau\right]$$

$$= \frac{\sigma_0 n}{G_0 \bar{h}} \sum_{j=1}^{n}\left[\phi_{1,j}^2 \left(1 - e^{\frac{G_0 \bar{h}^2}{\eta_0 \lambda_j n^2} t}\right)\right]. \tag{58}$$

Equation (58) shows that the discretized model provides a solution in terms of the sum of exponentials. The sum, tends asymptotically, by increasing the observation time, to the following limit:

$$\lim_{t\to\infty} \gamma(t) = \frac{\sigma_0 n}{G_0 \bar{h}} \sum_{j=1}^{n} \phi_{1,j}^2 = a(n, \bar{h}) \tag{59}$$

while the exact solution, given by Eq. (5) is the power-law type solution:

$$\gamma(t) = \sigma_0 J(t) = \frac{\sigma_0}{G_0 \Gamma(1.5)}\left(\frac{G_0}{\eta_0} t\right)^{0.5} \tag{60}$$

showing that Eq. (60) does not denote an asymptotic behavior for $t \to \infty$. It follows that Eq. (58) is able to represent the exact solution only for a certain interval of time t^* that depends on the number of laminae as well as on the depth $\bar{h}$ selected for the analysis.

In order to predict a reference time t^* such that the solution obtained by the discretization model and the exact one are nearly coincident, we may refer to Fig. 9 where the asymptotic value $a(n, \bar{h})$, the discretized solution and that obtained by creep function are reported. We observe that at time $\bar{t}_{n,\bar{h}}$ the exact solution reaches the asymptotic one. In more details placing the equality of the Eq. (60) and the asymptotic value $a(n, h)$ we obtain the following limit time $\bar{t}_{n,\bar{h}}$:

$$\bar{t}_{n,\bar{h}} = \left(\frac{\Gamma(1.5) a(n, \bar{h})}{\sigma_0}\right)^2 \eta_0 G_0. \tag{61}$$

The solution obtained by discretization is always smaller than its asymptotic value. It follows that the

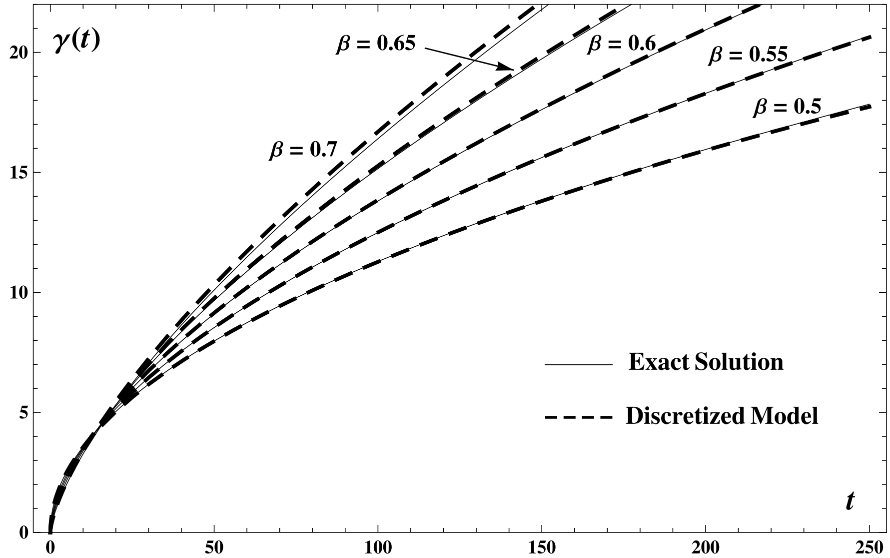


Fig. 8 Creep test of VE model with $\sigma_0 = 1$: comparison between the exact and approximate solution

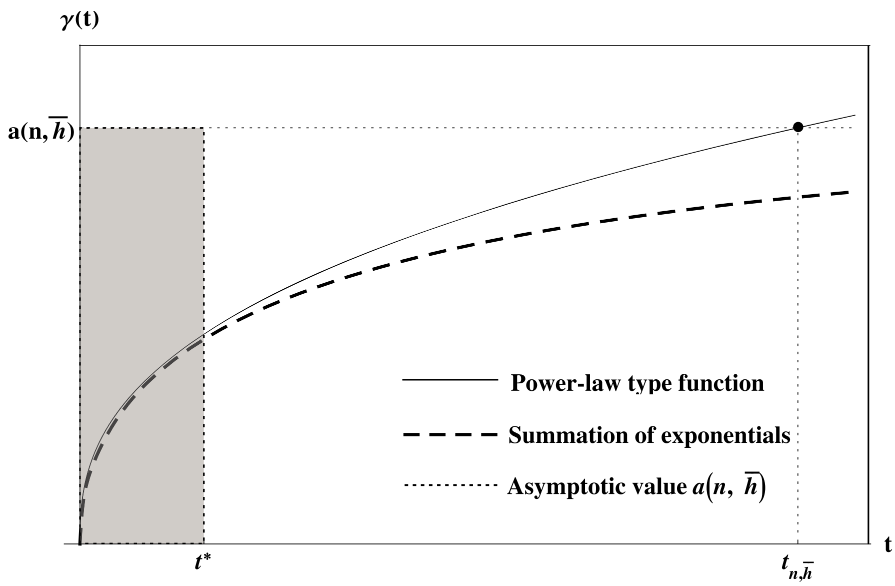


Fig. 9 Comparison between power-law function and summation of exponentials

reference time t^*, at which the response is well approximate, may be expressed in the form:

$$t^* = \nu \bar{t}_{n,\bar{h}} \tag{62}$$

with $\nu \ll 1$. From these observations follows that the larger is the observation time the more the number of layers n and the larger truncation depth $\bar{h}$ will be.

The second considered case consists in forcing the upper lamina with the stress history $\sigma(t) = \sigma_0 \sin(\omega t) U(t)$. The time-variation of the transverse displacement $\gamma(t)$ has been contrasted in Fig. 10 and Fig. 11, respectively for EV and VE case, with the exact representation of fractional integral expressed in Eq. (57).

The discretization of both models has been obtained with $\triangle z = 0.01$ and $n = 1000$.

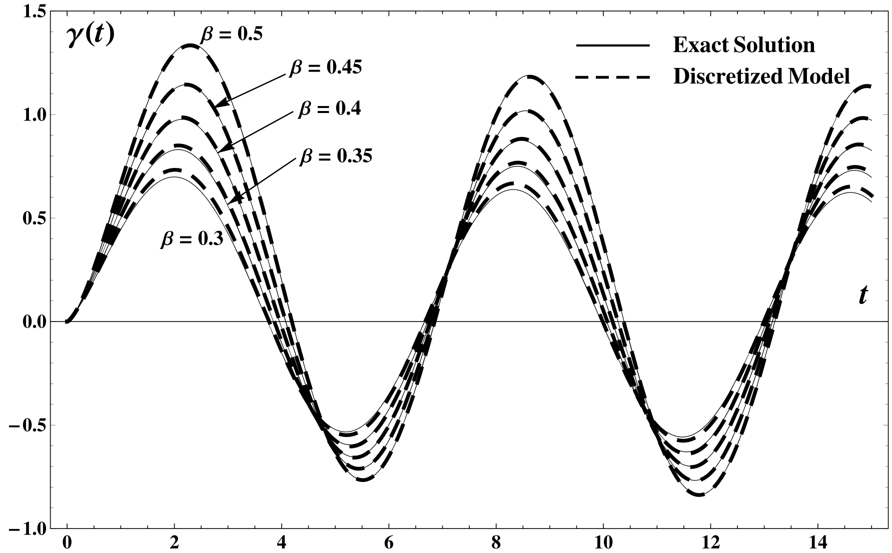


Fig. 10 Harmonic test of EV model with $\omega = 1$ and $\sigma_0 = 1$: comparison between the exact and approximate solution

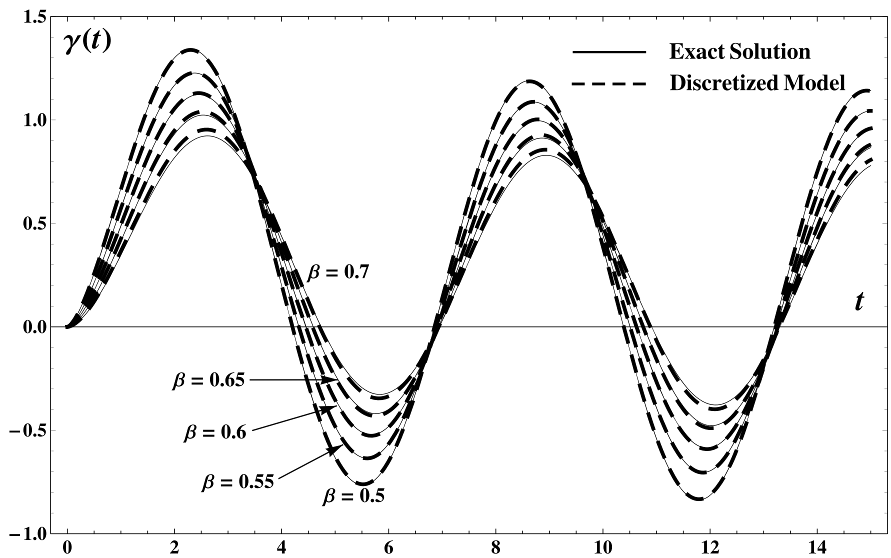


Fig. 11 Harmonic test of VE model with $\omega = 1$ and $\sigma_0 = 1$: comparison between the exact and approximate solution

Also in this case exact solution obtained by Eq. (57) and that obtained by the discretized model fit very well each another for all value of β exponent.

6 Conclusions

Fractional hereditary constitutive laws are characterized by fractional operators of order $\beta : 0 \leq \beta \leq 1$. The order β may be found by creep or relaxation test that exhibit a power-law with exponent β. Two different continuous mechanical models leading to fractional stress-strain relation have been found. For $0 \leq \beta \leq 1/2$ the predominant behavior is the elastic one as compared with the viscous behavior. Its mechanical correspondent is a massless undefined column of Newtonian fluid resting on a bed of independent springs.

For $1/2 \leq \beta \leq 2$ the viscous properties are predominant with respect to the elastic ones. Its mechanical correspondent is a massless undefined shear column resting on a bed of independent dashpots. In both cases the various coefficients decay with a power-law that is related to the characteristic value of β.

Discretization leads to an infinite set of coupled differential equations ruled by symmetric and positive definite matrices. It has been shown that:

(i) The matrices ruling the evolution of the discretized systems have the same form for the two models (EV and VE cases).
(ii) Truncation of the depth of the column leads to a finite set of coupled differential equations for which the modal analysis may be easily performed.
(iii) The eigenvectors for a fixed value of $\beta : 0 \leq \beta \leq 1/2$ and $\bar{\beta} = \beta + 0.5$ are the same as we assume for the two models the same truncation depth and the same discretization of the depth.
(iv) In both cases decoupled system are a set of Kelvin-Voigt elements. It follows that truncation leads to a finite asymptotic value for $\sigma(t) = \sigma_0 U(t)$.
(v) The case $\beta = 1/2$ may be obtained either from the mechanical system with $0 \leq \beta \leq 1/2$ or by that for $1/2 \leq \beta \leq 1$. For such a system the characteristic coefficients like viscosity or stiffness along the column remain constant. For such a particular case the eigenvectors and eigenvalues are given in analytical form once the discretization step along the column is fixed.

As a conclusion, truncation of the depth of the columns and discretization lead always to a set of Kelvin-Voigt elements. It follows that discretization always corresponds to asymptotic value in contrast with the power-law creep function of FHM. As a consequence, discretized model produces accurate results only for prescribed observation time. However numerical examples performed for different values of β and different forcing functions produce accurate results in a reasonable long time and then the procedure outlined in the paper may be used for fractional calculations.

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Appendix: Fractional calculus

In this appendix we introduce some fundamental concepts on fractional calculus.

The fractional calculus is the natural extension of ordinary differential calculus. In fact, it extends the concepts of derivation and integration to non-integer and complex order.

The fractional calculus was born in the 1695 when *G.W. Leibniz* introduced the half derivate concept in a note to *G. de l'Hôpital* [24]. Subsequent studies have focused by different mathematicians [25]: *J.B.J. Fourier*, *P.S. Laplace*, *L. Euler*, *S.F. Lacorix*, *N.H. Abel*, etc.

The first definition of fractional operator is probably attributable to *J. Liouville*, who in 1832 gave the impulse to research by formulating the definition of fractional derivative of exponential function. In 1847, an important contribution was given by *G.F.B. Riemann*, who introduced their own definition of fractional integral. Following, *N.Ya. Sonin* unified formulations of Liouville and Riemann from multiple Cauchy integration formula [25], obtaining the following expression of fractional integral:

$$\left(\mathrm{I}_{a^+}^{\beta} f\right)(t) = \frac{d^{-\beta} f(t)}{d(t-a)^{-\beta}} = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} f(\tau)\, d\tau \tag{A.1}$$

Equation (A.1) is known in literature as a *fractional integral of Riemann-Liouville*, since $\Re(\beta) > 0$, and it is valid for $\beta \in \mathbb{C}$.

To obtain the *Riemann-Liouville fractional derivative* just think that the derivative of order n can be considered as the derivative of order $n + m$ of the mth primitive function, and then generalizing, we have:

$$\left(\mathrm{D}_{a^+}^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}\, d\tau \tag{A.2}$$

valid for $(n-1) < \Re(\beta) < n$.

Another definition of fractional integro-differential operator was provided in 1967 by *M. Caputo* [26]. This definition is easier to handle for the solution of physical problems. The *Caputo fractional derivative* has the following expression:

$$\left({}_{\mathrm{C}}\mathrm{D}_{a^+}^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta+1-n}}\, d\tau \quad \text{(A.3)}$$

Equation (A.3) is valid for $(n-1) < \beta < n$. The expression obtained is the result of an interpolation between the integer order derivatives, in fact, for $\beta \to n$, the expression becomes an nth derivative of $f(t)$.

It can be observed that the expressions (A.3) and (A.2) coincide if we start from initial conditions zero ($f(a) = 0$).

Another definition of fractional operator, which is suitable for the techniques of discretization, it's known as *Grünwald-Letnikov differintegral* [25] and is given as

$$\left(\mathrm{D}_{\mathrm{a}^+}^{\beta} f\right)(t) = \lim_{N\to\infty} \left\{ \left(\frac{t-a}{N}\right)^{-\beta} \frac{1}{\Gamma(-\beta)} \times \sum_{r=0}^{N-1} \frac{\Gamma(r-\beta)}{\Gamma(r+1)} f\left[t - r\left(\frac{t-a}{N}\right)\right] \right\} \quad \text{(A.4)}$$

Equation (A.4) defines in the same time two different operators, fractional derivate (for $\beta > 0$) and fractional integral (for $\beta < 0$).

Many definitions of fractional operators exist but are not reported for brevity sake's. For in-depth studies look at previous citied books and [27–30].

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