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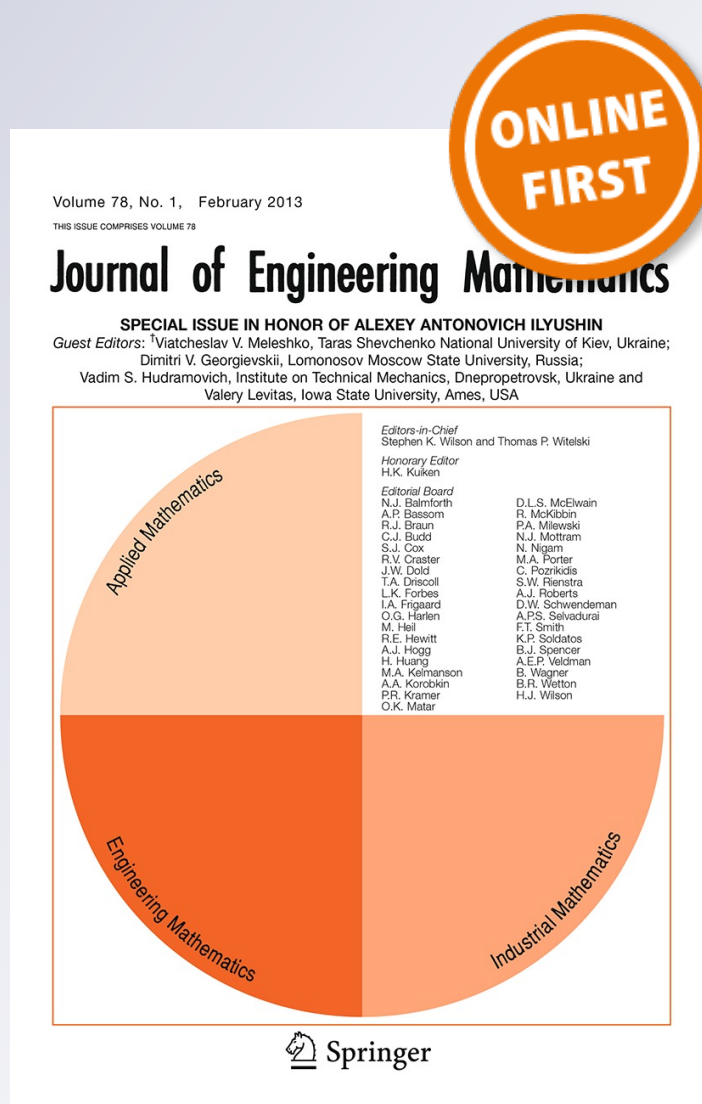
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Exact treatment of operator difference equations with nonconstant and noncommutative coefficients

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Abstract We study a homogeneous linear second-order difference equation with nonconstant and noncommuting operator coefficients in a vector space. We build its exact resolutive formula consisting of the explicit noniterative expression of a generic term of the unknown sequence of vectors. Some nontrivial applications are reported in order to show the usefulness and the broad applicability of the result.

Keywords Cauchy problem · Noncommuting operators · Operator difference equations

1 Introduction

Difference equations naturally occur and play a central role whenever the problem under scrutiny or the phenomenon under investigation allows a mathematical formulation traceable to the set of natural numbers. A broad variety of situations in biology, economics, dynamical systems, electrical circuit analysis, and other fields [1, Chap. 1] are indeed modeled by difference equations often demanding, however, quite different resolution methods due to peculiar aspects of their mathematical nature. Thus, as in the case of differential equations, a useful classification of difference equations has emerged.

The classical field of difference equations deals with linear or nonlinear equations where the unknown is a real or complex-valued function defined on a countable domain, which in turn may be thought of as coincident with \mathbb{N}^* without loss of generality. These equations are conveniently classified as scalar difference equations [2, Chaps. 2–4]–[6] to discriminate them from the so-called matrix difference equations [7, 8], where the unknowns constitute a sequence of $d \times d$ matrices with entries in \mathbb{C} as well as possibly all the given coefficients.

Quite recently a new class of difference equations has been introduced, namely, the class of linear operator difference equations [9]. The peculiarity of such equations with respect to matrix difference equations is that in this case the unknowns are elements of a given abstract vector space V , whereas the “coefficients” are linear operators

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acting on V . We clarify this important point by introducing the following linear second-order operator difference equation, which we will investigate and solve in this paper:

$$Y_{n+2} = \mathcal{L}_0(n)Y_n + \mathcal{L}_1(n)Y_{n+1}, \quad n \in \mathbb{N}^*. \quad (1)$$

This equation lives in a vector space V over a field \mathbb{F} , and the nature will not be further specified. The null vector of V is denoted by 0 , whereas $\mathcal{L}_0(n)$ and $\mathcal{L}_1(n)$, $n \in \mathbb{N}$, are two families of linear operators acting in V . To keep the investigation at the most general level possible, we introduce no constraints concerning the commutability of these operators in and between the two classes.

The aim of this paper is to build a general solution of Eq. (1) when its coefficients are nonconstant and noncommutative. This means giving a noniterative expression of Y_n , for a generic n , in terms of an appropriate ordered sequence of operators $\mathcal{L}_0(n)$ and $\mathcal{L}_1(n)$. Of course, this expression will contain two “summation constants” to be specified starting from given initial conditions.

The main result will be derived by mathematical induction in Sect. 3, and some applications will be presented in the two subsequent sections.

2 Mathematical preliminaries

We start by recalling that when $\forall n \in \mathbb{N}^*$, $\mathcal{L}_0(n) \equiv \mathcal{L}_0$ and $\mathcal{L}_1(n) \equiv \mathcal{L}_1$ the notation $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$ with $u, v \in \mathbb{N}^*$ expresses the sum of all the $\binom{u+v}{m}$ possible distinct permutations of u , \mathcal{L}_0 -factors and v , \mathcal{L}_1 -factors, where $m := \min(u, v)$ [10]. For example, $\{\mathcal{L}_0^{(1)} \mathcal{L}_1^{(1)}\} = \mathcal{L}_0 \mathcal{L}_1 + \mathcal{L}_1 \mathcal{L}_0$, and we define $\{\mathcal{L}_0^{(0)} \mathcal{L}_1^{(0)}\} := I$ setting consistently $\binom{0}{0} = 1$. It is possible to convince ourselves that any additive contribution to $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$ may be represented as a formal product of r ($r \geq 1$) powers of \mathcal{L}_0 alternated with r powers of \mathcal{L}_1 , of the form

$$\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r}. \quad (2)$$

The parameter r , henceforth called the length of the expression (2), is an integer number running from 1 to $m+1 \equiv r_M$. The $2r$ integer exponents τ_1, \dots, τ_r and s_1, \dots, s_r are numbers fulfilling the following conditions:

$$\left\{ \begin{array}{l} \sum_{i=1}^r \tau_i = u, \\ \sum_{i=1}^r s_i = v, \\ s_1 \geq 1, \quad \tau_1 \geq 0, \quad \tau_r \geq 1, \quad s_r \geq 0, \\ \tau_i \geq 1, \quad s_i \geq 1, \quad i = 2, \dots, r-1. \end{array} \right. \quad (3)$$

Hereafter the symbol $S_{u,v}^r$ represents the set of all the possible pairs of r -tuples $\bar{\tau} = (\tau_1, \dots, \tau_r)$ and $\bar{s} = (s_1, \dots, s_r)$ satisfying Eq. (3). Expanding the symbol $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$ in accordance with its definition we obtain contributions possessing all the compatible lengths and for any length all the possible terms obtained in accordance with Eq. (3). It is possible to show that the number of additive contributions of equal length r to $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$ is given by

$$\binom{u}{r-1} \binom{v}{r-1}, \text{ so that using the Vandermonde identity [11]}$$

$$\sum_{i=1}^{m+1} \binom{u}{i-1} \binom{v}{m+1-i} = \binom{u+v}{m} \quad (4)$$

we derive the following length-based representation of $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$

$$\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\} = \sum_{r=1}^{m+1} \sum_{(\bar{\tau}, \bar{s}) \in S_{u,v}^r} \mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r}, \quad (5)$$

where the summation

$$\sum_{(\bar{\tau}, \bar{s}) \in S_{u,v}^r} \mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \quad (6)$$

runs over all the pairs $(\bar{\tau}, \bar{s})$ compatible with the conditions (3).

The operator coefficients appearing in Eq. (1) depend on n and do not commute with each other in general. In such a case, powers like $\mathcal{L}_0^{\tau_i}$ and $\mathcal{L}_1^{s_j}$ become ambiguous as soon as τ_i or s_j exceeds 1. Thus, to retain as much as possible the advantages of the notation adopted in expression (2), we introduce the following descending products of operators:

$$\prod_{p=p_1}^{p_r} X_p := X(p_r) \dots X(p_{j-1}) \dots X(p_1), \quad p_1 \in \mathbb{N}^*, \quad (7)$$

postulating that $\prod_{p=p_1}^{p_r} X_p = I$ when $p_r < p_1$. For the sake of clarity, we stress that the symbol introduced on the left-hand side of Eq. (7) singles out the inverted ordered product of the noncommuting p_r operators X_{p_1}, \dots, X_{p_r} .

Exploiting Eq. (7) we get rid of the ambiguity possessed by expression (2) due to the noncommutativity of $\mathcal{L}_i, i \in \{0, 1\}$ by substituting it with the ordered operator $\left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_q$ defined as follows:

$$\begin{aligned} s \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_q \\ := \left[\prod_{i_1=0}^{\tau_1-1} \mathcal{L}_0 \left(k_q - 2i_1 \right) \right] \left[\prod_{j_1=0}^{s_1-1} \mathcal{L}_1 \left(k_q - 2\tau_1 - j_1 \right) \right] \dots \\ \times \left[\prod_{i_2=0}^{\tau_2-1} \mathcal{L}_0 \left(k_q - 2\tau_1 - s_1 - 2i_2 \right) \right] \left[\prod_{j_2=0}^{s_2-1} \mathcal{L}_1 \left(k_q - 2(\tau_1 + \tau_2) - s_1 - j_2 \right) \right] \dots \\ \times \left[\prod_{i_r=0}^{\tau_r-1} \mathcal{L}_0 \left(k_q - 2 \sum_{l=1}^{r-1} \tau_l - \sum_{l=1}^{s_{r-1}} s_l - 2i_r \right) \right] \left[\prod_{j_r=0}^{s_r-1} \mathcal{L}_1 \left(k_q - 2 \sum_{l=1}^r \tau_l - \sum_{l=1}^{s_{r-1}} s_l - j_r \right) \right], \end{aligned} \quad (8)$$

where

$$k_q := 2u + v - q, \quad q = 1, 2. \quad (9)$$

In the next section we shall see that the peculiar “order explosion” of Eq. (2) as given by Eq. (8) provides a very useful tool to write down the solution of a generic Cauchy problem associated to Eq. (1). In what follows the symbol $\left\{ \mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)} \right\}_q$ represents the sum (5), where each additive contribution $\{ \mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)} \}$ is replaced by the associated ordered operator, that is,

$$\left\{ \mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)} \right\}_q = \sum_{r=1}^{m+1} \sum_{(\bar{\tau}, \bar{s}) \in S_{u,v}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_q. \quad (10)$$

To clarify the notation, we develop, for example, $\{ \mathcal{L}_0^2 \mathcal{L}_1^1 \}_1$:

$$\begin{aligned} \left\{ \mathcal{L}_0^{(2)} \mathcal{L}_1^{(1)} \right\}_1 &= \sum_{r=1}^2 \sum_{(\bar{\tau}, \bar{s}) \in S_{2,1}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 \\ &= \sum_{(\bar{\tau}, \bar{s}) \in S_{2,1}^1} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \right]_1 + \sum_{(\bar{\tau}, \bar{s}) \in S_{2,1}^2} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \right]_1 \\ &= \left[\mathcal{L}_0^2 \mathcal{L}_1 \right]_1 + \left[\mathcal{L}_0^0 \mathcal{L}_1^1 \mathcal{L}_0^2 \mathcal{L}_1^0 \right]_1 + \left[\mathcal{L}_0^1 \mathcal{L}_1^1 \mathcal{L}_0^1 \mathcal{L}_1^0 \right]_1. \end{aligned} \quad (11)$$

Since $k_1 = 4$, the three terms acquire the following explicit form:

$$\begin{aligned} [\mathcal{L}_0^2 \mathcal{L}_1^1]_1 &= \left[\prod_{i_1=0}^1 \mathcal{L}_0(4 - 2i_1) \right] \left[\prod_{j_1=0}^0 \mathcal{L}_1(j_1) \right] = \mathcal{L}_0(4) \mathcal{L}_0(2) \mathcal{L}_1(0), \\ [\mathcal{L}_0^0 \mathcal{L}_1^1 \mathcal{L}_0^2 \mathcal{L}_1^0]_1 &= \left[\prod_{j_1=0}^0 \mathcal{L}_1(4 - j_1) \right] \left[\prod_{i_2=0}^1 \mathcal{L}_0(3 - 2i_2) \right] I = \mathcal{L}_1(4) \mathcal{L}_0(3) \mathcal{L}_0(1), \\ [\mathcal{L}_0^1 \mathcal{L}_1^1 \mathcal{L}_0^1 \mathcal{L}_1^0]_1 &= \mathcal{L}_0(4) \mathcal{L}_1(2) \mathcal{L}_0(1). \end{aligned}$$

We thus conclude that

$$\left\{ \mathcal{L}_0^2 \mathcal{L}_1^1 \right\}_1 = \mathcal{L}_0(4) \mathcal{L}_0(2) \mathcal{L}_1(0) + \mathcal{L}_1(4) \mathcal{L}_0(3) \mathcal{L}_0(1) + \mathcal{L}_0(4) \mathcal{L}_1(2) \mathcal{L}_0(1).$$

3 Resolution of a Cauchy problem associated with Eq. (1)

It is convenient to extend the definition of the operator $\left\{ \mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)} \right\}_q$ to negative integer values of u and v , simply setting, when u or v is a negative integer, $\left\{ \mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)} \right\}_q Y = \underline{0}$, for any Y that \mathcal{L}_0 , \mathcal{L}_1 can legitimately act on. We are now ready to prove the following theorem constituting the main result of the paper.

Theorem 1 *The solution of Eq. (1), given that $Y_0 = \underline{0}$ and $Y_1 = B$, may be written down as*

$$Y_n = \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \mathcal{L}_0^t \mathcal{L}_1^{n-1-2t} \right\}_1 B. \quad (12)$$

Proof We prove (12) by mathematical induction. For $n = 0$ and $n = 1$, Eq. (12) gives $Y_0 = \underline{0}$ and $Y_1 = B$, as expected. It is immediate to deduce $Y_2 = B$ from both Eqs. (1) and (12).

Let us suppose now that the first $(n + 1)$ terms of the sequence solution of Eq. (1) are representable by Eq. (12). We must prove that whatever $n > 0$ is,

$$\begin{aligned} Y_{n+2} &= \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \left\{ \mathcal{L}_0^t \mathcal{L}_1^{n+1-2t} \right\}_1 B \\ &= \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=1}^{m'+1} \sum_{(\bar{r}, \bar{s}) \in S_{t, n+1-2t}^r} \left[\mathcal{L}_0^{\bar{r}_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\bar{r}_r} \mathcal{L}_1^{s_r} \right]_1 B, \end{aligned} \quad (13)$$

where $m' = \min\{t, n + 1 - 2t\}$ satisfies Eq. (1). To this end, we start by appropriately transforming

$$\begin{aligned} \mathcal{L}_0(n) Y_n &= \mathcal{L}_0(n) \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \mathcal{L}_0^t \mathcal{L}_1^{n-1-2t} \right\}_1 B \\ &= \mathcal{L}_0(n) \sum_{t=1}^{\lfloor \frac{n-1}{2} \rfloor + 1} \left\{ \mathcal{L}_0^{t-1} \mathcal{L}_1^{n+1-2t} \right\}_1 B \\ &= \mathcal{L}_0(n) \sum_{t=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=1}^{m''+1} \sum_{(\bar{r}, \bar{s}) \in S_{t-1, n+1-2t}^r} \left[\mathcal{L}_0^{\bar{r}_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\bar{r}_r} \mathcal{L}_1^{s_r} \right]_1 B, \end{aligned} \quad (14)$$

where $m'' = \min\{t-1, n+1-2t\} \leq m'$, and we have used the fact that $\lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$. In what follows we will show that for every r , $1 \leq r \leq m'' + 1$ the following operator relation holds:

$$\mathcal{L}_0(n) \sum_{(\bar{\tau}, \bar{s}) \in S_{t-1, n+1-2t}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1 > 0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1, \quad (15)$$

where the right-hand-side summation symbol means that the ordered operators $\left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1$ are not included. We point out that $\mathcal{L}_0(n) \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1$ generates permutations of the same length r as $\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r}$, with τ_1 at least 1 necessarily. In addition, we stress that the range $\{1, 2, \dots, m'' + 1\}$ of r is compatible with the pair $u = t$, $v = n + 1 - 2t$. First, using Eq. (8) and taking into account that in this case $k_1 = n - 2$, the left-hand-side term of Eq. (15) may be developed as follows:

$$\begin{aligned} \mathcal{L}_0(n) \sum_{(\bar{\tau}, \bar{s}) \in S_{t-1, n+1-2t}^r} & \left[\prod_{i_1=0}^{\tau_1-1} \mathcal{L}_0(n-2-2i_1) \right] \left[\prod_{j_1=0}^{s_1-1} \mathcal{L}_1(n-2-2\tau_1-j_1) \right] \\ & \times \left[\prod_{i_2=0}^{\tau_2-1} \mathcal{L}_0(n-2-2\tau_1-s_1-2i_2) \right] \left[\prod_{j_2=0}^{s_2-1} \mathcal{L}_1(n-2-2(\tau_1+\tau_2)-s_1-j_2) \right] \dots \\ & \times \left[\prod_{i_r=0}^{\tau_r-1} \mathcal{L}_0\left(n-2-2\sum_{l=1}^{r-1} \tau_l - \sum_{l=1}^{s_r-1} s_l - 2i_r\right) \right] \left[\prod_{j_r=0}^{s_r-1} \mathcal{L}_1\left(n-2-2\sum_{l=1}^r \tau_l - \sum_{l=1}^{s_r-1} s_l - j_r\right) \right]. \end{aligned} \quad (16)$$

Shifting up the running index i_1 by 1, that is, setting $I_1 =: i_1 + 1$, we transform the first \mathcal{L}_0 -segment $\left[\prod_{i_1=0}^{\tau_1-1} \mathcal{L}_0(n-2-2i_1) \right]$ appearing in Eq. (16) into $\prod_{I_1=1}^{\tau_1} \mathcal{L}_0(n-2I_1)$, which in turn may be rewritten as $\prod_{I_1=0}^{\tau_1} \mathcal{L}_0(n-2I_1)$ through multiplication by $\mathcal{L}_0(n)$.

Introducing the positive index $T_1 =: \tau_1 + 1$ the previous expression (16) may be cast in the following form:

$$\begin{aligned} & \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1=T_1}^r} \left[\prod_{I_1=0}^{T_1-1} \mathcal{L}_0(n-2I_1) \right] \left[\prod_{j_1=0}^{s_1-1} \mathcal{L}_1(n-2T_1-j_1) \right] \\ & \times \left[\prod_{i_2=0}^{\tau_2-1} \mathcal{L}_0(n-2T_1-s_1-2i_2) \right] \left[\prod_{j_2=0}^{s_2-1} \mathcal{L}_1(n-2(T_1+\tau_2)-s_1-j_2) \right] \dots \\ & \times \left[\prod_{i_r=0}^{\tau_r-1} \mathcal{L}_0\left(n-2T_1-2\sum_{l=2}^{r-1} \tau_l - \sum_{r=1}^{s_r-1} s_r - 2i_r\right) \right] \left[\prod_{j_r=0}^{s_r-1} \mathcal{L}_1\left(n-2T_1-2\sum_{l=2}^r \tau_l - \sum_{l=1}^{s_r-1} s_l - j_r\right) \right], \end{aligned} \quad (17)$$

where each ordered contribution still has length r and the passage from $t-1$ to t arises from the “absorption” of $\mathcal{L}_0(n)$. Equation (17) is exactly the development of the right-hand-side term of Eq. (15), where T_1 plays the role of τ_1 and then Eq. (14) may be given the following aspect:

$$\mathcal{L}_0(n) Y_n = \sum_{t=1}^{\lceil \frac{n+1}{2} \rceil} \sum_{r=1}^{m''+1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1 > 0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 B. \quad (18)$$

To establish a connection between Eqs. (18) and (13), we extract from the latter expression the ordered operators beginning with $\mathcal{L}_0^{\tau_1}$, with $\tau_1 > 0$

$$\sum_{t=1}^{\lceil \frac{n+1}{2} \rceil} \sum_{r=1}^{m''+1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1 > 0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (19)$$

Here $t = 0$ is excluded since it implies $m' = 0$, and then necessarily $\tau_1 = 0$. If $m' = \min(t, n + 1 - 2t) = t$, then the corresponding highest length $r_M = t + 1$ is indeed incompatible with the condition $\tau_1 > 0$. Thus we are justified in substituting m' with $m'' = \min(t - 1, n + 1 - 2t)$ in expression (19). These arguments prove that $\mathcal{L}_0(n)Y_n$ captures all the ordered operators appearing in Eq. (13) effectively beginning with an \mathcal{L}_0 -segment ($\tau_1 > 0$).

In what follows, in view of Eq. (1), we will prove that $\mathcal{L}_1(n)Y_{n+1}$ coincides with the sum of all the other contributions in Eq. (13), namely, all those ordered operators beginning with $\mathcal{L}_0^{\tau_1}(n)$ with $\tau_1 = 0$. To this end, we exploit the inductive hypothesis writing the action of $\mathcal{L}_1(n)$ on Y_{n+1} as follows:

$$\begin{aligned}\mathcal{L}_1(n)Y_{n+1} &= \mathcal{L}_1(n) \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ \mathcal{L}_0^t \mathcal{L}_1^{n-2t} \right\}_1 B \\ &= \mathcal{L}_1(n) \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{m'''+1} \sum_{(\bar{\tau}, \bar{s}) \in S'_{t, n-2t}} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 B,\end{aligned}\quad (20)$$

where $m''' = \min\{t, n - 2t\}$.

We first consider the terms of increasing length r in Y_{n+1} , having $\tau_1 = 0$. Taking into account that $k_1 = n - 1$ when $r = 1$ we get from Eq. (8) the unique contribution ($t = 0$)

$$\begin{aligned}\mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S'_{0, n/\tau_1=0}} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \right]_1 &= \mathcal{L}_1(n) \left[\mathcal{L}_0^0 \mathcal{L}_1^n \right]_1 \\ &= \mathcal{L}_1(n) \left[\prod_{j_1=0}^{n-1} \mathcal{L}_1(n-1-j_1) \right] = \mathcal{L}_1(n) \left[\prod_{J_1=1}^n \mathcal{L}_1(n-J_1) \right] \\ &= \left[\prod_{J_1=0}^n \mathcal{L}_1(n-J_1) \right] = \left\{ \mathcal{L}_0^0 \mathcal{L}_1^{n+1} \right\}_1,\end{aligned}\quad (21)$$

where we replaced j_1 by $J_1 - 1$. This operator applied to B is present in Eq. (13), ($t = 0$).

Now we will prove that a generic term of Y_{n+1} having length $r > 1$ and still with $\tau_1 = 0$ satisfies the property

$$\mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S'_{t, n-2t/\tau_1=0}} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{(\bar{\tau}, \bar{s}) \in S'_{t, n+1-2t/\tau_1=0, s_1>1}} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (22)$$

Expanding the left-hand side of Eq. (22) we indeed get

$$\begin{aligned}\mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S'_{t, n-2t/\tau_1=0}} &\left[\prod_{i_1=0}^{\tau_1-1} \mathcal{L}_0(n-1-2i_1) \right] \left[\prod_{j_1=0}^{s_1-1} \mathcal{L}_1(n-1-2\tau_1-j_1) \right] \\ &\times \left[\prod_{i_2=0}^{\tau_2-1} \mathcal{L}_0(n-1-2\tau_1-s_1-2i_2) \right] \left[\prod_{j_2=0}^{s_2-1} \mathcal{L}_1(n-1-2(\tau_1+\tau_2)-s_1-j_2) \right] \dots \\ &\times \left[\prod_{i_r=0}^{\tau_r-1} \mathcal{L}_0\left(n-1-2\sum_{l=1}^{r-1} \tau_l - \sum_{l=1}^{s_{r-1}} s_l - 2i_r\right) \right] \left[\prod_{j_r=0}^{s_r-1} \mathcal{L}_1\left(n-1-2\sum_{l=1}^r \tau_l - \sum_{l=1}^{s_{r-1}} s_l - j_r\right) \right]\end{aligned}\quad (23)$$

which, setting $j_1 + 1 = J_1, s_1 + 1 = S_1$, yields

$$\begin{aligned} & \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1=0, s_1=S_1+1}^r} \left[\prod_{i_1=0}^{\tau_1-1} \mathcal{L}_0(n-2i_1) \right] \left[\prod_{J_1=0}^{S_1-1} \mathcal{L}_1(n-2\tau_1-J_1) \right] \\ & \times \left[\prod_{i_2=0}^{\tau_2-1} \mathcal{L}_0(n-2\tau_1-S_1-2i_2) \right] \left[\prod_{j_2=0}^{s_2-1} \mathcal{L}_1(n-2(\tau_1+\tau_2)-S_1-j_2) \right] \dots \\ & \times \left[\prod_{i_m=0}^{\tau_m-1} \mathcal{L}_0\left(n-2\sum_{l=1}^{\tau_{m-1}} \tau_l - S_1 - \sum_{l=2}^{s_{r-1}} s_l - 2i_r\right) \right] \left[\prod_{j_r=0}^{s_r-1} \mathcal{L}_1\left(n-2\sum_{l=1}^{\tau_r} \tau_l - S_1 - \sum_{q=2}^{s_{r-1}} s_l - j_r\right) \right]. \end{aligned} \quad (24)$$

Equation (24) is the expansion of the right-hand side of Eq. (22), where the increase from $(n-2t)$ to $(n+1-2t)$ is due to the “absorption” of $\mathcal{L}_1(n)$ as given by Eq. (24), and the condition $s_1 > 1$ stems directly from the fact that by construction $r > 1$ in Eq. (22). Summing both members Eq. (22) in view of Eq. (20) over $r > 1$ and $t > 1$ ($t = 0 \Rightarrow r = 1$) yields

$$\mathcal{L}_1(n) \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=2}^{m''' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1=0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=2}^{m''' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1=0, s_1>1}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (25)$$

Note that since $m''' < m'$, each ordered operator of length $r > 1$ compatible with $\{u = t, v = n - 2t\}$ generates an ordered operator compatible with $\{u = t, v = n + 1 - 2t\}$, increasing s_1 by 1, thereby rendering in the latter case $s_1 > 1$. This circumstance means that all the ordered operators appearing in Eq. (25) determine only the set of all ordered operators that have $r > 1$ and are compatible with the maximum length $m' + 1$ under the constraints $\tau_1 = 0$ and $s_1 > 1$. Thus putting together Eqs. (21) and (20) we claim to have proved that

$$\mathcal{L}_1(n) \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{m''' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1=0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=1}^{m' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1=0, s_1>1}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1, \quad (26)$$

where $\lfloor \frac{n}{2} \rfloor$ may be substituted with $\lfloor \frac{n+1}{2} \rfloor$ exploiting the fact that for any n odd the highest value of t gives rise to a contribution for which $\tau_1 > 0$. We concentrate now on the action of $\mathcal{L}_1(n)$ on the generic terms of Y_{n+1} in Eq. (20) for which $\tau_1 > 0$, that is,

$$\mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1>0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (27)$$

To this end, we observe that under the constraint $\tau_1 > 0$ the ordered operators as given by Eq. (20) in the length-based expansion of Y_{n+1} do not contribute to the first member of Eq. (27) for $r_M = t + 1, (m''' = t)$ in accordance with Eq. (12). When instead $m''' < t$, then $m''' + 2 = (n + 1 - 2t) + 1 = m' + 1$ since in such a condition $m' = \min(t, n + 1 - 2t) = n + 1 - 2t$. Considering that when $\tau_0 = 0$ and $s_0 = 1$ the operator $\mathcal{L}_1(n)$ may be represented as

$$\mathcal{L}_1(n) = \mathcal{L}_0^0(n) \mathcal{L}_1^1(n) = \left[\prod_{i_0=0}^{\tau_0-1} \mathcal{L}_0(n-2i_0) \right] \left[\prod_{j_0=0}^{s_0-1} \mathcal{L}_1(n-2\tau_0-j_0) \right] \quad (28)$$

Equation 27 may be rewritten as follows:

$$\mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1>0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1>0}^r} \left[\mathcal{L}_0^0 \mathcal{L}_1^1 \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1, \quad (29)$$

where r runs from 1 to $m''' + 1$ in accordance with Eq. (20). We are interested in summing on r both members of Eq. (29), getting

$$\sum_{r=1}^{m''' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1 > 0}^r} \left[\mathcal{L}_0^0 \mathcal{L}_1^1 \mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{r=2}^{m''' + 2} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1=0, s_1=1}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (30)$$

As was already discussed just before Eq. (28), if $m''' = t$, then r may be stopped at $m''' + 1 = m' + 1$, whereas if $m''' < t$, then $m''' + 2 = m' + 1$ as well. Thus we are justified in writing

$$\sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{m''' + 1} \mathcal{L}_1(n) \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n-2t/\tau_1 > 0}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1 = \sum_{t=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=2}^{m' + 1} \sum_{(\bar{\tau}, \bar{s}) \in S_{t, n+1-2t/\tau_1=0, s_1=1}^r} \left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1. \quad (31)$$

We have thus completed the demonstration of Theorem 1 since we have shown that $\mathcal{L}_0(n)Y_n(\mathcal{L}_1(n)Y_{n+1})$, when $Y_n, (Y_{n+1})$ is expressed by Eq. (12), generates all and only the ordered operators of Y_{n+2} as given by the same equation beginning with $\mathcal{L}_0^{\tau_1}(n)$ with $\tau_1 > 0, (\tau_1 = 0)$.

Remark 1 Using a treatment analogous to that used to demonstrate Theorem 1, it is possible to prove that the solution of Eq. (1) given that $Y_0 = A$ and $Y_1 = 0$ may be written, for any $n \geq 2$, as follows:

$$Y_n = \left[\sum_{t=0}^{\lfloor \frac{n-2}{2} \rfloor} \left\{ \mathcal{L}_0^{t+1} \mathcal{L}_1^{n-2-2t} \right\}_{2/\tau_r=1, s_r=0} \right] A, \quad (32)$$

where all the ordered contributions to be considered in the summation are those finishing with $\mathcal{L}_0^1 \mathcal{L}_1^0$.

To clarify the notation, we give the expression of Y_5 :

$$Y_5 = \left[\left\{ \mathcal{L}_0^1 \mathcal{L}_1^3 \right\}_{2/\tau_r=1, s_r=0} + \left\{ \mathcal{L}_0^2 \mathcal{L}_1^1 \right\}_{2/\tau_r=1, s_r=0} \right] A.$$

The constraints extract from the first term only one ordered operator of length 2, and from the second one, two ordered contributions only, of the same length 2:

$$Y_5 = [\mathcal{L}_1(3)\mathcal{L}_1(2)\mathcal{L}_1(1)\mathcal{L}_0(0) + \mathcal{L}_0(3)\mathcal{L}_1(1)\mathcal{L}_0(0) + \mathcal{L}_1(3)\mathcal{L}_0(2)\mathcal{L}_0(0)] A.$$

Since the two Cauchy problems we solved give rise to two independent solutions of Eq. (1), its general solution, for any $n \geq 2$, may be written using Eqs. (12) and (32):

$$Y_n = \left[\sum_{t=0}^{\lfloor \frac{n-2}{2} \rfloor} \left\{ \mathcal{L}_0^{t+1} \mathcal{L}_1^{n-2-2t} \right\}_{2/\tau_r=1, s_r=0} \right] A + \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \mathcal{L}_0^t \mathcal{L}_1^{n-1-2t} \right\}_1 B. \quad (33)$$

4 Applications

In this section we show the effectiveness of Eqs. (12) and (32) by exactly solving a nontrivial example of Eq. (1) formulated in a complex vectorial space V of even dimension N . Following Dirac, denote by $|v\rangle$ a generic vector of V and by $\langle v|w\rangle := \langle w|v\rangle^*$ the scalar product between the two vectors $|v\rangle$ and $|w\rangle$ belonging to V . If an orthonormal basis \mathcal{B} of V is composed of the N vectors $|1\rangle, |2\rangle, \dots, |i\rangle, \dots, |j\rangle, \dots, |N\rangle$ such that $\langle i|j\rangle = \delta_{ij}$, then let us introduce the transition operator $T_{ij} := |i\rangle\langle j|$ acting upon a generic vector $|i'\rangle$ of \mathcal{B} as follows: $T_{ij}|i'\rangle = \delta_{ji'}|i\rangle$. We define the following linear operators acting on V where the $N/2$ coefficients $\{c_i, i = 1, 2, \dots, N/2\}$ are complex numbers:

$$M_+ = \sum_{i=1}^{N/2} c_i T_{i, N-i+1} = \sum_{i=1}^{N/2} c_i |i\rangle\langle N-i+1|, \quad (34a)$$

$$M_- = (M_+)^{\dagger} = \sum_{i=1}^{N/2} c_i^* |N-i+1\rangle \langle i|, \quad (34b)$$

$$D_+ = \sum_{i=1}^{N/2} |c_i|^2 |i\rangle \langle i|, \quad (35a)$$

$$D_- = \sum_{i=N/2+1}^N |c_{N-i+1}|^2 |i\rangle \langle i|, \quad (35b)$$

$$M_0 = D_+ - D_-, \quad (36a)$$

$$D_N = D_+ + D_-. \quad (36b)$$

When $|c_i| = \rho_i = \rho$, whatever i is, it is easy to check the following properties:

$$M_{\pm}^2 |v\rangle = \underline{0}, \quad \forall |v\rangle \in V, \quad (37a)$$

$$[M_+, M_-] = M_0, \quad (37b)$$

$$M_{\pm} M_{\mp} = D_{\pm}, \quad (37c)$$

$$M_- D_+ = \rho^2 M_- \Rightarrow D_+ M_+ = \rho^2 M_+, \quad (38a)$$

$$M_+ D_+ |v\rangle = \underline{0} \Rightarrow D_+ M_- |v\rangle = \underline{0}, \quad \forall |v\rangle \in V, \quad (38b)$$

$$M_- D_- |v\rangle = \underline{0} \Rightarrow D_- M_+ |v\rangle = \underline{0}, \quad \forall |v\rangle \in V, \quad (38c)$$

$$M_+ D_- = \rho^2 M_+ \Rightarrow D_- M_- = \rho^2 M_-. \quad (38d)$$

We point out that Eqs. (37a–37c) and (38b–38c) hold without any restriction on the coefficients c_n and that when $N = 2$ and the corresponding unique coefficient $c_1 = 1$, M_+ , M_- and M_0 may be traced back to the well-known Pauli matrices. In the following analysis, we will make use only of properties (37a)–(38d) rather than use the explicit representation given by Eqs. (34b–36b). It is therefore worthwhile to underline that, generally speaking, other sets of M_- and D_- matrices fulfilling properties (34b)–(38d) exist.

The purpose of this section is to use Eqs. (12) and (32) to provide the closed form of the following general Cauchy problem:

$$Y_{n+2} = \mathcal{L}_0(n)Y_n + \mathcal{L}_1(n)Y_{n+1}, \quad n \in \mathbb{N}^*, \quad (39a)$$

$$Y_0 = \bar{Y}_0, \quad Y_1 = \bar{Y}_1, \quad (39b)$$

where Y_n is a vector of V , \bar{Y}_0 and \bar{Y}_1 are initial conditions fixed at will in V , and the generally noncommuting coefficient operators $\mathcal{L}_0(n)$ and $\mathcal{L}_1(n)$, $n \in \mathbb{N}^*$, are defined as functions of n as follows:

$$\mathcal{L}_0(n) = \begin{cases} M_+ & n \text{ even,} \\ M_- & n \text{ odd,} \end{cases} \quad (40)$$

$$\mathcal{L}_1(n) = \begin{cases} M_- & n \text{ even,} \\ M_+ & n \text{ odd.} \end{cases} \quad (41)$$

To this end, it is enough to consider the two initial conditions

$$(I) \quad Y_0 = \underline{0}, \quad Y_1 = \bar{Y}_1, \quad (42)$$

$$(II) \quad Y_0 = \bar{Y}_0, \quad Y_1 = \underline{0}. \quad (43)$$

4.1 Solving the Cauchy problem (I)

Consider the operator $\left[\mathcal{L}_0^{\tau_1} \mathcal{L}_1^{s_1} \mathcal{L}_0^{\tau_2} \mathcal{L}_1^{s_2} \dots \mathcal{L}_0^{\tau_r} \mathcal{L}_1^{s_r} \right]_1$, and observe that as a consequence of Eqs. (37a), (40), and (41) it vanishes as soon as one of the r integer exponents τ_i is greater than 1. In this case, Eq. (12) indeed exhibits

“ \mathcal{L}_0 -segments” containing M_{\pm}^2 . Moreover, if $r > 1$, then $s_1\tau_1 > 1$, and the operator $\left[\mathcal{L}_0^{\tau_1}\mathcal{L}_1^{s_1}\mathcal{L}_0^{\tau_2}\mathcal{L}_1^{s_2}\dots\mathcal{L}_0^{\tau_r}\mathcal{L}_1^{s_r}\right]_1$ vanishes because it includes the product

$$\dots\mathcal{L}_1(k-2\tau_1-s_1+1)\mathcal{L}_0(k-2\tau_1-s_1)\dots, \quad (44)$$

which coincides with M_{\pm}^2 in view of Eqs. (40) and (41). Summing up, when the length r of the operator $\left[\mathcal{L}_0^{\tau_1}\mathcal{L}_1^{s_1}\mathcal{L}_0^{\tau_2}\mathcal{L}_1^{s_2}\dots\mathcal{L}_0^{\tau_r}\mathcal{L}_1^{s_r}\right]_1$ exceeds 1, its ordered expression given by Eq. (8) vanishes. This property, of course, strictly related to the operators \mathcal{L}_0 and \mathcal{L}_1 , greatly simplifies Eq. (12) where the ordered values of t of interest become $t = 0$ (for any n) and $t = 1$ (for $n \geq 4$). The reason is that $t > 1$ and $r = 1$ requires $\tau_1 > 1$ and then leads to a vanishing operator, as previously discussed.

We thus focus on the evaluation of the expression of the two operators $\{\mathcal{L}_0^0\mathcal{L}_1^{n-1}\}_1$ and $\{\mathcal{L}_0^1\mathcal{L}_1^{n-3}\}_1$. For the first one we have $r = 1$ and

$$\left\{\mathcal{L}_0^0\mathcal{L}_1^{n-1}\right\}_1 = \left[\mathcal{L}_0^0\mathcal{L}_1^{n-1}\right]_1 = \mathcal{L}_1(n-2)\dots\mathcal{L}_1(0) = M_*\dots M_+M_-, \quad (45)$$

where $*$ is $-$ ($+$) if n is even (odd). The second and third equalities stem from the application of Eqs. (8) and (41), respectively. Taking into account Eq. (37c) and the constraints on the complex entries c_i , we finally get

$$\left\{\mathcal{L}_0^0\mathcal{L}_1^{n-1}\right\}_1 = \begin{cases} \rho^{n-3}D_+ & n \text{ odd}, \\ \rho^{n-2}M_- & n \text{ even}. \end{cases} \quad (46)$$

When $t = 1$ ($n \geq 4$), the maximal length r_M compatible with t is $r_M = 2$ and

$$\left\{\mathcal{L}_0^1\mathcal{L}_1^{n-3}\right\}_1 = \left[\mathcal{L}_0^1\mathcal{L}_1^{n-3}\right]_1. \quad (47)$$

Then it is easy to verify that

$$\left[\mathcal{L}_0^1\mathcal{L}_1^{n-3}\right]_1 = \mathcal{L}_0(n-2)\mathcal{L}_1(n-4)\dots\mathcal{L}_1(0) = \begin{cases} \rho^{n-4}D_+ & n \text{ even}, \\ \rho^{n-3}M_- & n \text{ odd}. \end{cases} \quad (48)$$

Substituting Eqs. (46) and (47) into Eq. (12), and taking into account that all the contributions from any $t > 1$ vanish, the solution of the Cauchy problem (15) may be cast in the following closed and explicit form:

$$Y_n^{(I)} = \begin{cases} 0 & n = 0, \\ \bar{Y}_1 & n = 1, \\ M_-\bar{Y}_1 & n = 2, \\ (\rho^{n-4}D_+ + \rho^{n-2}M_-)\bar{Y}_1 & n > 2, \text{ even}, \\ \rho^{n-3}(D_+ + M_-)\bar{Y}_1 & n > 2, \text{ odd}. \end{cases} \quad (49)$$

In passing we note that when $\rho_i = \rho = 1$ for any i , $Y_n = (D_+ + M_-)\bar{Y}_1$, whatever $n > 2$ is, which means that we obtain a very simple constant solution in this case.

4.2 Solving the Cauchy problem (II)

We make use of Eq. (32) observing that since for $n > 2$

$$\left\{\mathcal{L}_0^1\mathcal{L}_1^{n-2}\right\}_{2/\tau_r=1,s_r=0} = \left[\mathcal{L}_1^{n-2}\mathcal{L}_0^1 = \mathcal{L}_1(n-2)\dots\mathcal{L}_1(1)\mathcal{L}_0(0)\right]_2 = 0. \quad (50)$$

In addition we have for any $n \geq 4$

$$\left\{\mathcal{L}_0^2\mathcal{L}_1^{n-4}\right\}_{2/\tau_r=1,s_r=0} = \left[\mathcal{L}_0^1\mathcal{L}_1^{n-4}\mathcal{L}_0^1\right]_2 = \mathcal{L}_0(n-2)\mathcal{L}_1(n-4)\dots\mathcal{L}_1(1)\mathcal{L}_0(0) = 0. \quad (51)$$

Thus the solution of the Cauchy problem (II) under scrutiny assumes the following form:

$$Y_n^{(II)} = \begin{cases} \bar{Y}_0 & n = 0, \\ 0 & n = 1, \\ \mathcal{L}_0\bar{Y}_0 & n = 2, \\ 0 & n > 2. \end{cases} \quad (52)$$

By direct substitution it is easy to check that this very simple sequence of vectors of V is the (unique) solution of the Cauchy problem (II). Using Eqs. (49) and (52), we write the solution Y_n of the general Cauchy problem as follows:

$$Y_n = \begin{cases} \bar{Y}_0 & n = 0, \\ \bar{Y}_1 & n = 1, \\ \mathcal{L}_0(0)\bar{Y}_0 + M_-\bar{Y}_1 & n = 2, \\ (\rho^{n-4}D_+ + \rho^{n-2}M_-)\bar{Y}_1 & n > 2, \text{ even}, \\ \rho^{n-3}(D_+ + M_-)\bar{Y}_1 & n > 2, \text{ odd}. \end{cases} \quad (53)$$

Of course, if we interpret the initial conditions \bar{Y}_0 and \bar{Y}_1 as playing the role of “summation constants”, then we may refer to Eq. (53) as the general solution of difference Eq. (39a).

In conclusion, we stress that the object of this section was to find the general solution of a selected difference equation using Eqs. (12) and (32) and was reached by Eq. (53). We do believe that solving this nontrivial toy difference equation represents a good test to appreciate the potential and practical value of the theory we have reported in previous sections of the paper.

5 Concluding remarks

The applicative potentialities of Eq. (1) go beyond the already broad context of operator difference equations. We indeed emphasize that the treatment does not exclude that the unknowns might depend on some continuous variables and that the operator coefficients $\mathcal{L}_0(n)$ and $\mathcal{L}_1(n)$ may act also upon the single components of Y_n . To appreciate the value of this observation, consider the following difference–differential equation:

$$Y_{n+2}(t) = \tilde{\mathcal{L}}_0(n)\dot{Y}_n(t) + \tilde{\mathcal{L}}_1(n)Y_{n+1}, \quad (54)$$

where the dot denotes the first time derivative of $Y_n(t)$, which in turn means deriving each component of Y_n . The two generally noncommuting operators $\tilde{\mathcal{L}}_0$ and $\tilde{\mathcal{L}}_1$ are supposed, for simplicity, to linearly mix only the component of Y_n . Let us then introduce the following operator:

$$\mathcal{L}_0(n) = \tilde{\mathcal{L}}_0(n)\frac{d}{dt}; \quad \mathcal{L}_1(n) = \tilde{\mathcal{L}}_1(n). \quad (55)$$

By definition $\tilde{\mathcal{L}}_0(n)$ and d/dt commute, and if $\tilde{\mathcal{L}}_0(n)$ is represented by a matrix, then $\mathcal{L}_0(n)$ is a formal matrix as well, whose entries $(\mathcal{L}_0(n))_{ij}$ are differential operators defined as follows:

$$(\mathcal{L}_0(n))_{ij} = [\tilde{\mathcal{L}}_0(n)]_{ij}\frac{d}{dt}, \quad (56)$$

where $[\tilde{\mathcal{L}}_0(n)]_{ij}$ are the $\tilde{\mathcal{L}}_0(n)$ entries. With the help of this notation, Eq. (54) formally coincides with Eq. (1), whose resolutive formula given by Eq. (53) applies also to difference–differential Eq. (54), provided that the initial conditions are appropriately reinterpreted (each component of \bar{Y}_0 , \bar{Y}_1 must be thought of as functions of t). When $\tilde{\mathcal{L}}_0(n)$ and $\tilde{\mathcal{L}}_1(n)$ are defined in accordance with Eqs. (40) and (41), respectively, where the constant $\rho = 1$ is further adopted, the solution of the Cauchy problem (I) [$\bar{Y}_0(t) = 0$, $\bar{Y}_1 = \bar{Y}(t)$] can be cast as follows:

$$Y_n^I(t) = \begin{cases} M_-Y_1(t) & n = 2, \\ D_+\frac{d}{dt} + M_- & n > 2, \text{ even}, \\ D_+ + M_-\frac{d}{dt} & n > 2, \text{ odd}. \end{cases} \quad (57)$$

As a final remark, we wish to emphasize that the strategic and far-reaching value of the treatment of Eq. (1) leads to a resolutive formula achieved without requiring a priori the “mathematical nature” of the unknown “object” Y_n . Since, in the formal framework of a linear second-order difference equation, Eq. (1) incorporates further ingredients such as noncommutativity and is in addition compatible with a possible dependence of Y_n on continuous variables, we believe that our treatment of Eq. (1) should attract the interest of researchers in many fields from, for example, physical mathematics and engineering to economics, biology, and the social sciences, where discrete- or continuous-variable-based mathematical modeling plays a central investigational role.

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