ON GLOBALLY GENERATED VECTOR BUNDLES ON PROJECTIVE SPACES II

JOSÉ CARLOS SIERRA* AND LUCA UGAGLIA

ABSTRACT. Extending the main result of [12], we classify globally generated vector bundles on $\mathbb{P}^n$ with first Chern class equal to three.

1. Main result

The main result of the paper is the following:

Theorem 1.1. *Let $\mathcal{E}$ be a globally generated vector bundle of rank k on $\mathbb{P}^n$. If $c_1(\mathcal{E}) = 3$ and $c_2(\mathcal{E}) \leq 4$ then one of the following holds:*

(i) $c_2(\mathcal{E}) = 0$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(3)$;
(ii) $c_2(\mathcal{E}) = 2$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$;
(iii) $c_2(\mathcal{E}) = 3$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus T_{\mathbb{P}^n}(-1)$;
(iv) $c_2(\mathcal{E}) = 3$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$;
(v) $c_2(\mathcal{E}) = 4$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1)$;
(vi) $c_2(\mathcal{E}) = 4$ *and* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)$;
(vii) $c_2(\mathcal{E}) = 4$ *and* $\mathcal{E} = \Omega_{\mathbb{P}^4}(2)$;
(viii) $\mathcal{E}$ *is given by an exact sequence* $0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus s} \to \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \to \mathcal{E} \to 0$, *where* $h^0(\mathcal{E}^*) = r$, $h^1(\mathcal{E}^*) = s$ *and* $\mathcal{G}$ *is a bundle as above.*

Theorem 1.1 immediately implies the following:

Corollary 1.2. *Let $\mathcal{E}$ be a globally generated vector bundle of rank k on $\mathbb{P}^n$. If $c_1(\mathcal{E}) = 3$ then $\mathcal{E}$ is either as in Theorem 1.1, or one of the following holds:*

(i) $c_2(\mathcal{E}) = 5$ *and* $\mathcal{E} = \Omega^2_{\mathbb{P}^4}(2)^*$;
(ii) $c_2(\mathcal{E}) = 5$ *and* $\mathcal{E} = T_{\mathbb{P}^3}(-1) \oplus \Omega_{\mathbb{P}^3}(2)$;
(iii) $c_2(\mathcal{E}) = 5$ *and* $\mathcal{E} = T_{\mathbb{P}^n}(-1) \oplus T_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$;
(iv) $c_2(\mathcal{E}) = 6$ *and* $\mathcal{E} = T_{\mathbb{P}^n}(-1) \oplus T_{\mathbb{P}^n}(-1) \oplus T_{\mathbb{P}^n}(-1)$;
(v) $c_2(\mathcal{E}) = 6$ *and* $0 \to \mathcal{O}_{\mathbb{P}^n}(-2) \oplus \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus k+n+1} \to \mathcal{E} \to 0$;
(vi) $c_2(\mathcal{E}) = 7$ *and* $0 \to \mathcal{O}_{\mathbb{P}^n}(-2) \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus k+2} \to \mathcal{E} \to 0$;
(vii) $c_2(\mathcal{E}) = 9$ *and* $0 \to \mathcal{O}_{\mathbb{P}^n}(-3) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus k+1} \to \mathcal{E} \to 0$;
(viii) $\mathcal{E}$ *is given by* $0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus s} \to \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \to \mathcal{E} \to 0$, *where* $h^0(\mathcal{E}^*) = r$, $h^1(\mathcal{E}^*) = s$ *and* $\mathcal{G}$ *is a bundle as above.*

This note is a natural extension of [12]. Therefore, we still want to thank the referee of that paper for his help.

Globally generated vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ with $c_1(\mathcal{E}) = 3$ have also been studied independently, and using a different approach, in [10] and [1] (cf. Remark 2).

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2. Proof of Theorem 1.1

We work over the field of complex numbers. Let $\mathcal{E}$ be a globally generated vector bundle on $\mathbb{P}^n$ of rank k, and let $\mathcal{E}^*$ denote its dual bundle. In view of the following result, we will assume throughout the paper that $h^0(\mathcal{E}^*) = h^1(\mathcal{E}^*) = 0$.

Lemma 1 (First reduction)**.** *Let $\mathcal{E}$ be a globally generated vector bundle on $\mathbb{P}^n$. If $h^0(\mathcal{E}^*) = r$ and $h^1(\mathcal{E}^*) = s$ then there exists a globally generated vector bundle $\mathcal{G}$ such that $h^0(\mathcal{G}^*) = h^1(\mathcal{G}^*) = 0$, and an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus s} \to \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \to \mathcal{E} \to 0.$$

Proof. Just put together [12, Lemmas 3 and 4]. □

Let $c_1 := c_1(\mathcal{E})$ and $c_2 := c_2(\mathcal{E})$ denote the first and second Chern class of $\mathcal{E}$, respectively. We point out that $c_1^2 - c_2 \geq 0$ since $\mathcal{E}$ is globally generated. Furthermore, in order to classify globally generated vector bundles one can assume $c_2 \leq \frac{c_1^2}{2}$ thanks to the following:

Lemma 2 (Second reduction)**.** *Let $\mathcal{E}$ be a globally generated vector bundle with Chern classes c_1, c_2. If $c_2 > \frac{c_1^2}{2}$ then there exists a globally generated vector bundle $\mathcal{K}^*$, whose dual $\mathcal{K}$ is given by the exact sequence*

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus h^0(\mathcal{E})} \to \mathcal{E} \to 0.$$

In particular, $c_1(\mathcal{K}^*) = c_1$ and $c_2(\mathcal{K}^*) = c_1^2 - c_2 < \frac{c_1^2}{2}$.

Proof. Consider the kernel $\mathcal{K}$ of the epimorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus h^0(\mathcal{E})} \to \mathcal{E} \to 0$. □

Globally generated vector bundles with $c_1 \leq 2$ were classified in [12]. From now on we concentrate on the case $c_1 = 3$. We start by considering the cases in which $\mathcal{E}$ admits a global section whose zero locus is a hypersurface in $\mathbb{P}^n$:

Proposition 2.1. *If $h^0(\mathcal{E}(-3)) \neq 0$ then $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(3)$. Moreover, if $h^0(\mathcal{E}(-3)) = 0$ then $h^0(\mathcal{E}_K(-3)) = 0$ for every linear subspace $K \subset \mathbb{P}^n$ of dimension greater than one.*

Proof. The first statement was shown in [12, Lemma 5]. On the other hand, if $h^0(\mathcal{E}_K(-3)) \neq 0$ then $\mathcal{E}_K = \mathcal{O}_K(3) \oplus \mathcal{O}_K^{\oplus k-1}$ by the first assertion and Lemma 1. Therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus k-1}$ (see for instance [11, Ch. I, Theorem 2.3.2]), whence $h^0(\mathcal{E}(-3)) \neq 0$. □

Proposition 2.2. *Assume $h^0(\mathcal{E}(-3)) = 0$.*

(i) *If $h^0(\mathcal{E}(-2)) \neq 0$ then either $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$, or $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus T_{\mathbb{P}^n}(-1)$.*
(ii) *If $h^0(\mathcal{E}(-2)) = 0$ then $h^0(\mathcal{E}_K(-2)) = 0$ for every linear subspace $K \subset \mathbb{P}^n$ of dimension greater than one.*

Proof. To prove (i), we essentially argue as in [12, Proposition 3.2]. If $n = 1$ the result is trivial, so we assume $n \geq 2$. Let $s \in H^0(\mathcal{E}(-2))$ be a non-zero section. Consider the corresponding exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}(-2) \to \mathcal{F} \to 0,$$

and let $Z \subset \mathbb{P}^n$ be the zero locus of s. We claim that Z is a finite scheme of length at most one. To get a contradiction, let P, Q be two points (maybe infinitely close) where s vanishes and let $L \subset \mathbb{P}^n$ be the line joining P and Q. Restricting to L and twisting, we get

$$0 \to \mathcal{O}_L(2) \to \mathcal{E}_L \to \mathcal{F}_L(2) \to 0.$$

Since $\mathcal{E}$ is globally generated and $\mathcal{F}_L(2)$ is a quotient of $\mathcal{E}_L$, we deduce that $\mathcal{F}_L(2)$ is also globally generated. Furthermore

$$3 = c_1(\mathcal{E}_L) = c_1(\mathcal{O}_L(2)) + c_1(\mathcal{F}_L(2)) = 2 + c_1(\mathcal{F}_L(2))$$

and $P, Q \in Z$, so s vanishes on L. Let $\mathbb{P}^2 \subset \mathbb{P}^n$ be a general plane containing L. Then s does not vanish identically on $\mathbb{P}^2$ as otherwise $s \in H^0(\mathcal{E}(-2))$ would be the zero section. Let $V \subset \mathbb{P}^n$ be the hypersurface of degree 2 where s vanishes (considered as a section of $\mathcal{E}$). Then s vanishes on L and $V \cap \mathbb{P}^2$, whence $h^0(\mathcal{E}_{\mathbb{P}^2}(-3)) \neq 0$ contradicting Proposition 2.1. This proves the claim. Consider the restriction sequence

$$0 \to \mathcal{O}_H \to \mathcal{E}_H(-2) \to \mathcal{F}_H \to 0$$

to a hyperplane $H \subset \mathbb{P}^n$ not meeting Z. Then $\mathcal{F}_H$ is a vector bundle such that $\mathcal{F}_H(2)$ is globally generated and $c_1(\mathcal{F}_H(2)) = 1$. Therefore $\mathcal{F}_H(2)$ is either $\mathcal{O}_H(1) \oplus \mathcal{O}_H^{\oplus k-2}$ or $T_H(-1) \oplus \mathcal{O}_H^{\oplus k-n}$ by [12, Proposition 3.1]. As

$$\mathcal{E}_H(-2) \in \operatorname{Ext}^1(\mathcal{F}_H, \mathcal{O}_H) = H^{n-2}(\mathcal{F}_H(-n)) = 0,$$

we deduce that $\mathcal{E}_H$ is either $\mathcal{O}_H(2) \oplus \mathcal{O}_H(1) \oplus \mathcal{O}_H^{\oplus k-2}$ or $\mathcal{O}_H(2) \oplus T_H(-1) \oplus \mathcal{O}_H^{\oplus k-n}$. We claim that $h^{n-1}(\mathcal{F}(-n-1)) = 0$. Assume the claim proved. Then

$$\mathcal{E}(-2) \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}) = H^{n-1}(\mathcal{F}(-n-1)) = 0,$$

whence $\mathcal{E}(-2) = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{F}$ and $\mathcal{F}$ is a vector bundle such that $\mathcal{F}(2)$ is globally generated and $c_1(\mathcal{F}(2)) = 1$, so $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ or $\mathcal{O}_{\mathbb{P}^n}(2) \oplus T_{\mathbb{P}^n}(-1)$ by [12, Proposition 3.1] and Lemma 1. Let us prove the claim. Consider the restriction sequence

$$0 \to \mathcal{E}^* \to \mathcal{E}^*(1) \to \mathcal{E}^*_H(1) \to 0.$$

Since $h^1(\mathcal{E}^*) = 0$ by assumption and $h^1(\mathcal{E}^*_H(1)) = 0$, we get $h^1(\mathcal{E}^*(1)) = 0$. Now consider the restriction sequence

$$0 \to \mathcal{F}(-n-1) \to \mathcal{F}(-n) \to \mathcal{F}_H(-n) \to 0.$$

As $h^{n-2}(\mathcal{F}_H(-n)) = 0$ and $h^{n-1}(\mathcal{F}(-n)) = h^{n-1}(\mathcal{E}(-n-2)) = h^1(\mathcal{E}^*(1)) = 0$, we deduce $h^{n-1}(\mathcal{F}(-n-1)) = 0$.

We now prove (ii). It suffices to show it for every hyperplane $H \subset \mathbb{P}^n$. If $h^0(\mathcal{E}_H(-2)) \neq 0$ for some hyperplane $H \subset \mathbb{P}^n$, we deduce from (i) and Lemma 1 that either $\mathcal{E}_H = \mathcal{O}_H(2) \oplus \mathcal{O}_H(1) \oplus \mathcal{O}_H^{\oplus k-2}$, or $\mathcal{E}_H$ fits in an exact sequence

$$0 \to \mathcal{O}_H^{\oplus s} \to \mathcal{O}_H(2) \oplus T_H(-1) \oplus \mathcal{O}_H^{\oplus k+s-n} \to \mathcal{E}_H \to 0$$

Assume first $n \geq 4$. Then $h^i(\mathcal{E}_H(-j)) = 0$ for $i = 0, 1$ and every $j \geq 3$. Consider the restriction sequence $0 \to \mathcal{E}(-j-1) \to \mathcal{E}(-j) \to \mathcal{E}_H(-j) \to 0$. We deduce from Serre's vanishing theorem that $h^0(\mathcal{E}(-3)) = h^1(\mathcal{E}(-3)) = 0$, whence $h^0(\mathcal{E}(-2)) = h^0(\mathcal{E}_H(-2)) \neq 0$. Now assume $n = 3$. The Hirzebruch-Riemann-Roch theorem yields $\chi(\mathcal{E}) = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + c_1^2 - 2c_2 + \frac{22}{12}c_1 + k$. Since $c_1 = c_2 = 3$ we deduce $\chi(\mathcal{E}) = 8 + k + \frac{1}{2}(c_3 + 1)$. To get a contradiction, assume $h^0(\mathcal{E}(-2)) = 0$. Then the restriction sequence gives $h^0(\mathcal{E}(-1)) \leq 3$ and $h^0(\mathcal{E}) \leq 9 + k$. We deduce $h^3(\mathcal{E}) = h^0(\mathcal{E}^*(-4)) = 0$ and $h^2(\mathcal{E}) = h^1(\mathcal{E}^*(-4)) = 0$ from Serre duality and the exact sequence $0 \to \mathcal{E}^*(-j-1) \to \mathcal{E}^*(-j) \to \mathcal{E}^*_H(-j) \to 0$. Hence $-h^1(\mathcal{E}) \geq (c_3 - 1)/2$, that is, $c_3 = 1$ since c_3 is odd (see for instance [11, p. 113]). Therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}$, and we get a contradiction. □

Remark 1. We would like to thank Edoardo Ballico for pointing out the following gap in the proof of [12, Proposition 3.2]. The natural isomorphism between $H^{n-1}(\mathcal{F}(-n-1))$ and the dual of $H^1(\mathcal{F}^*)$ holds if the quotient $\mathcal{F}$ is a locally free sheaf, so we just have $\mathcal{E}(-1) \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}) = H^{n-1}(\mathcal{F}(-n-1))$. In order to show that $h^{n-1}(\mathcal{F}(-n-1)) = 0$, and hence $\mathcal{E}(-1) = \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^n}$, just note that

$h^{n-1}(\mathcal{F}(-n)) = h^{n-1}(\mathcal{E}(-n-1)) = h^1(\mathcal{E}^*) = 0$ and that $h^{n-2}(\mathcal{F}_H(-n)) = 0$ (cf. Lemma 3 below).

The cases $h^0(\mathcal{E}(-3)) \neq 0$ and $h^0(\mathcal{E}(-2)) \neq 0$ were described in Propositions 2.1 and 2.2, respectively. Now we study in detail the case $h^0(\mathcal{E}(-1)) \neq 0$. The following lemma, that somehow appeared in the proof of Proposition 2.2, will be used in the sequel:

Lemma 3. *Let $s \in H^0(\mathcal{E}(-1))$ be a non-zero section, and let $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}(-1) \to \mathcal{F} \to 0$ be the corresponding exact sequence of sheaves. If $h^{n-2}(\mathcal{F}_H(-n)) = 0$ for some hyperplane $H \subset \mathbb{P}^n$ then $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{F}(1)$. In particular, $\mathcal{F}(1)$ is a globally generated vector bundle with $c_1(\mathcal{F}(1)) = 2$.*

Proof. We deduce $h^{n-1}(\mathcal{F}(-n)) = h^{n-1}(\mathcal{E}(-n-1)) = h^1(\mathcal{E}^*) = 0$ from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-n) \to \mathcal{E}(-n-1) \to \mathcal{F}(-n) \to 0$, Serre duality, and the assumption $h^1(\mathcal{E}^*) = 0$ throughout the paper. Therefore, we get $h^{n-1}(\mathcal{F}(-n-1)) = 0$ from the restriction sequence $0 \to \mathcal{F}(-n-1) \to \mathcal{F}(-n) \to \mathcal{F}_H(-n) \to 0$. As $\mathcal{E}(-1) \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}) = H^{n-1}(\mathcal{F}(-n-1)) = 0$, we deduce $\mathcal{E}(-1) = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{F}$. $\square$

From now on we also assume $c_2 \leq 4$ (cf. Lemma 2).

Proposition 2.3. *If $h^0(\mathcal{E}(-2)) = 0$ and $h^0(\mathcal{E}(-1)) \neq 0$ then one of the following holds:*

- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$;
- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1)$;
- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)$.

Proof. For $n = 1$ the result is obvious, so we assume $n \geq 2$. Let $s \in H^0(\mathcal{E}(-1))$ be a non-zero section, and consider the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}(-1) \to \mathcal{F} \to 0.$$

Let $Z \subset \mathbb{P}^n$ be the zero locus of s. We claim that Z is a finite scheme of length at most two. To get a contradiction, let $T \subset Z$ be a subscheme of length three and let $\Pi \subset \mathbb{P}^n$ be a plane containing T. Consider the restriction $\mathcal{E}_\Pi$ and the quotient

$$0 \to \mathcal{O}_\Pi^{\oplus k-2} \to \mathcal{E}_\Pi \to \mathcal{Q} \to 0$$

(cf. [11, Ch. I, Lemma 4.3.1]). Then $\mathcal{Q}$ is a globally generated vector bundle of rank two, $c_1(\mathcal{Q}) = c_1(\mathcal{E}_\Pi) = 3$ and $c_2(\mathcal{Q}) = c_2(\mathcal{E}_\Pi) \leq 4$. The restriction to Π of the non-zero section $s \in H^0(\mathcal{E}(-1))$ yields a non-zero section in $H^0(\mathcal{E}_\Pi(-1))$ by Proposition 2.2(ii). Therefore, since $H^0(\mathcal{E}_\Pi(-1)) \cong H^0(\mathcal{Q}(-1))$, we get a non-zero section $\sigma \in H^0(\mathcal{Q}(-1))$ vanishing on $T \subset \Pi$. Since the zero locus of σ is finite as otherwise $\sigma \in H^0(\mathcal{Q}(-2)) \cong H^0(\mathcal{E}_\Pi(-2)) = 0$, we get $c_2(\mathcal{Q}(-1)) \geq 3$ contradicting the fact

$$c_2(\mathcal{Q}(-1)) = (-1)^2 - c_1(\mathcal{Q}) + c_2(\mathcal{Q}) = c_2(\mathcal{Q}) - 2 \leq 2,$$

and hence proving the claim. Now consider the restriction

$$0 \to \mathcal{O}_H \to \mathcal{E}_H(-1) \to \mathcal{F}_H \to 0$$

to a hyperplane $H \subset \mathbb{P}^n$ such that $Z \cap H = \emptyset$. Then $\mathcal{F}_H(1)$ is a globally generated vector bundle, $c_1(\mathcal{F}_H(1)) = 2$ and $c_2(\mathcal{F}_H(1)) \leq 2$. Therefore $\mathcal{F}_H(1)$ can be as in [12, Theorem 1.1], cases (i)-(iv). In case (i) we have $\mathcal{F}_H(1) = \mathcal{O}_H(2) \oplus \mathcal{O}_H^{\oplus k-2}$, so $h^{n-2}(\mathcal{F}_H(-n)) = 0$ and hence $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(2)$ by Lemma 3, giving a contradiction. In case (ii) we have $\mathcal{F}_H(1) = \mathcal{O}_H(1)^{\oplus 2} \oplus \mathcal{O}_H^{\oplus k-3}$, so $h^{n-2}(\mathcal{F}_H(-n)) = 0$ and therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ by Lemma 3. In case (iv) we also have $h^{n-2}(\mathcal{F}_H(-n)) = 0$. Therefore $\mathcal{F}(1)$ is a globally generated vector bundle by Lemma 3 such that $\mathcal{F}_H(1)$ is either $\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus k-4}$ or $\mathcal{N}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus k-3}$, and we get a contradiction by [12, Theorem 1.1]. If $\mathcal{F}_H(1)$ is as in case (iii), we remark that

$\mathcal{F}_H(1)$ is either $T_H(-1)\oplus\mathcal{O}_H(1)\oplus\mathcal{O}_H^{k-n-1}$ or $\mathcal{G}\oplus\mathcal{O}_H^{k-n}$, where $\mathcal{G}$ is a vector bundle of rank $n-1$ obtained as a quotient

$$0\to\mathcal{O}_H\to\mathcal{O}_H(1)\oplus T_H(-1)\to\mathcal{G}\to 0$$

(cf. [12, Remark 3]). If $\mathcal{F}_H(1)=T_H(-1)\oplus\mathcal{O}_H(1)\oplus\mathcal{O}_H^{k-n-1}$ then $h^{n-2}(\mathcal{F}_H(-n))=0$, and hence $\mathcal{E}=\mathcal{O}_{\mathbb{P}^n}(1)\oplus\mathcal{F}(1)$ by Lemma 3. Therefore, $\mathcal{F}(1)$ is either $T(-1)\oplus\mathcal{O}_{\mathbb{P}^n}(1)$ or $\Omega_{\mathbb{P}^3}(2)$ by [12, Theorem 1.1]. Let us see now that $\mathcal{F}_H(1)=\mathcal{G}\oplus\mathcal{O}_H^{k-n}$ yields a contradiction. Assume first $n=3$. Then $h^1(\mathcal{F}_H(-3))=h^1(\mathcal{G}(-4))=h^1(\mathcal{G}^*(-2))=0$, as $\mathcal{G}^*(-2)\cong\mathcal{G}(-4)$ since $\mathcal{G}$ is of rank two and $c_1(\mathcal{G}(-4))=-6$. Therefore $\mathcal{E}=\mathcal{O}_{\mathbb{P}^3}(1)\oplus\mathcal{F}(1)$ by Lemma 3, and hence $\mathcal{F}(1)=\mathcal{N}_{\mathbb{P}^3}(1)\oplus\mathcal{O}_{\mathbb{P}^3}^{\oplus k-3}$ by [12, Theorem 1.1]. This contradicts the assumption $h^1(\mathcal{E}^*)=0$. Assume now $n\geq 4$. To get a contradiction, we point out that $h^1(\mathcal{F}_H^*(-1))=h^1(\mathcal{G}^*)=1$. Then it follows from the exact sequence

$$0\to\mathcal{F}_H^*(-1)\to\mathcal{E}_H^*\to\mathcal{O}_H(-1)\to 0$$

that $h^1(\mathcal{E}_H^*)=1$. Hence the exact sequence

$$0\to\mathcal{E}^*(-1)\to\mathcal{E}^*\to\mathcal{E}_H^*\to 0$$

yields $h^2(\mathcal{E}^*(-1))\neq 0$, as we assume $h^1(\mathcal{E}^*)=0$. Let us see that $h^2(\mathcal{E}^*(-2))=0$. Consider the exact sequence

$$0\to\mathcal{F}_H^*(-1-j)\to\mathcal{E}_H^*(-j)\to\mathcal{O}_H(-1-j)\to 0.$$

Then $h^i(\mathcal{E}_H^*(-j))=h^i(\mathcal{F}_H^*(-1-j))=h^i(\mathcal{G}^*(-j))=0$ for $i\in\{1,2\}$ and every integer $j\geq 2$ (here we use $n\geq 4$). So we deduce from the exact sequence

$$0\to\mathcal{E}^*(-1-j)\to\mathcal{E}^*(-j)\to\mathcal{E}_H^*(-j)\to 0$$

and Serre's vanishing theorem that $h^2(\mathcal{E}^*(-2))=0$. Therefore, $h^2(\mathcal{E}^*(-2))=0$ and $h^2(\mathcal{E}^*(-1))\neq 0$ yields $h^2(\mathcal{E}_H^*(-1))\neq 0$, which is a contradiction as $h^2(\mathcal{E}_H^*(-1))=h^2(\mathcal{F}_H^*(-2))=h^2(\mathcal{G}^*(-1))=0$. □

Finally, we consider the case $h^0(\mathcal{E}(-1))=0$.

Corollary 2.4. *Assume $n\geq 3$. If $h^0(\mathcal{E}(-1))=0$ but $h^0(\mathcal{E}_H(-1))\neq 0$ for some hyperplane $H\subset\mathbb{P}^n$, then $n=4$ and $\mathcal{E}_{\mathbb{P}^3}$ is either $\mathcal{O}_{\mathbb{P}^3}(1)\oplus\Omega_{\mathbb{P}^3}(2)$, or a quotient $0\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_{\mathbb{P}^3}(1)\oplus\Omega_{\mathbb{P}^3}(2)\to\mathcal{E}_{\mathbb{P}^3}\to 0$ of rank three.*

Proof. Suppose first $n\geq 4$. If $h^0(\mathcal{E}_H(-1))\neq 0$ then it follows from Lemma 1 and Proposition 2.3 that $\mathcal{E}_H$ fits in an exact sequence $0\to\mathcal{O}_H^{\oplus s}\to\mathcal{G}\oplus\mathcal{O}_H^{\oplus r}\to\mathcal{E}_H\to 0$, where $r=h^0(\mathcal{E}_H^*)$, $s=h^1(\mathcal{E}_H^*)$ and either

(i) $\mathcal{G}=\mathcal{O}_H(1)^{\oplus 3}$, or
(ii) $\mathcal{G}=\mathcal{O}_H(1)^{\oplus 2}\oplus T_H(-1)$, or
(iii) $\mathcal{G}=\mathcal{O}_{\mathbb{P}^3}(1)\oplus\Omega_{\mathbb{P}^3}(2)$.

In cases (i) and (ii) we get $h^i(\mathcal{E}_H(-j))=h^i(\mathcal{G}(-j))=0$ for $i\in\{0,1\}$ and every integer $j\geq 2$ (here we use $n\geq 4$). Hence we deduce from the exact sequence

$$0\to\mathcal{E}(-j-1)\to\mathcal{E}(-j)\to\mathcal{E}_H(-j)\to 0$$

and Serre's vanishing theorem that $h^0(\mathcal{E}(-2))=h^1(\mathcal{E}(-2))=0$. Therefore

$$h^0(\mathcal{E}(-1))=h^0(\mathcal{E}_H(-1))\neq 0,$$

yielding a contradiction. Hence case (iii) holds and $n=4$. Furthermore, we claim that $h^0(\mathcal{E}_H^*)=0$. From the dual sequence $0\to\mathcal{E}_H^*\to\mathcal{G}^*\oplus\mathcal{O}_H^{\oplus r}\to\mathcal{O}_H^{\oplus s}\to 0$ we deduce that $h^i(\mathcal{E}_H^*(-j))=h^i(\mathcal{G}^*(-j))=0$ for $i\in\{0,1\}$ and every integer $j\geq 1$. From the exact sequence

$$0\to\mathcal{E}^*(-1-j)\to\mathcal{E}^*(-j)\to\mathcal{E}_H^*(-j)\to 0$$

and Serre's vanishing theorem we get $h^0(\mathcal{E}^*(-1)) = h^1(\mathcal{E}^*(-1)) = 0$, and hence $h^0(\mathcal{E}_H^*) = h^0(\mathcal{E}^*) = 0$. Therefore $\mathcal{E}_H$ is either $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)$, or a quotient $0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus s} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2) \to \mathcal{E}_H \to 0$ where, in the latter, $s = 1$ as $c_3(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)) \neq 0$.

Assume now $n = 3$. We argue as in Proposition 2.2. To get a contradiction, assume $h^0(\mathcal{E}(-1)) = 0$ and $h^0(\mathcal{E}_H(-1)) \neq 0$. Then we deduce from Proposition 2.3 that $\mathcal{E}_H$ is given by an exact sequence

$$0 \to \mathcal{O}_H^{\oplus s} \to \mathcal{O}_H(1)^{\oplus 2} \oplus T_H(-1) \oplus \mathcal{O}_H^{\oplus k+s-4} \to \mathcal{E}_H \to 0$$

As $h^0(\mathcal{E}(-1)) = 0$, we deduce from the restriction sequence that $h^0(\mathcal{E}) \leq k + 5$. We deduce $h^3(\mathcal{E}) = h^0(\mathcal{E}^*(-4)) = 0$ and $h^2(\mathcal{E}) = h^1(\mathcal{E}^*(-4)) = 0$ from Serre duality and the exact sequence $0 \to \mathcal{E}^*(-1-j) \to \mathcal{E}^*(-j) \to \mathcal{E}_H^*(-j) \to 0$. By the Hirzebruch-Riemann-Roch theorem we get $\chi(\mathcal{E}) = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + c_1^2 - 2c_2 + \frac{22}{12}c_1 + k$, and hence $h^0(\mathcal{E}) - h^1(\mathcal{E}) = k + 5 + c_3/2 \leq k + 5 - h^1(\mathcal{E})$, that is, $c_3 = 0$ giving a contradiction (see for instance [5, Theorem 1.1]). □

Let us see that only the first case in Corollary 2.4 actually occurs:

Proposition 2.5. *Assume $h^0(\mathcal{E}(-1)) = 0$ but $h^0(\mathcal{E}_H(-1)) \neq 0$ for some hyperplane $H \subset \mathbb{P}^4$. Then $\mathcal{E} \cong \Omega_{\mathbb{P}^4}(2)$.*

Proof. It follows from Corollary 2.4 that $\mathcal{E}_{\mathbb{P}^3}$ is either $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)$, or a quotient $0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2) \to \mathcal{E}_{\mathbb{P}^3} \to 0$ of rank three.

If $\mathcal{E}_H = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2)$ then we see from Serre's vanishing theorem and the restriction sequence

$$0 \to \mathcal{E}(-j-1) \to \mathcal{E}(-j) \to \mathcal{E}_H(-j) \to 0$$

that $h^1(\mathcal{E}(-2)) = h^1(\mathcal{E}_H(-2)) = 1$. Therefore we have a non-trivial extension

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{G} \to \mathcal{E}^*(2) \to 0$$

We claim that $\mathcal{G} = \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$. In view of [11, Ch. I, Theorem 2.3.2], it is enough to show that $\mathcal{G}_H = \mathcal{O}_H(1)^{\oplus 5}$. Let us see that $\mathcal{G}_H$ has no intermediate cohomology. From the exact sequence

$$0 \to \mathcal{O}_H \to \mathcal{G}_H \to \mathcal{O}_H(1) \oplus T_H \to 0,$$

we deduce that $h^1(\mathcal{G}_H(j)) = 0$ for every integer j and that $h^2(\mathcal{G}_H(j)) = 0$ for every integer $j \neq -4$. For $j = -4$, we have $h^2(\mathcal{G}_H(-4)) = h^1(\mathcal{G}_H^*)$. It follows from the exact sequence

$$0 \to \mathcal{O}_H(-1-j) \oplus \Omega_H(-j) \to \mathcal{G}_H^*(-j) \to \mathcal{O}_H(-j) \to 0$$

that $h^0(\mathcal{G}_H^*(-j)) = h^1(\mathcal{G}_H^*(-j)) = h^2(\mathcal{G}_H^*(-j)) = 0$ for every $j \geq 1$. Therefore Serre's vanishing theorem applied to the restriction sequence

$$0 \to \mathcal{G}^*(-j-1) \to \mathcal{G}^*(-j) \to \mathcal{G}_H^*(-j) \to 0$$

yields $h^1(\mathcal{G}^*(-1)) = h^2(\mathcal{G}^*(-1)) = 0$, and hence $h^1(\mathcal{G}_H^*) = h^1(\mathcal{G}^*) = 0$. Then Horrocks' theorem (see for instance [11, Ch. I, Theorem 2.3.1]) implies that $\mathcal{G}_H$ splits. Finally $c_1(\mathcal{G}_H) = 5$ and $h^0(\mathcal{G}_H(-2)) = 0$, so we get $\mathcal{G}_H = \mathcal{O}_H(1)^{\oplus 5}$. Then $\mathcal{E} = \Omega_{\mathbb{P}^4}(2)$.

Assume now that $\mathcal{E}_H$ is given by a quotient

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \Omega_{\mathbb{P}^3}(2) \to \mathcal{E}_{\mathbb{P}^3} \to 0$$

Then $c_t(\mathcal{E}) = c_t(\mathcal{E}_H) = 1 + 3t + 4t^2 + 2t^3$. Therefore, we get a contradiction by the Schwarzenberger condition (S_4^3) [11, p.113] for $s = 4$. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We can assume $h^0(\mathcal{E}^*) = h^1(\mathcal{E}^*) = 0$ by Lemma 1, otherwise we get case (viii). If $h^0(\mathcal{E}(-3)) \neq 0$ then we get case (i) by Proposition 2.1. If $h^0(\mathcal{E}(-3)) = 0$ but $h^0(\mathcal{E}(-2)) \neq 0$ then we get cases (ii) and (iii) by Proposition 2.2. If $h^0(\mathcal{E}(-2)) = 0$ but $h^0(\mathcal{E}(-1)) \neq 0$ then we get cases (iv), (v) and (vi) by Proposition 2.3. If $h^0(\mathcal{E}(-1)) = 0$ but $h^0(\mathcal{E}_H(-1)) \neq 0$ for some hyperplane $H \subset \mathbb{P}^n$ then we get case (vii) by Corollary 2.4 and Proposition 2.5. Furthermore, we claim that there is no vector bundle $\mathcal{E}$ on $\mathbb{P}^5$ such that $\mathcal{E}_H = \Omega_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus k-4}$. As $h^i(\mathcal{E}_H^*(-j)) = 0$ for $i \in \{0,1\}$ and every integer $j \geq 1$, we deduce from Serre's vanishing theorem and the restriction sequence

$$0 \to \mathcal{E}^*(-1-j) \to \mathcal{E}^*(-j) \to \mathcal{E}_H^*(-j) \to 0$$

that $h^i(\mathcal{E}^*(-1)) = 0$ for $i \in \{0,1\}$. Therefore $h^0(\mathcal{E}^*) = h^0(\mathcal{E}_H^*) = k-4$ and hence there exists a rank-4 vector bundle $\mathcal{G}$ such that $\mathcal{E} = \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus k-4}$. Then $c_t(\mathcal{G}) = c_t(\mathcal{E}_H) = 1 + 3t + 4t^2 + 2t^3 + t^4$ and we get a contradiction by the Schwarzenberger condition (S_5^4) [11, p.113] for $s = 5$. This proves the claim. Finally, if $h^0(\mathcal{E}(-1)) = 0$ and $h^0(\mathcal{E}_H(-1)) = 0$ for every hyperplane $H \subset \mathbb{P}^n$ then we get

$$h^0(\mathcal{E}) \leq h^0(\mathcal{E}_H) \leq \cdots \leq h^0(\mathcal{E}_{\mathbb{P}^2}) \leq h^0(\mathcal{E}_{\mathbb{P}^1}) = k+3.$$

Let us see that this is impossible. Consider the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus h^0(\mathcal{E}_{\mathbb{P}^2})} \to \mathcal{E}_{\mathbb{P}^2} \to 0$$

where $\mathcal{K}$ is a vector bundle on $\mathbb{P}^2$ with $h^0(\mathcal{K}) = h^1(\mathcal{K}) = 0$, $c_1(\mathcal{K}) = -3$ and $c_2(\mathcal{K}) = c_2(\mathcal{K}^*) = 9 - c_2 \geq 5$. Then the Hirzebruch-Riemann-Roch theorem

$$\chi(\mathcal{K}) = \frac{1}{2}(c_1(\mathcal{K})^2 - 2c_2(\mathcal{K}) + 3c_1(\mathcal{K})) + rk(\mathcal{K})$$

for vector bundles on $\mathbb{P}^2$ yields

$$0 \leq h^2(\mathcal{K}) = -c_2(\mathcal{K}) + h^0(\mathcal{E}_{\mathbb{P}^2}) - k \leq -5 + h^0(\mathcal{E}_{\mathbb{P}^2}) - k$$

i.e. $h^0(\mathcal{E}_{\mathbb{P}^2}) \geq k+5$, so we get a contradiction. □

As a consequence, we obtain the classification of globally generated vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ with $c_1 = 3$ and no restriction on c_2.

Proof of Corollary 1.2. It follows from Theorem 1.1 and Lemmas 1 and 2. □

Remark 2. Some well-known globally generated vector bundles seem to be *hidden* in Theorem 1.1(viii) (e.g. $T_{\mathbb{P}^2}$) and Corollary 1.2(viii) (e.g. the Tango bundle $\mathcal{T}$ given by the exact sequence $0 \to T_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 7} \to \mathcal{T} \to 0$, see for instance [11, Ch. I, §4]). They can be easily detected in our classification by means of [11, Ch. I, Lemmas 4.3.1 and 4.3.2]. In this context, we point out that the only globally generated vector bundle of rank k on $\mathbb{P}^n$ with $c_1 = 3$ and $k < n$ which does not split is the Tango bundle $\mathcal{T}$ of rank 3 on $\mathbb{P}^4$, as one immediately deduces from Theorem 1.1 and Corollary 1.2 that $c_n(\mathcal{E}) = 0$ if and only if $\mathcal{E} = \Omega_{\mathbb{P}^4}^2(2)^* \cong \wedge^2 T_{\mathbb{P}^4}(-2)$, giving the diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} & = & \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{\mathbb{P}^4}(-2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}^{\oplus 10} & \longrightarrow & \wedge^2 T_{\mathbb{P}^4}(-2) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{\mathbb{P}^4}(-2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}^{\oplus 7} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Remark 3. As in [12], one can easily deduce the classification of triple Veronese embeddings of $\mathbb{P}^r$ in a Grassmannian of $(k-1)$-planes from Theorem 1.1 and Corollary 1.2. The case $k = 2$ has been studied in [8]. Globally generated vector bundles and embeddings in Grassmannians are closely related to matrices of constant rank on projective spaces (see [9] and [7]), but we will not consider this matter in this note.

Remark 4. Following the research initiated in [12], globally generated vector bundles and reflexive sheaves with low first Chern class on projective spaces and quadric hypersurfaces have been recently studied by several authors (see [5], [6], [10], [1], [2], [3] and [4]).

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Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas (CSIC), Campus de Cantoblanco, 28049 Madrid, Spain

E-mail address: `jcsierra@icmat.es`

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, 90123, Palermo, Italy
E-mail address: luca.ugaglia@unipa.it