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Long-range interactions in 1D heterogeneous solids with uncertainty

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Abstract

In this paper, the authors aim to analyze the response of a one-dimensional non-local elastic solid with uncertain Young's modulus. The non-local effects are represented as long-range central body forces between non-adjacent volume elements. Following a non-probabilistic approach, the fluctuating elastic modulus of the material is modeled as an interval field. The analysis is conducted resorting to a novel formulation that confines the overestimation effect involved in interval models. Approximate closed-form expressions are derived for the bounds of the interval displacement field.

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Keywords: non-local elasticity; long-range interactions; interval field; upper bound and lower bound.

1. Introduction

Mechanical, thermal and electrical measure properties of condensed matter are strictly dependent on the observation scales. As an example, stiffness of covalent bonds in a nanotube structure is much lower than the overall stiffness of graphene nanotube. The scaling properties of material characteristics at different observation scales are among the most advanced and fascinating problems in physical and engineering sciences due to the very complex morphologic and physical material structure that may be observed at different scales. This means that, from an engineering perspective, the analysis of heterogeneous solids depends on the observation scale of the considered problem. Indeed, as the heterogeneity of material properties may be observed at comparable material lengths, then homogeneization methods and/or some recent variants may be used to define overall material properties. As far as, instead, materials are markedly inhomogeneous at very different observation scales, then homogeneization methods are no more capable to handle reliable description of material response since dispersion of travelling waves, plastic shear bands, voids nucleation and some other well-known phenomena cannot be detected. A theoretical framework to deal with such aspects has been provided in terms of proper enrichment of classical and advanced solid mechanics yielding the so-called non-local theories: the strong

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[1,2] and the weak non-local elastic models [3]. Basically, the common idea beyond these theories is the attempt to introduce the complex nature of the matter by means of additional terms in the stress-strain relations of the materials. Those terms may be of very different nature: integral terms of the strain field in strong non-local elasticity and gradients of the strain field in the weak formulations.

A quite different description of non-local effects in solid body mechanics, referred to as *mechanically-based model* of non-local elasticity theory has been recently proposed [4-6]. Non-local effects handled by means of this theory are modeled as central body forces that depend upon the product of interacting volumes, on the relative displacements of the centroids as well as on a proper distance-decaying function accounting for the decay of the inner body interactions as the distance between the centroids increases. In this context, a very important aspect is represented by the presence of uncertain local and non-local effects [7]. Indeed, almost all structural systems exhibit physical and geometrical uncertainties to some degree. These sources of uncertainty, which affect to a certain extent the structural response, are usually described following two contrasting points of view, known as probabilistic and non-probabilistic approaches [8]. Within a non-probabilistic context, the interval model turns out to be the most suitable approach when only the upper and lower bounds of a non-deterministic property are well defined. Indeed, this model is derived from the interval analysis [9-11] in which the number is treated as an interval variable ranging between its lower and upper bounds. Unfortunately, the "ordinary" interval analysis [9] suffers from the so-called *dependency phenomenon* which occurs when an expression contains multiple instances of one or more interval variables and often leads to an overestimation of the interval solution [11,12]. To limit the conservatism due to the dependency phenomenon, the so-called *generalized interval analysis* [13] and the *affine arithmetic* [14] have been introduced in the literature. Another shortcoming of the ordinary interval analysis is the inability to quantify any sort of dependency between two or more uncertain variables. To overcome such limitation, the so-called *interval field* concept [15, 16] has been introduced.

This study deals with the analysis of 1D non-local heterogeneous solids with uncertain Young's modulus. The problem is tackled in the context of the *mechanically-based model* of non-local elasticity by introducing a novel definition of the interval field to model the fluctuating material property. The *improved interval analysis*, recently proposed by the first two authors [17], is applied in conjunction with the approach proposed by Impollonia and Muscolino [18] to derive closed-form expressions for the bounds of the interval axial displacement field.

To assess the effectiveness of the proposed procedure, a non-local elastic bar under tension with uncertain Young's modulus is analyzed. For comparison purposes, a stochastic modeling of the uncertain Young's modulus is also considered.

2. The long-range interaction model in a 1D heterogeneous solid

In this section, the mechanically-based model of 1D solids with long-range interactions, recently proposed by Di Paola et al. [4-6], will be briefly summarized. Let us consider a 1D elastic bar of length L referred to a coordinate system $0-x$ positive rightward (Fig. 1a). In the context of the mechanically-based model of non-local elasticity, it is thought that the actions applied to a volume element $dV(x)$ at the abscissa x consist of three contributions: the well-known local Cauchy stress, $\sigma^{(l)}(x)$, the external body force field, $b(x)$, and the additional central body forces exerted by non-adjacent volumes $dV(\xi)$, located at the abscissas ξ (Fig. 1b). Moreover, it is assumed that the long-range interactions between volumes $dV(x)$ and $dV(\xi)$ depend on the product of the elementary interacting masses, $dM(x)=\rho(x)dV(x)$ and $dM(\xi)=\rho(\xi)dV(\xi)$, $\rho(x)$ and $\rho(\xi)$ being the mass density of the material at locations x and ξ, as well as on their relative axial displacement field $\eta(x,\xi)=u(\xi)-u(x)$, i.e.:

$$q(x,\xi)\,dM(x)\,dM(\xi)=c_q\,\rho^2 A(x)\,A(\xi)\,g(x,\xi)\,\eta(x,\xi)\,dx\,d\xi \tag{1}$$

where $q(x,\xi)=c_q\,g(x,\xi)\eta(x,\xi)$ is the specific long-range force; $[c_q]=F/LM^2$ is a physical material-dependent force constant; $A(x)$ is the cross-section at the abscissa x; $\rho=\rho(x)=\rho(\xi)$ denotes the mass density of the

material herein assumed constant along the bar; $g(x,\xi)$ is a material-dependent, symmetric real-valued scalar function which must be strictly positive to satisfy the Drucker stability criterion. Moreover, the function $g(x,\xi)$ is monotonically decreasing with the distance $d=|x-\xi|$ between interacting volume elements.

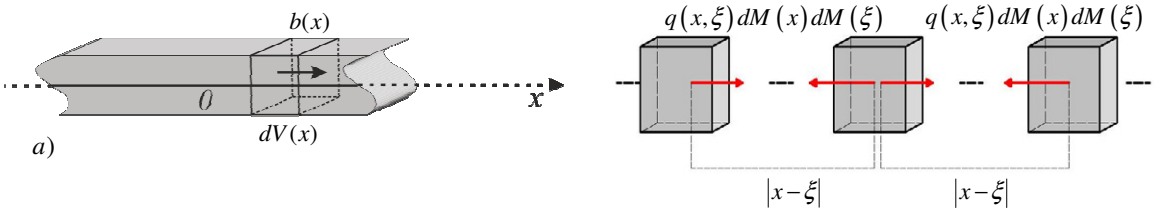

Fig. 1 (a) 1D elastic solid; (b) Long-range interactions.

Then, the equilibrium equation of the generic volume element located at the abscissa x can be written, for a 1D heterogeneous solid, in the following form [5, 7]:

$$\frac{d}{dx}\left[E^*(x)\,A(x)\frac{du(x)}{dx}\right]+c_q\,\rho^2 A(x)\int_0^L A(\xi)\,g(x,\xi)\eta(x,\xi)d\xi=-A(x)b(x) \tag{2}$$

where $E^*(x)$ is a non-local elastic modulus, related to the measure of the Young's modulus of the material $E(x)$ as $E^*(x)=E(x)\beta_1$, with $0\le\beta_1\le 1$ being a dimensionless real coefficient which weights the amount of local interactions. The non-local elastic modulus $E^*(x)$ represents the value of the elastic modulus measured in a specimen at a sufficient distance from the boundaries that any edge effect may be disregarded. Finally, the boundary conditions associated to the integro-differential equation in Eq.(2) read [5]:

$$\begin{aligned} &u(0)=u_0 \quad \text{or} \quad E^*(x)\,A(x)\left[du(x)/dx\right]\big|_{x=0}=-F_0;\\ &u(L)=u_L \quad \text{or} \quad E^*(x)A(x)\left[du(x)/dx\right]\big|_{x=L}=F_L. \end{aligned} \tag{3a,b}$$

A remarkable feature of the mechanically-based model of non-local elasticity is that the static boundary conditions in Eqs. (3a,b) involve only the local Cauchy stress. Indeed, the non-local effects, being represented by long-range body forces, vanish at the edges of the body where the applied external tractions are equilibrated only by the contact Cauchy stress (see e.g. [5, 6]).

The physical implications of the mechanically-based model of non-local elasticity ruled by the integro-differential equation (2) can be obtained in the context of the finite difference method by introducing a proper discretization of the bar into n volume elements $\Delta V_j\approx A_j\Delta x_j=A_j\Delta x$ with $j=1,2,...n$. In this setting, the equilibrium equation of the j-th volume element reads:

$$\frac{1}{\Delta x^2}\left[h_j\,u_{j+1}-\left(h_j+h_{j-1}\right)u_j+h_{j-1}u_{j-1}\right]+c_q A_j\rho^2\sum_{r=1}^{n}g\left(x_j,x_r\right)\left(u_r-u_j\right)A_r\Delta x=-A_j b\left(x_j\right) \tag{4}$$

where u_j is the axial displacement at the j-th grid point $x_j=(j-1)\Delta x$; $h_j=h(x_j)=E^*(x_j)A(x_j)=E_j^*A_j$ with $j=1,2,...,n$. Equation (4) may be cast in a more convenient form multiplying both sides by Δx and defining the elastic coefficients:

$$k_j^*=\frac{E_j^*A_j}{\Delta x}=\frac{h_j}{\Delta x};\quad k_{jr}^{(nl)}=c_q A_j A_r\left(\rho\,\Delta x\right)^2 g\left(x_j,x_r\right). \tag{5a,b}$$

So operating, the displacement field at the grid points, namely u_j ($j=1,2,...,n$), can be evaluated as solution of a set of linear algebraic equations that may be conveniently written in matrix form as:

$$\mathbf{K}_0\mathbf{u}=\left(\mathbf{K}^*+\mathbf{K}^{(nl)}\right)\mathbf{u}=\mathbf{F} \tag{6}$$

where $\mathbf{F}$ is the forcing vector whose j-th element, $F_j=b(x_j)A_j\Delta x$, is the resultant of the body force field applied to the grid point x_j ; $\mathbf{u}=[u_1 \ \ ... \ \ ... \ \ u_n]^T$ is the n-vector gathering the nodal displacements; and $\mathbf{K}_0$ is the stiffness matrix of the elastic problem, sum of the local and non-local stiffness matrices, $\mathbf{K}^*$ and $\mathbf{K}^{(nl)}$, respectively, defined as:

$$\mathbf{K}^*=\begin{bmatrix} k_1^* & -k_1^* & 0 & 0 & \cdots & 0 \\ -k_1^* & k_1^*+k_2^* & -k_2^* & 0 & \cdots & 0 \\ 0 & -k_2^* & k_2^*+k_3^* & -k_3^* & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_{n-1}^*+k_n^* \end{bmatrix};$$

$$\mathbf{K}^{(nl)}=\begin{bmatrix} k_{11}^{(nl)} & -k_{12}^{(nl)} & -k_{13}^{(nl)} & \cdots & -k_{1n}^{(nl)} \\ \cdots & k_{22}^{(nl)} & -k_{23}^{(nl)} & \cdots & -k_{2n}^{(nl)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \text{SYM} & \cdots & \cdots & -k_{n-1\,n}^{(nl)} \\ \cdots & \cdots & \cdots & \cdots & k_{nn}^{(nl)} \end{bmatrix} \tag{7a,b}$$

where the diagonal terms of the matrix $\mathbf{K}^{(nl)}$ read $k_{jj}^{(nl)}=\sum_{r=1,r\neq j}^{n}k_{jr}^{(nl)}$. In Eq.(7b), the main features of the distance-decaying function, $g(x,\xi)$, are fulfilled and the matrices $\mathbf{K}^*$ and $\mathbf{K}^{(nl)}$ turn out to be symmetric and positive-definite matrices also for heterogeneous materials.

The spatial discretization of the integro-differential Eq. (2) clearly shows that the discrete counterpart of the continuous mechanically-based model is equivalent to a point-spring network with contact forces represented by linear springs of axial stiffness k_j^* (see Eq. (5a)) and long-range interactions described by linear springs of distance-decaying stiffness $k_{jr}^{(nl)}$ (see Eq. (5b)) connecting all non-adjacent points.

3. Long-range interactions in presence of uncertain-but-bounded elastic modulus

3.1. Improved interval analysis

Following a non-probabilistic approach, the Young's modulus of the material $E^*(x)$, at a generic abscissa x , is herein modelled as a variable which can assume real values inside a real interval. According to the *interval analysis* [9, 11], let us denote by $\mathbb{IR}$ the set of all real interval numbers $\alpha^I\triangleq[\underline{\alpha},\overline{\alpha}]\in\mathbb{IR}$ such that $\underline{\alpha}\le\alpha\le\overline{\alpha}$. The symbols $\underline{\alpha}$ and $\overline{\alpha}$ denote the lower bound and the upper bound of the interval, respectively, while the apex I characterizes the interval variables. Mathematical derivations involving real numbers α bounded by intervals should be performed by means of the *"ordinary" interval analysis* [9].

Unfortunately, in the *"ordinary" interval analysis* the interval variables are assumed independent and any form of dependency cannot be taken into account. Indeed, it is well known that the "ordinary" interval analysis suffers from the so-called *dependency phenomenon*, [11, 12] which often introduces a high amount of conservatism leading to useless results for real sized structures. This is due to the inability of ordinary interval

arithmetic to keep track of the dependency between interval variables. Therefore, when an expression contains multiple instances of one or more interval variables, the operand interval numbers are erroneously treated as independent. In an attempt to limit the catastrophic effects of the dependency phenomenon, the *generalized interval analysis* [13] and the *affine arithmetic* [14] have been introduced in the literature. In these formulations, each intermediate result is represented by a linear function with a small remainder interval [19]. In the context of the stochastic analysis of structures with uncertain-but-bounded parameters under random excitation, following the philosophy of the *affine arithmetic*, Muscolino and Sofi [17] proposed an *improved interval analysis* based on the definition of the so-called *extra symmetric unitary interval* (EUI) variable $\hat{e}_i^I \triangleq [-1,+1]$ such that $\hat{e}_i^I - \hat{e}_i^I = 0$, $\hat{e}_i^I \times \hat{e}_i^I = [1,1]$ and $\hat{e}_i^I \times \hat{e}_j^I = [-1,+1],\ \ i \neq j$. The subscript i in the interval variable $\hat{e}_i^I$ indicates that this variable is associated to the i-th uncertain-but-bounded parameter α_i^I. Then, introducing the midpoint value (or mean), $\alpha_{0,i} = (\underline{\alpha}_i + \overline{\alpha}_i)/2$, and the deviation amplitude (or radius), $\Delta\alpha_i = (\overline{\alpha}_i - \underline{\alpha}_i)/2$, and according to the *improved interval analysis*, the interval variable α_i^I can be written in *affine form* as follows:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i \, \hat{e}_i^I . \tag{8}$$

The *improved interval analysis* has proved able to limit the conservatism due to the *dependency phenomenon* since by means of the EUI variables it allows to keep track of the dependency between interval variables throughout interval computations.

3.2. Interval fields

The second main shortcoming of the "ordinary" interval analysis, in the context of the analysis of structures with uncertain parameters, is that it does not allow for the quantification of any sort of dependency between two or more uncertain variables. In order to cope with this problem, Moens et al. [15] introduced the concept of interval field. An interval field is able to define a form of dependency between adjacent interval values that cannot differ as much as values that are further apart. In order to gain further insight into this concept, let us consider the case in which the variability of the uncertain elastic modulus along the 1D non-local continuum is represented by the following interval function:

$$E^{*I}(x) = \left[\underline{E}^*(x), \overline{E}^*(x) \right], \quad x \in [0, L] \tag{9}$$

with lower bound $\underline{E}^*(x)$ and upper bound $\overline{E}^*(x)$. Without loss of generality, it is assumed that $E^{*I}(x)$ can be expressed by introducing a dimensionless interval function $\Delta B^I(x)$ with zero midpoint and deviation $\Delta B(x) \ll 1$ for all $x \in [0, L]$, i.e.:

$$E^{*I}(x) = E_0^* \left[1 + \Delta B^I(x) \right], \quad x \in [0, L]. \tag{10}$$

Then, the midpoint value and deviation of the interval function $E^{*I}(x)$ are given, respectively, by:

$$\operatorname{mid}\left\{ E^{*I}(x) \right\} = \frac{\overline{E}^*(x) + \underline{E}^*(x)}{2} = E_0^* ; \quad \frac{\operatorname{dev}\left\{ E^{*I}(x) \right\}}{E_0^*} = \frac{\overline{E}^*(x) - \underline{E}^*(x)}{2E_0^*} = \Delta B(x) \tag{11}$$

where $E_0^* \in \mathbb{R}$. Let us assume now that the spatial dependency between adjacent interval values that cannot differ as much as values that are further apart is governed by the deterministic symmetric non-negative bounded function $\Gamma_{\Delta B}(x, \xi)$, defined as the midpoint of the dimensionless interval function $\Delta B^I(x)\Delta B^I(\xi)$ and related to the midpoint of $E^{*I}(x)E^{*I}(\xi)$ by the following relationship:

$$\Gamma_{\Delta B}(x, \xi) = \operatorname{mid}\left\{ \Delta B^I(x)\Delta B^I(\xi) \right\} \equiv \frac{\operatorname{mid}\left\{ E^{*I}(x)E^{*I}(\xi) \right\}}{\left(E_0^* \right)^2} - 1, \quad x, \xi \in [0, L]. \tag{12}$$

Based on Eq. (12), the function $\Gamma_{\Delta B}(x,\xi)$ may be viewed as the non-probabilistic counterpart of the autocorrelation function characterizing random fields. This analogy suggests to decompose the function $\Gamma_{\Delta B}(x,\xi)$ as:

$$\Gamma_{\Delta B}(x,\xi)=\sum_{i=1}^{\infty}\lambda_i\,\psi_i(x)\,\psi_i(\xi);\qquad \int_0^L \psi_i(x)\,\psi_j(x)\,\mathrm{d}x=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases} \tag{13a,b}$$

where λ_i, $(i=1,2,\ldots)$, is the i-th *eigenvalue* of the bounded symmetric non-negative function, $\Gamma_{\Delta B}(x,\xi)$, and $\psi_i(x)$ is the corresponding *eigenfunction*, which satisfies the orthogonality conditions (13b). The eigenproperties of $\Gamma_{\Delta B}(x,\xi)$ are found by solving the following homogeneous Fredholm integral equation of the second kind:

$$\int_0^L \Gamma_{\Delta B}(x,\xi)\,\psi_i(x)\,\mathrm{d}x=\lambda_i\,\psi_i(\xi). \tag{14}$$

The eigenvalues solutions of this eigenproblem are real positive numbers and the associated eigenfunctions are real functions. Notice that the expansion in Eq. (13a) is usually truncated after N terms to reduce the computational burden of the subsequent structural analysis. According to the philosophy of the approach proposed by Verhaeghe et al. [16] and adopting the *improved interval analysis*, once the function $\Gamma_{\Delta B}(x,\xi)$ has been decomposed as in Eq. (13), the interval function $E^{*I}(x)$ can be rewritten in *affine form* as:

$$E^{*I}(x)=E_0^*\left[1+\sum_{i=1}^{N}\sqrt{\lambda_i}\;\psi_i(x)\,\hat{e}_i^I\right],\quad x\in[0,L]. \tag{15}$$

Notice that the dimensionless interval function $\Delta B^I(x)$ in Eq. (10) is now expressed as the superposition of N deterministic functions multiplied by the corresponding EUI variables $\hat{e}_i^I$. Equation (15) may be regarded as a proper extension of the Karhunen-Loève decomposition to interval fields.

3.3. Bounds of the solution

Substituting the expression (15) of the interval elastic modulus $E^{*I}(x)$ into Eq.(2), the following interval integro-differential equation is obtained:

$$\begin{aligned}E_0^*\frac{d}{dx}\left[A(x)\frac{du^I(x)}{dx}\right]+E_0^*\sum_{i=1}^{N}\sqrt{\lambda_i}\;\hat{e}_i^I\,\frac{d}{dx}\left[A(x)\psi_i(x)\frac{du^I(x)}{dx}\right]\\ +c_q\rho^2A(x)\int_0^L A(\xi)\,g(x,\xi)\left[u^I(\xi)-u^I(x)\right]d\xi=-A(x)b(x)\end{aligned} \tag{16}$$

where $u^I(x)$ denotes the interval displacement function. Equation (16) must be supplemented by the pertinent kinematic and static boundary conditions, herein assumed deterministic, i.e.:

$$\begin{aligned}&u^I(0)=u_0\quad\text{or}\quad E^{*I}(x)\,A(x)\left[du^I(x)/dx\right]\Big|_{x=0}=-F_0;\\ &u^I(L)=u_L\quad\text{or}\quad E^{*I}(x)A(x)\left[du^I(x)/dx\right]\Big|_{x=L}=F_L.\end{aligned} \tag{17a,b}$$

The solution of Eq.(16) can be obtained by applying the finite difference method which leads to a discretized model analogous to the one derived in the previous section within a deterministic setting. To this aim, let us

introduce a discretization grid of the domain $[0, L]$ into n intervals Δx so that, after multiplying both sides by Δx, Eq.(16) is discretized in the form:

$$\frac{E_0^*}{\Delta x}\left[A_j\, u_{j+1}^I - \left(A_j + A_{j-1}\right) u_j^I + A_{j-1}\, u_{j-1}^I\right] + \sum_{i=1}^{N}\left[s_{ij}\; u_{j+1}^I - \left(s_{ij} + s_{ij-1}\right) u_j^I + s_{ij-1}\; u_{j-1}^I\right]\hat{e}_i^I + c_q\left(\rho\Delta x\right)^2 A_j \sum_{r=1}^{n} A_r\, g\left(x_j, x_r\right)\left(u_r^I - u_j^I\right) = -b\left(x_j\right) A_j \Delta x \tag{18}$$

where $u_j^I = u^I((j-1)\Delta x)$, $s_{ij} = E_0^*\, \psi_{ij}\, A_j \sqrt{\lambda_i}\,/\,\Delta x$ and $\psi_{ij} = \psi_i((j-1)\Delta x)$, $(j = 1, 2, ..., n;\; i = 1, 2, ..., N)$. The set of linear interval equations in Eq.(18) can be written in compact form as:

$$\mathbf{K}^I \mathbf{u}^I = \left(\mathbf{K}_0 + \Delta\mathbf{K}_B^I\right)\mathbf{u}^I = \mathbf{F} \tag{19}$$

where $\mathbf{u}^I$ is the vector of order n collecting the interval displacements u_j^I $\left(j = 1, 2, ...n\right)$ at the grid points; and $\mathbf{K}_0 = \mathbf{K}^* + \mathbf{K}^{(nl)}$ is the stiffness matrix of the nominal system which includes the contribution of both local and non-local stiffness (see Eq. (6)). Notice that the interval stiffness matrix $\mathbf{K}^I$ in Eq. (19) contains the local interval matrix $\Delta\mathbf{K}_B^I$ associated to the EUI variables, whereas the vector $\mathbf{F}$ is completely analogous to the one introduced in Section 2 for the nominal system. The additional interval stiffness matrix $\Delta\mathbf{K}_B^I$ can be written as:

$$\Delta\mathbf{K}_B^I = \sum_{i=1}^{N} \Delta\mathbf{S}_{B,i}\, \hat{e}_i^I \tag{20}$$

where $\Delta\mathbf{S}_{B,i}$ is a $n \times n$ tridiagonal symmetric and positive-definite matrix, given by:

$$\Delta\mathbf{S}_{B,i} = \begin{bmatrix} s_{i1} & -s_{i1} & 0 & 0 & \cdots & 0 \\ -s_{i1} & s_{i1} + s_{i2} & -s_{i2} & 0 & \cdots & 0 \\ 0 & -s_{i2} & s_{i2} + s_{i3} & -s_{i3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s_{in-1} + s_{in} \end{bmatrix}. \tag{21}$$

The problem is now to determine the narrowest interval $\mathbf{u}^I$ containing all possible vectors, $\mathbf{u}$, satisfying the set of linear interval equations (19). Preliminarily, it is useful to underline that the square interval matrix $\mathbf{K}^I$ is regular, that is each matrix $\mathbf{K} \in \mathbf{K}^I$ is non-singular [20], then the solution of Eq.(19) exists for all $\mathbf{K} \in \mathbf{K}^I$ and can be written, by adopting the interval formalism, as:

$$\mathbf{u}^I = \left(\mathbf{K}_0 + \Delta\mathbf{K}_B^I\right)^{-1}\mathbf{F} = \left(\mathbf{K}_0 + \sum_{i=1}^{N} \Delta\mathbf{S}_{B,i}\, \hat{e}_i^I\right)^{-1}\mathbf{F}. \tag{22}$$

Moreover, under the assumption of small dimensionless deviation of the interval elastic modulus, i.e. $\Delta B(x) \ll 1$ for all $x \in [0, L]$, the numerical solution of the set of linear interval equations (19), is herein evaluated starting from the following decomposition of the local interval matrix $\Delta\mathbf{K}_B^I$:

$$\Delta\mathbf{K}_B^I = \sum_{i=1}^{N} \Delta\mathbf{S}_{B,i}\, \hat{e}_i^I = \sum_{i=1}^{N}\sum_{\ell=1}^{n} \mathbf{s}_{B,i\ell}\, \mathbf{w}_\ell^T\, \hat{e}_i^I \tag{23}$$

where $\mathbf{s}_{B,i\ell}$ is the ℓ-th column of the $n\times n$ matrix $\Delta\mathbf{S}_{B,i}$ in Eq. (21) and $\mathbf{w}_\ell$ is a column vector of order n containing all zeros except the ℓ-th element which is equal to 1. It follows that, according to the formulation described by Impollonia and Muscolino [18] and after some algebra, the approximate inverse of the interval matrix $(\mathbf{K}_0+\Delta\mathbf{K}_B^I)^{-1}$, with the position (23), can be evaluated in explicit form as:

$$\left(\mathbf{K}_0+\Delta\mathbf{K}_B^I\right)^{-1}\approx\mathbf{K}_0^{-1}-\sum_{i=1}^{N}\sum_{\ell=1}^{n}\frac{\hat{e}_i^I}{1+\hat{e}_i^I d_{B,i\ell}}\mathbf{D}_{B,i\ell} \tag{24}$$

where the following quantities have been introduced:

$$d_{B,i\ell}=\mathbf{w}_\ell^T\mathbf{K}_0^{-1}\mathbf{s}_{B,i\ell};\quad \mathbf{D}_{B,i\ell}=\mathbf{K}_0^{-1}\mathbf{s}_{B,i\ell}\mathbf{w}_\ell^T\mathbf{K}_0^{-1}. \tag{25a,b}$$

Equation (24) holds if and only if $-1<d_{B,i\ell}<1$. Upon rewriting in affine form the ratio appearing in the summation in Eq. (24), the interval vector solution $\mathbf{u}^I\in\mathbb{IR}^n$ can be determined in closed-form as follows:

$$\mathbf{u}^I\approx\left[\mathbf{K}_0^{-1}+\sum_{i=1}^{N}\sum_{\ell=1}^{n}\left(a_{0,i\ell}+\Delta a_{i\ell}\,\hat{e}_i^I\right)\mathbf{D}_{B,i\ell}\right]\mathbf{F} \tag{26}$$

where the quantities $a_{0,i\ell}$ and $\Delta a_{i\ell}$, after some interval algebra, can be written as:

$$a_{0,i\ell}=\frac{d_{B,i\ell}}{1-d_{B,i\ell}^2};\quad \Delta a_{i\ell}=\frac{1}{1-d_{B,i\ell}^2}. \tag{27a,b}$$

From an engineering point of view, within the interval framework, the main goal of structural analysis is the evaluation of the narrowest interval which certainly contains the response. This interval is bounded by the lower and upper bounds, $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$, of the interval response vector $\mathbf{u}^I$ satisfying Eq.(26). Adopting the interval formalism, based on the explicit solution in Eq. (26), the vectors $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ can be evaluated as follows:

$$\underline{\mathbf{u}}=\mathbf{u}_0-\Delta\mathbf{u};\quad \overline{\mathbf{u}}=\mathbf{u}_0+\Delta\mathbf{u} \tag{28}$$

where

$$\mathbf{u}_0=\left(\mathbf{K}_0^{-1}+\sum_{i=1}^{N}\sum_{\ell=1}^{n}a_{0,i\ell}\mathbf{D}_{B,i\ell}\right)\mathbf{F};\quad \Delta\mathbf{u}=\sum_{i=1}^{N}\left|\sum_{\ell=1}^{n}\Delta a_{i\ell}\mathbf{D}_{B,i\ell}\mathbf{F}\right| \tag{29a,b}$$

are the midpoint and the deviation of the interval displacement vector $\mathbf{u}^I$. The symbol $|\bullet|$ in Eq. (29b) denotes the component wise absolute value.

4. Numerical application

The proposed procedure has been applied to the analysis of a non-local elastic bar with uncertain Young's modulus, fixed at $x=0$ and subjected to a tensile force $F=1$ kN at the free end $x=L$. Numerical simulations have been carried out assuming the following geometrical and mechanical properties: cross-section $A=1\ \text{cm}^2$, length $L=100$ cm, nominal Young's modulus $E_0=2.1\times10^7\ \text{N cm}^{-2}$, mass density $\rho=7850\times10^{-6}\ \text{Kg cm}^{-3}$. Furthermore, the distance-decaying function governing the long-range forces in Eq.(1) has been selected as:

$$g(x,\xi)=\exp\left(-|x-\xi|/l_0\right) \tag{30}$$

where l_0 denotes the internal length material scale. The material constant c_q in Eq. (1) has been set equal to $c_q = c/\rho^2 = E_0(1-\beta_1)/(2A^2 l_0 \rho^2)$ [7]. The parameters l_0 and β_1 governing the non-local behavior should be determined from experiments [21] or by fitting the theoretical model to molecular dynamics simulation results [22]. Such parameters are here selected so as to enhance non-local effects, say $l_0 = 5$ cm and $\beta_1 = 0.7$. For comparison purposes, the uncertain Young's modulus $E^*(x)$ has been modeled within both the interval and stochastic framework. In the latter case, the expression $E^*(x) = E_0^*[1+\tilde{B}(x)]$ has been assumed where $\tilde{B}(x) < 1$ denotes a homogeneous zero-mean Gaussian random field with autocorrelation function $R_{\tilde{B}\tilde{B}}(x,\xi) \equiv R_{\tilde{B}\tilde{B}}(|x-\xi|)$ herein assumed coincident with the deterministic symmetric non-negative bounded function $\Gamma_{\Delta B}(x,\xi)$ governing the spatial dependency between adjacent interval values of the dimensionless interval deviation function $\Delta B^I(x)$ in Eq. (10). In particular, the following exponential function has been selected:

$$\Gamma_{\Delta B}(x,\xi) \equiv R_{\tilde{B}\tilde{B}}(|x-\xi|) = \sigma_{\tilde{B}}^2 \exp\left(-|x-\xi|/l_{\tilde{B}}\right) \tag{31}$$

where $\sigma_{\tilde{B}}^2 = 0.05$ and $l_{\tilde{B}}$ is the correlation length herein taken variable to investigate the effects of spatial correlation on the response. Numerical results have been obtained by applying the finite difference method using a uniform grid with $n = 100$ subdivisions.

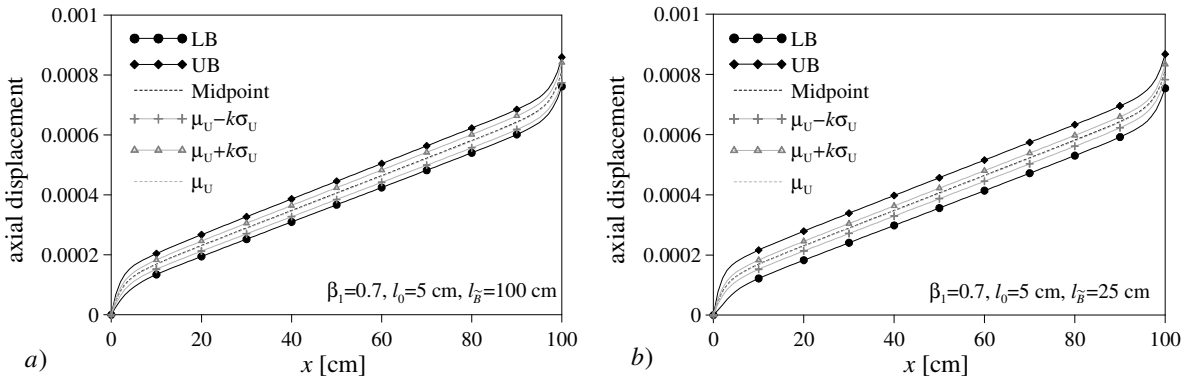

Fig. 2 Comparison between the interval and stochastic regions of the non-local axial displacement: (a) $l_{\tilde{B}} = 100$ cm ; (b) $l_{\tilde{B}} = 25$ cm .

In order to enlighten the main differences between the interval approach and the classical stochastic model, in Fig. 2, the region of the non-local displacement field provided by the proposed improved interval analysis is compared with the *confidence interval* of the stochastic response for two different values of the correlation length $l_{\tilde{B}}$. The confidence interval is bounded by the values $\mu_U - k\sigma_U$ and $\mu_U + k\sigma_U$, with μ_U and σ_U denoting the mean-value and the standard deviation of the random axial displacement, while $k = 1.645$. As expected, the correlation length $l_{\tilde{B}}$ (see Eq. 31) affects both the confidence interval and the region of the displacement field obtained via interval analysis. Notice that for both the values of $l_{\tilde{B}}$ herein considered the interval of confidence provided by stochastic analysis turns out to be tighter than the interval region, consistently with the meaning of the interval model.

5. Conclusions

An approach for the analysis of 1D heterogeneous non-local elastic solids with uncertain Young's modulus has been presented. Within the context of a recently proposed mechanically-based model, non-local effects have been represented as long-range interactions between non-adjacent volume elements which depend on the product

of interacting masses, as well as on their relative displacements by means of a proper material-dependent, distance-decaying function. Following a non-probabilistic approach, the fluctuations of the Young's modulus have been described by introducing a novel interval field definition which allows to account for the dependency between interval values at various locations thorough a deterministic symmetric non-negative bounded function. The latter may be viewed as the counterpart of the autocorrelation function characterizing random fields. In view of this analogy, by properly extending the Karhunen-Loève decomposition, the interval field is decomposed as superposition of deterministic functions multiplied by unitary interval variables which allow to keep track of the spatial dependency. Within a finite difference discretization of the interval integro-differential equation governing the displacement field, approximate closed-form expressions of the bounds of the interval response have been derived. For validation purposes, numerical results concerning a non-local elastic bar under tension with interval Young's modulus have been contrasted with those pertaining to a stochastic modeling of the uncertain material property. The confidence interval provided by the stochastic analysis has been found to be enclosed by the interval region, at least for the selected case study.

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