



A POSSIBLE QUANTIC MOTIVATION OF THE STRUCTURE OF QUANTUM GROUP

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Abstract

We propose a (first) simple natural model of a non-finitely generated braided non-commutative Hopf Algebra, suggested by the elementary quantum mechanics.

1. Introduction

The idea of quantum group has been introduced, in a pure mathematical context, independently by Drinfeld ([16], [17]) and Jimbo ([39], [40]), who used the adjective ‘quantum’ for the fact that a such structure is obtained quantizing (according to geometric quantization) a Poisson symplectic structure introduced on the algebra $\mathfrak{F}(G, \mathbb{C})$ of the differentiable \mathbb{C} -functions defined on a Lie group G . This structure of quantum group is the result of a non-commutative Hopf algebra achieved as non-commutative (non-trivial) deformations ([5]) of the initial Hopf algebra $\mathfrak{F}(G, \mathbb{C})$, or of the universal enveloping Hopf algebras of an arbitrary semisimple Lie algebra. A little later, Manin (see [53]) and Woronowicz ([64]) independently constructed non-commutative deformations of the algebra of functions on $SL_2(\mathbb{C})$ and SU_2 , respectively.

On the other hand, the study of quantum group was inspired by the works of

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physicists on integrable XYZ models with highest spin. Moreover, Faddeev and coworkers ([25], [26], [27]), had already used similar structures in mathematical physics, for the algebra of quantum inverse scattering transform of the theory of integrable models, as well Kulish and Reshetikhin ([43]) and Sklyanin ([44]).

However, from elementary quantum mechanics, it is possible constructs a simple model of a braided non-commutative Hopf algebra, through an algebraically modern re-examination of few elementary notions of the original papers of Heisenberg et al. [33] on matrix quantum mechanics.

The basic algebraic structure for our model is that of groupoid. Independently, Baer ([4]), and Brandt ([8]) in some of his researches on quadratic forms on \mathbb{Z} , introduced a well-defined new algebraic structure, that we will call Ehresmann-Baer-Brandt groupoid (or EBB-groupoid) (see [9], [15], [37], [45], [49]). It is a partial algebraic structure because is an algebraic system provided by a partial (or no total) composition law, that is, not defined everywhere.

Historically, Eilenberg and Steenrod ([23]) formulated the notion of abstract category on the basis of the previous notion introduced by Eilenberg and MacLane ([24]), considering a groupoid as a special category in which every morphism has an inverse (see also [50]). A little later, the notions of abstract category and groupoid appeared too in the work of Ehresmann ([21]).

However, retrospectively, the common substructure to all these notions of groupoid, is that of multiplicative system (or Eilenberg-Steenrod groupoid, or ES-groupoid), say $(G, \mathcal{D}_\diamond, \diamond)$, where \diamond is a partial composition law over G , defined on a domain $\mathcal{D}_\diamond \subseteq G \times G$. It was introduced by [23].

The further categorial (over) structure of an ES-groupoid, leads to the notion of *Eilenberg-MacLane groupoid* (or *EM-groupoid*), introduced in [22], and to the equivalent¹ notion of *Ehresmann groupoid* (or *E-groupoid*), introduced in [21]. We will use the notion of E-groupoid.

An *E-groupoid* is an algebraic system of the type $(G, G^{(0)}, r, s, \star)$, where $G, G^{(0)}$ are non-void sets such that $G^{(0)} \subseteq G$, $r, s : G \rightarrow G^{(0)}$ and $\star : G^{(2)} \rightarrow G$ with $G^{(2)} = \{(g_1, g_2) \in G \times G, s(g_1) = r(g_2)\}$, satisfying the following conditions:

¹See [62] for a proof of this equivalence.

- ₁ $s(g_1 \star g_2) = s(g_2), r(g_1 \star g_2) = r(g_1) \quad \forall (g_1, g_2) \in G^{(2)}$;
- ₂ $s(g) = r(g) = g \quad \forall g \in G^{(0)}$;
- ₃ $g \star \alpha(s(g)) = \alpha(r(g)) \star g = g \quad \forall g \in G$;
- ₄ $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3) \quad \forall g_1, g_2, g_3 \in G$;
- ₅ $\forall g \in G, \exists g^{-1} \in G : g \star g^{-1} = \alpha(r(g)), g^{-1} \star g = \alpha(s(g))$,

where $\alpha : G^{(0)} \hookrightarrow G$ is the immersion of $G^{(0)}$ into G . The maps r, s are called, respectively, *range* (or *target*) and *source*, G is the support and $G^{(0)}$ is the set of units of the groupoid. g^{-1} is said the (bilateral) inverse of g , so that we have an inversion map of the type $i_G : g \rightarrow g^{-1}$, defined on the whole of G . Instead, the map \star is a partial map defined on $G^{(2)} \subseteq G \times G$, and not on the whole of $G \times G$. However, from now on, for simplicity, we will suppress the symbol α in 3. and 5., writing only $r(g), s(g)$ instead of $\alpha(r(g)), \alpha(s(g))$. However, an E -groupoid is more general than the first notion of groupoid used by W. Brandt and A. Baer, that we call Baer-Brandt groupoid (or BB-groupoid); indeed, a BB-groupoid may be defined as an E -groupoid satisfying the further condition

$$\bullet_6 \quad \forall g, g'' \in G, \exists g' \in G \text{ such that } (g, g') \in G^{(2)}, (g', g'') \in G^{(2)},$$

so that, we will call an Ehresmann-Baer-Brandt groupoid (or EBB-groupoid²), an algebraic system satisfying \bullet_i $i = 1, \dots, 6$.

In view of the applications to quantum mechanics, it is important the following particular examples of EBB-groupoid (see [15], Section II.5).

If X is an abstract (non-empty) set, let $G = X^2$, $G^{(0)} = \Delta(X^2) = \{(x, x); x \in X\}$, $r = pr_1 : (x, y) \rightarrow x$, $s = pr_2 : (x, y) \rightarrow y \quad \forall x, y \in X$ and $(x, y) \star (y, z) = (x, z)$. Then, it is easy to verify that $(G, G^{(0)}, r, s, \star)$ is an EBB-

²Nijenhuis ([55]) sets the structures of groupoid in the general context of the group theory, whereas a topological characterization of such structures is due to Golab ([31]).

groupoid, with $(x, y)_d^{-1} = (x, y)_s^{-1} = (y, x)$. It is called³ the *natural EBB-groupoid* on X , and denoted by $\mathcal{G}_{EBB}(X)$.

On the other hand, there exists E-groupoids that are not EBB-groupoids. For instance, if Γ is a group that acts on a set X by the action $\psi : \Gamma \times X \rightarrow X$, if e is the identity of Γ , and if we put⁴ $G = \Gamma \times X$, $G^{(0)} = \{e\} \times X$, $r(g, x) = x$, $s(g, x) = \psi(g, x)$ and $(g_1, x) \star (g_2, y) = (g_1 g_2, x)$ if and only if⁵ $y = \psi(g_1, x)$, then we obtain an E-groupoid, with $(g, x)^{-1} = (g^{-1}, \psi(g, x))$, called the *semidirect product E-groupoid* of Γ by X and denoted by $\Gamma \times_E X$.

Nevertheless, such a structure may not be an EBB-groupoid if, for example, the action ψ is not transitive. In fact, for every $(g_1, x), (g_3, z) \in G$, then there exists $(g_2, y) \in G$ such that $(g_1, x) \star (g_2, y) \in G^{(2)}$ and $(g_2, y) \star (g_3, z) \in G^{(2)}$ if and only if $y = \psi(g_1, x)$ and $z = \psi(g_2, y)$, that is, if and only if there exists $g_2 \in G$ such that $z = \psi(g_2, \psi(g_1, x))$ for given $z, \psi(g_1, x) \in X$. Therefore, if ψ is a no transitive action, follows that $\Gamma \times_E X$ is an E-groupoid but not an EBB-groupoid. The following result puts in relation the two structures of E-groupoid and EBB-groupoid (see [62]).

I. A multiplicative system $(G, \mathcal{D}_\diamond, \diamond)$ is an E-groupoid if and only if there exists a unique partition \mathcal{P} of G such that $\mathcal{D}_\diamond \subseteq \bigcup_{A \in \mathcal{P}} A$ and that the induced multiplicative system (A, \diamond_A) be an EBB-groupoid for every $A \in \mathcal{P}$.

For instance, if the given multiplicative system is an E-groupoid, namely, $(G, G^{(0)}, r, s, \star)$ (with $\star = \diamond$), then we consider the relation

$$\sim \doteq \{(g, g'') \in G \times G; \exists g' \in G, (g, g') \in G^{(2)}, (g', g'') \in G^{(2)}\}.$$

³ Or pair EBB-groupoid, or coarse EBB-groupoid.

⁴ In the definition of r and s , it is necessary to consider the bijection $x \rightarrow (e, x) \forall x \in X$, that identifies X with $\{e\} \times X$.

⁵ Indeed, $G^{(2)} = \{(g_1, x), (g_2, y) \in G^2; s(g_1, x) = r(g_2, y)\}$ and $s(g_1, x) = \psi(g_1, x) = g_1 x = y = r(g_2, y)$, that is $y = g_1 x$.

Hence, it is possible to prove that \sim is an equivalence relation, so that $\mathcal{P} = G/\sim$ is a partition of G . If we set $A^{(2)} = G^{(2)} \cap A$, $A^{(0)} = G^{(0)} \cap A$, $r_A = r|_A$, $s_A = s|_A$ and $\star_A = \star|_{A^{(2)}}$, for every $A \in \mathcal{P}$, then $(A, A^{(0)}, r_A, s_A, \star_A)$ is an EBB-groupoid.

2. The Notion of Convolution Structure⁶

Let A be a unitary commutative ring, J be a set of indices and $X = \{x_j; j \in J\} = (x_j)_{j \in J}$ be a family of abstract symbols. Let

$$\langle X \rangle = \langle (x_j)_{j \in J} \rangle = \left\{ \sum_{j \in J} a_j x_j; a_j \in A, x_j \in X \right\}$$

be the set of the formal linear combinations of the elements of $(x_j)_{j \in J}$ with coefficients on A : $a_j x_j$ must be understood as the value of a map of the type $A \times X \rightarrow \langle X \rangle$, whereas $\sum_{j \in J} a_j x_j$ have all coefficients a_j zero except a finite number; $\langle X \rangle$ is said the *free set* generated by X . With the operations

$$\left(\sum_{j \in J} a_j x_j \right) + \left(\sum_{j \in J} b_j x_j \right) = \sum_{j \in J} (a_j + b_j) x_j, \quad (5)$$

$$a \cdot \left(\sum_{j \in J} a_j x_j \right) = \sum_{j \in J} (aa_j) x_j, \quad a \in A, \quad (6)$$

$\langle X \rangle$ is an A -module, said the free A -module on X , with base X . In this context, it is possible to prove that its elements admits a unique decompositions of the type $\sum_{j \in J} a_j x_j$. We will denote this free A -module by $\mathcal{M}_A(X) = (\langle X \rangle, +, \cdot)$, the operations $+$, \cdot being (5), (6). Furthermore, it is possible to prove that: an A -module $(M, +, \cdot)$ is free on $(\emptyset \neq) X \subseteq M$ if and only if, for any A -module $(M', +, \cdot)$ and

⁶See [32], [56], [3], [61].

for any map $\varphi : X \rightarrow M'$, there exists a unique A -homomorphism of A -modules $\psi : M \rightarrow M'$ such that $\varphi = \psi|_X$ (in such case, ψ is said A -extension of φ , whereas X is a base of M , that is $M = \langle X \rangle$). A base identifies a unique free A -module in the sense that, if M, M' are two free A -modules with respective bases X, X' and $f : X \rightarrow X'$ is a bijection, then M, M' are A -isomorphic. These last results are important in the processes of linear extension from a base.

On the free A -module $\mathcal{M}_A(X) = (\langle X \rangle, +, \cdot)$ it is possible establish a structure of unitary ring (in general, non-commutative) by a precise⁷ product $*$ (of convolution) of elements of $\langle X \rangle$. The resulting algebraic structure⁸ $\mathcal{C}_A(X) = (\langle X \rangle, +, \cdot, *)$, is said a *convolution structure* associated to the free A -module $\mathcal{M}_A(X)$.

It is important to point out that such structure is strictly related to the choice of the product $*$, and, to this purpose, the following remarks are meaningful. When the set X is already endowed with a given algebraic structure of the type (X, \diamond) (for example, that of EBB-groupoid), formal coherence principles imposes that such (convolution) product must be ‘predetermined’ by such preexistent structure. For instance, it is usually required such product to be the result of the linear A -extension of the operation \diamond , eventually taking into account further informal requirements (of physical nature).

In particular, in view of the next arguments, A will be a scalar field \mathbb{K} , so that the A -module $(\langle X \rangle, +, \cdot)$ is a \mathbb{K} -linear space that will be extended to a linear \mathbb{K} -algebra (often said the *convolution \mathbb{K} -algebra* of the given A -module), by means a convolution product.

Historically, the first structures of convolution \mathbb{K} -algebras were the so-called group algebra, convolution structures made on a given group. Group algebras were

⁷A priori, the formal choice of this convolution product may be arbitrary. Nevertheless, in the mathematical physics context, often there are cases where such choice is forced by some ad hoc physical reasons (as, for example, causality – see [61]).

⁸The unitary ring (in general, non-commutative) of this structure $\mathcal{C}_A(X)$, is $(\langle X \rangle, +, *)$, whereas the convolution product $*$ and the A -module product \cdot , are linked by the compatibility relation $a \cdot (x * y) = (a \cdot x) * y = x * (a \cdot y) \forall a \in A, \forall x, y \in \langle X \rangle$; as usual, the product \cdot is implicit.

introduced (for finite groups) by T. Molien and G. Frobenius, for investigations of representations of these groups. Subsequently, at the beginning of the 20th-century, Schur ([59]) and Weyl ([63]) used systematically the group algebras for investigations of compact groups and commutative locally compact groups.

For applications of group algebra structures to quantum mechanics, see, for example, [46].

3. The Notion of EBB-groupoid Algebra

With any EBB-groupoid, a unique, well defined, natural structure of linear \mathbb{K} -algebra is associated by means a precise convolution product $*$.

If \mathbb{K} is a field, let us consider the free \mathbb{K} -module generated by the support of the given EBB-groupoid, hence we will consider the related convolution structure provided by an adapted convolution product. Such a group algebra is more appropriately called the *EBB-groupoid algebra* associated to the given EBB-groupoid (although it is nothing else that a group algebra on the support of a EBB-groupoid).

If $\mathcal{G} = (G, G^{(0)}, r, s, \star)$ is an EBB-groupoid and \mathbb{K} is a scalar field, we put $X = G$, and thus $\langle G \rangle = \left\{ \sum_{g \in G} \lambda(g)g; \lambda(g) \in \mathbb{K}, \forall g \in G \right\}$ is the set of the formal combinations (with coefficients in \mathbb{K}) of elements of G , equipped with the operations

$$\left(\sum_{g \in G} \lambda(g)g \right) + \left(\sum_{g \in G} \mu(g)g \right) = \sum_{g \in G} (\lambda(g) + \mu(g))g, \quad (1)$$

$$\mu \cdot \left(\sum_{g \in G} \lambda(g)g \right) = \sum_{g \in G} (\mu\lambda(g))g \quad \forall \mu \in \mathbb{K}. \quad (2)$$

$\mathcal{M}_{\mathbb{K}}(G) = (\langle G \rangle, +, \cdot)$ is a free \mathbb{K} -module (that is a \mathbb{K} -linear space) with base G . We define the convolution product

$$\left(\sum_{g \in G} \lambda(g)g \right) * \left(\sum_{g \in G} \mu(g)g \right) = \sum_{g \in G} \xi(g)g, \quad (3)$$

where (with Cauchy)


$$\xi(g) = \sum_{g_1 * g_2 = g} \lambda(g_1) \mu(g_2) \quad \forall g \in G. \quad (4)$$

Therefore, it is immediate to verify that $\mathcal{C}_{\mathbb{K}}(G) = (\langle G \rangle, +, \cdot, *)$, as convolution structure associated to $\mathcal{M}(G)$, is a linear \mathbb{K} -algebra, that we will call the *EBB-groupoid algebra* (over the field \mathbb{K}) associated to the EBB-groupoid $\mathcal{G} = (G, G^{(0)}, r, s, \star)$. We will denote it by $\mathcal{A}_{\mathbb{K}}(\mathcal{G})$. As regards what has been said at the end of Section 2, it is possible a (unique) linear extension of the algebraic relations of the EBB-groupoid \mathcal{G} to $\mathcal{A}_{\mathbb{K}}(\mathcal{G})$.

4. Brief Outlines of Atomic Spectroscopy⁹

The physical phenomenology leading to the first theoretical formulations of quantum mechanics, were the atomic spectroscopy of emission and absorption of electromagnetic waves. Nevertheless, the classical physics failed in the interpretation of the structure of the observed atomic spectra of the various chemical elements. An atomic spectrum of emission [absorption] of an arbitrary chemical element is characterized by a well defined sequence of spectral lines. Each spectroscopic line is related to the emission [absorption] of an electromagnetic radiation of well precise frequency, from the atom of the given chemical element under examination. Hence, each spectral line is identified by that well-determined frequency ν of the e.m. radiation corresponding to it. Moreover, such lines appear organized into groups called (*spectral series*) (of the given spectrum).

From these experimental investigations, born the atomic spectroscopy of the beginning of the 20th-century. It was, mainly, a coherent set of qualitative and descriptive rules concerning the symmetry and regularity properties of these spectral lines. The new quantum theory was built on the basis of it. The first semi-quantitative spectroscopic rules has been formulated with the quantum theory of N. H. Bohr, A. Sommerfeld and W. Wilson in the years 1913-1914, on the basis of the pioneering works of M. Planck, A. Einstein, E. Rutherford, and others.

Nevertheless, in experimental spectroscopy, the works of J. R. Rydberg (1908) 

⁹ See [24], Chapt. 7, and [1], [20], [24], [30], [38], [47].



and W. Ritz (1910) got a prominent rule, and led to the formulation of their homonymous principle.

From the analysis of the (spectral) series of lines of several single atomic spectra, Rydberg established the following (Rydberg) combination principle:

- ₁ Each spectral line, of any series, can be described in terms of suitable spectral terms $T_{(s)}(n)$, $n \in \mathbb{N}$;

- ₂ The frequency (or the wave number) of a line is given by a relation of the type

$$(\spadesuit) \quad \nu_{(n, n')}^{(s, s')} = T_{(s')}(n') - T_{(s)}(n),$$

where s, s' are indices of spectral series (of lines), n, n' indices of lines (of spectral series), and $\lim_{n \rightarrow \infty} T_{(s)}(n) = 0$ for any fixed s ;

- ₃ There are precise *selection rules* that are imposed instructions¹⁰ on the possible values s, s' of (\spadesuit) in order that the frequency predicted by this formula represents observed spectral lines.

Successively, Ritz stated that:

(1) On the basis of the Rydberg's formula, found for the hydrogenoid atoms, according to

$$T_{(s)}(n) = R_H / (n - a_s)^2$$

(where R_H is a universal constant (of Rydberg) and a_0, a_1, a_2, \dots are real constants, typical of each hydrogenoid atom, and that characterizes the different spectral series of its spectrum), it is possible to formulate, for the spectral terms, the following more general expression

$$T_{(s)}(n) = \Phi_{(s)}(n)(K/n^2) \text{ (Ritz-Rydberg formula)}$$

where K is a universal constant, and $\lim_{n \rightarrow \infty} \Phi_{(s)}(n) = 1$ for each fixed s ;

¹⁰ For instance, a simple selection rule is $|s - s'| = 1$. In any case, according to these rules, not every possible combination of values of s, s' , corresponds to an effectively observed spectral lines, but only those that satisfy well defined algebraic relations.

(2) The frequencies of the lines of each spectral series (that is, for each fixed $s' = s$), verify the following Ritz's combination (or composition) law

$$\nu_{(i,j)}^{(s)} + \nu_{(j,k)}^{(s)} = \nu_{(i,k)}^{(s)}.$$

Subsequently, Ritz verified the validity of law (2) also for the lines of different series, reaching to a more general Ritz-Rydberg composition law¹¹:

$$(\clubsuit) \quad \nu_{(i,j)}^{(s,s')} + \nu_{(j,k)}^{(s,s')} = \nu_{(i,k)}^{(s,s')} \quad \forall s, s', \forall i, j, k.$$

The principle achieved by the above conditions (1) and (2), with the last extension due to Ritz, it is usually called the *Ritz-Rydberg composition principle* (as extension of the Rydberg's composition principle). The main interesting part of this principle is the Ritz-Rydberg composition law¹² $\nu_{(i,j)} + \nu_{(j,k)} = \nu_{(i,k)}$. The principle has been confirmed by a large class of spectroscopic phenomena, ranging from atomic spectra to the molecular ones, from those optical to the X-ray spectra, and so on.

The class of the fundamental Franck-Hertz experiments ([10]; [29], Chap. IV, Section 10; [36]), experimentally confirmed the existence of discrete energy levels in an atom (predicted by N. H. Bohr), each characterized by a certain frequency $\nu (= E/h, E$ being the typical energy value of the given level). Hence, the set of the energy levels of an atom is characterized by a well determined set of frequencies $\mathcal{F}_I = \{\nu_i; \nu_i \in \mathbb{R}^+, i \in I \subseteq \mathbb{N}\} \subseteq \mathbb{R}^+$, that the Franck-Hertz experiences proved to be finite or countable, hence labelled by a subset I of \mathbb{N} . Since I is a non-empty subset of $(\mathbb{N}; \leq)$ (with the usual order), \mathcal{F}_I can be ordered according to the increasing indices of I , so that we can assume that the map $\rho : i \rightarrow \nu_i \in \mathcal{F}_I \forall i \in I$ is monotonically increasing and thus $I \cong \mathcal{F}_I$. Moreover, at each energy level correspond the energy $E_i = h\nu_i$, where h is the Planck's constant. Furthermore, we recall another basic principle of the spectroscopy, namely the principle of A. Conway (see [54], Chapter 13, Section 11), according to any spectral line is

¹¹Independently by eventual selection rules on s, s' as, instead, required by the Rydberg's combination principle (see \clubsuit_3). Moreover, (\clubsuit) is verified by (\spadesuit) , under the suitable selection rules.

¹²Neglecting the series indices (s, s') .

produced, once at a time, by a unique (perinuclear) electron, with $\nu_{ij} \neq 0$ whenever $i \neq j$.

A spectral line correspond to the electromagnetic radiation involved in the transition of an electron from a given level to another. If, throughout such transition $i \rightarrow j$, the initial energy level is ν_i and the final is ν_j , with $i \neq j$, then the only *physical observable* (according to P. A. M. Dirac – see [19]) is the electromagnetic radiation of emission (if $i > j$), or of absorption (if $i < j$), with frequency $\nu_{i \rightarrow j} = \nu_j - \nu_i = (E_j - E_i)/h$. In the emission it is $\nu_{i \rightarrow j} < 0$, whereas in the absorption it is $\nu_{i \rightarrow j} > 0$. Let we put¹³ $\nu_{i \rightarrow j} = \nu_{ij}$, so that, if $\nu_{ij} = \nu_j - \nu_i < 0$, then we have an emission line, whereas, if $\nu_{ji} = -\nu_{ij} > 0$, then we have an absorption line. It follows the existence of a simple (opposite) symmetry, called the *Kirchhoff-Bunsen (inversion) symmetry* (or *KB-symmetry*), between the emission spectrum and the absorption spectrum of the same atom, given by $\nu_{ij} = -\nu_{ji}$ (and that follows from the Ritz-Rydberg composition law). Therefore, the lines of the atomic spectrum (of emission/absorption) of a chemical element, are represented (in \mathbb{R}) by the elements of the set $\Delta\mathcal{F}_I = \{\nu_{ij} = \nu_j - \nu_i; i, j \in I \subseteq \mathbb{N}\} \subseteq \mathbb{R}$, symmetric respect to the origin, whose positive part represents the absorption spectrum, whereas that negative represents the emission spectrum.

Finally, other spectroscopic principles prescribe symmetries and regularities of this set, from which it is possible infer further algebraic properties of it. However, as seen, between these principles, the Ritz-Rydberg combination principle has a prominent rule in justifies the intrinsic non-commutativity of the formal quantum theory.

5. The Heisenberg-Born-Jordan EBB-groupoid

In general, $(\Delta\mathcal{F}_I; +)$ is not a subgroup of the commutative group $(\mathbb{R}; +)$ because, if $\nu_{ij}, \nu_{lk} \in \Delta\mathcal{F}_I$, not even $\nu_{ij} \pm \nu_{lk}$ correspond to an observed spectral line, that is may be $\nu_{ij} \pm \nu_{lk} \notin \Delta\mathcal{F}_I$. The elements of $\Delta\mathcal{F}_I$ combines by means the

¹³ With $\nu_{(i,j)} = \nu_{ij}$.

Ritz-Rydberg combination principle, according to $v_{ij} + v_{lk} \in \Delta\mathcal{F}_I$ if and only if¹⁴ $j = l$, whence $v_{ij} + v_{lk} = v_{ik}$. It is just this last principle that make $+$ a partial law¹⁵ in $\Delta\mathcal{F}_I$, so that $(\Delta\mathcal{F}_I; +)$ not only is not a subgroup, but neither a groupoid in the sense of Universal Algebra (see [12], or¹⁶ [7]). We will prove, instead, that it is an EBB-groupoid. Indeed, if $G = \Delta\mathcal{F}_I$, we have $G^{(0)} = \mathcal{F}_I \times r(v_{ij}) = v_i$, $s(v_{ij}) = v_j$ with $v_{ij} \star v_{lk}$ defined on $G^{(2)} = \{(v_{ij}, v_{lk}) \in \Delta\mathcal{F}_I \times \Delta\mathcal{F}_I; s(v_{ij}) = r(v_{lk})\}$. Therefore, since $s(v_{ij}) = v_j = v_l = r(v_{lk})$, we have $v_j = v_l$, from which (by the bijectivity of $\rho : i \rightarrow v_i$ of Section 4) follows that $j = l$, hence $v_{ij} \star v_{lk} = v_{ij} \star v_{jk}$. Thus, if $\star = \tilde{+}$, we have¹⁷ $v_{ij} \star v_{jk} = v_{ij} \tilde{+} v_{jk} = v_{ik}$ by the Ritz-Rydberg composition law, and hence it is immediate to verify that GHBJ $\mathcal{G}_{HBJ}(\mathcal{F}_I) = (\Delta\mathcal{F}_I, \mathcal{F}_I, r, s, \tilde{+})$ is an EBB-groupoid, said¹⁸ the *Heisenberg-Born-Jordan EBB-groupoid* (or *HBJ EBB-groupoid*). There are many different representation (or equivalent models) of $\mathcal{G}_{HBJ}(\mathcal{F}_I)$. We will consider a first representation in \mathbb{N}^2 . As regard what has been said above, the map $\xi : v_{ij} \rightarrow (i, j) \in I^2 (\subseteq \mathbb{N}^2) \quad \forall v_{ij} \in \Delta\mathcal{F}_I$ is bijective, so that $\Delta\mathcal{F}_I \cong I^2$, that is I^2 is a Cartesian representation of $\Delta\mathcal{F}_I$. Considering the Cartesian lattice \mathbb{N}^2 , if $\Delta(\mathbb{N}^2) = \{(i, i); i \in \mathbb{N}\}$, then the points $(i, j) \in I^2$ with $i < j$ [$i > j$] represents the lines of the absorption [emission] spectrum, whereas the points with $i = j$ represents the energy levels of the atom because, being $\rho^{-1} : v_i \rightarrow i \in I \quad \forall v_i \in \mathcal{F}_I$ bijective and

¹⁴ This last condition suggests the formal presence of an EBB-groupoid structure. Indeed, this structure is endowed with a partial binary operation defined on a domain $G^{(2)} = \{(g_1, g_2) \in G \times G; s(g_1) = r(g_2)\}$, where the condition $s(g_1) = r(g_2)$ corresponds to $j = l$, as we shall see.

¹⁵ That is, do not defined for all possible pairs (v_{ij}, v_{lk}) .

¹⁶ This Author introduces a structure that he call magma, corresponding to the notion of groupoid of Universal Algebra.

¹⁷ $\tilde{+}$ denotes the usual addition of $(\mathbb{R}; +)$, only partially defined on $G^{(2)} \subset \mathbb{R}^2$, and that, substantially, represents the Ritz-Rydberg composition law.

¹⁸ The motivation for this terminology will be given at Section 6.

$\alpha : i \rightarrow (i, i)$ bijection of \mathbb{N} in $\Delta(\mathbb{N}^2)$, we have that $(\alpha \circ \rho^{-1})(\mathcal{F}_I) = \{(i, i); i \in I\} = \Delta(I^2) (\subseteq \Delta(\mathbb{N}^2))$ represents \mathcal{F}_I in \mathbb{N}^2 .

Thus, we have a Cartesian representation (in \mathbb{N}^2) of the atomic spectrum $\Delta\mathcal{F}_I$: the upper half plane respect to $\Delta(\mathbb{N}^2)$ represents the absorption spectrum, whereas the lower half plane represents the emission spectrum, and finally $\Delta(I^2)$ represents the set of energy levels of the atom. The points of the two emission and absorption half planes, are pairwise correlated by the (inversion) Kirchhoff-Bunsen symmetry, that it is a simple reflexive symmetry¹⁹ respect to $\Delta(\mathbb{N}^2)$.

The map $\xi^{-1} : (i, j) (\in I^2) \rightarrow v_{ij}$ provides the kinematical time evolution of a dynamic system through the coordinate q and the momentum p given by the (Hermitian) matrices

$$(\star) \quad q = [q_{ij} e^{2\pi i v_{ij} t}], \quad p = [p_{ij} e^{2\pi i v_{ij} t}] \quad (\text{Heisenberg's representation}),$$

from which follows that any other physical observable $g(q, p)$ can always be written in the form $g = [q_{ij} e^{2\pi i v_{ij} t}]$.

Finally, we have the identifications $\Delta\mathcal{F}_I \cong I^2 \cong \mathcal{F}_I^2$, that give a representation (in \mathbb{N}^2) of the algebraic system $\mathcal{G}_{HBJ}(\mathcal{F}_I)$. In fact, if we put $X = I$, it is immediate to verify that $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ is identifiable (since isomorphic to it) with the natural EBB-groupoid $\mathcal{G}_{Br}(X)$ (of the last part of Section 1) with $X = I$, $G = I^2$, $G^{(0)} = \Delta(I^2)$, $r : (i, j) \rightarrow i$, $s : (i, j) \rightarrow j$, $(i, j) \star (j, k) = (i, k)$; $\mathcal{G}_{Br}(I)$ is said the Heisenberg-Born-Jordan natural EBB-groupoid.

Two EBB-groupoids $\mathcal{G}_1 = (G_1, G_1^{(0)}, r_1, s_1, \star_1)$, $\mathcal{G}_2 = (G_2, G_2^{(0)}, r_2, s_2, \star_2)$ are said²⁰ isomorphic if there exists $\psi : G_1 \rightarrow G_2$, $\psi_0 : G_1^{(0)} \rightarrow G_2^{(0)}$ both bijective,

¹⁹From this symmetry it is possible to construct a second equivalent representation of the given EBB-groupoid.

²⁰This definition of isomorphism must be intended in the sense of category theory.

such that $r_2 \circ \psi = \psi_0 \circ r_1$, $s_2 \circ \psi = s_1 \circ \psi_0$ and $\psi(g \star_1 g') = \psi(g) \star_2 \psi(g') \forall g, g' \in G_1$. In this case, we will write $\mathcal{G}_1 \cong \mathcal{G}_2$.

If $\mathcal{G}_1 = \mathcal{G}_{HBJ}(\mathcal{F}_I)$, $\mathcal{G}_2 = \mathcal{G}_{Br}(I)$, the maps $\psi : v_{ij} \rightarrow (i, j) \in I^2 \forall v_{ij} \in \Delta\mathcal{F}_I$, $\psi_0 : v_i \rightarrow (i, i) \in \Delta(I^2) \forall v_i \in \mathcal{F}_I$, are bijective and make isomorphic the two given B-groupoid²¹, that is $\mathcal{G}_{HBJ}(\mathcal{F}_I) \cong \mathcal{G}_{Br}(I)$, hence they are identifiable.

6. The Heisenberg-Born-Jordan EBB-groupoid Algebra

Let us prove that the EBB-groupoid algebra of the Heisenberg-Born-Jordan EBB-groupoid, in general, is a non-commutative linear \mathbb{K} -algebra, isomorphic, in the finite dimensional case, to a well defined matrix \mathbb{K} -algebra.

First, from $\mathcal{G}_{Br}(I) \cong \mathcal{G}_{HBJ}(\mathcal{F}_I)$, follows that²² $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{Br}(I)) \cong \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$. These two isomorphic structures represent the so-called²³ Heisenberg-Born-Jordan EBB-groupoid algebra (or HBJ EBB-algebra).

It is $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = (\langle \Delta\mathcal{F}_I \rangle, +, \cdot, *)$ with

$$\begin{aligned} \langle \Delta\mathcal{F}_I \rangle &= \left\{ \sum_{g \in \Delta\mathcal{F}_I} \lambda(g)g; \lambda(g) \in \mathbb{K}, g \in \Delta\mathcal{F}_I \right\} \\ &= \left\{ \sum_{v_{ij} \in \Delta\mathcal{F}_I} \lambda(v_{ij})v_{ij}; \lambda(v_{ij}) \in \mathbb{K}, v_{ij} \in \Delta\mathcal{F}_I \right\}, \end{aligned}$$

the operations $+$, \cdot being definite as in (1), (2), whereas the convolution product $*$ is defined as follows. If we consider two arbitrary elements of $\langle \Delta\mathcal{F}_I \rangle$, their convolution product must be an element of this set, and hence it is expressible as

²¹For instance, we have $\psi(v_{ij} \tilde{+} v_{jk}) = \psi(v_{ik}) = (i, k) = (i, j) \star (j, k) = \psi(v_{ij}) \star (v_{jk})$, the remaining conditions being easily verified.

²²Indeed, the base of the free \mathbb{K} -modules constructed on such EBB-groupoid, that is to say their supports, are respectively I^2 and $\Delta\mathcal{F}_I$, and since $I^2 \cong \Delta\mathcal{F}_I$, follows that the respective \mathbb{K} -modules are \mathbb{K} -isomorphic (see Section 3), whence $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{Br}(I)) \cong \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$.

²³See Section 7.

unique linear combination of the base elements of $\Delta\mathcal{F}_I$. Hence, in general, we have

$$\left(\sum_{\mathbf{v}_{ij} \in \Delta\mathcal{F}_I} \lambda(\mathbf{v}_{ij}) \mathbf{v}_{ij} \right) * \left(\sum_{\mathbf{v}_{lk} \in \Delta\mathcal{F}_I} \mu(\mathbf{v}_{lk}) \mathbf{v}_{lk} \right) = \sum_{\mathbf{v}_{ps} \in \Delta\mathcal{F}_I} \xi(\mathbf{v}_{ps}) \mathbf{v}_{ps},$$

where $\xi(\mathbf{v}_{ps}) \in \mathbb{K}$ are uniquely determined by:

$$\xi(\mathbf{v}_{ps}) \doteq \sum_{\mathbf{v}_{ij} \tilde{+} \mathbf{v}_{lk} = \mathbf{v}_{ps}} \lambda(\mathbf{v}_{ij}) \mu(\mathbf{v}_{lk}),$$

and, since $\mathbf{v}_{ij} \tilde{+} \mathbf{v}_{lk}$ is defined if and only if $s(\mathbf{v}_{ij}) = r(\mathbf{v}_{lk})$, that is $j = l$, we have

$\mathbf{v}_{ij} \tilde{+} \mathbf{v}_{lk} = \mathbf{v}_{ij} \tilde{+} \mathbf{v}_{jk} = \mathbf{v}_{ik}$, and thus $\mathbf{v}_{ps} = \mathbf{v}_{ik}$, that is $p = i, s = k$, so that

$$(\diamond) \quad \xi(\mathbf{v}_{ik}) = \sum_{\mathbf{v}_{ij} \tilde{+} \mathbf{v}_{jk} = \mathbf{v}_{ik}} \lambda(\mathbf{v}_{ij}) \mu(\mathbf{v}_{jk}).$$

In the last sum, the indexes i, k are saturated while $j(\in I)$ is free, so that the sum is extended only to last index, that is we can write²⁴

$$(\heartsuit) \quad \xi(\mathbf{v}_{ik}) = \sum_{j \in I} \lambda(\mathbf{v}_{ij}) \mu(\mathbf{v}_{jk})$$

and this is the expression of the generic element of the product²⁵ matrix of the two formal matrices²⁶ $[\lambda(\mathbf{v}_{ij})], [\mu(\mathbf{v}_{lk})]$. Then, in general, the Heisenberg-Born-Jordan EBB-groupoid algebra $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, is non-commutative, and, if $\text{card } I < \infty$, then it is isomorphic to a matrix \mathbb{K} -algebra (see [57], Chap. 5, Theorem 3.2, or [7], [48], [51]).

Nevertheless, since, in general, I is not finite, it follows that $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ is not a finitely generated \mathbb{K} -algebra.

²⁴ All these convolution structures are based on the Rytz-Rydberg composition law, and, the latter, has been at the foundations of the works of W. Heisenberg, M. Born and P. Jordan on matrix mechanics (see [1], [6], [16], [17], [20], [28], [30], [33], [34], [35], [38], [47], [58]). For modern quantum theories see [60], [24], whereas for the geometric developments of the non-commutativity, and its physical applications, see [13], [14].

²⁵ Rows by columns.

²⁶ Of order $\text{card } I$.

7. The Notion of Hopf Algebra²⁷

Let $V_{\mathbb{K}}$ be a \mathbb{K} -linear space and id its identity map. A \mathbb{K} -algebra is an algebraic system $\mathcal{A} = (V_{\mathbb{K}}, m, \eta)$ with $m : V \otimes V \rightarrow V$ (*product*) and $\eta : \mathbb{K} \rightarrow V$ (*unit*) \mathbb{K} -linear maps such that: 1) (associativity) $m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$; 2) $m \circ (\eta \otimes \text{id}) = m \circ (\text{id} \otimes \eta) = \text{id}$. We have $m(a \otimes b) = ab$.

A \mathbb{K} -coalgebra is an algebraic system $c\mathcal{A} = (V_{\mathbb{K}}, \Delta, \varepsilon)$ with $\Delta : V \rightarrow V \otimes V$ (*coproduct*) and $\varepsilon : V \rightarrow \mathbb{K}$ (*counit*) \mathbb{K} -linear maps such that: (1') (coassociativity) $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$; (2') $(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}$. A \mathbb{K} -bialgebra is an algebraic system $b\mathcal{A} = (V_{\mathbb{K}}, \Delta, \varepsilon, m, \eta)$ such that $(V_{\mathbb{K}}, \Delta, \varepsilon)$ is a \mathbb{K} -coalgebra, $(V_{\mathbb{K}}, m, \eta)$ a \mathbb{K} -algebra and Δ, ε \mathbb{K} -algebras homomorphism (see [Ka], Chapt. III, Theorem III.2.1). Let $f \in \text{End}(V_{\mathbb{K}})$, and $\{v_i; i \in I\}$ a base of $V_{\mathbb{K}}$ with $\dim_{\mathbb{K}} V_{\mathbb{K}} = \text{card } I$; then $\{v_i \otimes v_j; (i, j) \in I^2\}$ is a base of $V \otimes V$, so that $v = \sum_{(i,j) \in I^2} \lambda_{ij}(v) v_i \otimes v_j$ for certain $\lambda_{ij}(v) \in \mathbb{K}$ and $I_v \subseteq I^2$, for any $v \in V \otimes V$. Then, we can use the following (Sweedler) sigma notation according to we will write simply $v = \sum_{(v)} v' \otimes v''$ for given $v', v'' \in V_{\mathbb{K}}$; in particular, since $\Delta(v) \in V \otimes V \quad \forall v \in V_{\mathbb{K}}$, we will have $\Delta(v) = \sum_{(v)} v' \otimes v''$. If $(V_{\mathbb{K}}, \Delta, \varepsilon)$ is a \mathbb{K} -coalgebra and $(W_{\mathbb{K}}, m, \eta)$ is a \mathbb{K} -algebra, for each $f, g \in \text{Hom}(V, W)$, we have $f \otimes g \in \text{Hom}(V \otimes V, W \otimes W)$, so that, by exploiting the relations

$$V \xrightarrow{\Delta} V \otimes V \xrightarrow{f \otimes g} W \otimes W \xrightarrow{m} W,$$

it is possible to consider the composition map $m \circ (f \otimes g) \circ \Delta \in \text{Hom}(V, W)$, denoted by $f \hat{*} g$ and given by

$$\begin{aligned} (f \hat{*} g)(v) &= (m \circ (f \otimes g) \circ \Delta)(v) = (m \circ (f \otimes g))(\Delta(v)) \\ &= (m \circ (f \otimes g)) \left(\sum_{(v)} v' \otimes v'' \right) = m \left(\sum_{(v)} (f \otimes g)(v' \otimes v'') \right) \end{aligned}$$

²⁷ We follow [41], [42], [11], [52], [53] and [2].

$$= m \left(\sum_{(v)} f(v') \otimes g(v'') \right) = \sum_{(v)} f(v') g(v'') \quad \forall v \in V,$$

defining the (internal) binary operation (also called *convolution*)

$$\hat{*} : \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W)$$

$$(f, g) \rightsquigarrow f \hat{*} g \quad \forall f, g \in \text{Hom}(V, W).$$

If $b\mathcal{A} = (V_{\mathbb{K}}, \Delta, \varepsilon, m, \eta)$ is a \mathbb{K} -bialgebra, what said above also subsists if we put $V = W$; an element $a \in \text{End}(V)$ is said an *antipode* of $b\mathcal{A}$ if $a \hat{*} \text{id} = \text{id} \hat{*} a = \eta \circ \varepsilon$, and the algebraic system $(b\mathcal{A}, a)$ is called a *Hopf \mathbb{K} -algebra*²⁸ of support $V_{\mathbb{K}}$.

The commutativity (or not) [cocommutativity (or not)] of such a structure, follows from the commutativity (or not) [cocommutativity (or not)] of the linear \mathbb{K} -algebra $(V_{\mathbb{K}}, m, \eta)$ [\mathbb{K} -coalgebra $(V_{\mathbb{K}}, \Delta, \varepsilon)$].

A \mathbb{K} -bialgebra $b\mathcal{A} = (V_{\mathbb{K}}, \Delta, \varepsilon, m, \eta)$ is said *quasi-cocommutative* if there exists an invertible element R (said a *universal R-matrix*) of $V_{\mathbb{K}} \otimes V_{\mathbb{K}}$ such that $\Delta^{op}(x) = R\Delta(x)R^{-1} \quad \forall x \in V_{\mathbb{K}}$, where $\Delta^{op} = \tau_{V_{\mathbb{K}}, V_{\mathbb{K}}} \circ \Delta$ is the opposite coproduct on $V_{\mathbb{K}}$ and $\tau_{V_{\mathbb{K}}, V_{\mathbb{K}}}$ is the flip switching the factors. We will denote such a quasi-cocommutative \mathbb{K} -bialgebra with $(V_{\mathbb{K}}, \Delta, \varepsilon, m, \eta, R)$.

Any cocommutative \mathbb{K} -bialgebra is also quasi-cocommutative with universal R-matrix $R = 1_{V_{\mathbb{K}}} \otimes 1_{V_{\mathbb{K}}}$. A Hopf \mathbb{K} -algebra whose underling \mathbb{K} -bialgebra has a universal R-matrix, is said a *quasi-cocommutative Hopf \mathbb{K} -algebra*.

For any $R \in V_{\mathbb{K}} \otimes V_{\mathbb{K}}$, we set $R_{12} = R \otimes 1_{V_{\mathbb{K}}} \in V_{\mathbb{K}} \otimes V_{\mathbb{K}} \otimes V_{\mathbb{K}} = \bigotimes^3 V_{\mathbb{K}}$, $R_{23} = 1_{V_{\mathbb{K}}} \otimes R \in \bigotimes^3 V_{\mathbb{K}}$, $R_{13} = (\text{id} \otimes \tau_{V_{\mathbb{K}}, V_{\mathbb{K}}})(R_{12}) = (\tau_{V_{\mathbb{K}}, V_{\mathbb{K}}} \otimes \text{id})(R_{23}) \in \bigotimes^3 V_{\mathbb{K}}$. A quasi-cocommutative \mathbb{K} -bialgebra [Hopf \mathbb{K} -algebra] $b\mathcal{A} = (V_{\mathbb{K}}, \Delta, \varepsilon, m, \eta, R)$ $[(b\mathcal{A}, a)]$ is *braided* (or *quasi-triangular*) if the universal R-matrix R satisfies the relations $(\Delta \otimes \text{id}_{V_{\mathbb{K}}})(R) = R_{13}R_{23}$ and $(\text{id}_{V_{\mathbb{K}}} \otimes \Delta)(R) = R_{13}R_{12}$. All cocommutative

²⁸ In a \mathbb{K} -bialgebra, may exist, at most, only one antipode (see [41], Def. III 3.2., p. 51).

bialgebras are braided with universal R -matrix $R = 1_{V_{\mathbb{K}}} \otimes 1_{V_{\mathbb{K}}}$. Often, a non-cocommutative braided Hopf \mathbb{K} -algebra is called a *quantum group*.

8. The HBJ EBBH-algebra

It is possible to associate a natural structure of (non-commutative) braided Hopf \mathbb{K} -algebra to the HBJ EBB-algebra. So, we will get an explicit example of non-commutative Hopf \mathbb{K} -algebra, that can be taken as basic structure of a (particular) quantum group.

On $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = (\langle \Delta \mathcal{F}_I \rangle, +, \cdot, *)$, let us consider the unique (see Section 2) linear extension i of the inversion $i_G : g \rightarrow g^{-1}$ (with $G = \Delta \mathcal{F}_I$) to $\langle \Delta \mathcal{F}_I \rangle$ and set²⁹ (if necessary, in the tensor product algebra $\mathcal{A}_{\mathbb{K}} \otimes \mathcal{A}_{\mathbb{K}} : V_{\mathbb{K}} = (\langle \Delta \mathcal{F}_I \rangle, +, \cdot, *)$, $m = *$, $\eta = \eta(1) (= 1$ the unit of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)))$, $\Delta(x) = x \otimes x$ (group-like elements), $\varepsilon(x) = 1 \forall x \in \langle \Delta \mathcal{F}_I \rangle$ and $a = \tilde{i} \in \text{End}(\langle \Delta \mathcal{F}_I \rangle)$).

Hence

$$\begin{aligned} \Delta(m(x, y)) &= \Delta(x * y) = (x * y) \otimes (x * y)^{30} \\ &= (x \otimes y) * (x \otimes y) = \Delta(x) * \Delta(y) = m(\Delta(x), \Delta(y)) \quad \forall x, y \in \langle \Delta \mathcal{F}_I \rangle, \\ \varepsilon(m(x, y)) &= \varepsilon(x * y) = \varepsilon(z) = 1 = 1 * 1 \\ &= \varepsilon(x) * \varepsilon(y) = m(\varepsilon(x), \varepsilon(y)) \quad \forall x, y \in \langle \Delta \mathcal{F}_I \rangle \end{aligned}$$

(where $z = m(x, y)$, and $\varepsilon(x) = 1 \forall x \in \Delta(\mathcal{F}_I)$), so that Δ, ε are homomorphism of \mathbb{K} -algebras.

Therefore, it is immediate to prove that $b\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = (\langle \Delta \mathcal{F}_I \rangle, \Delta, \varepsilon, m, \eta)$ is a (non-commutative) \mathbb{K} -bialgebra (see, also, [42], Chapt. I, Section 1.2.6., Example 7). In fact, we have

$$\begin{aligned} (a \hat{*} \text{id})(x) &= (m \circ (\tilde{i} \otimes \text{id}) \circ \Delta)(x) = m(\tilde{i} \otimes \text{id})(x \otimes x) = m(\tilde{i}(x) \otimes x) \\ &= m(x^{-1} \otimes x) = {}^{31} x^{-1} * x = 1 = (\eta \circ \varepsilon)(x) (= (\text{id} \hat{*} a)(x)) \end{aligned}$$

²⁹ In the notations of Section 7.

³⁰ See [41], Chapt. II, Section II.4.

for each $x \in \Delta(\mathcal{F}_I)$. Hence $a \hat{*} \text{id} = \text{id} \hat{*} a = \eta \circ \varepsilon$ and therefore, a is an antipode, that is, $(b\mathcal{A}_{HBJ}(\mathcal{F}_I), a)$ is a non-commutative Hopf \mathbb{K} -algebra³², naturally braided by $R = 1 \otimes 1$. We will denote it by $\mathcal{H}_{\mathbb{K}}(\mathcal{F}_I)$, and will be said the³³ *Heisenberg-Born-Jordan EBBH-algebra* (or *HBJ EBBH-algebra*). As already said, this last structure may be regarded as the basic structure of a quantum group that also got, a posteriori, a more physical motivation in its name. In conclusion, in the HBJ EBBH-algebra may be recognized the eventually quantic origins of the basilar structure of quantum group, although it has been obtained endowing the basic EBB-groupoid algebra with a trivial structure of braided Hopf \mathbb{K} -algebra. There are further, well known, (over) structures and properties on such EBB-groupoid algebra when I is finite (since we shall get a finitely generated algebra), but very few when I is infinite (that is, when the algebra is not finitely generated).

In this first paper, we have considered the only possible (although trivial) structure of braided non-commutative Hopf algebra on such EBB-groupoid algebra, when I is infinite (no finitely generation).

In a further paper, we shall try to find other non-trivial structure on it, in spite of its (interesting) no finitely (algebraic) generation.

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³¹It is enough consider the unique extention of \star to $\langle \Delta\mathcal{F}_I \rangle$, coherently with \star (see (\diamond) ; (\heartsuit)) and 5 of Section 1.

³²Furthermore, it is also a \ast -Hopf algebra with the matrix transposition, when $\mathbb{K} = \mathbb{C}$ and card $I < \infty$.

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³³HBJ EBBH-algebra stands for Heisenberg-Born-Jordan Ehresmann-Baer-Brandt-Hopf algebra.

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