ACTIVE MACRO-ZONES ALGORITHM VIA MULTIDOMAIN SBEM FOR STRAIN-HARDENING ELASTOPLASTIC ANALYSIS

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Abstract. *In this paper a strategy to perform strain-hardening elastoplastic analysis by using the Symmetric Boundary Element Method (SBEM) for multi-domain type problems is shown. The procedure has been developed inside Karnak.sGbem code by introducing an additional module.*

1 INTRODUCTION

A multi-domain SBEM strategy [1], based on an initial strain approach, is applied for the analysis of 2D structures, in the hypothesis of kinematic hardening behaviour, von Mises model, associated flow rules and strain plane state. Let us start from the discretization of the domain in substructures called bem-elements, where the plastic strain accumulation have to be computed. Then, in order to obtain the self-stresses equation governing the elastoplastic problem, let us impose the regularity conditions, in strong form on the displacements (nodal compatibility) and in weak form on the tractions (generalized equilibrium) both evaluated on the interface boundary, and let us effectuate a strong variable condensation. For the generic load increment, this equation permits to locate the active bem-elements which require correction techniques. Then, the trial solution is corrected by a return mapping algorithm, which is defined in according to the extremal paths theory [2], simultaneously in all the plastically active bem-elements. The proposed algorithm utilizes the same self-stresses equation in a nonlinear global system of 7xa equations in 7xa unknowns, where *a* is the active bem-elements number. In the present approach the approximate solution is easily obtained by using the well-known standard Newton-Raphson procedure, just used in elastoplastic problems within the Bem formulations [3]. Finally a numeral test, performed by the Karnak.sGbem code [4], is shown.

2 SELF-STRESSES EQUATION VIA MULTI-DOMAIN SBEM

For each bem-e let us start by imposing the following Dirichlet and Neumann conditions

$$\mathbf{u}_1 = \bar{\mathbf{u}}_1 \quad \text{on } \Gamma_1, \qquad \mathbf{t}_2 = \bar{\mathbf{f}}_2 \quad \text{on } \Gamma_2 \tag{1a,b}$$

and by evaluating the response in terms of the displacement $\mathbf{u}_0$ and traction $\mathbf{t}_0$ vectors on the interface boundary Γ_0. In addiction, let us introduce the stress vector at Gauss points

$$\begin{aligned}
&\mathbf{u}_1[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0]+\beta\cdot\mathbf{u}_1[\bar{\mathbf{f}}_2,-\bar{\mathbf{u}}_1,\bar{\mathbf{b}}]+\mathbf{u}_1[\boldsymbol{\varepsilon}^p]=\bar{\mathbf{u}}_1 && \text{on } \Gamma_1\\
&\mathbf{t}_2[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0]+\beta\cdot\mathbf{t}_2[\bar{\mathbf{f}}_2,-\bar{\mathbf{u}}_1,\bar{\mathbf{b}}]+\mathbf{t}_2[\boldsymbol{\varepsilon}^p]=\bar{\mathbf{f}}_2 && \text{on } \Gamma_2\\
&\mathbf{u}_0=\mathbf{u}_0[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0]+\beta\cdot\mathbf{u}_0[\bar{\mathbf{f}}_2,-\bar{\mathbf{u}}_1,\bar{\mathbf{b}}]+\mathbf{u}_0[\bar{\boldsymbol{\varepsilon}}^p] && \text{on } \Gamma_0\\
&\mathbf{t}_0=\mathbf{t}_0[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0]+\beta\cdot\mathbf{t}_0[\bar{\mathbf{f}}_2,-\bar{\mathbf{u}}_1,\bar{\mathbf{b}}]+\mathbf{t}_0[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_0\\
&\boldsymbol{\sigma}=\boldsymbol{\sigma}[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0]+\beta\cdot\boldsymbol{\sigma}[\bar{\mathbf{f}}_2,-\bar{\mathbf{u}}_1,\bar{\mathbf{b}}]+\boldsymbol{\sigma}[\boldsymbol{\varepsilon}^p] && \text{on } \Omega
\end{aligned} \tag{2a-e}$$

In the latter equations the Somigliana Identities (S.I.) of the displacements and of the tractions have been introduced.
The vector $\boldsymbol{\varepsilon}^{p}$ represents the inelastic strains due to thermal or plastic actions, whose presence requires domain integrals having singular kernels, suitably studied [3,5].

A boundary discretization into boundary elements is made by introducing the following modelling of all the known and unknown quantities:

$$\mathbf{f}_1 = \boldsymbol{\Psi}_{\mathrm{t}}\, \mathbf{F}_1, \bar{\mathbf{f}}_2 = \boldsymbol{\Psi}_{\mathrm{t}}\, \bar{\mathbf{F}}_2, \mathbf{t}_0 = \boldsymbol{\Psi}_{\mathrm{t}}\, \mathbf{F}_0, \mathbf{u}_2 = \boldsymbol{\Psi}_{\mathrm{u}}\, \mathbf{U}_2, \bar{\mathbf{u}}_1 = \boldsymbol{\Psi}_{\mathrm{u}}\, \bar{\mathbf{U}}_1, \mathbf{u}_0 = \boldsymbol{\Psi}_{\mathrm{u}}\, \mathbf{U}_0, \boldsymbol{\varepsilon}^{p} = \boldsymbol{\Psi}_{\mathrm{p}}\, \mathbf{p} \tag{3a-g}$$

where $\boldsymbol{\Psi}_{\mathrm{t}}$ and $\boldsymbol{\Psi}_{\mathrm{u}}$ are shape functions regarding the boundary quantities, while $\boldsymbol{\Psi}_{\mathrm{p}}$ are domain shape functions used to model plastic strains $\mathbf{p}$ connected to the Gauss points of the bem-e.
Besides, the capital letters $\mathbf{F}$ and $\mathbf{U}$ indicate the nodal vectors of the forces ($\mathbf{F}_1$ on Γ_1, $\bar{\mathbf{F}}_2$ on Γ_2 and $\mathbf{F}_0$ on Γ_0) and of the displacements ($\bar{\mathbf{U}}_1$ on Γ_1, $\mathbf{U}_2$ on Γ_2 and $\mathbf{U}_0$ on Γ_0) defined on the boundary elements.

Let us perform the weighting of all the coefficients of the eqs.(2a-d). At this aim, the same shape functions as those modelling the causes have been employed, but introduced in an energetically dual way in according to the Galerkin approach. In this way it is possible to obtain the following block system:

$$\left|\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \hline \mathbf{P}_0 \\ \hline \boldsymbol{\sigma} \end{array}\right| = \left|\begin{array}{ccc|c|c} \mathbf{A}_{\mathrm{u1,u1}} & \mathbf{A}_{\mathrm{u1,f2}} & \mathbf{A}_{\mathrm{u1,u0}} & \mathbf{A}_{\mathrm{u1,f0}} & \mathbf{A}_{\mathrm{u1,\sigma}} \\ \mathbf{A}_{\mathrm{f2,u1}} & \mathbf{A}_{\mathrm{f2,f2}} & \mathbf{A}_{\mathrm{f2,u0}} & \mathbf{A}_{\mathrm{f2,f0}} & \mathbf{A}_{\mathrm{f2,\sigma}} \\ \mathbf{A}_{\mathrm{u0,u1}} & \mathbf{A}_{\mathrm{u0,f2}} & \mathbf{A}_{\mathrm{u0,u0}} & \bar{\mathbf{A}}_{\mathrm{u0,f0}} & \mathbf{A}_{\mathrm{u0,\sigma}} \\ \hline \mathbf{A}_{\mathrm{f0,u1}} & \mathbf{A}_{\mathrm{f0,f2}} & \bar{\mathbf{A}}_{\mathrm{f0,u0}} & \mathbf{A}_{\mathrm{f0,f0}} & \mathbf{A}_{\mathrm{f0,\sigma}} \\ \hline \mathbf{a}_{\sigma,\mathrm{u1}} & \mathbf{a}_{\sigma,\mathrm{f2}} & \mathbf{a}_{\sigma,\mathrm{u0}} & \mathbf{a}_{\sigma,\mathrm{f0}} & \mathbf{a}_{\sigma,\sigma} \end{array}\right| \left|\begin{array}{c} \mathbf{F}_1 \\ -\mathbf{U}_2 \\ \mathbf{F}_0 \\ \hline -\mathbf{U}_0 \\ \hline \mathbf{p} \end{array}\right| + \beta \left|\begin{array}{c} \hat{\mathbf{W}}_1 \\ \hat{\mathbf{P}}_2 \\ \hat{\mathbf{W}}_0 \\ \hline \hat{\mathbf{L}}_0 \\ \hline \hat{\mathbf{l}}_\sigma \end{array}\right| \tag{4}$$

where the introduced coefficient β is the multiplier of the all external actions.

The eqs.(4) may be expressed in compact form in the following way:

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{X} + \mathbf{A}_0\left(-\mathbf{U}_0\right) + \mathbf{A}_\sigma \mathbf{p} + \beta \cdot \hat{\mathbf{L}} \\ \mathbf{P}_0 &= \tilde{\mathbf{A}}_0 \mathbf{X} + \mathbf{A}_{00}\left(-\mathbf{U}_0\right) + \mathbf{A}_{0\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{L}}_0 \\ \boldsymbol{\sigma} &= \mathbf{a}_\sigma \mathbf{X} + \mathbf{a}_{\sigma 0}\left(-\mathbf{U}_0\right) + \mathbf{a}_{\sigma\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{l}}_\sigma \end{aligned} \tag{5a-c}$$

where the vector $\mathbf{X}$ collects the sub-vectors $\mathbf{F}_1$, $\left(-\mathbf{U}_2\right)$ and $\mathbf{F}_0$, whereas the $\left(-\mathbf{U}_0\right)$ and $\mathbf{p}$ vectors characterize the displacements of the nodes in the interface zones, changed in sign, and the plastic strains at the Gauss points, respectively.

The vector $\mathbf{P}_0$ represents the generalized (or weighted) traction vector defined in the boundary elements of the interface zones, obtained as a weighted response to all the known, amplified by β, and unknown actions, regarding boundary and domain quantities. The vector $\boldsymbol{\sigma}$ represents the stress, valued at the Gauss points, due to the all the known and unknown actions.

By performing a variables condensation through the replacement of the $\mathbf{X}$ vector extracted from eq.(5a) into eqs.(5b,c), one obtains:

$$\begin{aligned} \mathbf{P}_0 &= \mathbf{D}_{00}\mathbf{U}_0 + \mathbf{D}_{0\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{P}}_0 \\ \boldsymbol{\sigma} &= \mathbf{d}_{\sigma 0}\mathbf{U}_0 + \mathbf{d}_{\sigma\sigma}\mathbf{p} + \beta \cdot \hat{\boldsymbol{\sigma}} \end{aligned} \tag{6a,b}$$

These latter are the characteristic equations of each bem-e. They relate the generalized (or weighted) tractions $\mathbf{P}_0$ defined on the interface zone Γ_0 and the stresses $\boldsymbol{\sigma}$ at the bem-e domain to the nodal displacements $\mathbf{U}_0$, to the plastic strains $\mathbf{p}$ and the two load terms $\hat{\mathbf{P}}_0$ and $\hat{\boldsymbol{\sigma}}$ amplified by β, respectively. These latter represent the elastic response in terms of the generalized tractions vector along the interface boundary and of the stresses vector in the domain with reference to each bem-e. Moreover, $\mathbf{D}_{00}$, $\mathbf{D}_{0\sigma}$, $\mathbf{d}_{\sigma 0}$, $\mathbf{d}_{\sigma\sigma}$ are the stiffness matrices of the bem-e, being $\mathbf{D}_{00}$ and $\mathbf{d}_{\sigma\sigma}$ square matrices, $\mathbf{D}_{0\sigma}$ and $\mathbf{D}_{\sigma 0}$ rectangular ones.

Let us subdivide the body in *m* bem-elements and consider the eqs.(6a,b) for each of these. Thus we obtain two global relations connecting all the generalized tractions and the stresses related to the bem-elements considered, formally equal to the same eqs.(6a,b), but regarding the constitutive equations of the assembled system.

Let us introduce the compatibility among the nodal displacements of the adjacent bem-elements $\mathbf{U}_0 = \mathbf{H}\boldsymbol{\xi}_0$ and the equilibrium condition $\mathbf{H}^T \mathbf{P}_0 = \mathbf{0}$ among generalized tractions at the interface boundaries, with $\mathbf{H}$ topological matrix and $\boldsymbol{\xi}_0$ nodal displacements vector of the assembled system. Using the previous regularity conditions, eqs.(6a,b) become:

$$\begin{aligned} &\mathbf{K}_{00}\boldsymbol{\xi}_0 + \mathbf{K}_{0\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{f}}_0 = \mathbf{0} \\ &\boldsymbol{\sigma} = \mathbf{k}_{\sigma 0}\boldsymbol{\xi}_0 + \mathbf{k}_{\sigma\sigma}\mathbf{p} + \beta \cdot \hat{\boldsymbol{\sigma}} \end{aligned} \tag{7a,b}$$

By performing a new variables condensation through the replacement of the $\boldsymbol{\xi}_0$ vector extracted from eq.(7a) into eq.(7b), the self-stresses equation is obtained:

$$\boldsymbol{\sigma} = \mathbf{Z}\mathbf{p} + \beta \cdot \hat{\boldsymbol{\sigma}}_s \tag{8}$$

This equation provides the stress at the strain points of each bem-e in function of the volumetric plastic strain $\mathbf{p}$ and of the external actions $\hat{\boldsymbol{\sigma}}_s$, the latter amplified by β. The matrix $\mathbf{Z}$, defined self-stresses influence matrix of the assembled system, is a square matrix having 3*m*x3*m* dimensions with *m* bem-elements number, full populated, non symmetric and semi-defined negative. The evaluation of this matrix involves only the elastic material characteristic and structure geometry knowledge.

3 ACTIVE MACRO-ZONES ANALYSIS

In this section the strategy to compute the plastic strains for each loading step and at every bem-e is shown. This approach utilizes eq.(8) both to evaluate the predictor phase and during the corrector one, here after shown. First eq.(8) provides all the predictors $\boldsymbol{\sigma}^*_{(n+1)}$, i.e. the purely elastic response, at the instant $n+1$ in each *m* bem-elements, as function of the plastic strain $\mathbf{p}_{(n)}$, stored up at the previous step and then imposed as volumetric distortions, and of load increment $\beta_{(n+1)}$:

$$\boldsymbol{\sigma}^*_{(n+1)} = \mathbf{Z}\mathbf{p}_{(n)} + \beta_{(n+1)} \cdot \hat{\boldsymbol{\sigma}}_s \tag{9}$$

where $\mathbf{Z}$ matrix regards all the bem-elements, obtained by the discretization. The check of the plastic consistency condition of the stresses computed on appropriately chosen points is performed by using the yield condition expressed in this context through the von Mises law for each bem-e:

$$F[\boldsymbol{\sigma}_{(n+1)}, \boldsymbol{\rho}_{(n+1)}] = \tfrac{1}{2}(\boldsymbol{\sigma}_{(n+1)} - \boldsymbol{\rho}_{(n+1)})^{\mathrm{T}}\, \mathbf{M}\, (\boldsymbol{\sigma}_{(n+1)} - \boldsymbol{\rho}_{(n+1)}) - \sigma_y^2 \le 0 \tag{10}$$

In the *a* bem-elements (with $a \le m$) where this latter inequality is violated, a return mapping phase occurs to evaluate the plastic strains and the direction of the plastic flow. Therefore the same eq.(8) is used to obtain the elastoplastic solution at every bem-e where the plastic consistency condition is violated. The vector $\boldsymbol{\sigma}$, representing the end step stress, the internal variable vector $\boldsymbol{\rho}$ as well as the volumetric plastic strain vector $\mathbf{p}$ are unknown quantities. This latter is the plastic strain to impose at every active plastically bem-e in order to have the stress on the yield boundary of the elastic domain, through which the direction of the plastic flow may be defined. Obviously, inside of each loading step the corrector phase has to be repeated until all the predictors do not satisfy the plastic consistency conditions.

In detail eq.(8), written for every *h* bem-elements ($h = 1,...,a$), is utilized to perform the elastoplastic analysis at $n+1$ load step simultaneously in all the plastically active macro-zones, defined as whole of active bem-elements, individuated in the previous predictor phase, i.e.:

$$\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^* - \sum_{k=1}^{a} \mathbf{Z}_{hk}\, \mathbf{p}_k = \mathbf{0} \tag{11}$$

where the subscript $n+1$ has been omitted for convenience.

The $\mathbf{Z}_{hk}$ sub-matrix with $h,k = 1,...,a$ derives from the $\mathbf{Z}$ matrix present in eq.(8), by extracting the blocks relative to the *a* plastically active bem-elements. The double index specifies the bem-elements *k* where the plastic strains (cause) and the *h* related stresses (effect) arise.

In the hypothesis that, for each *h*-th bem-e, the shape function definite in eq.(3g) is the same of the shape function related to the plastic multiplier, i.e. $\lambda_k = \psi_{pk}\Lambda_k$ with $\psi_p \geq 0$, the plastic strain for the *h*-th active bem-e is expressed as $\mathbf{p}_k = \Lambda_k \partial F_k / \partial \boldsymbol{\sigma}_k = \Lambda_k\, \mathbf{M}\,(\boldsymbol{\sigma}_k - \boldsymbol{\rho}_k)$.

The non linear solving system for all the active bem-elements is the following:

$$\begin{cases} \boldsymbol{F}_{Ih} \equiv \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^* - \sum_{k=1}^{a} \Lambda_k\; \mathbf{Z}_{hk}\; \mathbf{M}(\boldsymbol{\sigma}_k - \boldsymbol{\rho}_k) = \mathbf{0} \\ \boldsymbol{F}_{IIh} \equiv \boldsymbol{\rho}_h - \boldsymbol{\rho}_h^n - \Lambda_h\; \mathbf{H}_h\; \mathbf{M}\;(\boldsymbol{\sigma}_h - \boldsymbol{\rho}_h) = \mathbf{0} \qquad\qquad \textit{with} \quad h = 1,...,a \\ F_{IIIh} \equiv \tfrac{1}{2}(\boldsymbol{\sigma}_h - \boldsymbol{\rho}_h)^{\mathrm{T}}\, \mathbf{M}\; (\boldsymbol{\sigma}_h - \boldsymbol{\rho}_h) - \sigma_y^2 = 0 \end{cases} \tag{12}$$

where $\boldsymbol{\sigma}_h$ is the stress solution located on the yield surface of the elastic domain, $\boldsymbol{\sigma}_h^*$ the elastic predictor, $\Lambda_h\; \mathbf{Z}_{hh}\; \mathbf{M}(\boldsymbol{\sigma}_k - \boldsymbol{\rho}_k)$ the direct corrective component and $\sum_{k=1}^{a \neq h} \Lambda_k\; \mathbf{Z}_{hk}\; \mathbf{M}(\boldsymbol{\sigma}_k - \boldsymbol{\rho}_k)$ the indirect corrective component, respectively.

Eqs.(12) comprises a system of 7xa non linear equations in 7xa unknowns (three stress components $\boldsymbol{\sigma}_h$, three internal variables $\boldsymbol{\rho}_h$ and a plastic multiplier Λ_h for each active bem-e).

The approximate solution of this nonlinear problem involving all the plastically active bem-elements is here obtained by applying the Newton-Raphson procedure:

$$\left|\begin{array}{ccc:ccc:ccc} \mathbf{I} - \Lambda_1^j \mathbf{Z}_{11}\mathbf{M} & \cdots & -\Lambda_a^j \mathbf{Z}_{1a}\mathbf{M} & \Lambda_1^j \mathbf{Z}_{11}\mathbf{M} & \cdots & \Lambda_a^j \mathbf{Z}_{1a}\mathbf{M} & -\mathbf{Z}_{11}\mathbf{M}(\boldsymbol{\sigma}_1^j - \boldsymbol{\rho}_1^j) & \cdots & -\mathbf{Z}_{1a}\mathbf{M}\,(\boldsymbol{\sigma}_a^j - \boldsymbol{\rho}_a^j) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\Lambda_a^j \mathbf{Z}_{a1}\mathbf{M} & \cdots & \mathbf{I} - \Lambda_a^j \mathbf{Z}_{aa}\mathbf{M} & \Lambda_a^j \mathbf{Z}_{a1}\mathbf{M} & \cdots & \Lambda_a^j \mathbf{Z}_{aa}\mathbf{M} & -\mathbf{Z}_{a1}\mathbf{M}\,(\boldsymbol{\sigma}_1^j - \boldsymbol{\rho}_1^j) & \cdots & -\mathbf{Z}_{aa}\mathbf{M}\,(\boldsymbol{\sigma}_a^j - \boldsymbol{\rho}_a^j) \\ \hdashline -\Lambda_1^j \mathbf{H}_1\mathbf{M} & \cdots & \mathbf{0} & \mathbf{I} + \Lambda_1^j \mathbf{H}_1\mathbf{M} & \cdots & \mathbf{0} & -\mathbf{H}_1\mathbf{M}\,(\boldsymbol{\sigma}_1^j - \boldsymbol{\rho}_1^j) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & -\Lambda_a^j \mathbf{H}_a\mathbf{M} & \mathbf{0} & \cdots & \mathbf{I} + \Lambda_a^j \mathbf{H}_a\mathbf{M} & \mathbf{0} & \cdots & -\mathbf{H}_a\mathbf{M}\,(\boldsymbol{\sigma}_a^j - \boldsymbol{\rho}_a^j) \\ \hdashline (\boldsymbol{\sigma}_1^j - \boldsymbol{\rho}_1^j)^T\mathbf{M} & \cdots & \mathbf{0} & -(\boldsymbol{\sigma}_1^j - \boldsymbol{\rho}_1^j)^T\mathbf{M} & \cdots & \mathbf{0} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & (\boldsymbol{\sigma}_a^j - \boldsymbol{\rho}_a^j)^T\mathbf{M} & \mathbf{0} & \cdots & -(\boldsymbol{\sigma}_a^j - \boldsymbol{\rho}_a^j)^T\mathbf{M} & 0 & \cdots & 0 \end{array}\right| \left|\begin{array}{c} \boldsymbol{\sigma}_1^{j+1} - \boldsymbol{\sigma}_1^j \\ \vdots \\ \boldsymbol{\sigma}_a^{j+1} - \boldsymbol{\sigma}_a^j \\ \hdashline \boldsymbol{\rho}_1^{j+1} - \boldsymbol{\rho}_1^j \\ \vdots \\ \boldsymbol{\rho}_a^{j+1} - \boldsymbol{\rho}_a^j \\ \hdashline \Lambda_1^{j+1} - \Lambda_1^j \\ \vdots \\ \Lambda_a^{j+1} - \Lambda_a^j \end{array}\right| = \left|\begin{array}{c} -\boldsymbol{F}_{I1}^j \\ \vdots \\ -\boldsymbol{F}_{Ia}^j \\ \hdashline -\boldsymbol{F}_{II1}^j \\ \vdots \\ -\boldsymbol{F}_{IIa}^j \\ \hdashline -F_{III1}^j \\ \vdots \\ -F_{IIIa}^j \end{array}\right| \tag{13}$$

which, written in compact form, becomes:

$$\boldsymbol{X}_a^{j+1} = \boldsymbol{X}_a^j - \mathbf{J}_{aa}(\boldsymbol{X}_a^j)^{-1}\, \boldsymbol{f}\,(\boldsymbol{X}_a^j) \tag{14}$$

The Jacobian matrix $\mathbf{J}_a$ contains the derivatives of the functions defined in eqs.(12), $\boldsymbol{X}_a^{j+1}$ is the vector of the unknowns, $\boldsymbol{X}_a^j$ and $\boldsymbol{f}\,(\boldsymbol{X}_a^j)$ are the known vectors computed in the *j*-th step.

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