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ON A RESULT OF THOMASSEN

PÉTER KOMJÁTH

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Abstract. We give a new proof of Thomassen's theorem stating that if the chromatic (coloring) number of a graph X is $> \kappa$, then X contains a κ-edge-connected subgraph with similar properties.

Recently Thomassen proved ([5]) that if κ is an infinite cardinal, then each graph with chromatic number $> \kappa$ contains a κ-edge connected subgraph with chromatic number greater than κ. He also proved the analogous result for the coloring number. We give a different proof of these results.

Notation. Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal is identified with the least ordinal of that cardinality. For the notions of regular cardinals, closed unbounded and stationary sets, and Fodor's pressing down lemma, the textbook [2] is recommended.

If S is a set, κ a cardinal, we define

$$[S]^\kappa = \{x \subseteq S : |x| = \kappa\}, [S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}.$$

A *graph* is a pair (V, X), where V is an arbitrary set (the set of *vertices*) and $X \subseteq [V]^2$ (the set of *edges*). We sometimes write simply X rather than (V, X), i.e., identify the graph with its edge set. If (V, X) is a graph and $x \in V$ a vertex, then $N(x) = \{y \in V : \{x, y\} \in X\}$ is the *neighborhood of* x. If V is ordered by $<$, then $N^-(x) = \{y < x : \{x, y\} \in X\}$ and $N^+(x) = \{y > x : \{x, y\} \in X\}$. A *path* is a sequence $(v_0, v_1, \dots, v_n)$ such that $\{v_i, v_{i+1}\} \in X$ $(i < n)$. A graph is *connected* if any two vertices are connected by a path. A graph (V, X) is *κ-edge-connected* if $(V, X - U)$ is connected for any $U \subseteq X$, $|U| < \kappa$.

The *chromatic number* of some graph (V, X), $\mathrm{Chr}(V, X)$, is the least cardinal μ such that there is a function $f : V \to \mu$ with $f(x) \neq f(y)$ for $\{x, y\} \in X$.

The *coloring number* of a graph (V, X), $\mathrm{Col}(V, X)$ or $\mathrm{Col}(X)$, is the least cardinal μ such that there is a well ordering $<$ of V such that $|N^-(x)| < \mu$ for each $x \in V$. In [1] Fodor proved that if μ is an infinite cardinal and there is an orientation of (V, X) such that the outdegree of each vertex is $< \mu$, then $\mathrm{Chr}(X) \leq \mu$. The proof actually gives $\mathrm{Col}(X) \leq \mu$, so, as the other direction is straightforward, we have that $\mathrm{Col}(X) \leq \mu$ is equivalent to the existence of an orientation as above.

Theorem 1. *If (V, X) is a graph with $\mathrm{Col}(V, X) > \kappa$, then there is a κ-edge-connected subgraph Y of X with $\mathrm{Col}(Y) > \kappa$.*

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Proof. By possibly passing to a smaller subgraph we can assume that $\mu = |V|$ is such that each subgraph of size $< \mu$ has coloring number $\leq \kappa$. For simplicity we assume that $V = \mu$. A special case of Shelah's singular cardinal compactness theorem ([4]) states that μ is regular. As κ is infinite, μ is uncountable, so it is meaningful to speak of stationary subsets of μ.

Define

$$S = \{\delta < \mu : \exists \alpha(\delta) \geq \delta, |N^-(\alpha(\delta)) \cap \delta| \geq \kappa\}.$$

An argument of Shelah in [3] shows that $\mathrm{Col}(X) > \kappa$ is equivalent to S being stationary. Next, define the closed, unbounded set

$$E = \{\gamma < \mu : \delta < \gamma \longrightarrow \alpha(\delta) < \gamma\}.$$

The set $S^* = S \cap E$ is stationary, and if for every $\gamma \in S^*$ we interchange γ and $\alpha(\gamma)$, then we obtain a different well ordering such that for $\delta \in S^*$, $|N^-(\delta)| \geq \kappa$ holds.

Using the Fodor pressing down lemma, we can assume that there is $\beta^* < \mu$ such that $\beta^* \in N^-(\delta)$ for each $\delta \in S^*$.

Define $B'' = \{\alpha < \mu : N(\alpha) \text{ is nonstationary}\}$.

Claim 1. $S'' = \{\delta \in S^* : N^-(\delta) \cap B'' \neq \emptyset\}$ is nonstationary.

Proof. Assume that S'' is stationary. Pick for every $\delta \in S''$ an element $\beta(\delta) \in N(\delta) \cap B''$. Again using Fodor's pressing down lemma there is some β such that $S^{**} = \{\delta \in S'' : \beta(\delta) = \beta\}$ is stationary. This is a contradiction, as $\beta \in B''$, yet $S^{**} \subseteq N(\beta)$. □

Define $S' = S^* - S''$, $B' = \bigcup\{N^-(\delta) : \delta \in S'\}$, $Y = X|(S' \cup B')$.

Claim 2. Y is κ-edge-connected.

Proof. Assume that $U \subseteq Y$, $|U| < \kappa$.

First we show that if $\delta \in S'$, then δ is connected with β^* in $Y - U$. Pick $\beta \in N^-(\delta)$ such that $\{\beta, \delta\} \notin U$. Then choose $\delta' \in N^+(\beta)$ with $\{\beta, \delta'\}, \{\beta^*, \delta'\} \notin U$. These choices are possible as $|U| < \kappa$. Now the path $(\beta^*, \delta', \beta, \delta)$ connects β^* and δ in $Y - U$.

Finally we show that if $\beta \in B'$, then β and β^* are joined in $Y - U$. Pick $\delta \in N^+(\beta)$ such that $\{\beta^*, \delta\}, \{\beta, \delta\} \notin U$. Then the path (β^*, δ, β) connects β^* and β in $Y - U$. □

By Shelah's argument quoted above $\mathrm{Col}(Y) > \kappa$, and so the proof is concluded. □

Theorem 2. *If X is a graph with $\mathrm{Chr}(X) > \kappa$, then X has a κ-edge-connected subgraph Y with $\mathrm{Chr}(Y) > \kappa$.*

Proof. Let V be the vertex set of X.

By Zorn's lemma each $A \subseteq V$ which induces a κ-edge-connected subgraph of X can be embedded into a maximal such subset. One can immediately see that if $A \neq B \subseteq V$, where both are maximal subsets of V that induce κ-edge-connected graphs, then $A \cap B = \emptyset$.

This splits V as $V = V' \cup V''$, where $X|V''$ has no κ-edge-connected subgraphs, while V' is partitioned as $V' = \bigcup\{A_i : i \in I\}$, where each A_i is maximal such that $X|A_i$ is κ-edge-connected.

By Theorem 1 $\mathrm{Chr}(X|V'') \le \kappa$ (even $\mathrm{Col}(X|V'') \le \kappa$), so we can remove V'' and assume that $V = \bigcup\{A_i : i \in I\}$.

We indirectly assume that $\mathrm{Chr}(X|A_i) \le \kappa$ for every $i \in I$.

Define the following graph Y on J: if $i, j \in I$, then $\{i, j\} \in Y$ iff there is at least one edge between A_i and A_j.

Case 1. $\mathrm{Chr}(Y) \le \kappa$.

In this case, there is a mapping $f : I \to \kappa$ such that if $f(i) = f(j)$, then there is no edge between A_i and A_j. For each $\alpha < \kappa$, define

$$B_\alpha = \bigcup\{A_i : f(i) = \alpha\}.$$

For each $\alpha < \kappa$, the graph $X|B_\alpha$ is the union of κ-chromatic graphs on disjoint sets, with no crossing edges; therefore $\mathrm{Chr}(X|B_\alpha) \le \kappa$. As $V = \bigcup\{B_\alpha : \alpha < \kappa\}$, we conclude that $\mathrm{Chr}(X) \le \kappa$, a contradiction.

Case 2. $\mathrm{Chr}(Y) > \kappa$.

In this case, by Theorem 1, there is a subset $J \subseteq I$ with $|J| > 1$ such that $Y|J$ is κ-edge-connected. Set $B = \bigcup\{A_i : i \in J\}$.

Claim. $X|B$ is κ-edge-connected.

Proof. Let $U \subseteq X|B$ have $|U| < \kappa$. As each $X|A_i$ is κ-edge-connected, for every $i \in J$, the graph $(X - U)|A_i$ is connected, that is, any two vertices of A_i are connected with some path in $X - U$.

Let U^* contain those pairs $\{i, j\}$ for which there is an edge in U joining a vertex of A_i with a vertex of A_j. Clearly, $|U^*| \le |U| < \kappa$. As $Y - U^*$ is connected, for any $i, j \in J$, there is a path of $(X - U)|B$ between some vertex in A_i and some vertex in A_j.

Altogether, we obtain that all vertices of B are in the same component of $X - U$, which proves the Claim. □

From the Claim we get that if $i \in J$, then B properly extends the maximal κ-edge-connected A_i, a contradiction. □

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Institute of Mathematics, Eötvös University, Budapest, Pázmány P. s. 1/C, 1117, Hungary

E-mail address: kope@cs.elte.hu