Object, Structure, and Form* †

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Abstract

The main task of this paper is to develop the non-Platonist view of mathematics as a science of structures I have called, borrowing the label from Putnam, 'realism with the human face'.

According to this view, if by 'object' we mean what exists independently of whether we are thinking about it or not, mathematics is a science of patterns (structures), where patterns are neither objects nor properties of objects, but aspects (or aspects of aspects, etc.) of concrete objects which dawn on us when we represent objects (or aspects of . . .) within a given system (of representation).

Mathematical patterns, therefore, are real, because they ultimately depend on concrete objects, but are neither objects nor properties of objects, because they are dependent, both metaphysically and epistemically, on systems of representation.

Although the article has been written as a presentation of my view of mathematics, and of some of its advantages, the reader should keep in mind that this is essentially a 'reply paper', as is shown by the fact that much of it is dedicated to the discussion of some issues which have become the focus of critical attention. Such issues are well expressed by the following questions: am I right in asserting that mathematical patterns are neither objects nor properties of objects? What is the difference, if any, between mathematical patterns and other mind-dependent entities such as the Cleveland Symphony Orchestra? Can mathematical patterns be always assimilated to relations? Can what I call 'form of representation' be assimilated to structure? Can the standpoint I take on mathematics, which regards it as a science of patterns, be correctly described as Aristotelian?

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1 Patterns, aspects, and infinity

The main task of this paper is developing the non-Platonist view of mathematics as a science of structures I have called, borrowing the label from Putnam, 'realism with the human face'.

According to this view, mathematics is a science of patterns (structures), where (1) patterns are neither objects nor properties of objects (see on this §§2–4), but (2) are, rather, aspects (or aspects of aspects, etc.) of concrete objects which dawn on us when we represent objects (or aspects of...) within a given system of representation.

At this point, before going any further in presenting my view, it is important to clarify some basic notions involved in the two theses above.

First of all, let me say that the notion of representation is rather complex, and deserves much attention. However, for the purpose of this paper, it is sufficient to know that what I mean by 'representation of an object' is a description of the object which can be expressed by propositions.

Typical non-propositional descriptions of objects are those offered by diagrams and maps. Take, for instance, the table below of the Turing left-machine:[1]

0	a_0	l	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	a_n	l	1
1	a_0	h	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	a_n	h	1

Table 1: The Turing left-machine

This is a diagram—a $[2(n+1)] \times 4$ matrix, where $n \in \mathbb{N}$—representing a particular Turing machine,[2] a diagram which can be transformed into a description expressed by propositions of the type 'the left-machine is a Turing machine having the following characteristics: if the machine is in state 0 and the symbol showing in the observed cell is a_0, then the observing device of the machine moves along the tape to the cell which is immediately to the left of the observed cell and the machine goes into state 1, etc. etc.'

[1] The natural numbers 0 and 1 refer to the states of the machine, a_0 is the symbol for the empty cell, $a_1, \ldots, a_n$ are the so-called 'proper symbols', 'l' stands for 'move one cell to the left of the observed cell', and 'h' stands for 'halt'.

[2] In contrast with what some authors believe about this (see [Hermes 1975], ch. 1, §3.6, p. 44), my reason for saying that the matrix in Table 1 represents the Turing left-machine, and does not actually coincide with it, is that the permutations of the set of $< 0, \ldots, \ldots, 1 >$ rows, and of the set of $< 1, \ldots, \ldots, 1 >$ rows of the matrix produce $[(n+1)!]^2$ different $[2(n+1)] \times 4$ matrixes which are tables of the same Turing machine.

Here the concept of description is very important, because in order, for instance, to have a representation of a dog (or of a Turing machine) it is not sufficient to be acquainted with it, we also need a(n even non-linguistic) way of identifying and re-identifying the object represented as what is represented. Moreover, I say that a given representation of an object is *faithful* just in case the proposition (or propositions) expressing it is (are) true.

Secondly, a mathematical system of representation is most simply a mathematical theory. And my reason for saying this is that mathematical theories, besides being what we might call 'deductive engines', provide us with systems of representation.

To see this, consider that mathematical theories have an important rôle in providing (linguistic) systems of representation as it is clearly exemplified by a large number of cases like that of the representation, within analytic geometry in $\mathbb{R}^2$, of a straight line r lying in a plane α by means of the equation $y = mx + c$.

Furthermore, mathematical theories happen to give a substantial contribution to perceptual representations as well. This is made manifest by those very frequent cases in which some concepts belonging to a mathematical theory T, very much like the Kantian pure *a priori* intuitions of space and time, operate a pre-reflective structuring of perception which enables us to see something as a square or as a triangle, etc.

Now, with regard to the plausibility of thesis (2)—that mathematical patterns are aspects (or aspects of aspects, etc.) of concrete objects which dawn on us when we represent objects (or aspects of...) within a given system of representation—think about the following situation. On a table in a room there are three marbles which an observer $\mathcal{O}$, under normal conditions, can see, in one case, as the vertexes of an equilateral triangle, if he knows Euclidean geometry, and, in the other, as a set A of 3 elements, if he can count and knows set theory.

In both cases the mathematical aspect that dawns on $\mathcal{O}$ depends on how he relates to each other the marbles on the table: in the first case he does so by using concepts like 'x is a vertex', 'x is a triangle having sides equal to each other in length', etc.; and in the second case by using the concept '$x \in A$' and counting.

Notice that, in both the cases mentioned above, $\mathcal{O}$ has criteria of identity for the patterns that dawn on him. In the first case, the criterion of identity is congruence between triangles, and in the second case is the idea that two sets are equal if they have the same elements. It is very important that we have criteria of identity for mathematical patterns, because, as Quine put it: no entity without identity.

Of course, we can have patterns of patterns. If $\mathcal{O}$ subjects what he sees as an equilateral triangle to rigid motions and reflections, and knows some group theory, then he will be able to see the set of rigid motions and reflections of what he sees as an equilateral triangle as a group, i.e., the dihedral group of order 6.[3]

Moreover, if, in seeing the marbles on the table as a set A of 3 elements, $\mathcal{O}$ considers the possible permutations of the elements of such a set, then, if he knows some group theory, he will be able to see the set of permutations of what he sees as a set of 3 elements as the symmetric group of degree 3.

Also in the case of the two patterns of patterns mentioned above, $\mathcal{O}$ has a criterion of identity: group isomorphism. Indeed, the two groups described are isomorphic to each other, i.e., they are the same group.

If (2) is correct, it follows that mathematical patterns are real, because, once I choose 2-dimensional Euclidean geometry as system of representation, the faithfulness of representing what is on the table as the vertexes of an equilateral triangle is something that would depend on the concrete objects I am representing, and their relative positions.

But, having said so, it might be worth clarifying that the dependence of mathematical patterns on concrete objects I refer to above is that typical of a structure on the set of concrete objects, and relations defined on such a set, that realize it. It, therefore, makes no sense to say that if one accepts my position—that mathematical patterns are real, because they are ultimately dependent on concrete objects—then, since the fictional characters responding to the names of 'Hamlet', 'Ophelia' and 'Falstaff', depend on the concrete object William Shakespeare who invented them, he should also concede that Hamlet, Ophelia, and Falstaff are real.

A second question that needs to be addressed here, and which I have already discussed in [Oliveri, 2007], Ch. 5, §12, is: if mathematical patterns depend ultimately on concrete objects (and on mathematical systems of representation), how do I deal with transfinite sets?

This is a very important question for the acceptability of my account of mathematics, because, since the concept of infinity has a paramount rôle in mathematics, if there were only a finite number of concrete objects in the world, it would not be easy to see how 'infinite mathematical patterns' might be ultimately dependent on concrete objects, and be given to us.

The solution to the problem above originates from the consideration that some patterns are patterns of patterns. Indeed, if I say 'There are three marbles on the table', what I express in my use of the term 'three' captures a numerical pattern of a set of concrete objects that is objectively given in the sense that: (i) I can *prove* that there are three marbles on the table, and that (ii) I have identity conditions for numerical patterns of sets (cardinal numbers), identity conditions represented by bi-univocal correspondences between sets.

Now, the fact that the pattern denoted by 'three' is objectively given

[3]The dihedral group of order 6 can also be individuated by means of a multiplication table, and as the non-abelian group of order 6.

is what, among other things, provides the necessary and sufficient conditions for referring to the pattern denoted by 'three' as a member of another pattern.

This is because, on the one hand, it justifies treating the number three as an object and, on the other, it provides both a necessary and sufficient condition for assigning a truth-value to the expression '$3 \in A$', where A is a given set.

Therefore, we have that whereas numerical patterns such as: one, two, three, etc., may be considered as aspects of sets of concrete objects, or as the outcome of operations on such ones,[4] the introduction of the concept of (actual) infinity comes about as a consequence of being able to see the collection of *all* those numerical aspects as a whole.

It is important to notice that the phenomenon described here as 'seeing a collection as a whole' is an instance of aspect-seeing in which we are dealing with a pattern of patterns. The correctness of this view is confirmed by the consideration that the ability to see the collection of (the set-theoretical representations of) the natural numbers as a whole (complete totality or set) depends on the system of representation adopted.

In fact, it is only through a modification of the Euclidean concept of 'whole'—the whole is greater than the part (Euclid's axiom 5)—and the axiomatization of set theory which eliminates the known paradoxes, that it becomes mathematically meaningful to see infinite collections not simply as entities of unbounded growth, i.e., as instances of potentially infinite collections, but as infinite totalities, i.e., as entities such that, given any two of them, it makes sense to ask whether the number of elements of one is lesser, equal, or greater than the number of elements of the other.

As a confirmation of this we have the exemplary case of Galileo. Galileo found himself in an embarrassing situation when he observed that, if we consider $\mathbb{N}$ as a completed totality, there appear to be as many squares of natural numbers as there are natural numbers. This is the case, he argued, because each natural number is the root of a square number and there are as many squares of natural numbers as their roots.

The observation above was a source of embarrassment for Galileo, because, besides contradicting Euclid's axiom 5, it seemed to conflict with the other observation that the occurrence of perfect squares in the natural ordering of $\mathbb{N}$ becomes more and more rare the larger the initial segment of $\mathbb{N}$ we consider.[5] For, the latter observation suggested to him that most natural numbers are not squares of natural numbers.[6]

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[4]Interesting examples of the latter type of numerical patterns are the numbers 0 and $10^{10^{10^{10}}}$.

[5]In the first 100 natural numbers there are only 10 squares, in the first 10000 only 100, in the first 1000000 only 1000, etc.

[6][Galileo 1638], Giornata Prima, pp. 78–79:

Salv. Ma se io domanderò, quante siano le radici, non si può negare che elle non siano

At this point Galileo, unable to deny Euclid's axiom 5 and unwilling to consider number theory to be inconsistent, to save the day, decided to reject the notion of actual infinity in favour of the old Aristotelian concept of potential infinity, and adopt an *ad hoc*, monster barring, deliberation whereby:[7]

> **Salv.** ... attributes of equal greater and lesser are not applicable to infinities, but only to finished [terminating, completed] quantities.

A couple of centuries later, armed with a different mathematical system of representation which, among other things, got rid of Euclid's axiom 5, Dedekind operated a truly Kuhnian *Gestalt* switch when he saw the phenomenon contemplated by Galileo not as a mathematical monstrosity engendered by the perfidy of infinite collections, but as a particular instance of a mathematically fruitful property which allows us to characterize infinite sets as those collections which can be put in bi-univocal correspondence with a proper subset.

At this point, taking for granted that the account given so far explains how to deal with $\mathbb{N}$, it is not difficult to see how to extend my account to transfinite sets of any cardinality following the traditional limited comprehension principles of ZFC. I will here illustrate only the case regarding what is known as the power set axiom.

If A is a set of mathematical patterns which ultimately depend on concrete objects (I am no longer going to repeat 'and on a system of representation'), a subset B of A would exemplify a pattern of mathematical patterns which ultimately depend on concrete objects and, therefore, the pattern exemplified by B would ultimately depend on concrete objects as well. (Take $A = \mathbb{N}$ and $B = \{x \mid x \in A \text{ and } x \text{ is even}\}$.)

Now, if this is the case, the ability to see the collection of *all* the subsets of A, $\mathcal{P}(A)$, as a whole/set presents us with the exemplification of a pattern$_2$ (of a pattern$_1$ of a pattern$_0$), which ultimately depends on concrete objects, because pattern$_1$ (and pattern$_0$) ultimately depends on concrete objects.

quante tutti i numeri, poichè non vi è numero alcuno che non sia radice di qualche quadrato; e stante questo, converrà dire che i numeri quadrati siano quanti tutti i numeri, poichè tanti sono quante le lor radici, e radici son tutti i numeri: e pur da principio dicemmo, tutti i numeri esser assai più che tutti i quadrati, essendo la maggior parte non quadrati. E pur tuttavia si va la moltitudine de i quadrati sempre con maggior proporzione diminuendo, quanto a maggior numeri si trapassa; perchè sino a cento vi sono dieci quadrati, che è quanto a dire la decima parte esser quadrati; in dieci mila solo la centesima parte son quadrati, in un milione solo la millesima: e pur nel numero infinito, se concepir lo potessimo, bisognerebbe dire, tanti essere i quadrati quanti tutti i numeri insieme.

[7] See [Galileo 1638], Giornata Prima, p. 79.

Therefore, we can conclude that, if $\mathbb{N}$ is a set of mathematical patterns which ultimately depend on concrete objects, we can say the same thing of each element of the sequence:

$$\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \ldots$$

Notice that also what I called 'the ability to see the collection of *all* the subsets of A, $\mathcal{P}(A)$, as a whole/set' strongly depends on the system of representation you adopt. For, if instead of ZFC, you work within Gödel's system of constructible sets, then if X is a set in L (the universe of constructible sets), the power-set of X is the set of all definable subsets of X, $D(X)$, which is usually smaller than $\mathcal{P}(X)$.

In bringing this section to a close, there is an important thing we should notice which emphasizes the purely relational character of mathematical patterns. If, in the thought experiment sketched at the beginning of this section, our observer $\mathcal{O}$ had 3 coins, or 3 dice, etc. on the table instead of three marbles, and if the coins, or the dice, etc. were to each other in positions similar to the positions in which the marbles were to each other, then, under normal circumstances, he could still see the coins, or the dice, etc. as the vertexes of an equilateral triangle, in one case, and as the elements of a set of 3 elements, in the other, etc.

This last consideration shows that what is relevant to the dawning of a mathematical pattern is not the nature of certain objects, but the relation(s) in which these objects are to one another.

2 Patterns, and systems of representation

In my book I defended the view that, if by 'object' we mean what exists independently of whether we are thinking about it or not, then, given my rejection of the Platonist belief in abstract objects, mathematical patterns are neither objects nor properties of objects. (Thesis (1), §1, p. 2.)

The main reason why mathematical patterns are not objects is that they depend, both metaphysically and epistemically, on systems of representation produced by mathematicians.

To see this consider, first, that since no concrete object is a perfect triangle, circle, square, etc. seeing something as—the process at the root of the dawning of mathematical aspects—can neither be construed as an act whereby a property, attribute of a concrete object is abstracted; nor as the consequence of that type of selective attention paid to a concrete object whose precondition is the existence of a system of representation.[8]

[8]Indeed, if, for the sake of argument, we assume that abstraction is a viable route to concept formation, then properties of concrete objects that might be good candidates for being 'abstracted' are those, for instance, of being transparent or opaque. In this case we can see without the aid of any instrument—whose way of functioning would have to be

Secondly, since, as I have already said, there are no squares in nature, a necessary condition for seeing something as a square has to be knowing what a square is or, to put it in a different way, knowing what falls under the concept 'x is a square'.

Now, given that concepts may be distinguished into well founded (sharp) and vague (fuzzy), and that in this paper I deal essentially with mathematical concepts, I am going to consider in what follows only well founded concepts.

For the concept 'x is a square' to be well founded, besides knowing the conditions that an object $\mathfrak{O}$ must satisfy for $\mathfrak{O}$ to be a square, we also need to know when two squares are the same, i.e., we need to have a criterion of identity for squares. An obvious candidate as criterion of identity for squares is square congruence, i.e., the idea that two squares are equal if and only if they can be made to coincide without deforming them.[9]

However, the very interesting thing that happens at this point is that, as soon as we introduce square congruence as a criterion of identity for squares, we produce a partition of all things we see as squares into equivalence classes—for square congruence is an equivalence relation—and each such an equivalence class is uniquely associated to an abstract structure,[10]

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justified by a theory—that certain bodies let light through them, whereas others do not.

An example of a property of concrete objects of the second kind mentioned above is, instead, that of having spin $\frac{1}{2}$. Clearly, in this case, it being impossible for human beings to be directly acquainted with subatomic particles, we need a theory (system of representation) for both postulating the existence of such entities and interpreting the experimental evidence confirming their existence and properties, experimental evidence deriving from the unavoidable use of instruments of observation.

Now, taking for granted the presence of a common agreement among the community of experts about the fact that observer $\mathcal{O}$ is not dreaming, hallucinating, etc. when he sees an object X, notice that an interesting philosophical difference 'between things that we can see without instruments and things that we can see only with the help of instruments' is that, if $\mathcal{O}$ is not directly acquainted with X, but needs, say, a telescope to see X, he might still be asked, like Galileo once was, how does he know that X is not a mere 'product' of the telescope generated by a defect in the lenses (or in the mirror) used etc. Of course, a satisfactory answer to such a question can only be obtained from a theory which explains, in particular, the 'workings' of the telescope.

Moreover, in contrast with what happens when $\mathcal{O}$ is directly acquainted with an object X, scientists have to learn to look through a telescope, a microscope, and 'into' a bubble-chamber. And such a leaning process is not like training the eye to see different types of snow, but it involves, among other things, learning to take (interpreting) certain spots of light to be stars, other images to be planets, etc. (The same considerations apply to learning to use microscopes and bubble-chambers).

[9]Notice that neither knowing what a square is nor having identity conditions for squares imply that there must be effective procedures able to determine in a finite number of steps whether or not any object $\mathfrak{O}$ is a square, or whether or not any two given squares α and β are the same.

[10]The reason why the geometrical structure common to all the elements of a particular equivalence class of squares is abstract is that, as it has been remarked several times, there are no squares in the external world, i.e., in space-time.

i.e., to the abstract geometrical structure common to all the elements of the equivalence class.

Such an abstract geometrical structure uniquely associated with and determined by the equivalence class of congruent squares is the mathematical pattern we are going to call 'the square of side r', for $r \in \mathbb{R}^+$. This is very much like what happens in set theory when we use the expression 'the cardinal number 3' to refer to that abstract entity—mathematical pattern—uniquely associated with and determined by the equivalence class of all sets which can be put in bi-univocal correspondence with the set $\{*, **, * * *\}$.

If this is correct, then, besides having here the beginning of an account of the emergence of abstract mathematical patterns such as 'the square of side r' and 'the cardinal number 3' from the cloud of our representations, we can also say that, since without 2-dimensional Euclidean geometry there is no concept of square, and, consequently, there are no criteria of identity for squares; and since with no criteria of identity for squares there are no equivalence classes of 'congruent squares' and, therefore, no mathematical square-patterns uniquely associated with and determined by them; it follows that without 2-dimensional Euclidean geometry there are no mathematical square-patterns.

At this point, given that (1) seeing something as a square can neither be construed as the outcome of an act of abstraction nor as that type of selective attention paid to a concrete object whose precondition is the existence of a system of representation; and that (2) without 2-dimensional Euclidean geometry there are no mathematical square-patterns, we can conclude that the dependence of mathematical patterns on appropriate systems of representation cannot simply be epistemic, but must also be metaphysical.

Lastly, with regard to the notion of mathematical system of representation, i.e., mathematical theory, it is important to observe that these mathematical systems of representation are not given *a priori* in the mind, but are rather the product of human activity. It is precisely this feature of mathematical systems of representation (of being the product of human activity) that led me to say that considering mathematics as a science of patterns is a form of realism with the human face.

3 Patterns are not self-standing objects

Julian Cole objects to my definition of an object—as what exists independently of whether we are thinking about it ot not—that:

> First, it makes nonsense of a well established tradition of referring to such items as choirs, countries, and legal corporations as social objects . . . Second, it flies in the face of a much more widely accepted characterization of an object—roughly, an item that

> may legitimately fall within the range of the first-order quantifiers of an appropriately formalized statement. Third, it fails to distinguish between (mind-independent) objects and (mind-independent) properties. ([Cole, 2008], p. 11.)

In reply to the points raised by Cole, I would like to say that the rationale behind my idea that if something is an object then it must exist independently of whether we are thinking about it or not is the concern about distinguishing what there is into two big classes: (A) what exists independently of human activity, and (B) what does not. Such a distinction is very important for my project, because it is preliminary to, and supports the '(structural) realism with the human face' I advocate about mathematics.

Therefore, my attitude towards objects should not be seen as part of an attempt to weaken the classical logico-linguistic Fregean distinction between saturated and unsaturated parts of a proposition, etc. For, it is only meant to add a metaphysical flavour to it by saying that an object, besides being the possible reference of a saturated part of a proposition, exists independently of human activity.

However, for the sake of clarity, let us ask ourselves whether the predicates 'x is an object', 'x is soluble in water', and 'x is prime' are analogous to one another or not.

Well, at first sight we notice that whereas the predicates 'x is soluble in water', and 'x is prime' are true, the first, of some concrete entities, and, the other, of some abstract entities, the predicate 'x is an object' is not only topic neutral, but is also ontologically neutral, because it is applicable to abstract, and concrete entities alike.

The predicate 'x is an object' is, indeed, applicable to anything about which we can say true or false things or, to put it in a different way, to anything that falls under a concept.

The generality of the predicate 'x is an object', which is part and parcel of its topic and ontological neutrality, speaks of its eminently logical nature. This is the reason why, from now on, I will take an object to be simply an element of the domain of discourse.

Furthermore, to dispel any possible source of misunderstanding on this issue, I will say that a self-standing object is an object that exists independently of whether we are thinking about it or not; and that mathematical patterns are neither self-standing objects nor properties of self-standing objects.

If my remarks concerning the meaning of 'a is an object'—in terms of 'a is an element of the domain of discourse D'—are correct, there is not only a way of distinguishing, as we have already seen, between mathematical patterns and self-standing objects, but there also is the opportunity to differentiate between objects and mathematical patterns. This is an opportunity that becomes available to us through the exploitation of the relativized notion of an object given above.

Indeed, through an opportune restriction to a given domain D, it is, now, possible to distinguish between objects in D, i.e., all the as such that $a \in$ D, and the n-ary relations on D (for $n \in \mathbb{N}$ and $n > 1$), i.e., the sets of ordered n-tuples of elements of D (subsets of D^n). Clearly, n-ary relations on D, for $n \in \mathbb{N}$ and $n > 1$, are not objects in D.

Such a distinction between objects in D, and relations on D, is philosophically relevant, because it keeps different things apart, and grounds the transferability of relations on D to domains different from D—transferability manifested, for instance, by the existence of different isomorphic models of a consistent mathematical theory T—showing that what is mathematically relevant, the study of relations, is independent of the nature of the objects belonging to a particular domain.

A beautiful example illustrating, in a simple setting, the dramatic difference concerning the mathematical importance of the 'geometrical' relations on D with respect to the purely perfunctory rôle of the objects in D is provided by the following formal system.

Let a formal system T be given such that the language of T contains a primitive binary relation 'x belongs to a set X' ($x \in X$), and its inverse 'X contains an element x' ($X \ni x$).

Furthermore, let us assume that D is a set of countably many undefined elements $a_1, a_2, \ldots$; call 'm-set' a subset X of D; and consider the following as the axioms of T:

Axiom 1 If x and y are distinct elements of D there is at least one m-set containing x and y;

Axiom 2 If x and y are distinct elements of D there is not more than one m-set containing x and y;

Axiom 3 Any two m-sets have at least one element of D in common;

Axiom 4 There exists at least one m-set.

Axiom 5 Every m-set contains at least three elements of D;

Axiom 6 All the elements of D do not belong to the same m-set;

Axiom 7 No m-set contains more than three elements of D.[11]

Now, since the language of T contains two different sorts of variables—x, y and X—if x, y range over $\mathrm{D} = \mathrm{D}_1 = \{\mathrm{A}, \ldots, \mathrm{G}\}$, where $a_1 = \mathrm{A}, \ldots, a_7 = \mathrm{G}$; and X over D_1^*, where D_1^* is a set whose elements are the subsets of D_1 which appear as the columns of the matrix below; we have that $\mathrm{D}_1 \cup \mathrm{D}_1^*$ is the domain of the model of T represented in Figure 1.

[11] These axioms have been taken, with some minor alterations, from [Tuller, 1967], §2.10, p. 30.

A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C

Figure 1: Model 1

On the other hand, if the variables x, y range over the set $\mathrm{D} = \mathrm{D}_2 = \{\mathrm{P}_1, \ldots, \mathrm{P}_7\}$, where $a_1 = \mathrm{P}_1, \ldots, a_7 = \mathrm{P}_7$, the elements of which are 7 distinct points in a Euclidean plane α; whereas X ranges over the set D_2^* whose elements are the sets of P_i points, for $i \in \{1, \ldots, 7\}$, lying on the sides of, the bisectrices of, and on the circle inscribed in, the triangle below, we have that $\mathrm{D}_2 \cup \mathrm{D}_2^*$ is also the domain of a model of T, a model represented in Figure 2.

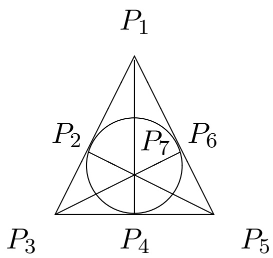

Figure 2: Model 2

Looking at these two models of T, it is clear that: $(\mathrm{D}_1 \cup \mathrm{D}_1^*) \cap (\mathrm{D}_2 \cup \mathrm{D}_2^*) = \emptyset$; the elements of $\mathrm{D}_1 \cup \mathrm{D}_1^*$ are not even homogeneous with the elements of $\mathrm{D}_2 \cup \mathrm{D}_2^*$; and that the two models are isomorphic to each other.[12]

The case of these two models of T brings out very clearly that the patterns described by the axioms and theorems of T are independent of the nature of the objects present in $\mathrm{D}_1 \cup \mathrm{D}_1^*$ (the first seven letters of the English alphabet plus ...), and in $\mathrm{D}_2 \cup \mathrm{D}_2^*$ (seven distinct points in a Euclidean plane α plus ...), showing that they have a different *onto-logical* status from that of the elements of $\mathrm{D}_1 \cup \mathrm{D}_1^*$ and of $\mathrm{D}_2 \cup \mathrm{D}_2^*$. What this means is that such patterns differ from the elements of $\mathrm{D}_1 \cup \mathrm{D}_1^*$ and $\mathrm{D}_2 \cup \mathrm{D}_2^*$ not only from

[12]The function $\Psi : D_1 \bigcup D_1^* \mapsto D_2 \bigcup D_2^*$ such that $\Psi(x) = f(x)$, and $\Psi(X) = g(X)$—where $f : D_1 \mapsto D_2$ and $g : D_1^* \mapsto D_2^*$ such that $f(A) = P_6, f(B) = P_2, f(C) = P_5, f(D) = P_4, f(E) = P_7, f(F) = P_3, f(G) = P_1$ and $g(x_i, x_j, x_k) = (f(x_i), f(x_j), f(x_k))$, for $i, j, k \in \{1, \ldots, 7\}$—shows that the two models are isomorphic to one another.

In fact, Ψ induces a bi-univocal correspondence between $D_1 \bigcup D_1^*$ and $D_2 \bigcup D_2^*$, and preserves the (two primitive) relations ($\in$ and $\ni$), that is:

$$(x, X) \in \; \in^{D_1 \cup D_1^*} \quad \text{iff} \quad (\Psi(x), \Psi(X)) \in \; \in^{D_2 \cup D_2^*} \tag{1}$$

$$(X, x) \in \; \ni^{D_1 \cup D_1^*} \quad \text{iff} \quad (\Psi(X), \Psi(x)) \in \; \ni^{D_2 \cup D_2^*} . \tag{2}$$

a logical point of view, but also with regard to what kind of entities they are (ontic difference).

With regard to the logical difference existing between the patterns described by T, and the objects (elements of $D_1 \cup D_1^*$ or $D_2 \cup D_2^*$), we need to observe that whereas the relations that generate these patterns are expressed by unsaturated parts of statements,[13] the objects (elements of $D_1 \cup D_1^*$ or $D_2 \cup D_2^*$) are expressed by saturated parts of statements (designators).

Concerning the difference relating to what kind of entities patterns described by T and objects (elements of $D_1 \cup D_1^*$ or of $D_2 \cup D_2^*$) respectively are, it is sufficient to consider that, for $n \in \mathbb{N}$ and $n > 1$, an n-ary relation $\Re^n$ on $D_1 \cup D_1^*$ does not belong to $D_1 \cup D_1^*$, but is an element of $\mathcal{P}((D_1 \cup D_1^*)^n)$, and that the first occurrence of $D_1 \cup D_1^*$ in von Neumann's cumulative hierarchy is lower down with respect to the first occurrence in the hierarchy of $\mathcal{P}((D_1 \cup D_1^*)^n)$, i.e., they are two different types of entities.

Therefore, although the relations that generate such patterns might be elements of a domain D^*—and could be said to be objects in D^*—considering them simply as objects blurs the vital *onto-logical* distinction between objects in $D_1 \cup D_1^*$ (or $D_2 \cup D_2^*$), and relations on $D_1 \cup D_1^*$ (or $D_2 \cup D_2^*$) I mentioned above.

It is such a distinction that provides with genuinely original content the view of mathematics as a science of patterns (and any other form of structural realism about mathematics) with regard to more traditional non-structuralist philosophies of mathematics.

4 Patterns are not properties of self-standing objects

Having made a case in favour of the idea that mathematical patterns are not self-standing objects (from here on I shall write '*object*' for 'self-standing object'), I am, now, going to show that mathematical patterns are not properties of *objects* either, by arguing that it is possible, in representing an *object* (a given set of *objects*), to switch from a particular mathematical pattern to a different one without any corresponding change taking place in the *object*(*s*) represented.[14]

To see this latter point consider the following expression:[15]

[13] In contrast with what Frege says on these matters, I am going to use the concept of statement rather than the metaphysically loaded concept of proposition.

[14] Notice that this is a generalization to all mathematical patterns of a point made by Frege in [Frege, 1980] (Ch. II, §§21–25, pp. 27^e–33^e) when he answered in the negative the question: Is Number a property of external things? The argument that follows in the main text is also 'Fregean'.

[15] Although, strictly speaking, the expression labelled by (*) is an artifact and, therefore, it is not an *object* let me consider it such, for the sake of argument. We could easily substitute for it a more cumbersome example of, say, five copses such that one of these

(*) *Quoque tu Brute fili mi.*

If we want to describe numerically what is labelled by (*), i.e., if we want to attribute a cardinal number to what is labelled by (*), we need to specify the set of things that we intend to count. In order to do this, we have to fix the property that characterizes such a set or list its elements.

Now, if within our system of representation we choose 'Latin word'[16] as property/concept, then it will be true to say that the set whose elements are the Latin words labelled by (*), and only those, contains 5 elements.

On the other hand, if within our system of representation we choose as property/concept 'letter belonging to the Latin alphabet',[17] then it will be true to say that the set of letters belonging to the Latin alphabet containing those, and only those, labelled by (*) has 19 elements.

It is interesting to notice that although, by changing concept, we can say something true about what is labelled by (*) switching at will from 5 to 19, this phenomenon is not correlated to any physical change taking place in what is labelled by (*).

However, if, to describe what is labelled by (*), we were to choose 'colour' or the 'leftmost character occurring in what is labelled by (*)', we would be using concepts which refer to actual properties of what is labelled by (*). For, we could not switch from one colour to another, or from one character to a different one, and say something true about what is labelled by (*), without any physical change occurring in the entity labelled by (*).

In spite of the arguments offered above in favour of the idea that mathematical patterns are neither *objects* nor properties of *objects*, an empiricist philosopher of mathematics might object that 'our systems of representation can reflect reality, and indeed that we are justified in thinking they do reflect reality if they are empirically confirmed'.

First of all, let me say that the systems of representation I am considering are mathematical systems of representation, i.e., mathematical theories; and that, for a system of representation to be applied to reality, you need much interpreting of the mathematical terms and concepts involved, as well as a whole host of non-mathematical concepts such as mass, energy, charge, etc.

Moreover, since, as we have already stated, there are no concrete *objects* that are perfect circles, squares, etc. the representations of a reality of concrete *objects* that make use of circles, squares, etc. cannot in any case be thought of as reflections (pictures) of concrete *objects*.

Indeed, such representations are, in my opinion, the outcome of the production of models of a reality of concrete *objects*; these are models that

consists of six trees, one of five trees, one of four trees, and two of two trees each.

[16]In the copse example the property/concept playing the rôle of 'Latin word' would be 'copse'.

[17]In the copse example the property/concept playing the rôle of 'letter belonging to the Latin alphabet' would be 'tree'.

arrange/order the phenomena in such a way as to simulate the portion of reality that is the object of study allowing, at the same time, the possibility of carrying out measurements, calculations, predictions, etc.

Therefore, the success of a model involving circles, squares, etc. in simulating a certain portion of a reality of concrete objects should not be taken as a confirmation of the fact that circles, squares, etc. are embodied in concrete *objects*, but, rather, as what corroborates the correctness of the more modest claim according to which circles, squares, etc. are good approximations of the concrete *objects* they are modeling.

Furthermore, against the argument I produced above in support of the view that mathematical patterns are not properties of *objects*, someone might say that 'numbers are properties of sets, so that, before assigning a number, one has to specify a set. The phrase "the entity labelled by (*)" does not specify a set because of the ambiguities pointed out, and so, unless these ambiguities are resolved, we cannot assign a number to (*)'.

True, the impossibility of individuating one and only one number to be associated to the entity labelled by (*) depends on the fact that numbers are properties of sets, and on the fact that the phrase "the entity labelled by (*)" does not specify a set, because it is a designator, not a concept.

However, I fail to understand how this would be incompatible with my claim that numerical patterns are not properties of *objects*. Quite the opposite, I should think.

For, in the case of *Quoque tu Brute fili mi*, what we call 'set' (of Latin words present in ..., of characters belonging to the Latin alphabet present in ...) is the product of an activity of segmentation of an object, activity of segmentation which consists either in an application of brute force—i.e., we produce a list of the elements of our set, a list that is justified neither by a concept nor by a rule—or in the use of a concept (x is a Latin word, x is a character belonging to the Latin alphabet) or of a rule.

Such an activity of segmentation of an *object* $\mathfrak{O}$ aims at seeing $\mathfrak{O}$ as a totality of elements; in other words, it is a way of describing $\mathfrak{O}$ as a set of $a, b, c, \ldots$; and cardinality is, therefore, a property of a description (it is some kind of second-order property), and not of an *object* (it is not a first-order property).

At this point some considerations are in order:

First, an immediate consequence of this discussion is that there is no 'natural' numerical description of what is labelled by (*). This conclusion is very important, because it is the deep reason behind the view that any abstractionist account of numbers (and of mathematical entities in general) has to be wrong.

Secondly, my assertion that mathematical patterns are neither *objects* nor properties of *objects* (they are relations) aims at expressing that (i) mathematical patterns (relations) are always defined on a domain D, and that (ii) although mathematical patterns are real, they depend, both epistemically and metaphysically, on mathematical systems of representation (mathematical theories).

5 Patterns, structure, and form

In defending my view that mathematics is a science of patterns, where patterns are aspects of concrete objects, or aspects of aspects of concrete objects, etc. I argued that mathematical patterns are neither *objects* nor properties of *objects*. I claimed that mathematical patterns are, rather, forms of either perceptual or linguistic representations.

In what I have been saying about these things, I have also tried to emphasize that such forms of representation are abstract, relational entities, and that my position on mathematics, which sees it as a science of patterns, is a version of structural realism.

But, having stated this much, it is legitimate to wonder about the plausibility of saying that abstract entities—mathematical patterns—might dawn on us.

To address the worry above, we need to draw a distinction between iconic and symbolic representations. A representation is iconic if it is a picture of the object represented. On the other hand, a representation is symbolic if it refers in a non-pictorial way to the object(s) represented. Examples of iconic representations are diagrams and maps, whereas a typical example of symbolic representation is offered by the way a variable represents the objects of its domain.

Now, if a set P_A is a picture (an iconic representation) of a set A, there must exist an isomorphism, Ψ, between A and P_A. For P_A needs to preserve the geometrical, topological, algebraic, etc. form of A.

Therefore, if P_A is both one of our perceptual representations and a picture of A, what is common to A and P_A is precisely that abstract entity, the form, which is the mathematical pattern that dawns on us when, for instance, we see something as a square of side r, for $r \in \mathbb{R}^+$ (see on this §2, p. 8). This is what, in particular, justifies saying that the mathematical pattern that dawns on us is the form of our representation.

Furthermore, the possibility of dawning on us of the abstract, general, square-pattern, i.e., the possibility of simply seeing something as a square, is given by the ability to switch from a purely iconic/pictorial representational function of, say, a diagram $\mathfrak{D}$ to a situation in which $\mathfrak{D}$ has both an iconic and a symbolic representational function.[18]

[18]What is at work when we draw a diagram $\mathfrak{D}$ is one of the most important functions of language. Such a function consists in giving us the opportunity to free ourselves from our subjective, perceptual representations by means of artifacts (diagrams) whose representational function is not the outcome of natural connections like those obtaining between, say, a fire and the smoke produced by it, or between a concrete macroscopic object and the shadow projected by it, etc. but of convention.

This is precisely what happens when we use a diagram $\mathfrak{D}$ to prove a result which is true of any triangle, or of any square, etc. Indeed, in such a case the diagram $\mathfrak{D}$ we draw happens to represent iconically a particular triangle, or a particular square, etc. even though it is used in the proof to represent symbolically any triangle, or any square, etc.

The crucial thing to notice here is that the abstract, general, triangle-pattern, or square-pattern, etc. becomes 'visible', dawns on us as a consequence of the interplay between these two different modes of representation.

Having provided an explanation of how abstract mathematical patterns dawn on us, what I need to do now in the remaining part of this section is to show how the characterizations I offered of mathematical patterns in terms of forms of perceptual or linguistic representations, and of abstract structural entities, harmonize with one another.

It has been known for a long time that, in what we call 'perceptual representation', we can distinguish between the perceptual content of the representation—colours, sounds, smells, etc.—and the form of the representation, which is the outcome of a pre-reflective partition of the perceptual input into what we might call 'potential objects of attention'.

It is interesting to notice that the phenomenon known as aspect-seeing is a form of perceptual representation in which the potential objects of attention are what we see as triangles, squares, ducks, rabbits, etc. In the case of aspect-seeing the form of our perceptual representation is the aspect that dawns on us.

Linguistic representations, like perceptual representations, have form and content. When I see a linguistic representation such as

$$(\ddagger) \qquad y = \frac{1}{2p}\, x^2$$

its form—the way the symbols are arranged within it—imposes, through unique readability, an ordering among the meanings of the symbols present in the expression. It is the form of the linguistic representation—the intended aspect of the expression—which, revealing how the meanings of the elementary parts of the expression should be composed with one another, leads us to the understanding of the meaning of the expression.

Furthermore, all aspects are relational entities, i.e., all aspects are characterized by a domain of objects $\mathfrak{O}$, and by a set of relations $\Re$ defined on $\mathfrak{O}$.

To see this take the example of the duck-rabbit figure below.
Here the duck aspect becomes salient (emerges), if we relate the parts of the drawing (the objects of attention) to each other in such a way as to take the drawing as facing left, the part protruding to the left as a bill, etc. In fact, if we relate these objects to each other in a different way, i.e., if we take the drawing as if it were facing right, consider the part protruding to the left as a pair of ears, etc. the aspect that dawns on us is that of a rabbit.

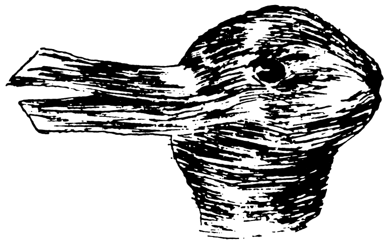

Figure 3: The duck-rabbit

These considerations apply to mathematical aspects/patterns as well, and we can see this immediately in the case of the equation (‡), where the intended aspect of the expression that dawns on us is a particular type of binary relation in which the xs are to the ys.

A different type of argument in favour of the relational nature of mathematical aspects/patterns (from here on I will say just 'patterns') is the following. Take a very simple mathematical pattern like the one that dawns on us when we observe the sequence below

$$(i) \qquad 01010101\cdots$$

Such a pattern is very interesting for us, because it is invariant under uniform substitution in (i) of any two symbols a and b respectively for 0 and 1.

From this it follows that such a mathematical pattern has a relational nature, because it is independent of what kind of things are the elements that appear to be related to one another in such-and-such a way. This argument can be easily generalized to any mathematical pattern using the (appropriate) concept of isomorphism.

These considerations, if correct, provide an answer in the affirmative to Bombieri's question: Can mathematical patterns be always assimilated to relations?

Another interesting consequence of what has been shown by the argument above is that, since the mathematical patterns that dawn on us are independent of the nature of the objects represented as being related to one another in such-and-such a way, the mathematical patterns that dawn on us have to do only with the form of the representation.

Lastly, as is well known, mathematical structures are relational entities, and usually, the properties of the relations defined on the domain of the structure are specified by an appropriate set of axioms. Not only groups, rings, fields, vector spaces, topological spaces, etc. are relational entities, but so is also the finite geometry structure characterized by Axioms 1 to 7 of §4.[19]

[19]Such a structure can be expressed as the triple $(D_1 \cup D_2, \in, \ni)$ such that $x \in X$ and

Therefore, also in the case of mathematical structures, what really matters from a mathematical point of view is the formal aspect of it, that is, the relations defined on a given domain D not (the nature of) the objects belonging to D.

This allows us to conclude that, in the mathematical case, the form of a perceptual or linguistic representation—the mathematical aspect that dawns on us when we see something as—ought to be understood in structural terms.

Such a conclusion is very important, because, besides substituting a precise concept—structure—for a vague one—form (of representation)—it can be used to draw a clear distinction between my position and any kind of Aristotelianism about mathematics along the following lines.

According to Aristotle, form is not, as for Plato, an abstract entity that exists independently of being thought and separately from its (partial) realizations in sensible objects. For Aristotle, form is an activity that individuates, specifies, actualizes, something that otherwise would exist only potentially in matter.[20]

On the other hand, the idea I defend that mathematical patterns are forms of perceptual or linguistic representations differs deeply from Aristotle's position on form. For, even though, according to such an idea, mathematical patterns are, both metaphysically and epistemologically, mind-dependent entities, they are, nevertheless, abstract structures. And this implies that mathematical patterns cannot be construed as activities leading to the actualization of potentialities.

However, independently of the considerations above, for Gillies,[21] my view of mathematical patterns commits me to a form of Aristotelianism in the broad sense. According to Gillies, a philosophy of mathematics is Aristotelian in the broad sense if it implies that 'mathematical entities are embodied in the material world, and so exist there'.

The reason that leads Gillies into thinking that my position on mathematical patterns commits me to Aristotelianism in the broad sense must be

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$X \ni x$ are defined if and only if x is an element of D_1 (x is an element of D) and X is an element of D_2 (X is an m-set).

[20][De Ruggiero, 1967], vol. II, Ch. V, §4, pp. 35-36:

> ...the form of an entity is the act by means of which the entity is individuated and determined; its matter is, instead, what can be subjected to such an act, and is in itself mere potentiality, that is, that lack of determination upon which the determining force of the act is exercised. Thus the dyad matter-form intersects the dyad potentiality-act (δύναμις-'ενέϱγεια), in which form, in contrast with Platonism, is revealed as an activity which *specifies* matter.

[21]The points relating to Gillies's criticism of my view of mathematics which I will be addressing in what follows are contained in an unpublished manuscript of his on mathematical realism and in private correspondence we have exchanged over the years.

that, if mathematical patterns are, as I say, aspects, or aspects of aspects, etc. of concrete objects, it seems to follow that the aspect of k that dawns on me when I see k as a λ cannot exist separately from k.

Although I agree with Gillies that what he means by 'Aristotelianism in the broad sense' captures an essential feature of Aristotle's view of mathematics, I do not think that what I call 'mathematical pattern' exists embodied in a material object. And the reason for this is that, since mathematical patterns are neither *objects* nor properties of *objects*, they cannot be embodied in material objects.

It is possible to convince oneself that this is, indeed, the case, if we reflect on the fact that, when we deal with mathematical patterns, the material object we see as a square is not a (perfect) square, because its sides are not perfectly straight, etc.; neither 19 nor 5 are properties of the material object *Quoque tu Brute fili mi*; etc.

Therefore, although mathematical patterns depend on material objects, or on aspects of material objects, or on aspects of aspects of material objects, etc.—as well as on systems of representation—they are never embodied in material objects.

To see this more clearly consider (‡). (‡) is a material object, because it consists of ink on paper; and what I called 'the intended aspect of (‡)' depends on (‡), and can be determined through the unique readability of the formula.

However, to be able to read the formula (with understanding), we need to know the meaning of its component parts, i.e., we need to know that 'y' is a symbol for a variable, '$\ldots = \ldots$' is a symbol for the equality relation, etc. and we also need to have mastered a reading algorithm such as: start reading the formula from its leftmost symbol then ...

Now, since being-a-symbol-for-a-variable is not a natural property of the material object y, i.e., to use Searle's efficacious terminology, it is not a brute fact that y is the symbol for a variable, but it is, rather, something that has been established by convention (likewise for $=, 1, 2,$ etc.), and since also the reading algorithm that must be applied to (‡) is based on a given convention relative to how to write well-formed formulae,[22] the intended aspect that dawns on us when we read (‡) (with understanding) depends on (‡), but is neither embodied in it nor in any other material entity, because the relation ... refers to ... that is at the root of any kind of convention (coding) is not *ipso facto* natural, i.e., it is not a brute fact that 'x refers to $\bar{x}$' is true.

A very important consequence of what we have just shown, namely, that mathematical patterns, although dependent on material objects (or on aspects of material objects, or on aspects of aspects of material objects, etc.), are not embodied in material objects; and of the previously established fact

[22]Writing formulae in Polish notation leads to changes in the reading algorithm.

that mathematical patterns are both metaphysically and epistemically dependent on systems of representation (mathematical theories) is that mathematical reality is not made of Searlean brute facts, but it consists of mind and world-dependent, abstract entities. These are entities that emerge as a consequence of the invention of opportune systems of representation (mathematical theories).

But, if mathematical patterns are mind and world-dependent entities, they are *a fortiori* mind-dependent entities, and since the realm of mind-dependent entities contains different types of things, perhaps, as Cole observes, saying something about what distinguishes mathematical patterns from other mind-dependent entities might help us in gaining a better understanding of the nature of mathematical patterns.

The problem of what distinguishes mathematical patterns from other mind-dependent entities will be at the very heart of the next section where, as a by-product of the discussion of such an issue, it should become clearer how it is possible to reconcile the abstract nature of mathematical patterns with the idea that they do not exist independently of us.

6 Mathematical patterns, and other mind-dependent entities

Within the realm of mind-dependent things, we come across all sorts of entities: money, countries, political parties, symphony orchestras, etc. But not all these entities bear a strong resemblance to mathematical patterns, because: (i) mathematical patterns become perspicuous to us as a consequence of the purely representational function of language, whereas money, countries, political parties, symphony orchestras, etc., are entities which come into being as a consequence of the performative (and not simply representational) function of language; (ii) mathematical patterns are universals, whereas money, countries, political parties are particulars; (iii) we can causally interact with money, countries, political parties, and symphony orchestras, but we cannot causally interact with mathematical patterns; (iv) the truth of statements about the existence and properties of mathematical patterns can be justified independently of experience, not so the truth of statements relating to the existence and properties of mind-dependent entities like money, countries, political parties, symphony orchestras, etc.

With regard to point (i) above, consider that the representational function of language is not sufficient for entities like money, countries, political parties, symphony orchestras, etc. to come into existence. The reason for this is that the mind-dependent entities mentioned above are, among other things, consequences of social construction, social construction in which the performative function of language, in the way of formal or informal, implicit or explicit, agreements attributing certain functions to certain entities is essential.[23]

To see this more clearly consider that, since performing the rôle of money is independent of the nature of the actual medium/vehicle chosen—paper, metal coins, traces in computers' memories, etc.—it follows that for something like money to exist people have to agree (formally or informally, implicitly or explicitly) to attribute the function of money to certain bits of paper or to . . . , and treat these things accordingly. (Similar considerations apply to countries, political parties, symphony orchestras, etc.)

In the case of mathematical patterns, things are quite different. In fact, although to see the object below as a square I need to know in advance what a square is, the dawning of the square-aspect on me is not the outcome of a formal or informal, implicit or explicit, agreement concerning the attribution of squareness to the object below (see Figure 3). For, seeing something as a square, besides being dependent on a system of representation being in place, is also dependent on an object. In other words, under normal circumstances, knowing what a square and a circle are, I could not see the object represented in Figure 3 as a circle, and such an impossibility would be determined by the object.

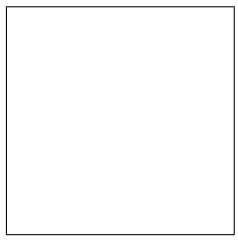

Figure 4:

To this someone might reply that, in contrast with what I have just argued, 'a given mathematical system of representation, and associated mathematical structure(s), is always something to which the relevant portion of the mathematical community agrees, even if only implicitly, much as a relevant group of people must agree to take a particular piece of paper to be a $20 bill'.

In responding to this objection, I want to dispute that what brings about agreement in mathematics is *much as* what leads to 'accepting a particular piece of paper to be a $20 bill'. For, in mathematics, in most cases, the source of agreement is proof, i.e., it is a consequence of the belief in the truth of shared assumptions, and of the hardness of the logical 'must'. And in those situations where proof is not forthcoming—because, for instance, we are dealing with axioms—paraphrasing what Gödel once said, the truth

[23]See on this [Searle, 1995], Ch. 1, The Assignment of Function, pp. 13-23.

of such statements imposes itself upon us even in the face of strong, and apparently overwhelming, opposition and/or consensus.[24]

I want to mention here two very well known cases. The first concerns the introduction of a new axiom and the other he elimination of an old and celebrated axiom.

The first case is that of the Axiom of Choice. As is well known, when Zermelo introduced it in 1904 to prove the well ordering theorem the vast majority of the mathematical community rejected it. But, after a while, the opinion of most members of the mathematical community concerning the Axiom of Choice changed as a consequence of the fact that the Axiom of Choice, and some of its equivalents such as Zorn's Lemma, unpredictably proved to be invaluable in a large number of mathematical theories.

The unexpected mathematical 'success' of the Axiom of Choice and of some of its equivalents showed to the mathematical community that, rather than simply being a non-self-evidently true *ad hoc* hypothesis introduced to fix the problem of proving the well ordering theorem, the Axiom of Choice describes some important features of a very general mathematical structure, i.e., it is true in such a structure.

The second case is that of Euclid's fifth axiom: the whole is greater than the part. This is a particularly intriguing case, because in spite of thousands of years of unanimous and unquestioning support received by our axiom from the mathematical community, in the 19^{th} century it was eventually discovered that Euclid's fifth axiom is false in general and that, therefore, it cannot be part of the set of principles which lie at the foundations of mathematics.

What the two examples above show is that in mathematics it makes sense to think of a situation in which, within the same mathematical community using the same mathematical system of representation (mathematical theory), one man can be right about, for instance, the introduction of the Axiom of Choice or the rejection of Euclid's fifth axiom while everyone else is wrong.

From this it follows that whereas in mathematics the agreement to accept/reject certain principles as an integral part of mathematical theories is the consequence of the truth/falsity of the statements expressing these principles, on the other hand, the truth of statements like 'That is a $20 bill' is a consequence of the community's agreement to treat that particular piece of paper as a $20 bill.

Furthermore, what I called 'the logical must' is the source of what we might call 'mathematical obligation', i.e., in mathematics we are obligated to accept statement C, because we accept statements A and B, and because C is a (proven) logical consequence of A and B. But, when it comes to

[24] What makes a statement belonging to a given system of representation true is, of course, the structure associated to the system of representation.

money, the source of the obligation to accept certain pieces of paper as $20 bills has nothing to do with truth and the relation of logical consequence, but rather with the prescriptive nature of the relevant legislation which is founded upon the community's implicit/explicit agreement on the government's attribution of a normative function to what something must be like to be a $20 bill.

One thing we should notice here is that this last point takes good care of worries like the following: if proofs are only possible, because there is prior agreement about what represents what, then this may have more in common with the establishment of notes as currency than the case made here can cope with.

To see this consider that, if the sources of obligation in mathematics are truth and the relation of logical consequence, whereas in the case under examination—'That is a $20 bill'—the source of obligation is ultimately the community's implicit/explicit agreement on the government's attribution of a normative function to what something must be like to be a $20 bill, it follows that, whatever mathematics and the regulated practice of dealing with $20 dollar bills may have in common, this has to do neither with the reasons why we correctly agree that, in one case, the statement 'That is a $20 bill' is true and, in the other, that the statement 'There are infinitely many primes' is true nor with the direction of the arrow between agreement and truth in these two cases.

In fact, as I have argued above, whereas in the case of 'That is a $20 bill' the community's implicit/explicit agreement on the government's attribution of a normative function to what something must be like to be a $20 bill is a necessary condition for the statement 'That is a $20 bill' to be true, this is not the case for mathematical statements, e.g., for statements like the Axiom of Choice or like 'There are infinitely many primes'.

A second objection runs as follows, 'we should acknowledge that just as accepting such systems of representation as the US Constitution, and the articles of incorporation of the Microsoft Corporation or the Cleveland Symphony Orchestra are performative actions that are responsible for these social-institutional entities existing, so, too, should we acknowledge that accepting a particular mathematical theory that characterizes a certain type of mathematical structure is a performative action that is responsible for that structure existing'.

Now, it seems to me that the US Constitution is not a system of representation, but rather a social contract in which basic functions are attributed to certain offices, and parties, basic rights and obligations are established, etc. The system of representation relevant to the US Constitution is that part of the theory of justice that deals with what is touched upon in the US Constitution.

Of course, the theory of justice relevant to social contracts need not be expressed in text-book form or in academic publications. It can simply appear as part of a (an even oral) tradition which comes together with religious myths, and other things as, for instance, in Sophocles's *Antigone*.

If this is correct, it follows that in accepting the theory of justice relative to the subjects touched upon in the US Constitution, and in the articles of incorporation of the Microsoft Corporation and of the CSO there is no performance of an action leading to the coming into existence of the US, of the Microsoft Corporation, or of the CSO. These—the US Constitution, etc.—are separate things, they are explicit acts or deeds.

With this I do not intend to deny that mathematical language is an institution. The thing I want to say is that, in contrast with what happens in social-institutional cases, once we have taken on board a particular mathematical system of representation, say classical number theory, the dawning upon us of a particular aspect of, for example, the set of prime numbers—that this is an infinite set—does not depend on anybody's will, that is, it depends neither on a contract nor on a majority's vote or on a jury's decision.

On the contrary, for such entities as the US, the Microsoft Corporation, the CSO to come into existence you need the consensus, the agreement to attribute certain functions to certain people, to respect such functions (and people) in your daily dealings, the instruments (and institutions) to enforce the respect of ... etc.

To see this, consider a situation in which the consensus is withdrawn from some such social construct, even before the relevant system of representation, and the relevant legislation, have been modified. Well, in such a situation the social construct *de facto* ceases to exist. Take the social construct of Public Executioner in a country where the death penalty is officially contemplated in current criminal law, but where no court of justice dares condemning someone to death, because the vast majority of the citizens of this country has come to abhor such a thing, etc.

The last objection to my argument in favour of (i) that I am going to examine here is that 'all that there is to there being a symphony orchestra in front of me is there being a group of people in front of me who meet certain criteria—those specified by the social practices surrounding symphony orchestras—just as, all that there is to there being a square in front of me is there being a two-dimensional figure in front of me that meets certain criteria—those specified within certain mathematical practices'.

Indeed, for an orchestra to be there in front of me, rather than being there a group of worshippers of a strange musical religion whose behaviour is indistinguishable from that of an orchestra, many things need to be in place. These are things like shared decisions about the functions attributed (conventionally) to some of the actions the players perform, etc. functions which need to have nothing to do with the observable structure of their behaviour. (Notice here how conventionally attributing functions to actions that the players perform takes good care of the otherwise mysterious notion of meaning.)

Therefore, for an orchestra to be there in front of me, or for an orchestra to exist, it would not be sufficient to have a group of people in front of me whose behaviour satisfies certain criteria specified by the social practices surrounding symphony orchestras. Not so with circles and squares. To see something as a square, i.e., for the pattern 'square' to exist, it is sufficient, under normal circumstances, to know what a square is, and to have a visual experience of an object like the one in Figure 3.

Concerning point (ii) above, *if by 'particular' we mean what occupies a unique portion of space-time*,[25] it is clear that money, countries, political parties, symphony orchestras, are particulars, because they occupy a unique portion of space-time. Money is represented by physical objects such as banknotes and coins, countries have a territory and borders, political parties and symphony orchestras have members.

On the other hand, groups, rings, fields, vector spaces, topological spaces, etc. are aspects (or aspects of aspects, etc.) of domains of objects which, besides differing from one another in terms of the nature of their elements, can occupy simultaneously different portions of space-time.

Imagine seeing a certain drawing on a piece of paper as an equilateral triangle, and considering its possible rigid motions and reflections. Then imagine seeing marbles on a table as the elements of a set of cardinality 3, and examining the possible permutations of this set. It makes sense to say that you can see the rigid motions and reflections of what you see as an equilateral triangle, and the permutations of what you see as a set of three marbles, as two different (also spatially) simultaneous realizations of the elements of the same abstract group, a group which has, therefore, the status of a universal.

It is possible to attack this argument saying that ' "one dollar bill", "symphony orchestra", etc. are abstract universals such that each of them would not exist were it not for the existence of certain social practices. Thus, they are within an abstract part of social-institutional reality'.

As a reply to the objection above, notice, first, that my claim made in point (ii) is not: social-institutional reality consists of particulars; but the more modest conjunction of the following: (1) money, countries, political parties, and symphony orchestras are part of social-institutional reality; (2) they are particulars;[26] (3) although squares and prime numbers are—both

[25]Although I have used a conditional formulation, someone may still want to object to my characterization of particulars that there are particulars which do not occupy a unique portion of space-time. Unfortunately, there is no room here to argue this point.

[26]Someone might want to object to my treatment of money as a particular that 'it does not make sense to talk about the exact position in space-time of the money I have in my bank deposit, because the bank has invested it elsewhere, i.e., exchanged it for who-knows-what'.

Now, the answer to this objection is that when you pool your money together with the money of the other bank account holders, the expression 'your money at time t' refers to the actual banknotes and coins, that, all being well, will be eventually paid to you at metaphysically and epistemically—dependent on systems of representation, they cannot be conceived as particulars, because they do not occupy a portion of space-time.

Secondly, of course 'one dollar bill' is a universal! It is the property common to all and only the elements of the set $\mathbf{1DB} = \{x \mid x \text{ is a one dollar bill}\}$. But what is the relevance of this to the discussion?

It seems to me that the only thing that really matters here is that the property of being a one dollar bill, although neither physical nor chemical, exists *in re* (and not *ante rem*), and is realized by those concrete objects which, as a consequence of the attribution of certain functions to them, have become the pieces of one of our social games.

With regard to the fact that certain concrete objects actually realize (have) the property of being a one dollar bill, consider that there are public and objective criteria set in place by the monetary authorities which establish whether or not a certain piece of paper is a one dollar bill; and that severe measures are taken to prevent the circulation of fake one dollar bills.

In contrast with what happens with properties like being a one dollar bill, the properties of being a square, a prime number, etc. are realized by no concrete object.

Thirdly, I might want to object to the idea that 'one dollar bill', 'symphony orchestra', etc. belong in the abstract part of social-institutional reality that the only thing that such abstract universals do is providing a certain level of description of concrete objects, i.e., the concept 'one dollar bill' provides a level of description of a concrete object as a piece of a particular social game, but, nevertheless, the same concrete object could also be described in a different way (at a different level), if we chose, for instance, concepts belonging to chemistry or (at yet another level) to physics.

Notice that inventing a social game in which we call certain concrete things 'one dollar bill', i.e., stating the rules of the game, etc. neither, in and of itself, conjures up or appeals to abstract entities—the Platonic one-dollar-bill-form—nor does it obtain the one dollar bill concept via abstraction from concrete entities, because being a one dollar bill is neither a physical nor a chemical property of a concrete object. As Wittgenstein pointed out, inventing a social game is simply matter of establishing public and objective criteria for the correct use of the pieces/expressions of the game, etc.

A second objection against point (ii) goes as follows, 'it is wrong to identify countries with their citizens and landmasses, and political parties with their members, etc. etc. (something which is a consequence of the claim present in (ii) that they are particulars).'

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time t, if you ask the bank to do so. Where such banknotes and coins are before time t, and whether there is a way of determining their exact position in space-time before you actually collect them (at time t) are intriguing, but, all the same, purely epistemologic questions, which have nothing to do with the fact that, at any time t, such things occupy a scattered but, nevertheless, unique portion of space-time.

Now, it seems to me that a country may be regarded as a domain D of different sorts of entities, a domain on which the game-relations belonging to a given set $\Re$, i.e., the conditions for the correct use of the elements of D within a game G, are defined. But, the fact that a country is an entity which can be described by an expression having the form '$(\mathrm{D}, \Re, \mathrm{G})$' does not imply that a country is an abstract entity, if the structure $(\mathrm{D}, \Re, \mathrm{G})$ exists only *in re*. (A particular, besides consisting of a domain of more or less scattered objects, has what I might call a 'form'.)[27]

However, there is an important difference between *in re* structures, and mathematical structures, besides that concerning the fact that mathematical structures are neither *in re* nor *ante rem*. Such a difference has to do with the consideration that whereas mathematical structures are stable over time (they are not subject to change), *in re* structures are stable (internally and externally) only for a certain interval of time.

'But', someone might ask, 'if mathematical structures are stable over time, whereas systems of representation are not, how can the latter sustain the former?'. The long and the short of the reply to such a question is that systems of representation do not change either. For, when we modify a given system of representation $\mathfrak{S}$, not simply in the sense of making explicit all the implicit assumptions used in proving the theorems of $\mathfrak{S}$, but, rather, by generating extensions $\mathfrak{S}^+$ (or restrictions $\mathfrak{S}^-$) of $\mathfrak{S}$ which alter the set of theorems we can prove in $\mathfrak{S}^+$ (or in $\mathfrak{S}^-$) with respect to $\mathfrak{S}$, we actually produce a new system of representation which then exists 'alongside' $\mathfrak{S}$.

One more difference—point (iii) above—existing between entities like money, countries, political parties, symphony orchestras, etc. and mathematical patterns is that you can causally interact with the former entities, but not with the latter. For instance, you can employ or sack members of an orchestra, and can donate money to a political party or vandalize some of its buildings,[28] but can in no way interact causally with mathematical patterns, for instance, by altering their properties.

At this point someone might say 'But, if we cannot causally interact with mathematical patterns, how can we say that they are dependent, among other things, on human activity?' Well, they are dependent on human activity, because they are dependent, both metaphysically and epistemically, on systems of representation which are the product of human activity, i.e., if there were no systems of representation there would not be mathemati-

[27]This idea—that countries, Supreme Courts etc. are *in re* structures—provides an elegant and satisfactory answer to both Uzquiano's questions: (a) what sort of object is the Supreme Court? and (b) what is the relation in which the set of justices serving as Supreme Court Justices is to the Supreme Court? (See [Uzquiano, 2004], §**1**, p. 137.) This is an answer that is along the same lines as the usual considerations paid to entities such as groups, rings, fields, vector spaces, etc. in mathematics.

[28]It is interesting to notice that the interactions with this type of entities work both ways. We can go to a concert given by the Cleveland Symphony Orchestra, buy things with money, receive the citizenship of a country, etc.

cal patterns either (see §2) and, therefore, there would be nothing to know about them.

Having said so, we need to keep in mind that such a relation of dependence, though a necessary condition for seeing something as ..., is by no means sufficient. It cannot, therefore, be conceived as what causes us to see X as ..., but as one of the reasons why we see X as ...

Now, it is important to observe that, although we can causally interact with our perceptual representations in all sorts of ways—by making pressure on our eye balls, taking LSD, etc.—there is no conflict whatsoever between this fact and my assertion that we cannot causally interact with mathematical patterns.

In fact, whereas a particular perceptual representation is an extensional entity—it is realized by the firing of a certain cluster of neurons, etc.—the mathematical pattern realized by the form of the perceptual representation is, instead, an abstract entity, i.e., it is the geometric, algebraic, topological, etc. abstract structure associated with the equivalence class of entities which are isomorphic to the form of our perceptual representation (see §§2 and 5).

These considerations explain the common phenomenon according to which, if we consider, for example, analytic geometry, we know that, within an analytic geometry system of representation, we can learn to see things as spheres, cylinders, etc. and we can easily prove that, for instance, the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$; that the volume of a sphere of radius r is $\frac{2}{3}$ the volume of the circumscribed equilateral cylinder, etc. and although drugs, powerful dictators, media moguls and mafia men, etc. have the power to interfere with individual representations in all manners of ways (causing us hallucinations, etc.), there is nothing they can do about the mathematical patterns themselves, and their properties.

An objection against (iii) is that 'we cannot causally interact, in a strict sense, with countries, political parties, and symphony orchestras; we causally interact with the members of political parties and symphony orchestras and this causal interaction might result in a change in the properties of these political parties and symphony orchestras'.

It seems to me that the claim that we do not causally interact with countries, political parties, and symphony orchestras is clearly false, as witnessed, for example, by the dramatic cultural, political, economical, etc. changes caused to several European countries by the Napoleonic wars, and by the solid, scholarly work of many reputable historians aimed at finding the causes of the rise and fall of the Roman Empire, of the separation of Pakistan from India, of Germany into West and East Germany at the end of the Second World War, etc.

The programmes of political parties consist precisely of promises (not) to cause certain changes to symphony orchestras, the military, financial institutions, the university, etc. and the reason why we vote for such parties is to make (im)possible for such changes to be brought about. It is common, daily experience that what governments do causes actual changes to such entities as I have mentioned above.

Moreover, how come that a relevant interaction with the members of the Cleveland Symphony Orchestra (hiring or sacking them) has no causal efficacy on the orchestra? Surely sacking the whole lot of them, and stopping any form of funding of the CSO, would cause (be sufficient to bring about) the end of the orchestra, wouldn't it?

According to another objection 'we can interact with mathematical structures. We do so by causally interacting with the portions of the mathematical community that are responsible for developing or sustaining the mathematical theory that characterizes the mathematical entities in question. As is well known, the initial form of such theories/systems of representation frequently differs significantly from their final form'.

To the above objection it is possible to reply that the fact that 'the initial form of such theories/systems of representation frequently differs significantly from their final form' is a consequence of the struggle that the mathematical system of representation must engage against the anomalies and puzzles generated by its introduction. This phenomenon has nothing to do with psycho-sociological considerations, even though such factors play the important rôle of catalysts.

Lastly—point (iv)—as is well known, the truth of assertions like (α) :

$$(\alpha) \quad e^{\pi i} = \cos\pi + i\sin\pi,$$

can be justified, independently of experience, by means of a mathematical proof.

But, the truth of assertions of type (β):

(β) The Cleveland Symphony Orchestra had more than one member on the first of September 1999

can only be justified by procedures from which experience cannot be expunged—you need to access the relevant records, etc.

To the claim that the truth of assertions like (α) can be justified independently of experience, an empiricist philosopher might reply that 'of course we can give proofs of assertions like (α), but such proofs will always proceed from some assumptions or axioms. So (α) will only be true if the assumptions/axioms used in its proof are true, and these assumptions/axioms will need to be justified empirically'.

First, in §4 of my paper I argue (along Fregean lines) that any abstractionist account of numbers has to be wrong, because, in particular, there is no natural numerical description of *Quoque tu Brute fili mi*.

But, now, if, by means of a generalization of the argument above—a generalization based on the idea that mathematical patterns are neither *objects* nor are they properties of *objects*—it is correct to say that there is no natural mathematical description of objects of our experience, it follows that neither the axioms nor any other statement belonging to a mathematical theory can be justified empirically.

Secondly, according to the position I defend, the axioms of a mathematical theory describe (are true in) an abstract structure which exists neither *in re* nor *ante rem*. Such a structure is one of the inhabitants of a third realm which very much resembles Popper's Third World.[29]

Now, since the Peano axioms are verifiably true independently of experience, and thus justified independently of experience, in the abstract structure realized by the set of natural numbers (the so-called 'standard model'), the group axioms are verifiably true independently of experience, and thus justified independently of experience, in the abstract structure realized by the set of rigid motions and reflections of an equilateral triangle, the field axioms are verifiably true independently of experience, and thus justified independently of experience, in the abstract structure realized by the set of complex numbers, etc. we have that the logical consequences of the Peano axioms, of the group axioms, of the field axioms, etc. will also be true in the same respective structures.[30]

If, in the light of what I have argued above, we return to the point from which our discussion originated, we can conclude that the justification for saying that certain statements are logical consequences of the Peano axioms (or of the group-axioms, etc.), and, therefore, the justification for saying that these very statements are true in all the structures in which the Peano axioms are true (or the group-axioms are true, etc.), is provided by means of mathematical proofs, i.e., it is independent of experience.

It is interesting to notice how points (iii) and (iv) strengthen each other. In fact, on the one hand, the abstract nature of mathematical patterns, and their being causally inert (point (iii)), explain why it is possible to establish the truth (or the falsity) of statements about them by means of procedures which do not appeal to experience (point (iv)).

Indeed, since the entities we are talking about are abstract, they do not occupy portions of space-time and, therefore, cannot be perceived; from this it follows that they cannot be objects of experience either, because experience contains a non-eliminable perceptual residuum.[31]

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[29]Concerning Popper's notion of Third World see footnote 34.

[30]The remarks in the main text might give the impression that I take mathematics to be 'a collection of axiomatized theories' or, to put it in a different way, that for me the axiomatic method is *the* mathematical method. I, actually, entirely agree with what Yehuda Rav says at the end of [Rav, 2008] where, after having examined the dispute between Carlo Cellucci and me over the importance of the axiomatic method in mathematics (see [Cellucci, 2005] and [Oliveri, 2005]), concludes that 'as Cellucci has maintained, the axiomatic method is not *the* mathematical method *pace* Hilbert. But, as Oliveri has argued, we do need axioms; it only depends on what type of axioms and where and when do they come in'.

[31]The notion of experience I use here is deeply rooted in the philosophical tradition. In-

Moreover, since these entities are causally inert, there is no question of investigating them indirectly through the study of their interactions with entities of which we have experience, as it happens, instead, in astrophysics with objects like the black holes.

Consequently, appealing to experience cannot be an essential ingredient present in arguments aimed at establishing the existence and properties of mathematical patterns.

On the other hand, if we can obtain knowledge concerning the existence and properties of mathematical patterns independently of experience, it is unreasonable to think that they must be causally efficacious extensional entities. For, if they were causally efficacious extensional entities, they would, in particular, have to occupy portions of space-time and, thus, there would be at least one question about them that could not be answered independently of experience: what is their exact location in space-time?

Having argued points (i)–(iv), we are now in possession of some of the reasons why mathematical patterns differ from other mind-dependent entities. And since our discussion of some of the differences existing between mathematical patterns and other mind-dependent entities has increased our knowledge of the nature of mathematical patterns, it is important to take stock of what we have found so far about such matters before tackling other issues.

According to the information gathered so far, mathematical patterns are: (a) neither *objects* nor properties of *objects* (§§2–4); they are (b) universals not instantiated in portions of space-time (point (ii) above).

Furthermore, if by 'natural world' we mean the set of *objects* that occupy a portion of space-time and their properties,[32] it follows that (c) mathematical patterns are not elements of the natural world, even though (d) they

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deed, for Kant, 'experience is knowledge by means of connected perceptions' ([Kant, 1990], Deduction of the pure concepts of the understanding, Section 2, §26, p. 171). This type of knowledge radically differs from the knowledge we obtain when we study 'relations of ideas', as it happens when we show that $7 + 5 = 12$.

Having said so, it is important to notice that I do not intend to deny that, as Husserl, Gödel, Parsons, and others have suggested, mathematical intuition produces genuine pre-reflective representations of mathematical entities, e.g., of essences (for Husserl), of relations between concepts (for Gödel), of types (for Parsons), etc. The important point, though, is that, since for me mathematical entities (patterns) are abstract entities, the pre-reflective intuition we have of them does not represent sensible objects and, therefore, the knowledge we obtain about such entities cannot be obtained 'by connecting perceptions', as it happens with entities we are acquainted with. From this it follows that, for me, mathematical knowledge is not empirical, under the technical Kantian notion of experience I adopt.

[32]To put it in a different way, one could say that what I mean by 'natural world' is a subset of the set of what Searle calls 'brute facts'. According to Searle, brute facts are portions of the real world which are totally independent of human agreement/opinion, whereas this is not the case for institutional facts. An example of brute fact is that the speed of light in the void is 299.790 km/s; whereas an example of institutional fact is that Italy is a member-state of the European Union. See on this [Searle, 1995], Ch. 1, p. 2.

depend on elements of the natural world (see point (i)).

Other intriguing properties of mathematical patterns that we have singled out in our discussion are that: (e) mathematical patterns are mind-dependent entities which, in spite of being (f) forms of (possible) representations, (g) are causally inert intensional objects.

At this point, if we take into account the properties of mathematical patterns listed above, we realize that properties (b), (c), and (g) imply that mathematical patterns belong to a realm of reality which differs from the natural world (property (c)), and that the relations obtaining among mathematical patterns, and between mathematical patterns and elements of the natural world, are not causal (property (g)).

Moreover, although mathematical patterns are mind-dependent entities, the communicability and objectivity of statements about their existence and properties—objectivity sustained by shared proof procedures—are sufficient to demarcate between mathematical patterns and sensations.

Therefore, what all these considerations seem to suggest is that mathematical patterns are the inhabitants of what Frege famously called 'the third realm'.[33]

However, properties (a), (d), and (f) make us realize that mathematical patterns do not exist independently of the natural world and of the mind. And this, of course, implies that, in contrast with what Frege thought about this, the third realm does not exist independently of the first and the second realms. On the contrary, the third realm comes into existence as a consequence of that part of human activity directed to the production of representations of regions of the first and of the second realm, and, with this, to the creation of the so-called 'space of reasons'.[34]

7 Some advantages of the main view

The position I defend about mathematics has many advantages with regard to more traditional versions of structural realism. I shall mention here only some of them.

First, since it is not an instance of Platonism about structures, like the structural realism of Resnik and Shapiro,[35] it is not affected by the

[33]For Frege, the first realm is populated by concrete objects, the second realm is the object of description of the so-called 'private language' or 'language of sensations', and the third is the realm of thoughts.

[34]My view of the third realm is similar to Popper's ideas on what he calls the 'Third World' (see on this [Popper, 1974a] and [Popper, 1974b]). Unfortunately, considerations relating to the already forbidding length of this paper make me realize that this is not the right place to engage in a discussion of my position on the third realm in relation to Popper's Third World.

[35][Shapiro, 2000], Ch. 8, §4, p. 261:

> It is surely correct to maintain that if there had never been any language (or

traditional friction between an ontology of independently existing abstract entities, and the various unhappy attempts to provide a satisfactory epistemology for such things.

Indeed, being patterns (aspects) forms of perceptual or linguistic representations, they are given to us together with the representations. This allows me to show the existence of a beautiful harmony between the ontology of mathematical patterns, and the account of how we come to know them.

Secondly, in contrast with the views of Resnik and Shapiro on mathematical structures, my position—as a consequence of considerations similar to those produced in §2, and in §5, p. 16, concerning, in particular, the interaction of iconic and symbolic modes of representation—can be easily applied to structures described by non-categorical axiom systems like, for instance, the group-axioms and the topology-axioms.

For, on the one hand, seeing the set of rigid motions and reflections of an equilateral triangle as a group is very much like what I described in §§2 and 5 in terms of 'seeing something as a square of side r', for $r \in \mathbb{R}^+$.

Indeed, being the concept of group well founded there are, in particular, identity conditions for groups. Now, since the identity conditions for groups are provided by the relation of group-isomorphism this, being an equivalence relation, partitions the class of all things we see as groups into equivalence classes each of which is associated to (individuates) a unique, abstract, algebraic structure that, in the case mentioned above, is the dihedral group of order 6. Moreover, such a unique, abstract, algebraic structure is common to all the elements of the same equivalence class, etc. etc.

On the other hand, the dawning on us of the abstract, general, group-pattern (structure) is made possible by switching from a purely iconic/pictorial representational function of, for example, the multiplication table of the dihedral group of order 6, to a situation in which the multiplication table of the dihedral group of order 6 is used as having both an iconic and a symbolic representational function.

Thirdly, my view of patterns provides a plausible account of mathematical intuition (pre-reflective representation), and of the applicability of mathematics to the empirical sciences.

It provides a plausible account of mathematical intuition, because since patterns are real, and are forms of representation, we can also have pre-reflective representations of them, and formulate conjectures about patterns, independently of the possibility of proving them.

Concerning the problem of explaining the applicability of mathematics to the empirical sciences, it is important to observe that: (1) the relational nature of mathematical patterns, being independent of the nature of the

any people), there would be trees, planets, and stars. There would also be numbers, sets of numbers, and Klein groups, if not baseball defenses. Such is the nature of *ante rem* structures.

objects belonging to the domain on which the relations are defined, provides a plausible explanation of the possibility of applying these patterns to other domains; (2) since patterns are aspects of concrete objects, or aspects of aspects of concrete objects, etc. the link between mathematical patterns, and the realm of concrete objects (applicability to the empirical sciences) is established as part of the very act of bringing into existence mathematical patterns.[36]

Lastly, an important objection against the traditional Platonist,[37] whose Platonism extends to cover set theory, goes as follows. For the traditional Platonist, the Continuum Hypothesis (CH) must be either true or false of set-theoretical reality. For, according to him, such a reality exists independently of whether anybody thinks about it or not, and is the object of study of mathematical theories in general, and of set theory in particular.

But since, as a consequence of the independence of CH from the axioms of ZFC, it is perfectly legitimate to develop systems of set theory for which CH is true, and systems of set theory for which CH is false, it follows that traditional Platonism, besides being at a loss in dealing with propositions like CH, conflicts also with mathematical practice. (The same considerations apply to what has happened in geometry with Euclid's fifth postulate, and the introduction of non-Euclidean geometries.)

Traditional Platonism has great difficulties in dealing with propositions like CH, because there is an insurmountable obstacle in its way of explaining why and how CH and $\neg$CH give us information about mathematical reality, an insurmountable obstacle represented by the inconsistency of the set $\{\text{CH}, \neg\text{CH}\}$.

Moreover, given that, for the traditional Platonist, CH and $\neg$CH cannot be both true of mathematical reality, it follows that, for him, the practice of developing both ZFC + CH and ZFC + $\neg$CH cannot be accounted for satisfactorily.

Indeed, such a practice seems, rather, to support a formalist view of mathematics according to which contributing to a mathematical theory is like playing a formal game based on conventionally established rules: the ZFC + CH game, or the ZFC + $\neg$CH game.

The traditional Platonist's standard reply to the objection above is that given by Gödel: the reason why CH is independent of ZFC is that this formal system does not express all the essential features of set-theoretical reality.

[36]With regard to points (1) and (2) above, we should notice that there are views of mathematics alternative to mine which are entitled to share in the merits which find expression in (1) and/or (2). For example, whereas any form of mathematical structuralism—and, therefore, in particular a version of mathematical structuralism that differs from mine—benefits from (1), a mathematical Aristotelianism *à la* Gillies benefits from (2).

[37]The term 'traditional Platonism' refers to all forms of Platonism, i.e., structural and non-structural Platonism, alike with the exception of Plentiful Platonism. On Plentiful Platonism see footnote 40.

What mathematicians should do is discover new axioms that describe relevant properties of set theoretical reality, add them to ZFC, and prove CH from the extension so obtained.

In spite of its immediate appeal, I find this counter objection rather unsatisfactory, because, taking for granted that the operation suggested by Gödel can be carried out, there is no obvious reason why proving CH from an extension of ZFC should make illegitimate, or detract mathematical interest from, developing systems like ZFC + ¬CH.

On the other hand, the objection above not only can be easily answered by my version of mathematical realism, but, in actual fact, provides a strong support for it.

Since patterns, besides being dependent on concrete objects or on aspects of concrete objects (or on aspects of aspects of . . .) depend also (epistemically and metaphysically) on systems of representation (mathematical theories), there is no difficulty whatsoever in accounting for the reality of both patterns describable by a system of representation in which CH is taken to be true, and patterns describable by a system of representation in which CH is taken to be false.

To see this consider that, if ZFC is consistent, the consistency of ZFC + CH and of ZFC+¬CH ensures the possibility of describing the set-theoretical reality which emerges in one case under the assumption that $2^{\aleph_0} = \aleph_1$, and in the other under the assumption that $2^{\aleph_0} \neq \aleph_1$.

As a confirmation of the correctness of talking here about the emergence of set-theoretical reality under the assumption that $2^{\aleph_0} = \aleph_1$ and under the assumption that $2^{\aleph_0} \neq \aleph_1$, we have the existence of the forcing models produced by Cohen, forcing models which are at the heart of his proof of the independence of CH from ZFC. This—on the relevance of models to the emergence of set-theoretical reality—is a deep point on which I must dwell a little longer.

If we study the history of mathematics, we realize that, time and again, models have been fundamental for the acceptance of a new mathematical theory.[38]

[38]As is well known, the technique of finding models of mathematical theories was explicitly developed only in the 19$^{\text{th}}$ century as a consequence, among other things, of the controversies generated by the introduction of the non-Euclidean geometries (relative consistency proofs), and of the development of the axiomatic method (independence proofs).

However, there are important pre-19$^{\text{th}}$ century examples of this phenomenon which, using an implicit notion of a model, aim at producing examples of entities postulated by a new mathematical theory in terms of entities postulated by an already accepted mathematical theory. Some notorious cases exemplifying this point are: (1) the representations given by the Pythagoreans of the positive integers, and of operations defined on positive integers like + and × in terms of, respectively, configurations of points (the so-called 'triangular', 'square', etc. numbers), and operations defined on configurations of points; (2) the representations given by the ancient Greek mathematicians of the time of Euclid of the positive integers, and of operations defined on positive integers like + and × in terms

There are very good reasons for this. One such reason is that (1) models provide consistency proofs of the new theory relative to an already accepted mathematical theory; and another important reason is that (2) models exhibit examples of (a) domains whose elements are already accepted mathematical entities, and of (b) (already accepted) mathematical relations defined on such domains, that satisfy the axioms of the theory in question supplying in this way some kind of existence proof for the entities postulated by the new theory.

It is the above condition (2) that is particularly important for us, because the existence of a model of a new mathematical theory T shows that the theorems of T are not a mere *flatus vocis*, but that they actually describe features of the mathematical reality represented by the model. But, of course, without a mathematical theory T there is no model of T! (Metaphysical dependence of mathematical patterns on systems of representation.)

There is, therefore, no conflict whatsoever between the version of mathematical realism I advocate, and the coexistence within mathematics of systems like ZFC + CH and ZFC + ¬CH, of systems of Euclidean and non-Euclidean geometries, of systems of real analysis in which the Archimedean property is true, and systems of real analysis in which the Archimedean property is false, etc.[39]

However, in closing this section and the paper, it is important to realize that this feature of my view of mathematics does in no way show that I defend some form of Plentiful Platonism.[40] For, an essential characteristic of my position is that, in contrast to a tenet common to all forms of Platonism, mathematical patterns do not exist independently of mathematical systems of representation.

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of lengths of line segments, and of operations involving line segments (as in geometric algebra); (3) the (much later) representations of the reals in terms of points of a line; (4) the representations of the complex numbers as points in $\mathbb{R}^2$; etc.

[39]Notice that, since a mathematical formal system (like ZFC) can be seen as a mathematical (set-theoretical) extension of first-order, second-order, etc. logic, it follows that, in choosing a particular mathematical formal system, we, among other things, fix the limits of its deductive apparatus, i.e., the limits of its logic.

[40]According to P. Maddy, Plentiful Platonism is '[T]he view that there exists an objective world of sets corresponding to each and every consistent theory in a first-order language with $\in$ as its sole non-logical symbol.' 1n [Maddy, 1998], p. 162.

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