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ON A MIXED BOUNDARY VALUE PROBLEM INVOLVING THE p-LAPLACIAN

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In this paper we prove the existence of infinitely many solutions for a mixed boundary value problem involving the one dimensional p-Laplacian. A result on the existence of three solutions is also established. The approach is based on multiple critical points theorems.

1. Introduction

We want to study the following mixed boundary value problem involving the one-dimensional p-Laplacian

$$\begin{cases} -(r|u'|^{p-2}u')' + s|u|^{p-2}u = \lambda f(t,u) \text{ in } \mathrm{I} =]\mathrm{a},\mathrm{b}[\\ u(a) = u'(b) = 0 \end{cases} \qquad (RD_\lambda)$$

where $p > 1$, λ is a positive parameter, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a L^2-Carathéodory function and $r, s \in L^\infty([a,b])$ such that

$$r_0 := \operatorname*{ess\,inf}_{t\in[a,b]} r(t) > 0, \;\; s_0 := \operatorname*{ess\,inf}_{t\in[a,b]} s(t) \geq 0.$$

In this paper we generalize the results obtained in [1] and [6] with $p = 2$.

Our main tool to investigate the existence of infinitely many solutions for mixed boundary value problems is the infinitely many critical points theorem due to Ricceri ([7]). Here, we recall it as given in [3].

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Theorem 1.1. (see [7, Theorem 2.5] and [3, Theorem 2.1]) *Let X be a reflexive Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional, $\Psi : X \to \mathbb{R}$ be sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional.*

Put, for each $r > \inf_X \Phi$

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty,r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}, \tag{1}$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \ \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

One has

- *(α) For every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.*
- *(β) If $\gamma < \infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*
  - *(β_1) $\Phi - \lambda\Psi$ possesses a global minimum, or*
  - *(β_2) there is a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.*
- *(ζ) If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*
  - *(ζ_1) there is a global minimum of Ψ which is a local minimum of $\Phi - \lambda\Psi$, or*
  - *(ζ_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi$ which weakly converges to a global minimum of Φ.*

Now, we recall a result which ensures the existence of three critical points; the result has been obtained in [5], it is a more precise version of theorem 3.2 of [4]

Theorem 1.2. (see [5, Theorem 3.6]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\Phi(0) = \Psi(0) = 0$$

and that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

(a_1) $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$;

(a_2) for each $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)} \right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

2. Preliminaries

Now, consider problem (RD_λ).

We recall that a function $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a L^2-Carathéodory function if

- $(\cdot)$ $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;
- $(\cdot)$ $x \to f(t,x)$ is continuous for every $t \in [a,b]$;
- $(\cdot)$ for every $\rho > 0$, $\quad \sup_{|x|\le\rho} |f(t,x)| \in L^2([a,b])$.

Put

$$F(t,x) := \int_0^x f(t,\xi)d\xi \qquad \forall (t,x) \in [a,b]\times\mathbb{R},$$

$$k := \frac{2(p+1)r_0}{2^p(p+1)||r||_\infty + (p+2)(b-a)^p||s||_\infty}, \tag{2}$$

where

$$||r||_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} r(t), \qquad ||s||_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} s(t)$$

$$A := \liminf_{\xi\to+\infty} \frac{\int_a^b \max_{|x|\le\xi} F(t,x)dt}{\xi^p}, \qquad B := \limsup_{\xi\to+\infty} \frac{\int_{\frac{a+b}{2}}^b F(t,\xi)dt}{\xi^p}, \tag{3}$$

$$\lambda_1 := \frac{r_0}{p(b-a)^{p-1}kB}, \qquad \lambda_2 := \frac{r_0}{p(b-a)^{p-1}A}. \tag{4}$$

where we suppose $\lambda_1 = 0$ if $B = \infty$, and $\lambda_2 = +\infty$ if $A = 0$.
Denote by X the Sobolev space $\{u \in W^{1,p}([a,b]), \quad u(a) = 0\}$ endowed with the following norm

$$||u|| := \left(\int_a^b r(t)|u'(t)|^p dt + \int_a^b s(t)|u(t)|^p dt \right)^{\frac{1}{p}}.$$

We observe that the norm $||\cdot||$ is equivalent to the usual one.

A function $u : [a,b] \to \mathbb{R}$ is said a generalized solution to problem (RD_λ) if $u \in C^1([a,b])$, $r|u'|^{p-2}u' \in AC([a,b])$, $u(a) = u'(b) = 0$, and

$$-(r(t)|u'(t)|^{p-2}u'(t))' + s(t)|u(t)|^{p-2}u(t) = \lambda f(t,u)$$

for almost every $t \in I =]a,b[$.
A function $u \in X$ is said a weak solution of problem (RD_λ) if

$$\int_a^b r(t)|u'(t)|^{p-2}u'(t)v'(t)dt + \int_a^b s(t)|u(t)|^{p-2}u(t)v(t)dt =$$

$$= \lambda \int_a^b f(t,u(t))v(t)dt \qquad \forall v \in X.$$

Standard methods show that generalized solutions to problem (RD_λ) coincides with weak ones when f is a L^2-Caratheodory function.

It is well known that $(X, ||\cdot||)$ is compactly embedded in $(C^0([a,b]), ||\cdot||_\infty)$ and one has

$$||u||_\infty \leq \sqrt[p]{\frac{(b-a)^{p-1}}{r_0}}||u|| \qquad \forall u \in X. \tag{5}$$

In order to study problem (RD_λ), we will use the functionals Φ, $\Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u) := \frac{1}{p}||u||^p, \quad \Psi(u) := \int_a^b F(t,u(t))dt \qquad \forall u \in X, \tag{6}$$

Φ is continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits a continuous inverse. On the other hand, Ψ is Gâteaux differentiable with compact derivative and one has

$$\Phi'(u)(v) = \int_a^b r(t)|u'(t)|^{p-2}u'(t)v'(t)dt + \int_a^b s(t)|u(t)|^{p-2}u(t)v(t)dt,$$

$$\Psi'(u)(v) = \int_a^b f(t,u(t))v(t)dt \quad \forall v \in X,$$

moreover

$$\Phi(0) = \Psi(0) = 0.$$

A critical point for the functional $\Phi - \lambda\Psi$ is any $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \qquad \forall v \in X.$$

We can observe that each critical point for functional $\Phi - \lambda\Psi$ is a generalized solution for problem (RD_λ).

3. Main results

Our main results are the following theorems

Theorem 3.1. *Assume that*

(h_1) $\int_a^{\frac{a+b}{2}} F(t,\xi)dt \geq 0 \qquad \forall \xi \geq 0,$

(h_2) $\liminf\limits_{\xi \to +\infty} \dfrac{\int_a^b \max_{|x| \leq \xi} F(t,x)dt}{\xi^p} < k \limsup\limits_{\xi \to +\infty} \dfrac{\int_{\frac{a+b}{2}}^b F(t,\xi)dt}{\xi^p}$

where k is given by (2).

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (4)*, the problem* (RD_λ) *has a sequence of weak solutions which is unbounded in X.*

Proof. Our goal is to apply Theorem 1.1. Consider the Sobolev space X and the operators defined in (6). Pick $\lambda \in]\lambda_1, \lambda_2[$.

By using (3), let $\{c_n\}$ be a real sequence such that $\lim_{n \to +\infty} c_n = +\infty$ and

$$\lim_{n \to +\infty} \frac{\int_a^b \max_{|\xi| \leq c_n} F(t,\xi)dt}{c_n^2} = A.$$

Put $r_n = \frac{r_0}{p(b-a)^{p-1}} c_n^p$ for all $n \in \mathbb{N}$, taking into account (5), one has $||v||_\infty \leq c_n$ for all $v \in X$ such that $||v||^p \leq pr_n$.
Hence, for all $n \in \mathbb{N}$, one has

$$\varphi(r_n) = \inf_{u \in \Phi^{-1}(]-\infty, r_n[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r_n[)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)}$$

$$\leq \frac{\sup_{||v||^p < pr_n} \int_a^b F(t,v(t))dt}{r_n}$$

$$\leq \frac{\int_a^b \max_{|\xi| \leq c_n} F(t,\xi)dt}{r_n} = \frac{p(b-a)^{p-1}}{r_0} \frac{\int_a^b \max_{|\xi| \leq c_n} F(t,\xi)dt}{c_n^p}$$

therefore, since from (h_2) one has $A < \infty$, we obtain

$$\gamma := \liminf_{n \to \infty} \varphi(r_n) \leq \frac{p(b-a)^{p-1}}{r_0} A < \infty.$$

Now, we claim that the functional $I_\lambda = \Phi - \lambda \Psi$ is unbounded from below.

By using (3), let $\{d_n\}$ be a real sequence such that $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{\frac{a+b}{2}}^{b} F(t,d_n)dt}{d_n^p} = B. \tag{7}$$

For all $n \in \mathbb{N}$ define

$$\omega_n(t) := \begin{cases} \frac{2d_n}{b-a}(t-a) \text{ if } t \in [a, \frac{a+b}{2}[\\ d_n \text{ if } t \in [\frac{a+b}{2}, b]. \end{cases}$$

Clearly $\omega_n \in X$ and

$$||\omega_n||^p \leq \frac{d_n^p}{2(p+1)(b-a)^{p-1}} \left(2^p(p+1)||r||_\infty + (p+2)(b-a)^p||s||_\infty\right) \tag{8}$$

therefore

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) = \frac{1}{p}||\omega_n||^p - \lambda \int_a^b F(t,\omega_n(t))dt \tag{9}$$

$$\leq \frac{d_n^p}{2(p+1)(b-a)^{p-1}} (2^p(p+1)||r||_\infty +$$

$$(p+2)(b-a)^p||s||_\infty) - \lambda \int_a^b F(t,\omega_n(t))dt.$$

Taking into account (h_1), we have

$$\int_a^b F(t,\omega_n(t))dt \geq \int_{\frac{a+b}{2}}^{b} F(t,d_n)dt. \tag{10}$$

Then, for all $n \in \mathbb{N}$

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \leq \frac{d_n^p}{2(p+1)(b-a)^{p-1}} (2^p(p+1)||r||_\infty + \tag{11}$$

$$(p+2)(b-a)^p||s||_\infty) - \lambda \int_{\frac{a+b}{2}}^{b} F(t,d_n)dt = \frac{d_n^p r_0}{p(b-a)^{p-1}k} - \lambda \int_{\frac{a+b}{2}}^{b} F(t,d_n)dt.$$

Now, if $B < \infty$, we fix $\varepsilon \in \left]\frac{r_0}{p\lambda(b-a)^{p-1}kB}, 1\right[$, from (7) there exists $\nu_\varepsilon \in \mathbb{N}$ such that

$$\int_{\frac{a+b}{2}}^{b} F(t,d_n)dt > \varepsilon B d_n^p \qquad \forall n > \nu_\varepsilon$$

therefore

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \leq \left[\frac{r_0}{p(b-a)^{p-1}k} - \lambda\varepsilon B\right] d_n^p \qquad \forall n > \nu_\varepsilon$$

by the choice of ε, one has

$$\lim_{n\to\infty}[\Phi(\omega_n)-\lambda\Psi(\omega_n)]=-\infty.$$

On the other hand, if $B=+\infty$, we fix

$$M>\frac{r_0}{p\lambda(b-a)^{p-1}k},$$

from (7) there exists $\nu_M\in\mathbb{N}$ such that

$$\int_{\frac{a+b}{2}}^{b}F(t,d_n)dt>Md_n^p \qquad \forall n>\nu_M$$

therefore

$$\Phi(\omega_n)-\lambda\Psi(\omega_n)\leq\left[\frac{r_0}{p(b-a)^{p-1}k}-\lambda M\right]d_n^p \qquad \forall n>\nu_M$$

by the choice of M, one has

$$\lim_{n\to\infty}[\Phi(\omega_n)-\lambda\Psi(\omega_n)]=-\infty.$$

Hence, our claim is proved.
Since all assumptions of Theorem 1.1 are verified, the functional $I_\lambda=\Phi-\lambda\Psi$ admits a sequence $\{u_n\}$ of critical points such that $\lim_{n\to\infty}||u_n||=+\infty$ and the conclusion is achieved. □

Now, we point out the following consequence of Theorem 3.1

Corollary 3.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function, $s\in C^0([a,b])$, $r\in C^1([a,b])$ and put $F(x)=\int_0^x f(\xi)d\xi$ $\forall x\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}<\frac{k}{2}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}.$$

Then, for each

$$\lambda\in\left]\frac{2r_0}{pk(b-a)^p\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}},\frac{r_0}{p(b-a)^p\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}}\right[$$

the problem

$$\begin{cases}-(r|u'|^{p-2}u')'+s|u|^{p-2}u=\lambda f(u) \text{ in } \mathrm{I}=]\mathrm{a},\mathrm{b}[\\ u(a)=u'(b)=0\end{cases}$$

possesses a sequence of pairwise distinct classical solutions.

Remark 3.3. In Theorem 3.1 we can replace $\xi \to +\infty$ by $\xi \to 0^+$, applying in the proof part (ζ) of Theorem 1.1 instead of (β). In this case a sequence of pairwise distinct solutions to the problem (RD_λ) which converges uniformly to zero is obtained.

Example 3.4. Put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$, and define the nonnegative continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(\xi) = \begin{cases} \frac{32(n+1)!^2[(n+1)!^p - n!^p]}{\pi} \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2} & \text{if } \xi \in \bigcup_{n\in\mathbb{N}}[a_n, b_n] \\ 0 & otherwise. \end{cases}$$

By a simple computation, we obtain $\liminf\limits_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0$, $\limsup\limits_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 2^p$, so

$$0 = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \frac{2(p+1)}{e(2^p(p+1)+(p+2))} =$$

$$\frac{(p+1)}{2^p e(2^p(p+1)+(p+2))} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}.$$

Hence, from Corollary 3.2, for each $\lambda > \frac{2^p(p+1)+(p+2)}{2^p p(p+1)}$, the problem

$$\begin{cases} -(e^{-t}|u'|^{p-2}u')' + e^{-t}|u|^{p-2}u = \lambda f(u) & \text{in }]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

has a sequence of pairwise distinct generalized solutions.

Now we point out the following result of three generalized solutions.

Theorem 3.5. *Assume that there exist three positive constants* c, d *and* ν *with* $c < d$, $\nu < p$ *and a function* $\mu \in L^1([a,b])$ *such that*

(i_1) $\int_a^{\frac{a+b}{2}} F(t,\xi)dt > 0 \quad \forall \xi \in [0,d]$,

(i_2) $\frac{\int_a^b \max_{|\xi|\le c} F(t,\xi)dt}{c^p} < k \frac{\int_{\frac{a+b}{2}}^b F(t,d)dt}{d^p}$
where k *is given by* (2),

(i_3) $F(t,\xi) \le \mu(t)(1+|\xi|^\nu)\, \forall t \in [a,b] \quad \forall \xi \in \mathbb{R}$.

Then, for each

$$\lambda \in \left]\frac{r_0 d^p}{pk(b-a)^{p-1}\int_{\frac{a+b}{2}}^{b} F(t,d)dt}, \frac{r_0 c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\le c} F(t,\xi)dt}\right[$$

the problem (RD_λ) has at least three generalized solutions.

Proof. Our goal is to apply Theorem 1.2. Consider the Sobolev space X and the operators defined in (6).

Now, we claim that (i_2) ensures (a_2) of Theorem 1.2. In fact, set $r = \frac{r_0 c^p}{p(b-a)^{p-1}}$ and consider the function $\bar{u} \in X$ defined by putting

$$\bar{u}(t) := \begin{cases} \frac{2d}{b-a}(t-a) \text{ if } \mathrm{t} \in [\mathrm{a}, \frac{\mathrm{a+b}}{2}[\\ d \text{ if } \mathrm{t} \in [\frac{\mathrm{a+b}}{2}, \mathrm{b}]. \end{cases} \tag{12}$$

We observe that

$$p\Phi(\bar{u}) = \frac{2^p d^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} r(t)dt + \frac{2^p d^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} (t-a)^2 s(t)dt + d^p \int_{\frac{a+b}{2}}^b s(t)dt$$

from $0 < c < d$ by using the previous relation and (2) we have

$$0 < r < \Phi(\bar{u}) < \frac{r_0 d^p}{pk(b-a)^{p-1}}.$$

In virtue of (i_1) we have

$$\Psi(\bar{u}) \ge \int_{\frac{a+b}{2}}^b F(t,d)dt.$$

Therefore, one has

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{p(b-a)^{p-1}k}{r_0 d^p}\int_{\frac{a+b}{2}}^b F(t,d)dt. \tag{13}$$

From (5) if $\Phi(u) \le r$, we have $\max_t |u(t)| \le c$ therefore

$$\sup_{\Phi(u)\le r} \Psi(u) \le \int_a^b \max_{|\xi|\le c} F(t,\xi)dt. \tag{14}$$

Hence, owing to (13), (14) and (i_2) condition (a_1) of Theorem 1.1 is verified.

We prove that the operator $\Phi - \lambda\Psi$ is coercive, in fact, for each $u \in X$, by using (i_3) one has

$$\Phi(u) - \lambda\Psi(u) = \frac{1}{p}||u||^p - \lambda \int_a^b F(t,u(t))dt \ge$$

$$\frac{1}{p}||u||^p - \lambda \int_a^b \mu(t)(1+|u(t)|^\nu)dt \geq$$

$$\frac{1}{p}||u||^p - \lambda \int_a^b \mu(t)dt - \lambda \int_a^b \mu(t)|u(t)|^\nu dt$$

and, by using (5)

$$\int_a^b |\mu(t)||u(t)|^\nu dt \leq ||u||_\infty^\nu \int_a^b |\mu(t)|dt \leq \left(\frac{(b-a)^{p-1}}{r_0}\right)^{\frac{\nu}{p}} ||u||^\nu ||\mu||_1,$$

we have

$$\Phi(u) - \lambda\Psi(u) \geq \frac{1}{p}||u||^p - \lambda||\mu||_1 - \lambda\left(\frac{(b-a)^{p-1}}{r_0}\right)^{\frac{\nu}{p}} ||u||^\nu ||\mu||_1 \qquad (15)$$

hence condition (a_2) of Theorem 1.2 is verified. All assumptions of Theorem 1.2 are satisfied and the proof is complete. □

Now, we point out the following consequence of Theorem 3.2.

Corollary 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $r \in C^1([a,b])$, $s \in C^0([a,b])$. Assume that there exist positive constants a, c, d and ν, with $c < d$ and $\nu < p$, such that*

(i_1') $\frac{F(c)}{c^p} \leq \frac{k}{2}\frac{F(d)}{d^p}$
where k is given by (2),

(i_2') $F(\xi) \leq a(1+|\xi|^\nu)\ \forall \xi \in \mathbb{R}$.

Then, for each $\lambda \in \left]\frac{2r_0 d^p}{pk(b-a)^p F(d)}, \frac{r_0 c^p}{p(b-a)^p F(c)}\right[$ the problem

$$\begin{cases} -(r|u'|^{p-2}u')' + s|u|^{p-2}u = \lambda f(u) \text{ in } \mathrm{I} =]\mathrm{a,b}[\\ u(a) = u'(b) = 0 \end{cases}$$

has at least three classical solutions.

Example 3.7. The problem

$$\begin{cases} -(|u'|u')' + (\frac{2+t}{3})|u|u = \lambda t^2[2e^{-t^2}u^{17}(9-u^2)+1] \text{ in } \mathrm{I} =]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions for each $\lambda \in \left]\frac{128e^4}{7(2^{18}+e^4)}, \frac{3e}{2(e+1)}\right[$.

In fact, if we choose, for example, $c = 1$ and $d = 2$, hypotheses of Theorem 3.1 are satisfied.

Remark 3.8. In ([2]), has been studied a mixed boundary problem involving the one dimensional p-Laplacian of type

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + |u|^{p-2}u = \lambda f(t,u) \text{ in } \mathrm{I} =]\mathrm{a}, \mathrm{b}[\\ u(a) = u'(b) = 0 \end{cases}.$$

We observe that our equation gives back that case with $r = s = 1$.

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