QUASI J-IDEALS OF COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity. In this paper, we introduce the concept of quasi J-ideal which is a generalization of J-ideal. A proper ideal of R is called a quasi J-ideal if its radical is a J-ideal. Many characterizations of quasi J-ideals in some special rings are obtained. We characterize rings in which every proper ideal is quasi J-ideal. Further, as a generalization of presimplifiable rings, we define the notion of quasi presimplifiable rings. We call a ring R a quasi presimplifiable ring if whenever $a, b \in R$ and a = ab, then either a is a nilpotent or b is a unit. It is shown that a proper ideal I that is contained in the Jacobson radical is a quasi J-ideal (resp. J-ideal) if and only if R/I is a quasi presimplifiable (resp. presimplifiable) ring.

1. INTRODUCTION

Throughout this paper, we shall assume unless otherwise stated, that all rings are commutative with non-zero identity. We denote the nilradical of a ring R, the Jacobson radical of R, the set of unit elements of R, the set of zero-divisors and the set of all elements that are not quasi-regular in R by N(R), J(R), U(R), Z(R), and NZ(R), respectively. In [11], the concept of n-ideals in commutative rings is defined and studied. A proper ideal I of R is said to be a n-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin N(R)$, then $b \in I$. Recently, as a generalization of n-ideals, the notion of J-ideals is introduced and investigated in [10]. A proper ideal I of R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$.

The aim of this article is to extend the notion of *J*-ideals to quasi *J*-ideals. For the sake of thoroughness, we give some definitions which we will need throughout this study. For a proper ideal *I* a ring *R*, let $\sqrt{I} = \{r \in R : \text{there exists } n \in \mathbb{N} \text{ with} r^n \in I\}$ denotes the radical of *I* and (I : x) denotes the ideal $\{r \in R : rx \in I\}$. Let *M* be a unitary *R*-module. Recall that the idealization $R(+)M = \{(r,m) : r \in R, m \in M\}$ is a commutative ring with the addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. For an ideal *I* of *R* and a submodule *N* of *M*, it is well-known that I(+)N is an ideal of R(+)M if and only if $IM \subseteq N$ [4, Theorem 3.1]. We recall also from [4, Theorem 3.2] that $\sqrt{I(+)N} = \sqrt{I}(+)M$, and the Jacobson radical of R(+)M is J(R(+)M) = J(R)(+)M. For the other notations and terminologies that are used in this article, the reader is referred to [5].

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We summarize the content of this article as follows. In Section 2, we study the basic properties of quasi J-ideals of a ring R. Among many results in this section, we first start with an example of a quasi J-ideal that is not a J-ideal. In Theorem 1, we give a characterization for quasi J-ideals. In Theorem 2, we conclude some equivalent conditions that characterize quasi-local rings. The relations among primary, δ_1 -n-ideal and quasi J-ideals are clarified (Proposition 2). Moreover, Example 2 and Example 3 are presented showing that the converses of the used implications are not true in general. Further, in Theorem 3, we show that every maximal quasi J-ideal is a J-ideal. In Theorem 4, we characterize quasi J-ideals of zero-dimensional rings in terms of quasi primary ideals. Moreover, the behavior of quasi J-ideals in polynomial rings, power series rings, localizations, direct product of rings, idealization rings are investigated (Proposition 13, Proposition 8, and Proposition 9, Remark 1 and Proposition 15).

In Section 3, we introduce quasi presimplifiable rings as a new generalization of presimplifiable rings. We call a ring R quasi presimplifiable if whenever $a, b \in R$ with a = ab, then $a \in N(R)$ or $b \in U(R)$. Clearly, the classes of presimplifiable and quasi presimplifiable reduced rings coincide. However, in Example 5, we show that in general this generalization is proper. In Proposition 10, it is shown that a ring R is quasi presimplifiable if and only if $NZ(R) \subseteq J(R)$. The main objective of the section is to characterize a J-ideal (resp. a quasi J-ideal) of R as the ideal I for which R/I is a presimplifiable (resp. quasi presimplifiable) ring. This characterization is used to justify more results concerning the class of J-ideals (resp. quasi J-ideals). For example, in Theorem 6, it is shown that if $\{I_{\alpha} : \alpha \in \Lambda\}$ is a family of J-ideals (resp. quasi J-ideals) over a system of rings $\{R_{\alpha} : \alpha \in \Lambda\}$, then $I = \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(I_{\alpha})$ is a J-ideal (resp. quasi J-ideal) of $R = \varinjlim_{\alpha \in \Lambda}$.

2. Properties of Quasi J-ideals

Definition 1. Let R be a ring. A proper ideal I of R is said to be a quasi J-ideal if \sqrt{I} is a J-ideal.

It is clear that every J-ideal is a quasi J-ideal. However, this generalization is proper and the following is an example of a quasi J-ideal in a certain ring which is not a J-ideal.

Example 1. Consider the idealization ring $R = \mathbb{Z}(+)\mathbb{Z}$. Then $I = 0(+)\mathbb{Z}$ is a *J*-ideal of *R* since 0 is a *J*-ideal of \mathbb{Z} by [10, Proposition 3.12]. Now, $\sqrt{0(+)2\mathbb{Z}} = \sqrt{0}(+)\mathbb{Z} = 0(+)\mathbb{Z}$ is a *J*-ideal of *R*, and thus $0(+)2\mathbb{Z}$ is a quasi *J*-ideal of *R*. However, $0(+)2\mathbb{Z}$ is not a *J*-ideal of *R* since for example $(0,1), (2,0) \in R$ with $(2,0) \cdot (0,1) = (0,2) \in 0(+)2\mathbb{Z}$ and $(2,0) \notin J(R) = J(\mathbb{Z})(+)\mathbb{Z} = 0(+)\mathbb{Z}$ but $(0,1) \notin 0(+)2\mathbb{Z}$.

Our starting point is the following characterization for quasi *J*-ideals.

Theorem 1. Let I be a proper ideal of a ring R. Then the following statements are equivalent:

- (1) I is a quasi J-ideal of R.
- (2) If $a \in R$ and K is an ideal of R with $aK \subseteq I$, then $a \in J(R)$ or $K \subseteq \sqrt{I}$.
- (3) If K and L are ideals of R with $KL \subseteq I$, then $K \subseteq J(R)$ or $L \subseteq \sqrt{I}$.
- (4) If $a, b \in R$ and $ab \in I$, then $a \in J(R)$ or $b \in \sqrt{I}$.

Proof. (1) \Rightarrow (2) Suppose that *I* is a quasi *J*-ideal of *R*, $aK \subseteq I$ and $a \notin J(R)$. Since \sqrt{I} is a *J*-ideal, $\sqrt{I} = (\sqrt{I} : a)$ by [10, Proposition 2.10]. Thus $K \subseteq (I : a) \subseteq (\sqrt{I} : a) = \sqrt{I}$.

 $(2) \Rightarrow (3)$ Suppose that $KL \subseteq I$ and $K \nsubseteq J(R)$. Then there exists $a \in K \setminus J(R)$. Since $aL \subseteq I$ and $a \notin J(R)$, we have $L \subseteq \sqrt{I}$ by our assumption.

 $(3) \Rightarrow (4)$ Suppose that $a, b \in R$ and $ab \in I$. The result follows by letting $K = \langle a \rangle$ and $L = \langle b \rangle$ in (3).

 $(4) \Rightarrow (1)$ We show that \sqrt{I} is a *J*-ideal. Suppose that $ab \in \sqrt{I}$ and $a \notin J(R)$. Then there exists a positive integer *n* such that $a^n b^n \in I$ and $a \notin J(R)$. It follows clearly that $a^n \notin J(R)$ and so $b^n \in \sqrt{I}$ by (4). Therefore, $b \in \sqrt{\sqrt{I}} = \sqrt{I}$ and *I* is a quasi *J*-ideal.

As a consequence of Theorem 1, we have the following.

Corollary 1. Let L be an ideal of a ring R such that $L \nsubseteq J(R)$. Then

(1) If I and K are quasi J-ideals of R with IL = KL, then $\sqrt{I} = \sqrt{K}$.

(2) If for an ideal I of R, IL is a quasi J-ideal, then $\sqrt{IL} = \sqrt{I}$.

Let I be a proper ideal of R. We denote by J(I), the intersection of all maximal ideals of R containing I. Next, we obtain the following characterization for quasi J-ideals of R.

Proposition 1. Let I be an ideal of R. Then the following statements are equivalent:

- (1) I is a quasi J-ideal of R.
- (2) $I \subseteq J(R)$ and if whenever $a, b \in R$ with $ab \in I$, then $a \in J(I)$ or $b \in \sqrt{I}$.

Proof. (1) \Rightarrow (2) Suppose I is a quasi J-ideal of R. Since \sqrt{I} is a J-ideal, then $I \subseteq \sqrt{I} \subseteq J(R)$ by [10, Proposition 2.2]. Now, (2) follows clearly since $J(R) \subseteq J(I)$. (2) \Rightarrow (1) Suppose that $ab \in I$ and $a \notin J(R)$. Since $I \subseteq J(R)$, we conclude that $J(I) \subseteq J(J(R)) = J(R)$ and so we get $a \notin J(I)$. Thus, $b \in \sqrt{I}$ and I is a quasi J-ideal of R.

In the following theorem, we characterize rings in which every proper (principal) ideal is a quasi J-ideal.

Theorem 2. For a ring R, the following statements are equivalent:

- (1) R is a quasi-local ring.
- (2) Every proper principal ideal of R is a J-ideal.
- (3) Every proper ideal of R is a J-ideal.
- (4) Every proper ideal of R is a quasi *J*-ideal.
- (5) Every proper principal ideal of R is a quasi *J*-ideal.
- (6) Every maximal ideal of R is a quasi J-ideal.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ is clear by [10, Proposition 2.3].

Since $(3) \Rightarrow (4) \Rightarrow (5)$ is also clear, we only need to prove $(5) \Rightarrow (6)$ and $(6) \Rightarrow (1)$.

 $(5) \Rightarrow (6)$ Assume that every proper principal ideal of R is a quasi J-ideal. Let M be a maximal ideal of R. Suppose that $ab \in M$ and $a \notin \sqrt{M} = M$. Since $\langle ab \rangle$ is proper in R, (ab) is a quasi J-ideal by our assumption. Since $ab \in \langle ab \rangle$ and clearly $a \notin \sqrt{\langle ab \rangle}$, we conclude that $b \in J(R)$, as required.

 $(6) \Rightarrow (1)$ Let M be a maximal ideal of R. Then M is a quasi J-ideal by (6) which implies $M = \sqrt{M} \subseteq J(R)$ by [10, Proposition 2.2]. Thus, J(R) = M; and so R is a quasi-local ring.

Let R be a ring and denote the set of all ideals of R by L(R). D. Zhao [13] introduced the concept of expansions of ideals of the ring R. A function $\delta : L(R) \to L(R)$ is called an ideal expansion if the following conditions are satisfied for any ideals I and J of R:

- (1) $I \subseteq \delta(I)$.
- (2) Whenever $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$.

For example, $\delta_1 : L(R) \to L(R)$ defined by $\delta_1(I) = \sqrt{I}$ is an ideal expansion of a ring R. For an ideal expansion δ defined on a ring R, the class of δ -n-ideals has been defined and studied recently in [12]. A proper ideal I of R is called a δ -n-ideal if whenever $a, b \in R$ and $ab \in I$, then $a \in N(R)$ or $b \in \delta(I)$.

Proposition 2. Let I be a proper ideal of R.

- (1) If I is a δ_1 -n-ideal, then I is a quasi J-ideal of R.
- (2) If I is a primary ideal of R and $I \subseteq J(R)$, then I is a quasi J-ideal of R.

Proof. (1) Suppose that $ab \in I$ and $a \notin J(R)$. Then $a \notin N(R)$ as $N(R) \subseteq J(R)$. Since I is a δ_1 -n-ideal, we have $b \in \delta_1(I) = \sqrt{I}$. By Theorem 1, we conclude that I is a quasi J-ideal of R.

(2) Suppose that $ab \in I$ and $a \notin J(R)$. If $b \notin \sqrt{I}$, then $a \in I$ since I is a primary ideal of R which contradicts the assumption that $I \subseteq J(R)$. Therefore, $b \in \sqrt{I}$ and I is a quasi J-ideal by Theorem 1.

However, the converses of the implications in Proposition 2 are not true in general as we can see in the following two examples.

Example 2. Consider the quasi-local ring $\mathbb{Z}_{\langle 2 \rangle} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b \right\}$. Then $J(\mathbb{Z}_{\langle 2 \rangle}) = \langle 2 \rangle_{\langle 2 \rangle} = \left\{ \frac{a}{b} : a \in \langle 2 \rangle, 2 \nmid b \right\}$ is a quasi J-ideal of $\mathbb{Z}_{\langle 2 \rangle}$ by Theorem 2. On the other hand, $\langle 2 \rangle_{\langle 2 \rangle}$ is not a δ_1 -n-ideal. Indeed, if we take $\frac{2}{3}, \frac{3}{5} \in \mathbb{Z}_{\langle 2 \rangle}$, then $\frac{2}{3}, \frac{3}{5} = \frac{6}{15} \in \langle 2 \rangle_{\langle 2 \rangle}$ but $\frac{2}{3} \notin N(\mathbb{Z}_{\langle 2 \rangle}) = 0_{\mathbb{Z}_{\langle 2 \rangle}}$ and $\frac{3}{5} \notin \sqrt{\langle 2 \rangle_{\langle 2 \rangle}} = \langle 2 \rangle_{\langle 2 \rangle}$.

Example 3. Consider the ring $C(\mathbb{R})$ of all real valued continuous functions and let $M = \{f \in C(\mathbb{R}) : f(0) = 0\}$. Then M is a maximal ideal of $C(\mathbb{R})$. Consider the quasi-local ring $R = (C(\mathbb{R}))_M$ and let $I = \langle x \sin x \rangle_M$. Then I is a quasi J-ideal by Theorem 2. On the other hand I is not primary since for example $x \sin x \in I$ but $x^n \notin I$ and $\sin^n x \notin I$ for all integers n.

Recall that a ring R is said to be semiprimitive if J(R) = 0.

Proposition 3. Let R be a semiprimitive ring.

- (1) R is an integral domain if and only if the only quasi J-ideal of R is the zero ideal.
- (2) If R is not an integral domain, then R has no quasi J-ideals.

Proof. (1) Suppose that R is an integral domain. Then it is easy to show that 0 is a quasi J-ideal of R. If I is a non-zero quasi J-ideal, then by Proposition 1 we have $I \subseteq J(R) = 0$ which is a contradiction.

(2) Suppose that I is a quasi J-ideal of R. Then $I \subseteq \sqrt{I} \subseteq J(R) = 0$. But since R is not integral domain, then 0 is not a prime ideal of R and so clearly it is not a quasi J-ideal.

Let R be a ring and S be a non-empty subset of R. Then clearly $(I:S) = \{r \in R : rS \subseteq I\}$ is an ideal of R. Now, while it is clear that $\sqrt{(I:S)} \subseteq (\sqrt{I}:S)$, the reverse inclusion need not be true in general. For example, consider $S = \{2\} \subseteq \mathbb{Z}$ and the ideal $I = \langle 12 \rangle$ of Z. Then $\sqrt{(I:S)} = \sqrt{\langle 6 \rangle} = \langle 6 \rangle$ while $(\sqrt{I}:S) = \langle 3 \rangle$.

Lemma 1. If I is a quasi J-ideal of a ring R and $S \nsubseteq J(R)$ is a subset of R, then $\sqrt{(I:S)} = (\sqrt{I:S}).$

Proof. If $a \in (\sqrt{I}:S)$, then $sa \in \sqrt{I}$ for all $s \in S$. Choose $s \notin J(R)$ such that $sa \in \sqrt{I}$. Then $a \in \sqrt{I}$ as I is a quasi J-ideal and so clearly, $a \in \sqrt{(I:S)}$. The other inclusion is obvious.

Lemma 2. Let S be a subset of a ring R with $S \nsubseteq J(R)$ and I be a proper ideal of R. If I is a quasi J-ideal, then (I:S) is a quasi J-ideal.

Proof. We first note that (I : S) is proper in R since otherwise if $1 \in (I : S)$, then $S \subseteq I \subseteq J(R)$, a contradiction. Suppose that $ab \in (I : S)$ and $a \notin J(R)$ for $a, b \in R$. Then $abS \subseteq I$ and $a \notin J(R)$ which imply that $bS \subseteq \sqrt{I}$ by Theorem 1. Thus, $b \in (\sqrt{I} : S) = \sqrt{(I : S)}$ by Lemma 1 and we are done.

A quasi J-ideal I of a ring R is called a maximal quasi J-ideal if there is no quasi J-ideal which contains I properly. In the following proposition, we justify that any maximal quasi J-ideal is a J-ideal.

Theorem 3. Let I be a maximal quasi J-ideal of R. Then I is a J-ideal of R.

Proof. Suppose I is a maximal quasi J-ideal of R. Let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. Then (I:a) is a quasi J-ideal of R by Lemma 2. Since I is a maximal quasi J-ideal and $I \subseteq (I:a)$, then $b \in (I:a) = I$. Therefore, I is a J-ideal of R.

If J(R) is a quasi J-ideal of a ring R, then clearly it is the unique maximal quasi J-ideal of R. In this case, J(R) is a prime ideal of R as can be seen in the following corollary.

Corollary 2. Let R be a ring. The following are equivalent:

- (1) J(R) is a J-ideal of R.
- (2) J(R) is a quasi J-ideal of R.
- (3) J(R) is a prime ideal of R.

Recall from [9] that a proper ideal of a ring R is called a quasi primary ideal if its radical is prime. We prove in the following theorem that under a certain condition on R, quasi primary ideals and quasi J-ideal are the same.

Theorem 4. Let R be a zero-dimensional ring and I be an ideal of R with $I \subseteq J(R)$. Then the following are equivalent:

(1) I is a quasi J-ideal of R.

- (2) I is a quasi primary ideal of R.
- (3) $I = P^n$ for some prime ideal P of R and some positive integer n.
- (4) (R, \sqrt{I}) is a quasi-local ring.

Proof. $(1) \Rightarrow (2)$ Suppose that $ab \in \sqrt{I}$ and $a \notin \sqrt{I}$. Then there exists a positive number n such that $a^n b^n \in I$. Since R is zero-dimensional, then every prime ideal is maximal and so $\sqrt{I} = J(I)$. Since I is a quasi J-ideal and clearly $a^n \notin J(I)$, we conclude $b^n \in \sqrt{I}$ by Theorem 1. Thus $b \in \sqrt{I}$ which shows that \sqrt{I} is prime as needed.

 $(2) \Rightarrow (3)$ Suppose that I is a quasi primary ideal of R. Then \sqrt{I} is prime. Since R is zero-dimensional, \sqrt{I} is a maximal ideal and clearly $I = P^n$ for some prime ideal P of R and some positive number n.

 $(3) \Rightarrow (4)$ Suppose that $I = P^n$ for some prime ideal P of R and some positive integer n. Then $\sqrt{I} = P$ is also a maximal ideal. Hence our assumption $I \subseteq J(R)$ implies that $\sqrt{I} = P = J(R)$ and so (R, \sqrt{I}) is a quasi-local ring.

 $(4) \Rightarrow (1)$ It follows directly by Theorem 2.

Since every principal ideal ring is zero-dimensional, we have the following corollary of Theorem 4.

Corollary 3. Let R be a principal ideal ring and I be a proper ideal of R. Then I is a quasi J-ideal of R if and only if $I = p^n R$ for some prime element p of R with $p \in J(R)$ and $n \ge 1$.

Let I be a proper ideal of R. Then I is said to be superfluous if whenever K is an ideal of R such that I + K = R, then K = R.

Proposition 4. If I is a quasi J-ideal of a ring R, then I is superfluous.

Proof. Suppose that I+K = R for some ideal K of R. Then $\sqrt{I}+\sqrt{K} = \sqrt{I+K} = R$. From [10, Proposition 2.9], we conclude that $\sqrt{K} = R$ which means K = R and we are done.

Proposition 5. (1) If $I_1, I_2, ..., I_k$ are quasi J-ideals of a ring R, then $\bigcap_{i=1}^k I_i$ is a quasi J-ideal of R.

(2) Let $I_1, I_2, ..., I_k$ be quasi primary ideals of a ring R in which their radicals are not comparable. If $\bigcap_{i=1}^{k} I_i$ is a quasi J-ideal of R, then I_i is a quasi J-ideal of R for i = 1, 2, ..., k.

Proof. (1) Since $\sqrt{\bigcap_{i=1}^{k} I_i} = \bigcap_{i=1}^{k} \sqrt{I_i}$, the claim is clear by [10, Proposition 2.25].

(2) Without loss of generality, we show that I_1 is a quasi *J*-ideal. Suppose that $ab \in I_1$ and $a \notin J(R)$. By assumption, we can choose an element $c \in \left(\prod_{i=2}^k I_i\right) \setminus \sqrt{I_1}$ and then we have $abc \in \bigcap_{i=1}^k I_i$. It follows that $bc \in \sqrt{\bigcap_{i=1}^k I_i} = \bigcap_{i=1}^k \sqrt{I_i} \subseteq \sqrt{I_1}$ as $\bigcap_{i=1}^{n} I_i \text{ is a quasi } J\text{-ideal. Since } I_1 \text{ is quasi primary, } \sqrt{I_1} \text{ is prime which implies that} \\ b \in \sqrt{I_1}. \text{ Thus } I_1 \text{ is a quasi } J\text{-ideal of } R.$

Proposition 6. (1) Let $I_1, I_2, ..., I_k$ be quasi J-ideals of a ring R. Then $\prod_{i=1}^k I_i$ is a quasi J-ideal of R.

(2) Let I₁, I₂, ..., I_k be quasi primary ideals of R in which their radicals are not comparable. If ∏^k_{i=1} I_i is a quasi J-ideal of R, then I_i is a quasi J-ideal of R for i = 1, 2, ..., k.

Proof. (1) Let $a, b \in R$ such that $ab \in \prod_{i=1}^{k} I_i$ and $a \notin J(R)$. Then clearly for all $i = 1, 2, ..., k, b \in \sqrt{I_i}$ since I_i is a quasi *J*-ideal of *R*. Now, for all *i*, there is an integer n_i such that $b^{n_i} \in I_i$. Thus, $b^{n_1+n_2+\cdots+n_k} \in \prod_{i=1}^{k} I_i$ and so $b \in \sqrt{\prod_{i=1}^{k} I_i}$. Therefore, $\prod_{i=1}^{k} I_i$ is a quasi *I*-ideal.

Therefore, $\prod_{i=1}^{k} I_i$ is a quasi *J*-ideal.

(2) Similar to the proof of Proposition 5 (2).

However, the J-ideal property can not pass to the product of ideals as can be seen in the following example.

Example 4. Consider the ring $\mathbb{Z}(+)\mathbb{Z}_2$. Then $0(+)\mathbb{Z}_2$ is a *J*-ideal since 0 is a *J*-ideal of \mathbb{Z} . But $(0(+)\mathbb{Z}_2)(0(+)\mathbb{Z}_2) = 0(+)\overline{0}$ is not a *J*-ideal of $\mathbb{Z}(+)\mathbb{Z}_2$ since for example, $(2,\overline{0})(0,\overline{1}) = (0,\overline{0})$ and $(2,\overline{0}) \notin J(\mathbb{Z})(+)\mathbb{Z}_2 = J(\mathbb{Z}(+)\mathbb{Z}_2)$ but $(0,\overline{1}) \neq (0,\overline{0})$.

Proposition 7. Let R_1 and R_2 be two rings and $f : R_1 \to R_2$ be an epimorphism. Then the following statements hold:

- (1) If I_1 is a quasi J-ideal of R_1 with $K \operatorname{erf} \subseteq I_1$, then $f(I_1)$ is a quasi J-ideal of R_2 .
- (2) If I_2 is a quasi *J*-ideal of R_2 and $K \operatorname{erf} \subseteq J(R)$, then $f^{-1}(I_2)$ is a quasi *J*-ideal of R_1 .

Proof. (1) Suppose that I_1 is a quasi *J*-ideal of R_1 . Since $\sqrt{I_1}$ is a *J*-ideal of R_1 and $K \operatorname{erf} \subseteq I_1 \subseteq \sqrt{I_1}$, then $f(\sqrt{I_1})$ is a *J*-ideal of R_2 by [10, Proposition 2.23]. Now, if $a, b \in R_2$ such that $ab \in \sqrt{f(I_1)}$ and $a \notin J(R_2)$, then $a^n b^n \in f(I_1) \subseteq f(\sqrt{I_1})$ for some integer n. Since $a^n \notin J(R_2)$, then $b^n \in f(\sqrt{I_1}) \subseteq \sqrt{f(I_1)}$. Therefore, $b \in \sqrt{f(I_1)}$ and $\sqrt{f(I_1)}$ is a *J*-ideal of R_2 . So, $f(I_1)$ is a quasi *J*-ideal of R_2 .

(2) Suppose that I_2 is a quasi *J*-ideal of R_2 . Since $\sqrt{I_2}$ is a *J*-ideal of R_2 and $K \operatorname{erf} \subseteq J(R)$, then $f^{-1}(\sqrt{I_2})$ is a *J*-ideal of R_1 by [10, Proposition 2.23]. Now, let $x, y \in R_1$ such that $xy \in \sqrt{f^{-1}(I_2)}$ and $x \notin J(R_1)$. Then $x^m y^m \in f^{-1}(I_2) \subseteq f^{-1}(\sqrt{I_2})$ for some integer m. But $x^m \notin J(R_1)$ implies that $y^m \in f^{-1}(\sqrt{I_2}) \subseteq \sqrt{f^{-1}(I_2)}$. It follows that $y \in \sqrt{f^{-1}(I_2)}$; and so $\sqrt{f^{-1}(I_2)}$ is a *J*-ideal of R_1 . \Box

Corollary 4. Let I and K be proper ideals of R with $K \subseteq I$. If I is a quasi J-ideal of R, then I/K is a quasi J-ideal of R/K.

Proof. Consider the natural epimorphism $\pi : R \to R/K$ with $Ker(\pi) = K \subseteq I$. By Proposition 7, $\pi(I) = I/K$ is a quasi J-ideal of R/K.

Let I be a proper ideal of R. In the following, the notation $Z_I(R)$ denotes the set of $\{r \in R | rs \in I \text{ for some } s \in R \setminus I\}$.

Proposition 8. Let S be a multiplicatively closed subset of a ring R such that $J(S^{-1}R) = S^{-1}J(R)$. Then the following hold:

- (1) If I is a quasi J-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a quasi J-ideal of $S^{-1}R$.
- (2) If $S^{-1}I$ is a quasi *J*-ideal of $S^{-1}R$ and $S \cap Z_I(R) = S \cap Z_{J(R)}(R) = \emptyset$, then *I* is a quasi *J*-ideal of *R*.

Proof. (1) Suppose that I is a quasi J-ideal of R. Since \sqrt{I} is a J-ideal of R, then by [10, Proposition 2.26], we conclude that $\sqrt{S^{-1}I} = S^{-1}\sqrt{I}$ is a J-ideal of R and we are done.

(2) Let $a, b \in R$ and $ab \in I$. Hence $\frac{a}{1}\frac{b}{1} \in S^{-1}I$. Since $S^{-1}I$ is a quasi *J*-ideal of $S^{-1}R$, we have either $\frac{a}{1} \in J(S^{-1}(R)) = S^{-1}J(R)$ or $\frac{b}{1} \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$ by Theorem 1. If $\frac{b}{1} \in S^{-1}\sqrt{I}$, then there exist $u \in S$ and a positive integer n such that $u^n b^n \in I$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $b^n \in I$ and so $b \in \sqrt{I}$. If $\frac{a}{1} \in S^{-1}J(R)$, then there exist $v \in S$ and a positive integer m such that $v^m a^m \in J(R)$. Since $S \cap Z_{J(R)}(R) = \emptyset$, we conclude that $a^m \in J(R)$ and so $a \in J(R)$. Therefore, I is a quasi J-ideal of R by Theorem 1.

Next, we justify that decomposable rings have no *J*-ideals.

Remark 1. Let R_1 and R_2 be two rings and $R = R_1 \times R_2$. Then there are no quasi *J*-ideal in *R*. Indeed, for every proper ideal $I_1 \times I_2$ of *R* we have $(1,0)(0,1) \in I_1 \times I_2$ but neither $(1,0) \in J(R)$ nor $(0,1) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$.

Lemma 3. Let I be an ideal of a Noetherian ring R. Then $\sqrt{I[|x|]} = \sqrt{I[|x|]}$.

Proof. See
$$[1]$$
.

Proposition 9. Let I be a proper ideal of a Noetherian ring R. Then I[|x|] is a quasi J-ideal of R[|x|] if and only if I is a quasi J-ideal of R.

Proof. Follows by [10, Proposition 2.18] and Lemma 3.

3. Quasi presimplifiable rings

Recall that a ring R is called presimplifiable if whenever $a, b \in R$ with a = ab, then a = 0 or $b \in U(R)$. This class of rings has been introduced by Bouvier in [7]. Then many of its properties are studied in [2] and [3]. Among many other characterizations, it is well known that R is presimplifiable if and only if $Z(R) \subseteq J(R)$. As a generalization of presimplifiable property, we introduce the following class of rings.

Definition 2. A ring R is called quasi presimplifiable if whenever $a, b \in R$ with a = ab, then $a \in N(R)$ or $b \in U(R)$.

It is clear that any presimplifiable ring R is quasi presimplifiable and that they coincide if R is reduced. The following example shows that the converse is not true in general.

Example 5. Let $R = \mathbb{Z}(+)\mathbb{Z}_2$ and let $(a, m_1), (b, m_2) \in R$ such that $(a, m_1)(b, m_2) = (a, m_1)$ and $(a, m_1) \notin N(R) = N(\mathbb{Z})(+)\mathbb{Z}_2$. Then ab = a with $a \notin N(R)$ and so we must have $b = 1 \in U(\mathbb{Z})$. It follows that $(b, m_2) \in U(\mathbb{Z})(+)\mathbb{Z}_2 = U(\mathbb{Z}(+)\mathbb{Z}_2) = U(R)$ and R is quasi presimplifiable. On the other hand, R is not presimplifiable. For example $(0, \overline{1}), (3, \overline{1}) \in R$ and $(0, \overline{1})(3, \overline{1}) = (0, \overline{1})$ but $(0, \overline{1}), (3, \overline{1}) \neq (0, \overline{0})$ and $(0, \overline{1}), (3, \overline{1}) \notin U(R)$.

A non-zero element a in a ring R is called quasi-regular if $Ann_R(a) \subseteq N(R)$. We denote the set of all elements of R that are not quasi-regular by NZ(R). As a characterization of quasi presimplifiable rings, we have the following.

Proposition 10. A ring R is quasi presimplifiable if and only if $NZ(R) \subseteq J(R)$.

Proof. Suppose R is quasi presimplifiable, $a \in NZ(R)$ and $r \in R$. Then $ra \in NZ(R)$ and so there exists $b \notin N(R)$ such that rab = 0. Hence, (1 - ra)b = b and so by assumption, $1 - ra \in U(R)$. It follows that $a \in J(R)$ and so $NZ(R) \subseteq J(R)$. Conversely, suppose $NZ(R) \subseteq J(R)$ and let $a, b \in R$ with a = ab. Then a(1-b) = 0. If $a \in N(R)$, then we are done, otherwise, $1 - b \in NZ(R) \subseteq J(R)$. Therefore, $b \in U(R)$ as required.

The main result of this section is to clarify the relationship between quasi J-ideals (resp. J-ideals) and quasi presimplifiable (resp. presimplifiable) rings.

Theorem 5. Let I be a proper ideal of a ring R. Then

- (1) I is a J-ideal of R if and only if $I \subseteq J(R)$ and R/I is presimplifiable.
- (2) I is a quasi J-ideal of R if and only if $I \subseteq J(R)$ and R/I is quasi presimplifiable.
- Proof. (1) Suppose I is a J-ideal of R. Then $I \subseteq J(R)$ by [10, Proposition 2.2]. Now, let $a + I \in Z(R/I)$. Then there exists $I \neq b + I \in R/I$ such that (a + I)(b + I) = I. Now, $ab \in I$ and $b \notin I$ imply that $a \in J(R)$ as I is a J-ideal of R. Thus, $a + I \in J(R)/I = J(R/I)$ and so R/I is presimplifiable. Conversely, suppose R/I is presimplifiable and let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. Then $a + I \notin J(R)/I = J(R/I)$ and by assumption, $a + I \notin Z(R/I)$. As (a + I)(b + I) = I, we conclude that b + I = I and so $b \in I$ as needed.
 - (2) Suppose I is a quasi J-ideal of R and note that $I \subseteq J(R)$ by Proposition 1. Let $a+I \in NZ(R/I)$ and choose $b+I \notin N(R/I)$ such that (a+I)(b+I) = I. Then $ab \in I$ and $b \notin \sqrt{I}$ which imply that $a \in J(R)$ as I is a quasi J-ideal of R. Hence, $a + I \in J(R)/I = J(R/I)$ and R/I is quasi presimplifiable by Proposition 10. Conversely, suppose R/I is quasi presimplifiable and let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. Then $a + I \notin J(R)/I = J(R/I)$ and so $a+I \notin NZ(R/I)$. As (a+I)(b+I) = I, we must have $b+I \in N(R/I)$ and so $b \in \sqrt{I}$. Therefore, I is a quasi J-ideal.

In view of Theorem 5, we deduce immediately the following characterization of presimplifiable (resp. quasi presimplifiable) rings.

Corollary 5. A ring R is presimplifiable (resp. quasi presimplifiable) if and only if 0 is a J-ideal (resp. quasi J-ideal) of R.

Recall that a ring R is said to be von Neumann regular if for every $a \in R$, there exists an element $x \in R$ such that $a = a^2 x$.

Lemma 4. If R is a quasi presimplifiable von Neumann regular ring, then R is a field.

Proof. Let a be a non-zero element of R. Since R is von Neumann regular, $a = a^2x$ for some element x of R. Observe that $a \notin N(R)$ as every von Neumann regular ring is reduced. Since a = a(ax) and R is quasi presimplifiable, we conclude that $ax \in U(R)$ and so $a \in U(R)$. Thus, R is a field.

We call an ideal I of a ring R regular if R/I is a von Neumann regular ring.

Proposition 11. Any regular quasi J-ideal in a ring R is maximal.

Proof. Suppose I is a regular quasi J-ideal of R. Then R/I is a von Neumann regular ring. Moreover, as $I \subseteq J(R)$, then R/I is quasi presimplifiable by Theorem 5. It follows by Lemma 4 that R/I is a field and so I is maximal in R.

For a ring R, we recall that $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ is a unit if and only if $a_0 \in U(R)$ and $a_1, a_2, ..., a_n \in N(R)$. In [2], it has been proved that R[x] is presimplifiable if and only if R is presimplifiable and 0 is a primary ideal of R.

Proposition 12. Let R be a ring. Then R[x] is quasi presimplifiable if and only if R is quasi presimplifiable and 0 is a δ_1 -n-ideal of R.

Proof. Suppose that R[x] is presimplifiable and let $a, b \in R \subseteq R[x]$ such that a = ab and $a \notin N(R)$. Then $a \notin N(R[x])$ and so by our assumption $b \in U(R[x])$. It follows that $b \in U(R)$ and so R is quasi presimplifiable. Now, let $a, b \in R$ such that ab = 0 and $a \notin N(R)$. Then we have a = a(1 - bx) and so $1 - bx \in U(R[x])$.

Hence $b \in N(R)$ and 0 is a δ_1 -*n*-ideal. For the converse, let $f(x) = \sum_{i=0}^n a_i x^i$,

$$g(x) = \sum_{j=0} b_j x^j \in R[x]$$
 such that $f(x) = f(x)g(x)$ and $f(x) \notin N(R[x])$. Then

 $a_i \notin N(R)$ for some *i*. Now, $a_i = g(x)a_i$ implies that $a_i = b_0a_i$ and so $b_0 \in U(R)$ since *R* is quasi presimplifiable. Moreover, for all $j \neq 0$, we have $a_ib_j = 0$. So, $b_j \in N(R)$ for all $j \neq 0$ as 0 is a δ_1 -*n*-ideal of *R*. Therefore, $g(x) \in U(R[x])$ and R[x] is quasi presimplifiable.

Recall that a ring R is called a Hilbert ring if every prime ideal of R is an intersection of maximal ideals. Moreover, it is well known that R is a Hilbert ring if and only if $M \cap R$ is a maximal ideal of R whenever M is a maximal ideal of R[x], see [8]. In this case, we have $J(R)[x] \subseteq J(R[x])$. Indeed, if M is a maximal ideal of R[x], then $M \cap R$ is a maximal ideal of R. Hence, $J(R)[x] \subseteq (M \cap R)[x] \subseteq M$.

In the following proposition, we determine conditions under which the extension I[x] in R[x] is a quasi J-ideal.

Proposition 13. Let I be an ideal of a ring R.

- (1) If I[x] is a quasi J-ideal of R[x], then I is a quasi J-ideal of R.
- (2) If R is Hilbert, $I \subseteq J(R)$ and I is a δ_1 -n-ideal of R, then I[x] is a quasi J-ideal of R[x].

- *Proof.* (1) Suppose I[x] is a quasi *J*-ideal of R[x] and let $a, b \in R \subseteq R[x]$ such that $ab \in I \subseteq I[x]$ and $a \notin J(R)$. Then clearly $a \notin J(R[x])$ and so $b \in \sqrt{I[x]}$. It follows clearly that $b \in \sqrt{I}$ and so *I* is a quasi *J*-ideal of *R*.
 - (2) Suppose I is a δ_1 -n-ideal of R. Then I is a quasi J-ideal by Proposition 2. Thus, R/I is quasi presimplifiable by Theorem 5 (2). By Proposition 12, we conclude that $R[x]/I[x] \cong (R/I)[x]$ is also quasi presimplifiable. Moreover, since R is Hilbert, then $I[x] \subseteq J(R)[x] \subseteq J(R[x])$. Therefore, I[x] is a quasi J-ideal of R[x] again by Theorem 5 (2).

Recall that (Λ, \leq) is called a directed quasi-ordered set if \leq is a reflexive and transitive relation on Λ and for $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. A system of rings over (Λ, \leq) is a collection $\{R_{\alpha} : \alpha \in \Lambda\}$ of rings, together with ring homomorphisms $\varphi_{\alpha,\beta} : R_{\alpha} \to R_{\beta}$ for all $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ such that $\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$ whenever $\alpha \leq \beta \leq \gamma$ and such that $\varphi_{\alpha,\alpha} = Id_{R_{\alpha}}$ for all α . A direct limit of $\{R_{\alpha} : \alpha \in \Lambda\}$ is a ring R together with ring homomorphisms $\varphi_{\alpha} : R_{\alpha} \to R$ such that $\varphi_{\beta} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha}$ for all $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ and such that following property is satisfied: For any ring S and collection $\{f_{\alpha} : \alpha \in \Lambda\}$ of ring maps $f : R_{\alpha} \to S$ such that $f_{\beta} \circ \varphi_{\alpha,\beta} = f_{\alpha}$ for all $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$, there is a unique ring homomorphism $f : R \to S$ with $f \circ \varphi_{\alpha} = f_{\alpha}$ for all $\alpha \in \Lambda$. This direct limit is usually denoted by $R = \varinjlim R_{\alpha}$.

Lemma 5. [6] Let $\{R_{\alpha} : \alpha \in \Lambda\}$ be a system of rings and let $R = \varinjlim R_{\alpha}$. If $\{I_{\alpha} : \alpha \in \Lambda\}$ is a family of ideals over $\{R_{\alpha} : \alpha \in \Lambda\}$, then $I = \sum_{\alpha \in \Lambda} \varphi_{\alpha}(I_{\alpha}) = \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(I_{\alpha})$ is an ideal of R. Moreover, $R/I = \varinjlim R_{\alpha}/I_{\alpha}$.

In [3], it is proved that if $\{R_{\alpha} : \alpha \in \Lambda\}$ is a system of presimplifiable rings, then so is $R = \underset{\alpha}{\lim} R_{\alpha}$. In the following proposition, we generalize this result to quasi presimplifiable case.

Proposition 14. Let (Λ, \leq) be a directed quasi-ordered set and let $\{R_{\alpha} : \alpha \in \Lambda\}$ be a direct system of rings. If each R_{α} is quasi presimplifiable, then the direct limit $R = \lim_{\alpha \to \infty} R_{\alpha}$ is quasi presimplifiable.

Proof. Let $x, y \in R$ with x = xy and $x \notin N(R)$. For $\alpha \in \Lambda$, let $\varphi_{\alpha} : R_{\alpha} \to R$ be the natural map. Then there exist $\alpha_0 \in \Lambda$ and $x_{\alpha_0}, y_{\alpha_0} \in R_{\alpha_0}$ such that $\varphi_{\alpha_0}(x_{\alpha_0}) = x$, $\varphi_{\alpha_0}(y_{\alpha_0}) = y$ and $x_{\alpha_0}y_{\alpha_0} = x_{\alpha_0}$. Since $x \notin N(R)$, then $x_{\alpha_0} \notin N(R_{\alpha_0})$, see [5], and so $y_{\alpha_0} \in U(R_{\alpha_0})$ as R_{α_0} is quasi presimplifiable. Therefore, $y = \varphi_{\alpha_0}(y_{\alpha_0}) \in U(R)$ and so R is quasi presimplifiable.

Theorem 6. Let (Λ, \leq) be a directed quasi-ordered set and let $\{R_{\alpha} : \alpha \in \Lambda\}$ be a direct system of rings. If $\{I_{\alpha} : \alpha \in \Lambda\}$ is a family of J-deals (resp. quasi Jideals) over $\{R_{\alpha} : \alpha \in \Lambda\}$, then $I = \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(I_{\alpha})$ is a J-ideal (resp. quasi J-ideal)

of $R = \underline{\lim} R_{\alpha}$.

Proof. For all $\alpha \in \Lambda$, we have $I_{\alpha} \subseteq J(R_{\alpha})$. Hence, $I = \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(I_{\alpha}) \subseteq \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(J(R_{\alpha})) \subseteq J(\varinjlim R_{\alpha}) = J(R)$. Indeed, let $x \in \bigcup_{\alpha \in \Lambda} \varphi_{\alpha}(J(R_{\alpha}))$ and $r \in R$. Then there exist $\alpha_0 \in \Lambda$ and $x_{\alpha_0}, r_{\alpha_0} \in R_{\alpha_0}$ such that $\varphi_{\alpha_0}(x_{\alpha_0}) = x$ and $\varphi_{\alpha_0}(r_{\alpha_0}) = r$. Now, $1 - rx = \varphi_{\alpha_0}(1_{R_{\alpha_0}} - r_{\alpha_0}x_{\alpha_0}) \in \varphi_{\alpha_0}(U(R_{\alpha_0})) \subseteq U(R)$ and so $x \in J(R)$. Since for

all $\alpha \in \Lambda$, I_{α} is a *J*-ideal (quasi *J*-ideal), then R_{α}/I_{α} is a presimplifiable (quasi presimplifiable) ring by Theorem 5. This implies that $R/I = \varinjlim R_{\alpha}/I_{\alpha}$ is presimplifiable (quasi presimplifiable) by Proposition 14. It follows again by Theorem 5 that *I* is a *J*-ideal (quasi *J*-ideal) of *R*.

Finally, for a ring R, an ideal I of R and an R-module M, we determine when is the ideal I(+)M quasi J-ideal in R(+)M.

Proposition 15. Let I be an ideal of a ring R and let M be an R-module. Then I(+)M is a quasi J-ideal of R(+)M if and only if I is a quasi J-ideal of R.

Proof. We have $I(+)M \subseteq J(R(+)M)$ if and only if $I \subseteq J(R)$ and $R/I \cong R(+)M/I(+)M$. Therefore, the result follows directly by Theorem 5.

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