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ORIGINAL RESEARCH

Differential and integrodifferential equations for Gould–Hopper–Frobenius–Euler polynomials

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Abstract
The fundamental aim of this paper is to derive the recurrence relation, shift operators, differential, integrodifferential and partial differential equations for Gould–Hopper–Frobenius–Euler polynomials using factorization method, which may be utilised in solving some emerging problems in different branches of science and technology.

Keywords Recurrence relation · Shift operators · Differential equation · Integro-differential equation · Partial differential equation

Mathematics Subject Classification 45J05 · 65Q30 · 65R20

Introduction and preliminaries

The investigation of factorization method has been seen as the relationship among Maxwell's and Dirac's equations. These equations are Lorentz invariant because of both being linear in nature and every one of them containing partial derivatives of first order. It might be commented that on account of Maxwell's equations, the linearity might be a distortion which prompts the troubles with infinite self-energies. An operational system which gives an answer to the inquiries regarding eigenvalue issues and which are of critical significance to physicists is known as the factorization method [1]. The fundamental thought is to think about a pair of first-order differential equations, which on working provides an equivalent second-order differential equation. The assembling cycle is additionally utilized for the figuring of transition probabilities. The strategy is summed up so it will deal with perturbation problems.

Let $\{l_m(\mu)\}_{m=0}^{\infty}$ be a polynomial sequence of degree m. Two operators η_m^- and η_m^+, for $m = 0, 1, 2, \ldots$ are defined by

$$\eta_m^-(l_m(\mu)) = l_{m-1}(\mu) \tag{1.1}$$

and

$$\eta_m^+(l_m(\mu)) = l_{m+1}(\mu), \tag{1.2}$$

which satisfies

$$(\eta_{m+1}^- \eta_m^+)\{l_m(\mu)\} = l_m(\mu), \tag{1.3}$$

which is known as factorization method. The fundamental intention behind the factorization method is to get the derivative operator η_m^- and multiplicative operator η_m^+ with the end goal that Eq. (1.3) holds.

The iterations of η_m^- and η_m^+ to $l_m(\mu)$ provide the following relations:

$$(\eta_{m-1}^+ \eta_m^-) l_m(\mu) = l_m(\mu), \tag{1.4}$$

$$(\eta_1^- \eta_2^- \eta_3^- \cdots \eta_{m-1}^- \eta_m^-) l_m(\mu) = l_0(\mu), \tag{1.5}$$

$$(\eta_{m-1}^+ \eta_{m-2}^+ \eta_{m-3}^+ \cdots \eta_1^+ \eta_0^+) l_0(\mu) = l_m(\mu). \tag{1.6}$$

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The operational relations referenced above permit us to determine higher-order differential equations fulfilled by some polynomials. The old-style factorization method acquainted in [1] was utilized with study of second-order differential equations.

Frobenius (see [2, 3]) studied in detail Frobenius–Euler polynomials $\mathcal{F}_m(x, \mu)$ defined by the generating relation:

$$\frac{1-\mu}{e^t-\mu}e^{xt} = \sum_{m=0}^{\infty} \mathcal{F}_m(x, \mu)\frac{t^m}{m!}, \quad \mu \in \mathcal{C}, \quad \mu \neq 1,$$

particularly

$$\mathcal{F}_m(0, \mu) = \mathcal{F}_m(\mu),$$

which are called Frobenius–Euler numbers given by

$$\frac{1-\mu}{e^t-\mu} = \sum_{m=0}^{\infty} \mathcal{F}_m(\mu)\frac{t^m}{m!}.$$

And the coefficients $e_k^F(\mu)$ related to Frobenius–Euler polynomials $\mathcal{F}_m(x, \mu)$ is defined as

$$e_k^F(\mu) = -\sum_{i=0}^{k} \frac{1}{2^i}\binom{k}{i}\mathcal{F}_{k-i}\left(\frac{1}{2}, \mu\right)$$

where

$$e_0^F = -1, \quad e_1^F(\mu) = -1 - \frac{1}{\mu-1}.$$

The Gould–Hopper polynomials $g_n^{(m)}(x, y)$ [4] are defined by the following series expansion

$$g_n^{(m)}(x, y) = n! \sum_{r=0}^{[\frac{n}{m}]} \frac{x^{n-mr}y^r}{(n-mr)!r!}$$

where m is a positive integer. These polynomials hold the following generating function:

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} g_n^{(m)}\frac{t^n}{n!}$$

Let GH-FEP be Gould–Hopper–Frobenius–Euler polynomials denoted by $G_m^{(j)}(x, y;\mu)$. It can be represented by the following generating function (see [5]):

$$\frac{1-\mu}{e^t-\mu}\exp(xt+yt^m) = \sum_{m=0}^{\infty} G_m^{(j)}(x, y;\mu)\frac{t^m}{m!}, \tag{1.7}$$

or equivalently

$$G_m^{(j)}(x, y;\mu) = \sum_{k=0}^{m} \binom{m}{k}\mathcal{F}_k(\mu)g_{m-k}^{(j)}(x, y) \tag{1.8}$$

and

$$G_m^{(j)}(x, y;\mu) = m! \sum_{k=0}^{[\frac{m}{j}]} \frac{\mathcal{F}_{m-k}(x;\mu)\, y^k}{k!\,(m-jk)!} \tag{1.9}$$

respectively.

These polynomials are significant in light of the fact that they have useful properties, for example, generating function, differential equations, recurrence relations, series definition, integral representations, summation formulae, and so on. In [6, 7], the differential equations for the Appell polynomials are determined by utilizing factorization method [1]. This methodology is additionally reached out to determine the integrodifferential equations for the hybrid, 2D extraordinary and mixed type polynomials identified with the Appell family, see for instance [8–18]. This gives inspiration to set up the differential equations for the Gould–Hopper–Frobenius–Euler polynomials.

Recurrence relations and Shift operators

In this section, the recurrence relation and shift operators for the Gould–Hopper–Frobenius–Euler polynomials (GH-FEP) are derived.

Theorem 2.1 *Let μ be a parameter, j is a positive integer, thus 2-variable GH-FEP, $G_m^{(j)}(x, y, \mu)$ satisfy the recurrence relation*

$$\begin{aligned} G_{m+1}^{(j)}(x, y;\mu) &= (x - \frac{1}{1-\mu})G_m^{(j)}(x, y;\mu) \\ &+ \frac{1}{1-\mu}\sum_{k=0}^{m-1}\binom{m}{k}e_{m-k}^F(\mu)G_{m-k}^{(j)}(x, y;\mu) \\ &+ jy\frac{m!}{(m-j+1)!}\, G_{m-j+1}^{(j)}(x, y;\mu). \end{aligned} \tag{2.1}$$

Proof Taking derivatives with respect to t of expression (1.7) on both sides and on simplification and using equations

$$\frac{1-\mu}{e^t-\mu} = \sum_{k=0}^{\infty} \mathcal{F}_k(\mu)\frac{t^k}{k!}$$

and

$$e_k^F(\mu) = -\sum_{i=0}^{k} \frac{1}{2^i}\binom{k}{i}\mathcal{F}_{k-i}\left(\frac{1}{2}, \mu\right)$$

where

$$e_0^F = -1, \quad e_1^F(\mu) = -1 - \frac{1}{\mu-1},$$

it follows that

$$\sum_{m=0}^{\infty} G_m^{(j)}(x,y;\mu)\frac{t^{m-1}}{(m-1)!} = x\sum_{m=0}^{\infty} G_m^{(j)}(x,y;\mu)\frac{t^m}{m!} - \frac{1}{1-\mu}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{F}_k(\mu)G_m^{(j)}(x,y;\mu)\sum_{m=0}^{\infty}\frac{t^{m+k}}{m!\,k!} + jy\sum_{m=0}^{\infty} G_m^{(j)}(x,y;\mu)\frac{t^{m+j-1}}{m!}. \tag{2.2}$$

On further simplification and using Cauchy-product rule in the right side, we get

$$\sum_{m=0}^{\infty} G_{m+1}^{(j)}(x,y;\mu)\frac{t^m}{m!} = x\sum_{m=0}^{\infty} G_m^{(j)}(x,y;\mu)\frac{t^m}{m!} - \frac{1}{1-\mu}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{m}{k}\mathcal{F}_k(\mu)G_{m-k}^{(j)}(x,y;\mu)\frac{t^m}{m!} + jy\sum_{m=0}^{\infty}\frac{m!}{(m-j+1)!}G_{m-j+1}^{(j)}(x,y;\mu)\frac{t^m}{m!}. \tag{2.3}$$

By comparing the same coefficients in the Eq. (2.3), we arrive at the desired result. □

To find the shift operators for the 2-variable GH-FEP, $G_m^{(j)}(x,y;\mu)$, we state the following theorem.

Theorem 2.2 *The shift operators for the 2-variable GH-FEP $G_m^{(j)}(x,y;\mu)$ are given by*

$${}_x\pounds_m^- := \frac{1}{m}D_x, \tag{2.4}$$

$${}_y\pounds_m^- := \frac{1}{m}D_x^{1-j}D_y, \tag{2.5}$$

$${}_x\pounds_m^+ := \left(x-\frac{1}{1-\mu}\right) + \frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{m-k} + jy\,D_x^{j-1}, \tag{2.6}$$

$${}_y\pounds_m^+ := \left(x-\frac{1}{1-\mu}\right) + \frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{(m-k)(1-j)}D_y^{m-k} + jy\,D_x^{-(j-1)^2}D_y^{j-1}, \tag{2.7}$$

where

$$D_x := \frac{\partial}{\partial x},\quad D_y := \frac{\partial}{\partial y},\quad \text{and}\quad D_x^{-1} := \int_0^x f(\xi)d\xi. \tag{2.8}$$

Proof Taking derivatives with respect to x in (1.7) and upon equating same exponents of the parameter t for resultant expression yields

$$\frac{\partial}{\partial x}\{G_m^{(j)}(x,y;\mu)\} = mG_{m-1}^{(j)}(x,y;\mu). \tag{2.9}$$

Consequently, we have

$${}_x\pounds_m^-\{G_m^{(j)}(x,y;\mu)\} = \frac{1}{m}\frac{\partial}{\partial x}\{G_m^{(j)}(x,y;\mu)\} = G_{m-1}^{(j)}(x,y;\mu), \tag{2.10}$$

thus proving assertion (2.4).

Further, taking derivatives with respect to y in (1.7) and upon equating same exponents of t of resultant expression yields

$$\frac{\partial}{\partial y}\{G_m^{(j)}(x,y;\mu)\} = \frac{m!}{(m-j)!}G_{m-j}^{(j)}(x,y;\mu) \tag{2.11}$$

$$= m\frac{\partial^{j-1}}{\partial x^{j-1}}G_{m-1}^{(j)}(x,y;\mu). \tag{2.12}$$

From here we have

$${}_y\pounds_m^-\{G_m^{(j)}(x,y;\mu)\} = \frac{1}{m}D_x^{1-j}D_y\{G_m^{(j)}(x,y;\mu)\} = G_{m-1}^{(j)}(x,y;\mu). \tag{2.13}$$

Thus, the proof is completed. □

Next, the succeeding expression is used to find raising operator (2.6):

$$G_k^{(j)}(x,y;\mu) = ({}_x\pounds_{k+1}^-\,{}_x\pounds_{k+2}^-\cdots{}_x\pounds_{m-1}^-\,{}_x\pounds_m^-)\{G_m^{(j)}(x,y;\mu)\}, \tag{2.14}$$

which can be further simplified on using expression (2.10) as:

$$G_k^{(j)}(x,y;\mu) = \frac{k!}{m!}D_x^{m-k}\{G_m^{(j)}(x,y;\mu)\}. \tag{2.15}$$

Inserting preceding expression in relation (2.1) gives

$$G_{m+1}^{(j)}(x,y;\mu) = \left(\left(x-\frac{1}{1-\mu}\right) + \frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{m-k} + jy\,D_x^{j-1}\right)G_m^{(j)}(x,y;\mu). \tag{2.16}$$

This yields expression of raising operator (2.6).

Furthermore, the succeeding expression is considered to derive the raising operator (2.7):

$$G_k^{(j)}(x,y;\mu) = ({}_y\pounds_{k+1}^-\,{}_y\pounds_{k+2}^-\cdots{}_y\pounds_{m-1}^-\,{}_y\pounds_m^-)\{G_m^{(j)}(x,y;\mu)\}, \tag{2.17}$$

simplifying preceding expression in view of expression (2.13) gives

$$G_k^{(j)}(x,y;\mu)=\frac{k!}{m!}D_x^{(m-k)(1-j)}D_y^{m-k}\{G_m^{(j)}(x,y;\mu)\}. \tag{2.18}$$

Inserting preceding expression in relation (2.1), it becomes

$$G_{m+1}^{(j)}(x,y;\mu)=\Bigg(\Big(x-\frac{1}{1-\mu}\Big)+\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!} D_x^{(m-k)(1-j)}D_y^{m-k}+jy\,D_x^{-(j-1)^2}D_y^{j-1}\Bigg) G_m^{(j)}(x,y;\mu). \tag{2.19}$$

This yields expression of raising operator (2.7).

Differential equations

In this section, we derive the differential, integro-differential and partial differential equations for the 2-variable GH-FEP, $G_m^{(j)}(x,y;\mu)$, via factorization method.

Theorem 3.1 *The 2-variable GH-FEP, $G_m^{(j)}(x,y;\mu)$, satisfy the following differential equation*:

$$\Bigg(\Big(x-\frac{1}{1-\mu}\Big)D_x+\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{m-k+1} +jy\,D_x^{j}-m\Bigg)G_m^{(j)}(x,y;\mu)=0. \tag{3.1}$$

Proof Using the factorization relation given by

$${}_x\pounds_{m+1}^{-}\,{}_x\pounds_m^{+}\{G_m^{(j)}(x,y;\mu)\}=G_m^{(j)}(x,y;\mu), \tag{3.2}$$

and inserting the expressions (2.4) and (2.6) of the shift operators in the left hand side yields assertion (3.1). □

Theorem 3.2 *The 2-variable GH-FEP, $G_m^{(j)}(x,y;\mu)$, satisfy the following integro-differential equations*:

$$\Bigg(\Big(x-\frac{1}{1-\mu}\Big)D_y+\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!} D_x^{(m-k)(1-j)}D_y^{m-k+1}+j\,D_x^{-(j-1)^2}D_y^{j-1}+jy\,D_x^{-(j-1)^2}D_y^{j} -(m+1)D_x^{j-1}\Bigg)G_m^{(j)}(x,y;\mu)=0 \tag{3.3}$$

and

$$\Bigg(\Big(x-\frac{1}{1-\mu}\Big)D_x+\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!} D_x^{(m-k)(1-j)+1}D_y^{m-k} +jy\,D_x^{-(j-1)^2+1}D_y^{j-1}-m\Bigg)G_m^{(j)}(x,y;\mu)=0. \tag{3.4}$$

Proof Since

$${}_y\pounds_{m+1}^{-}\,{}_y\pounds_m^{+}\{G_m^{(j)}(x,y;\mu)\}=G_m^{(j)}(x,y;\mu) \tag{3.5}$$

and the shift operators (2.5) and (2.7), one can easily see Eq. (3.3).

Combining (2.4) and (2.7) of the shift operators with the factorization relation

$${}_x\pounds_{m+1}^{-}\,{}_y\pounds_m^{+}\{G_m^{(j)}(x,y;\mu)\}=G_m^{(j)}(x,y;\mu) \tag{3.6}$$

yields our assertion (3.4). □

Theorem 3.3 *The 2-variable GH-FEP, $G_m^{(j)}(x,y;\mu)$, satisfy the following partial differential equations*

$$\Bigg(\Big(x-\frac{1}{1-\mu}\Big)D_x^{m(j-1)}D_y+m(j-1)D_yD_x^{(m(j-1)-1)} +\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{k(j-1)}D_y^{m-k+1} +j\,D_x^{(j-1)(m-j+1)}D_y^{j-1}+jy\,D_x^{(j-1)(m-j+1)} D_y^{j}-(m+1)D_x^{(m+1)(j-1)}\Bigg)G_m^{(j)}(x,y;\mu)=0 \tag{3.7}$$

and

$$\Bigg(\Big(x-\frac{1}{1-\mu}\Big)D_x^{(m(j-1)+1)}+m(j-1)D_x^{m(j-1)} +\frac{1}{1-\mu}\sum_{k=0}^{m-1}\frac{e_{m-k}^F(\mu)}{(m-k)!}D_x^{(k(j-1)+1)}D_y^{m-k} +jy\,D_x^{(j-1)(m-j+1)}D_y^{(j-1)}-mD_x^{m(j-1)}\Bigg)G_m^{(j)}(x,y;\mu)=0, \tag{3.8}$$

respectively.

Proof Differentiating $m(j-1)$ times with respect to x of integro-differential Eqs. (3.3) and (3.4), partial differential Eqs. (3.7) and (3.8) are obtained. □

Conclusion

The problems arising in different areas of science and engineering are usually expressed in terms of differential equations, which in most of the cases have special functions as their solutions. The differential and integral equations satisfied by these hybrid type special polynomials may be used to solve new emerging problems in different branches of science. To study the combination of operational representations with the factorization method and their applications to the theory of differential equations for other hybrid type special polynomials and for their relatives will be taken in further investigation.

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