THE DELIGNE–SIMPSON PROBLEM FOR CONNECTIONS ON $\mathbb{G}_m$ WITH A MAXIMALLY RAMIFIED SINGULARITY

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ABSTRACT. The classical additive Deligne–Simpson problem is the existence problem for Fuchsian connections with residues at the singular points in specified adjoint orbits. Crawley-Boevey found the solution in 2003 by reinterpreting the problem in terms of quiver varieties. A more general version of this problem, solved by Hiroe, allows additional unramified irregular singularities. We apply the theory of fundamental and regular strata due to Bremer and Sage to formulate a version of the Deligne–Simpson problem in which certain ramified singularities are allowed. These allowed singular points are called toral singularities; they are singularities whose leading term with respect to a lattice chain filtration is regular semisimple. We solve this problem in the special case of connections on $\mathbb{G}_m$ with a maximally ramified singularity at 0 and possibly an additional regular singular point at infinity. Examples of such connections arise from Airy, Bessel, and Kloosterman differential equations. They play an important role in recent work in the geometric Langlands program. We also give a complete characterization of all such connections which are rigid, under the additional hypothesis of unipotent monodromy at infinity.

CONTENTS

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1. Introduction

1.1. **The classical Deligne–Simpson problem.** A fundamental concern in the study of meromorphic connections is the existence problem for connections with specified singularities. More precisely, this problem poses the question: given points $a_1, \dots, a_m$ in $\mathbb{P}^1$ and formal connections $\widehat{\nabla}_1, \dots, \widehat{\nabla}_m$, does there exist a meromorphic connection ∇ which is regular away from the a_i's and satisfies $\nabla_{a_i} \cong \widehat{\nabla}_i$ for all i? The classical Deligne–Simpson problem is a variant of this problem for Fuchsian connections.

From now on, we assume that the underlying vector bundles of all connections on $\mathbb{P}^1$ are trivializable. Without loss of generality, we assume that the collection of singular points does not include ∞. A Fuchsian connection with singular points $a_1, \dots, a_m$ is defined by

$$d + \Big(\sum_{i=1}^{m} \frac{A_i}{z - a_i}\Big) dz,$$

where $A_i \in \mathfrak{gl}_n(\mathbb{C})$ for all i. Note that the adjoint orbit of A_i determines the formal isomorphism class at a_i. Since ∞ is not a singularity, the residue theorem forces $\sum A_i = 0$. We say that the collection of matrices $A_1, \dots, A_m$ is irreducible if they have no common invariant subspaces besides $\{0\}$ and $\mathbb{C}^n$. We can now state the (additive) Deligne–Simpson problem:

> *Given adjoint orbits $\mathscr{O}_1, \dots, \mathscr{O}_m$, determine whether there exists an irreducible m-tuple $(A_1, \dots, A_m)$ with $A_i \in \mathscr{O}_i$ satisfying $\sum A_i = 0$* [Kos03].

In other words, when is there an irreducible Fuchsian connection with residues in the given orbits? Note that the original problem considered by Deligne and Simpson was the multiplicative version, where one looks for Fuchsian connections with monodromies in specified conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$ [Sim91]. The additive version stated above was originally formulated by Kostov, who solved it in the "generic" case. Crawley-Boevey gave a complete solution by reinterpreting the problem in terms of quiver varieties [CB03]. We remark that while there is an obvious analogue of this problem for arbitrary reductive G, little is known about the solution outside of type A.

1.2. **The unramified Deligne–Simpson problem.** In order to generalize the Deligne–Simpson problem to allow for irregular singularities, one considers connections with higher order principal parts at the singularities:

$$d + \Big(\sum_{i=1}^{m} \sum_{\nu=0}^{r_i} \frac{A_\nu^{(i)}}{(z - a_i)^\nu}\Big) \frac{dz}{z}. \tag{1}$$

Again, we assume that ∞ is not a singular point, so $\sum_{i=1}^{m} A_0^{(i)} = 0$. We now require that the singularity at each a_i has a certain specified form called a "formal type".

Most previous work on the irregular Deligne–Simpson problem has restricted attention to the "unramified case" [Kos10, Boa08, HY14, Hir17]. This means that at each singularity, the slope decomposition of the corresponding formal connection only involves integer slopes. More concretely, each such formal connection has Levelt–Turrittin (LT) normal form

$$d + (D_r z^{-r} + \cdots + D_1 z^{-1} + R)\frac{dz}{z}, \tag{2}$$

where the D_i's are diagonal, $D_r \neq 0$, and the residue term R is upper triangular and commutes with each D_i. We view the 1-form $(D_r z^{-r} + \cdots + D_1 z^{-1} + R)\frac{dz}{z}$ as an unramified formal type. In the regular singular case, one can take R to be in Jordan canonical form and view the formal type as $R\frac{dz}{z}$.

For Fuchsian connections, the principal part at a singular point is just the residue. Hence, in the classical Deligne–Simpson problem, one requires that the principal part agrees with the formal type after conjugation by a constant matrix (i.e., an element of $\mathrm{GL}_n(\mathbb{C})$). In other words, the principal part lies in the adjoint orbit of the formal type. For unramified formal types of positive slope, one instead requires the principal part to lie in the orbit of the formal type under a certain action of the group $\mathrm{GL}_n(\mathbb{C}[\![z]\!])$. Let $\mathcal{B}_r = \{(B_r z^{-r} + \cdots + B_0)\frac{dz}{z} \mid B_i \in \mathfrak{gl}_n(\mathbb{C})\}$ denote the space of principal parts of order at most r. The group $\mathrm{GL}_n(\mathbb{C}[\![z]\!])$ acts on $\mathcal{B}_r$ by conjugation followed by truncation at the residue term. Note that this action factors through the finite-dimensional group $\mathrm{GL}_n(\mathbb{C}[\![z]\!]/z^r\mathbb{C}[\![z]\!])$. If $\mathscr{A}$ is an unramified formal type of slope r, we call the orbit $\mathscr{O}_{\mathscr{A}}$ under this action the *truncated orbit* of $\mathscr{A}$. If $\mathscr{A}$ has slope 0, $\mathscr{O}_{\mathscr{A}}$ may be identified with the usual adjoint orbit of $\mathscr{A}/\frac{dz}{z}$.

We can now state the unramified irregular Deligne–Simpson problem: Given points a_i and unramified formal types $\mathscr{A}_i$ of slope r_i, determine when there exists an irreducible connection as in (1) whose principal part at each a_i lies in $\mathscr{O}_{\mathscr{A}_i}$. This problem can also be restated in the language of moduli spaces. Given the formal types $\mathscr{A}_i$, one can consider the moduli space of "framable" connections on a rank n trivial bundle whose singularities have the specified formal types [HY14]. The construction generalizes that of Boalch [Boa01], who assumes that the $\mathscr{A}_i$ are all nonresonant, i.e., that the leading term of each $\mathscr{A}_i$ is regular semisimple. This moduli space is not necessarily well-behaved, but it is a complex manifold if one restricts to the *stable* moduli space, i.e., the open subset consisting of irreducible connections. The unramified Deligne–Simpson problem is simply the question of when such a stable moduli space is nonempty.

This problem was solved in 2017 by Hiroe [Hir17], building on earlier work of Boalch [Boa08] and Hiroe and Yamakawa [HY14]. As in the Fuchsian case, the proof involves quiver varieties. Hiroe uses the collection of unramified formal types to define a certain quiver variety and identifies the stable moduli space with a certain open subspace of the quiver variety. He then finds necessary and sufficient conditions for this open subspace to be nonempty. As a corollary, Hiroe shows that the stable moduli space is a connected manifold as long as it is nonempty.

1.3. **The ramified Deligne–Simpson problem for toral connections.** In this paper, we introduce the study of the *ramified Deligne–Simpson problem*, where ramified singularities are allowed. A singularity is called *ramified* if the associated formal connection can only be expressed in LT normal form after passing to a ramified cover. The LT normal form is thus no longer a suitable notion of formal type for ramified singularities. It is possible to formulate the ramified Deligne–Simpson problem by replacing the LT normal form with a "rational canonical form" for connections. Such a form may be obtained from Sabbah's refined Levelt–Turrittin decomposition [Sab08]; we will discuss this in a future paper.

Here, we only sketch the setup of the ramified Deligne–Simpson problem for a special class of irregular connections called *toral connections*. Roughly speaking, a formal connection is called toral if its leading term with respect to an appropriate filtration satisfies a graded version of regular semisimplicity. (The precise definition involves the theory of fundamental and regular strata for connections introduced by Bremer and Sage [BS13b, BS18, BS13a].) The terminology reflects the fact that toral connections can be "diagonalized" into a (not necessarily split) Cartan subalgebra of the loop algebra.

First, we describe formal types for toral connections. A rank n toral connection has slope r/b, where b is a divisor of n or $n-1$ and $\gcd(r,b)=1$. If $b>1$, define $\omega_b \in \mathfrak{gl}_b(\mathbb{C}((z)))$ by $\omega_b = \sum_{i=1}^{b-1} e_{i,i+1} + z e_{b,1}$; i.e., ω_b is the matrix with 1's in each entry of the superdiagonal, z in the lower-left entry, and 0's elsewhere. If $b=1$, set $\omega_b = z$. Note that $\omega_b^b = z\mathrm{id}$. Given such a b with $b\ell = n$ (resp. $b\ell = n-1$), we define a block-diagonal Cartan subalgebra $\mathfrak{s}^b = \mathbb{C}((\omega_b))^\ell$ (resp. $\mathfrak{s}^b = \mathbb{C}((\omega_b))^\ell \oplus \mathbb{C}((z))$). There is a natural $\mathbb{Z}$-filtration $\mathfrak{s}^b = \bigcup_i (\mathfrak{s}^b)^i$ induced by assigning degree i to ω_b^i. Let S^b denote the corresponding maximal torus in the loop group.

An S^b-*formal type* of slope r/b (with $\gcd(r,b)=1$) is a 1-form $A\frac{dz}{z}$, where $A \in (\mathfrak{s}^b)^{-r}$ has regular semisimple term in degree $-r$ and no terms in positive degree. It is a fact that any toral connection of slope r/b is formally isomorphic to a connection $d + A\frac{dz}{z}$ with $A\frac{dz}{z}$ an S^b-formal type; the formal type is unique up to an action of the relative affine Weyl group of S^b [BS13b, BS13a].

In the case of unramified toral connections, $S^1 = T(\mathbb{C}((z)))$ is the usual diagonal maximal torus, and the S^1-formal types of slope r are those connection matrices in LT normal form (2) with D_r regular (so that R is necessarily 0). At the opposite extreme, $\mathcal{C} := S^n = \mathbb{C}((\omega_n))^*$ is a "Coxeter maximal torus".[1] The $\mathcal{C}$-formal types of slope r/n are the 1-forms $p(\omega_n^{-1})\frac{dz}{z}$, where p is a polynomial of degree r.

The unramified Deligne–Simpson problem involves global connections which satisfy a stronger condition than just having specified formal types at the singularities. One also needs the local isomorphisms transforming the matrices of the formal connections into the given formal types to

[1] Under the bijection between classes of maximal tori in $\mathrm{GL}_n(\mathbb{C}((z)))$ and conjugacy classes in the Weyl group $\mathfrak{S}_n$ [KL88], $\mathcal{C}$ corresponds to the Coxeter class consisting of n-cycles.

satisfy a global compatibility condition called “framability”. We now explain how this condition can be generalized to toral formal types.

Recall (see, e.g., [Sag00, BS13b]) that the *parahoric subgroups* of $\mathrm{GL}_n(\mathbb{C}(\!(z)\!))$ are the local field analogues of the parabolic subgroups of $\mathrm{GL}_n(\mathbb{C})$. A parabolic subgroup is the stabilizer of a partial flag of subspaces in $\mathbb{C}^n$, and a parahoric subgroup is the stabilizer of a ”lattice chain” of $\mathbb{C}[\![z]\!]$-lattices in $\mathbb{C}(\!(z)\!)^n$. If P is a parahoric subgroup with associated lattice chain $\{L^j\}_j$, then there is an associated ”lattice chain filtration” $\{\mathfrak{p}^i\}_{i\in\mathbb{Z}}$ on $\mathfrak{gl}_n(\mathbb{C}(\!(z)\!))$ defined by $\mathfrak{p}^i = \{X \mid X(L^j) \subset L^{j+i}\ \forall j\}$.

To each maximal torus S^b, there is a unique ”standard parahoric subgroup” $P^b \subset \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ with the property that the corresponding filtration $\{(\mathfrak{p}^b)^i\}_i$ is compatible with the filtration on $\mathfrak{s}^b$, in the sense that $(\mathfrak{s}^b)^i = (\mathfrak{p}^b)^i \cap \mathfrak{s}^b$ for all i [BS13b]. In the unramified case, we have $P^1 = \mathrm{GL}_n(\mathbb{C}[\![z]\!])$. For the Coxeter maximal torus S^n, the corresponding parahoric subgroup is the standard “Iwahori subgroup” $I := P^n$; i.e., I is the preimage of the upper-triangular Borel subgroup B (consisting of all upper-triangular matrices in $\mathrm{GL}_n(\mathbb{C})$) via the map $\mathrm{GL}_n(\mathbb{C}[\![z]\!]) \to \mathrm{GL}_n(\mathbb{C})$ induced by the “evaluation at zero” map $z \mapsto 0$.

The most natural way of describing framability involves coadjoint orbits. One can view the principal part at 0 of a connection as a continuous functional on $\mathfrak{gl}_n(\mathbb{C}[\![z]\!])$ via $Y \mapsto \mathrm{Res}(\mathrm{Tr}(YX\frac{dz}{z}))$. Similarly, an S^b-formal type can be viewed as a functional on $\mathfrak{p}^b$. The global connection ∇ then is *framable at* 0 with respect to the S^b-formal type $A\frac{dz}{z}$ at 0 if for some global trivialization, the restriction to $\mathfrak{p}^b$ of the principal part at 0 lies in the P^b-coadjoint orbit of $A\frac{dz}{z}$. (See Definition 5.1.)

However, one can also give a description more reminiscent of the definition in the unramified case.

Definition 1.1. Let ∇ be a global connection on $\mathbb{P}^1$ with a singular point at 0, and let $\mathscr{A} = A\frac{dz}{z}$ be a toral formal type of slope r/b. We say that ∇ is *framable at* 0 *with respect to* $\mathscr{A}$ if

(1) under some global trivialization ϕ, the matrix form $\nabla = d + [\nabla]_\phi \frac{dz}{z}$ satisfies $[\nabla]_\phi \in (\mathfrak{p}^b)^{-r}$, and $[\nabla]_\phi - A \in (\mathfrak{p}^b)^{1-r}$; and
(2) there exists an element $p \in (P^b)^1$ such that the nonpositive truncation of $\mathrm{Ad}(p)[\nabla]_\phi$ equals A.

Recall that $\mathrm{GL}_n(\mathbb{C})$ acts simply transitively on the space of global trivializations. If one starts with a fixed trivialization ϕ', then the choice of trivialization ϕ in the definition above corresponds to an element $g \in \mathrm{GL}_n(\mathbb{C})$; i.e., there is a unique g such that $\phi = g \cdot \phi'$. This matrix g is called a *compatible framing* (or simply, a *framing*) of ∇ at 0. Framability with respect to a formal type at an arbitrary point $a \in \mathbb{P}^1$ is defined similarly, by simply replacing z by $z - a$ if a is finite, and by z^{-1} if $a = \infty$.

We can now state the Deligne–Simpson problem for connections whose irregular singularities are all toral. Note that the statement below can easily be extended to allow for arbitrary unramified singular points.

Toral Deligne–Simpson Problem. Let $\mathbf{A} = (\mathscr{A}_1, \dots, \mathscr{A}_m)$ be a collection of toral formal types at the points $a_1, \dots, a_m \in \mathbb{P}^1$, and let $\mathbf{O} = (\mathscr{O}_1, \dots, \mathscr{O}_\ell)$ be a collection of adjoint orbits at other points $b_1, \dots, b_\ell \in \mathbb{P}^1$. Does there exist an irreducible rank n connection ∇ such that

(1) ∇ is regular away from the a_i's and b_j's;
(2) ∇ is framable at a_i with respect to the formal type $\mathscr{A}_i$; and
(3) ∇ is regular singular at b_j with residue in $\mathscr{O}_j$?

If such a connection exists, we call it a "framable connection" with the given formal types.

This problem can be restated in terms of moduli spaces of connections. Suppose that each $\mathscr{O}_j$ is nonresonant; i.e., suppose that no pair of the eigenvalues of the orbit differ by a nonzero integer. Further assume that $m \geq 1$, so that there is at least one irregular singular point. In [BS13b], Bremer and Sage constructed the moduli space $\mathcal{M}(\mathbf{A}, \mathbf{O})$ of connections satisfying all the above hypotheses except irreducibility. Let $\mathcal{M}_{\text{irr}}(\mathbf{A}, \mathbf{O})$ be the subset of the moduli space consisting of irreducible connections. In this language, the toral Deligne–Simpson problem poses the question of when $\mathcal{M}_{\text{irr}}(\mathbf{A}, \mathbf{O})$ is nonempty.

1.4. **Coxeter connections.** We now restrict attention to a simple special case: connections with a *maximally ramified* irregular singularity and (possibly) an additional regular singular point. Without loss of generality, we will view such connections as connections on $\mathbb{G}_m$ with the irregular singularity at 0. Following [KS21b], we refer to such connections as *Coxeter connections*. Well-known classical examples arise from the Airy differential equation and a modified version of the Bessel equation.[2] Another important class of examples consists of the generalized Kloosterman connections studied by Katz [Kat88, Kat90]. These hypergeometric connections are the geometric incarnations of certain exponential sums called Kloosterman sums, which are of great importance in number theory.

Coxeter connections and their G-connection analogues (for G a simple algebraic group) have played a significant role in recent work in the geometric Langlands program. For example, Frenkel and Gross [FG09], building on work of Deligne [Del70] and Katz [Kat88, Kat96], constructed a rigid G-connection of this type. This connection, which may be viewed as a G-version of a modified Bessel connection, was the first connection with irregular singularities for which the geometric Langlands correspondence was understood explicitly [HNY13, Zhu17]. This connection also arises in Lam and Templier's proof of mirror symmetry for minuscule flag varieties [LT17]. Other examples include the Airy G-connection and more general rigid "Coxeter G-connections"

[2] In these connections and others described below, the irregular singularity is at ∞.

constructed in [KS21b]. The Airy G-connection and its ℓ-adic analogue have also been studied in [JKY21].

Recall that if $\widehat{\nabla}$ is a rank n formal connection, then every slope of $\widehat{\nabla}$ has denominator (when the slope is expressed in lowest form) between 1 and n. We say $\widehat{\nabla}$ is maximally ramified if all such denominators (or equivalently, at least one) is n. In this case, all the slopes are the same — say r/n with $\gcd(r,n)=1$ — and equal to the slope of the connection. More concretely, the leading term of the LT normal form is of the form $D_r z^{-r/n}$ with D_r a constant diagonal matrix, and is necessarily regular.

It is shown in [KS21a] that maximally ramified connections are toral connections with respect to a Coxeter maximal torus. Thus any maximally ramified connection of slope r/n has a rational canonical form $d + p(\omega_n^{-1})\frac{dz}{z}$, where p is a polynomial of degree r, and the set of formal types is given by $\{p(\omega_n^{-1})\frac{dz}{z} \mid p \in \mathbb{C}[x], \deg(p) = r\}$. Moreover, any such connection is irreducible.

In this paper, we solve the ramified Deligne–Simpson problem for Coxeter connections. More precisely, let $\mathscr{A}$ be a maximally ramified formal type, and let $\mathscr{O}$ be an adjoint orbit (which we will always assume to be nonresonant). We determine necessary and sufficient conditions for the existence of a meromorphic connection ∇ on $\mathbb{P}^1$ which is framable at 0 with formal type $\mathscr{A}$, is regular singular with residue in $\mathscr{O}$ at ∞, and is otherwise nonsingular. Note that any such connection is automatically irreducible, since its formal connection at 0 is irreducible. Thus, in the language of moduli spaces, we determine when $\mathcal{M}(\mathscr{A},\mathscr{O})$ is nonempty.

In order to state our result, we need some facts about adjoint orbits. Fix a monic polynomial $q = \prod_{i=1}^{s}(x-a_i)^{m_i}$ of degree n with the a_i's distinct complex numbers. The set $\{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \operatorname{char}(X) = q\}$ of matrices with characteristic polynomial q is a closed subset of $\mathfrak{gl}_n(\mathbb{C})$ which is stable under conjugation. We denote the set of orbits with characteristic polynomial q by π_q. This set is partially ordered under the usual Zariski closure ordering: $\mathscr{O} \preceq \mathscr{O}'$ if and only if $\mathscr{O} \subset \overline{\mathscr{O}'}$. The theory of the Jordan canonical form makes it clear that π_q can be identified with the Cartesian product $\prod_{i=1}^{s} \operatorname{Part}(m_i)$, where $\operatorname{Part}(m_i)$ denotes the set of partitions of m_i. Moreover, this identification defines a poset isomorphism between the closure ordering and the direct product of the dominance orders.

Given positive integers r and m, there exists a unique smallest partition $\lambda^{m,r} \in \operatorname{Part}(m)$ with at most r parts. Define $\mathscr{O}_q^r$ to be the orbit in π_q corresponding to the element

$$(\lambda^{m_1,r}, \lambda^{m_2,r}, \ldots, \lambda^{m_s,r}) \in \prod_{i=1}^{s} \operatorname{Part}(m_i).$$

This tuple of partitions is the (unique) smallest element of $\prod_{i=1}^{s} \operatorname{Part}(m_i)$ such that each component partition has at most r parts. Note that $\mathscr{O}_q^1$ is just the regular orbit in π_q. On the other extreme, if $r \geq m_i$ for all i (as is the case when $r \geq n$), then $\mathscr{O}_q^r$ is the semisimple orbit, the unique minimal orbit in π_q. Let $\langle \mathscr{O}_q^r \rangle$ denote the principal filter generated by $\mathscr{O}_q^r$ in π_q, i.e., $\mathscr{O} \in \pi_q$ satisfies $\mathscr{O} \in \langle \mathscr{O}_q^r \rangle$

if and only if $\mathscr{O} \succeq \mathscr{O}^r_q$. This filter is proper unless $r \geq m_i$ for all i. As we will see in Theorem 2.3, the collection of orbits $\mathscr{O}^r_q$ for each fixed r satisfies a generalization of one characterization of regular orbits.

We can now give the solution to the Deligne–Simpson problem for Coxeter connections. Given a rank n maximally ramified formal type $\mathscr{A}$ and a monic polynomial q of degree n that is nonresonant (i.e., no two roots differ by a nonzero integer), let

$$\mathrm{DS}(\mathscr{A}, q) = \{\mathscr{O} \in \pi_q \mid \mathcal{M}(\mathscr{A}, \mathscr{O}) \neq \varnothing\}.$$

Theorem 5.4. Let r and n be positive integers with $\gcd(r, n) = 1$, let $\mathscr{A}$ be a maximally ramified formal type of slope r/n, and let $q = \prod_{i=1}^s (x - a_i)^{m_i} \in \mathbb{C}[x]$ with $a_1, \dots, a_s \in \mathbb{C}$ distinct modulo $\mathbb{Z}$. Then

$$\mathrm{DS}(\mathscr{A}, q) = \begin{cases} \langle \mathscr{O}^r_q \rangle & \text{if } \operatorname{Res}(\operatorname{Tr}(\mathscr{A})) = -\sum_{i=1}^s m_i a_i, \\ \varnothing & \text{else.} \end{cases}$$

In other words, given $\mathscr{A} = p(\omega^{-1})\frac{dz}{z}$, then $\mathcal{M}(\mathscr{A}, \mathscr{O})$ is nonempty if and only if $np(0) = -\operatorname{Tr}(\mathscr{O})$ and $\mathscr{O} \succeq \mathscr{O}^r_{\mathrm{char}(\mathscr{O})}$. Concretely, the condition $\mathscr{O} \succeq \mathscr{O}^r_{\mathrm{char}(\mathscr{O})}$ means that $\mathscr{O}$ has at most r Jordan blocks for each eigenvalue.

Note that the solution depends only on the slope and the residue of the formal type.

Remark 1.2. If $r \geq m_i$ for all i, then $\mathrm{DS}(\mathscr{A}, q) = \pi_q$ as long as the trace condition is satisfied. In particular, this is the case if $r > n$.

There is an obvious analogue of this problem for SL_n-connections (as opposed to GL_n-connections). Here, maximally ramified formal types are of the form $p(\omega^{-1})\frac{dz}{z}$ with $p(0) = 0$ and $\operatorname{Tr}(\mathscr{O}) = 0$. Thus, the trace condition becomes vacuous, and the Deligne–Simpson problem has a positive solution if and only if $\mathscr{O} \succeq \mathscr{O}^r_{\mathrm{char}(\mathscr{O})}$.

One can define Coxeter G-connections for any simple group G (or for any reductive group with connected Dynkin diagram) [KS21b]. For such a G, Coxeter toral connections have slope r/h, where h is the Coxeter number for G and $\gcd(r, h) = 1$. Moreover, there is an analogue of the Deligne–Simpson problem in this more general context. We restrict to the case where the regular singularity at ∞ has nilpotent residue (and thus has unipotent monodromy).

Conjecture 1.3. Let G be a simple complex group with Lie algebra $\mathfrak{g}$. Fix a Coxeter G-formal type $\mathscr{A}$ of slope r/h with $\gcd(r, h) = 1$. Then there exists a nilpotent orbit $\mathscr{O}^r \subset \mathfrak{g}$ such that the Deligne–Simpson problem for Coxeter G-connections with initial data $\mathscr{A}$ and the nilpotent orbit $\mathscr{O}$ has a positive solution if and only if $\mathscr{O} \succeq \mathscr{O}^r$. Moreover, if $r > h$, then $\mathscr{O}^r = 0$, so the Deligne–Simpson problem always has a positive solution.

1.5. **Rigidity.** Our results have applications to the question of when Coxeter connections are rigid. Let $U \subset \mathbb{P}^1$ be a nonempty open set, and let $j : U \hookrightarrow \mathbb{P}^1$ denote the inclusion. A G-connection ∇ on U is called *physically rigid* if it is uniquely determined by the formal isomorphism class at each point of $\mathbb{P}^1 \setminus U$. It is called *cohomologically rigid* if $H^1(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla) = 0$. For irreducible connections, cohomological rigidity implies that ∇ has no infinitesimal deformations. For $G = \mathrm{GL}_n(\mathbb{C})$, it is a result of Bloch and Esnault that cohomological and physical rigidity are the same [BE04].

In [KS21b], Kamgarpour and Sage investigated the question of rigidity for "homogeneous" Coxeter G-connections with unipotent monodromy. A homogeneous Coxeter G-formal type of slope r/h (i.e., for GL_n, a formal type of the form $a\omega_n^{-r}\frac{dz}{z}$ with $a \neq 0$) gives rise to a Coxeter G-connection on $\mathbb{G}_m$ with nilpotent residue at infinity. They determined precisely when these connections are (cohomologically) rigid, thus generalizing the work of Frenkel and Gross [FG09]. For GL_n, it turns out that such connections are rigid precisely when r divides $n+1$ or $n-1$.

We can now generalize the results of [KS21b] to give a classification of rigid Coxeter connections in type A.

Theorem 6.1. Let $\mathscr{A}$ be a rank n maximally ramified formal type of slope r/n, and let $\mathscr{O}$ be any nilpotent orbit with $\mathscr{O} \succeq \mathscr{O}^r_{x^n}$. Then there exists a rigid connection with the given formal type and and unipotent monodromy determined by $\mathscr{O}$ if and only if $\mathscr{O} = \mathscr{O}^r_{x^n}$ and $r|(n \pm 1)$.

We expect that the analogous statement is true for Coxeter G-connections.

1.6. **Organization of the paper.** In §2, we discuss some facts about the poset of adjoint orbits in $\mathfrak{gl}_n(\mathbb{C})$ that will be needed in our applications. In particular, we introduce and characterize a sequence of orbits which generalize regular orbits. In §3, we provide a brief review of lattice chain filtrations. In §4, we describe the role of these filtrations in studying formal connections, following earlier work of Bremer and Sage [BS13b, BS12, Sag17]. In particular, we discuss toral connections and characterize maximally ramified formal connections as Coxeter toral connections. In §5, we describe moduli spaces of connections with toral singularities and then state and prove our main result on the Deligne–Simpson problem for Coxeter connections. We conclude the paper in §6 by characterizing rigid Coxeter connections with unipotent monodromy at the regular singular point.

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2. The poset of adjoint orbits

The solution to the Deligne–Simpson problem for Coxeter connections involves certain distinguished orbits for the adjoint action of the general linear group $\mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{gl}_n(\mathbb{C})$ (i.e., similarity classes of $n \times n$ complex matrices). We will need some facts about these adjoint orbits.

The set π of adjoint orbits in $\mathfrak{gl}_n(\mathbb{C})$ is partially ordered via the closure order: $\mathscr{O} \preceq \mathscr{O}'$ if $\mathscr{O} \subseteq \overline{\mathscr{O}'}$. Let char be the map sending a matrix to its characteristic polynomial. Given a monic degree n polynomial q, $\mathrm{char}^{-1}(q)$ is closed and $\mathrm{GL}_n(\mathbb{C})$-stable. If we let π_q be the set of adjoint orbits in $\mathrm{char}^{-1}(q)$, it is immediate that as a poset,

$$\pi = \bigsqcup_{\substack{q \text{ monic} \\ \deg(q)=n}} \pi_q.$$

The theory of Jordan canonical forms allows us to identify the posets π_q with posets involving partitions. If n is a positive integer, a *partition of* n is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of positive integers that sum to n. Each integer appearing in this sequence is called a part of λ. The total number of parts is denoted by $|\lambda|$. It will sometimes be convenient to use exponential notation for partitions: if the b_i's are the distinct parts, each appearing with multiplicity k_i, we will also denote by λ the multiset $\{b_1^{k_1}, \dots, b_s^{k_s}\}$. Let $\mathrm{Part}(n)$ be the set of partitions of n. We view $\mathrm{Part}(n)$ as a poset via the dominance order

$$\lambda \succeq \mu \iff |\lambda| \le |\mu| \text{ and } \sum_{i=1}^{j} \lambda_i \ge \sum_{i=1}^{j} \mu_i \text{ for all } j \in [1, |\lambda|]. \tag{3}$$

Write $q = \prod_{i=1}^{s}(x - a_i)^{m_i}$ for distinct $a_1, \dots, a_s \in \mathbb{C}$ and $m_1, \dots, m_s \in \mathbb{Z}_{>0}$. The set π_q can be identified with $\prod_{i=1}^{s} \mathrm{Part}(m_i)$, where the partition of m_i is given by the sizes of the Jordan blocks with eigenvalue a_i. It is well-known that the closure order corresponds to the product of the dominance orders under this identification.

The unique maximal orbit in π_q is the orbit with a single Jordan block for each eigenvalue. This is the regular orbit with characteristic polynomial q, i.e., the unique orbit in π_q of codimension n. We now define a sequence $\{\mathscr{O}_q^r\}_r$ of orbits in π_q which generalize the regular orbit.

Fix $r \in \mathbb{Z}_{>0}$, and consider the subset $\mathcal{F}_q^r \subset \pi_q$ consisting of orbits with at most r Jordan blocks for each eigenvalue. It is immediate from (3) that $\mathcal{F}_q^r$ is a filter in the poset π_q. This means that if $\mathscr{O} \in \mathcal{F}_q^r$ and $\mathscr{O} \preceq \mathscr{O}'$, then $\mathscr{O}' \in \mathcal{F}_q^r$. It turns out that $\mathcal{F}_q^r$ is a principal filter; i.e., there exists a (unique) element $\mathscr{O}_q^r \in \pi_q$ such that $\mathcal{F}_q^r = \langle \mathscr{O}_q^r \rangle := \{\mathscr{O} \in \pi_q \mid \mathscr{O} \succeq \mathscr{O}_q^r\}$. We will also set $\mathcal{F}^r = \bigcup_q \mathcal{F}_q^r$; it is a filter in π. Note that $\mathcal{F}^1$ is the set of all regular (or equivalently, maximal) adjoint orbits.

Proposition 2.1. Given $n, r \in \mathbb{Z}_{>0}$, write $n = kr + r'$ with $k, r' \in \mathbb{Z}$ and $0 \le r' < r$. Then the partition

$$\lambda^{n,r} = \{(k+1)^{r'}, k^{r-r'}\}$$

is the (unique) smallest partition of n with at most r parts.

Proof. Let λ be any partition of n with at most r parts, say with biggest part u and smallest part v. We will show that there exists a strictly smaller partition with at most r parts unless $\lambda = \lambda^{n,r}$.

If $r \geq n$, then $\lambda^{n,r} = \{1^n\}$, the smallest element of $\mathrm{Part}(n)$, so the statement is trivial. We thus may assume that $r < n$. First, suppose that $|\lambda| < r$. It follows that $u \geq 2$, so one obtains a strictly smaller partition with $|\lambda| + 1 \leq r$ parts by replacing one u with $u - 1$ and adjoining a new part with value 1. We may thus assume without loss of generality that $|\lambda| = r$.

Next, suppose that $u - v \geq 2$. Define a partition μ with the same parts as λ except one u is replaced by $u - 1$ and one v is replaced by $v + 1$. It is obvious that $|\lambda| = |\mu|$ and that λ is strictly bigger than μ.

It remains to consider the case $u - v \leq 1$, so $\lambda = \{(v + 1)^s, v^{r-s}\}$ for some s with $0 \leq s < r$. We then have $n = s(v + 1) + (r - s)v = vr + s$, so $v = k$ and $s = r'$. Thus, $\lambda = \lambda^{n,r}$. □

Now, define $\mathscr{O}_q^r$ to be the orbit in π_q corresponding to the element

$$(\lambda^{m_1,r}, \lambda^{m_2,r}, \dots, \lambda^{m_s,r}) \in \prod_{i=1}^{s} \mathrm{Part}(m_i).$$

Corollary 2.2. The filter $\mathcal{F}_q^r$ is principal with generator $\mathscr{O}_q^r$, i.e., $\mathcal{F}_q^r = \langle \mathscr{O}_q^r \rangle$.

Proof. It suffices to show that the corresponding filter in $\prod_{i=1}^{s} \mathrm{Part}(m_i)$ is principal with generator $(\lambda^{m_1,r}, \lambda^{m_2,r}, \dots, \lambda^{m_s,r})$. This is immediate from the proposition. □

Recall that the semisimple orbit in π_q is the unique minimal orbit. If $r \geq m_i$ for all i (as is the case when $r \geq n$), then $\mathscr{O}_q^r$ is the semisimple orbit so $\langle \mathscr{O}_q^r \rangle = \pi_q$.

We can now give a Lie-theoretic interpretation of $\mathcal{F}^r$ which will be important in our applications. Let V^r be the set of matrices with nonzero entries on, and 0's below, the rth subdiagonal:

$$V^r = \{(x_{ij}) \in \mathfrak{gl}_n(\mathbb{C}) \mid x_{ij} = 0 \text{ if } i - j > r \text{ and } x_{ij} \neq 0 \text{ if } i - j = r\}.$$

Note that $V^r = \mathfrak{gl}_n(\mathbb{C})$ if $r \geq n$. It is well-known that every element of V^1 is regular and that every regular orbit has a representative in V^1. (It is a famous result of Kostant that the analogous statement holds for any complex simple group [Kos59].)

We now prove a generalization of this result.

Theorem 2.3. The adjoint orbits which intersect V^r are precisely the orbits in $\mathcal{F}^r$. The minimal such orbits are the $\mathscr{O}_q^r$'s.

We begin by proving that $\mathscr{O}_q^r$ intersects V^r. Let $N_{r,n}$ be the $n \times n$ matrix with 1's on the rth subdiagonal and 0's elsewhere. We usually omit n from the notation.

Proposition 2.4. Fix a positive integer $r < n$. Let $D_q = \mathrm{diag}(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_s, \dots, a_s)$, where the eigenvalue a_i appears with multiplicity m_i. Let U_r be any matrix with all entries on the rth subdiagonal nonzero and all other entries 0. Then $U_r + D_q \in \mathscr{O}_q^r$.

Proof. It is easy to see that there exists an invertible diagonal matrix t such that $\mathrm{Ad}(t)(U_r + D_q) = N_r + D_q$, so we may assume without loss of generality that $U_r = N_r$. We prove the proposition by induction on the number of distinct eigenvalues s (for arbitrary n).

If $s = 1$, the partition for the single eigenvalue is $\lambda^{n,r}$. Indeed, the Jordan strings for $N_r + D_q - a_1\mathrm{id} = N_r$ are given by $e_i \mapsto e_{i+r} \mapsto e_{i+2r} \mapsto \cdots$ for $1 \le i \le r$, with the strings of length $k+1$ for $1 \le i \le r'$ and k otherwise.

Now assume $s > 1$. Let V be the span of $e_{m_1+1}, \dots, e_n$. Note that $N_r + D_q$ stabilizes V; in fact, $(N_r + D_q)|_V = N_{r,n-m_1} + D_{\hat{q}}$, where $\hat{q} = \prod_{i=2}^{s}(x - a_i)^{m_i}$. It is clear that the Jordan strings for $(N_r + D_q)|_V$ are also Jordan strings for $N_r + D_q$ corresponding to the eigenvalues $a_2, \dots, a_s$. Also, $N_r + D_q$ induces the endomorphism $N_{r,m_1} + D_{(x-a_1)^{m_1}}$ on $\mathbb{C}^n/V$. It remains to show that each Jordan string $\bar{e}_i \mapsto \bar{e}_{i+r} \mapsto \cdots \mapsto \bar{e}_{i+(\ell-1)r} \mapsto 0$ for N_{r,m_1} on $\mathbb{C}^n/V$ lifts to a Jordan string for the zero eigenvalues of $N_r + D_q - a_1\mathrm{id}$. (Here, ℓ is the smallest integer such that $i + \ell r > m_1$.) Let $f = (N_r + D_q - a_1\mathrm{id})|_V$. This map is invertible, so $e_i - f^{-\ell}(e_{i+\ell r}) \mapsto \cdots \mapsto e_{i+(\ell-1)r} - f^{-1}(e_{i+\ell r}) \in \ker(N_r + D_q - a_1\mathrm{id})$ is the desired lift. □

Proof of Theorem 2.3. We first show that V^r consists of elements whose Jordan forms have at most r blocks for each eigenvalue. Let $X \in V^r$, and let a be any eigenvalue of X. The bottom $n - r$ rows of the matrix $X - a\mathrm{id}$ are linearly independent, so $\mathrm{rank}(X - a\mathrm{id}) \ge n - r$. We conclude that $\dim\ker(X - a\mathrm{id}) \le r$. Since this dimension is the number of Jordan blocks for the eigenvalue a, the claim follows.

We have shown that the set of adjoint orbits intersecting V^r is contained in $\mathcal{F}^r$. Now take $\mathscr{O} \in \mathcal{F}^r$. Let q be its characteristic polynomial, so $\mathscr{O} \succeq \mathscr{O}_q^r$. Take any $X \in \mathscr{O}_q^r$ as in Proposition 2.4. Then $X \in V^r$. By a theorem of Krupnik [Kru97, Theorem 1], there exists a strictly upper triangular matrix $Z \in \mathfrak{gl}_n(\mathbb{C})$ such that $X + Z \in \mathscr{O}$. Since $X + Z \in V^r$, this proves the theorem. □

Remark 2.5. Krupnik's approach does not lead to explicit constructions of orbit representatives in V^r. However, at least for nilpotent orbits, it is possible to find explicit, simple representatives by means of an algorithm described in [KLM+21].

3. Lattice chain filtrations

Let $F = \mathbb{C}((z))$ denote the field of formal Laurent series, and let $\mathfrak{o} = \mathbb{C}[[z]]$ denote the ring of formal power series. An $\mathfrak{o}$-*lattice* L in F^n is a finitely generated $\mathfrak{o}$-module with the property that $L \otimes_{\mathfrak{o}} F \cong F^n$. A *lattice chain* in F^n is a collection $\{L^i\}_{i \in \mathbb{Z}}$ of lattices satisfying the following properties:

(1) $L^i \supsetneq L^{i+1}$ for all i; and
(2) there exists a positive integer e, called the *period*, such that $L^{i+e} = zL^i$ for all i.

A *parahoric subgroup* $P \subset \mathrm{GL}_n(F)$ is the stabilizer of a lattice chain; i.e., $P = \{g \in \mathrm{GL}_n(F) \mid gL^i = L^i \text{ for all } i\}$. We write e_P to denote the period of the lattice chain stabilized by P. A lattice chain in F^n is called *complete* if its period is as large as possible, i.e., if its period equals n. The parahoric subgroups associated to complete lattice chains are called *Iwahori subgroups*.

Each lattice chain $\{L^i\}_{i\in\mathbb{Z}}$ — say with corresponding parahoric P — determines a filtration $\{\mathfrak{p}^i\}_{i\in\mathbb{Z}}$ of $\mathfrak{gl}_n(F)$ defined by $\mathfrak{p}^i = \{X \in \mathfrak{gl}_n(F) \mid XL^j \subset L^{j+i} \text{ for all } j\}$; in particular, $\mathfrak{p}^0 = \mathfrak{p} := \mathrm{Lie}(P)$. One also gets a filtration $\{P^i\}_{i\in\mathbb{Z}_{\geq 0}}$ of P defined by $P^0 = P$ and $P^i = 1 + \mathfrak{p}^i$ for $i > 0$. In the special case that each lattice in the lattice chain stabilized by P admits an $\mathfrak{o}$-basis of the form $\{z^{k_j}e_j\}_{j=1}^n$, the corresponding filtration $\{\mathfrak{p}^i\}_i$ is induced by a grading $\{\mathfrak{p}(i)\}_i$ on $\mathfrak{gl}_n(\mathbb{C}[z, z^{-1}])$.

Remark 3.1. The "lattice chain filtrations" described above may also be obtained through a more general construction. Let $\mathcal{B}$ be the (reduced) Bruhat–Tits building associated to the loop group $\mathrm{GL}_n(F)$. This is a simplicial complex whose simplices are in bijective correspondence with the parahoric subgroups of the loop group. For each point in the building, there is an associated "Moy–Prasad filtration" on the loop algebra [MP94]. Up to rescaling, the lattice chain filtration determined by a parahoric subgroup P is the Moy–Prasad filtration associated to the barycenter of the simplex in the Bruhat–Tits building corresponding to P [BS18].

The building $\mathcal{B}$ is the union of $(n-1)$-dimensional real affine spaces called apartments, which are in one-to-one correspondence with split maximal tori in $\mathrm{GL}_n(F)$. The parahoric subgroups described in the "special case" above — i.e., the parahoric subgroups P where each lattice in the lattice chain stabilized by P admits an $\mathfrak{o}$-basis of the form $\{z^{k_j}e_j\}_{j=1}^n$ — are precisely the parahoric subgroups corresponding to the simplices in the "standard apartment", i.e., the apartment corresponding to the diagonal torus.

We will focus our attention on two particular parahoric subgroups of $\mathrm{GL}_n(F)$: $\mathrm{GL}_n(\mathfrak{o})$ and the "standard Iwahori subgroup" I (defined below). The parahoric subgroup $\mathrm{GL}_n(\mathfrak{o})$ is the stabilizer of the 1-periodic lattice chain $\{z^i\mathfrak{o}^n\}_i$. The associated filtration on $\mathfrak{gl}_n(F)$ is the *degree filtration* $\{z^i\,\mathfrak{gl}_n(\mathfrak{o})\}_{i\in\mathbb{Z}}$.

The *standard Iwahori subgroup* I is the stabilizer of the *standard complete lattice chain* in F^n, i.e., the lattice chain

$$\{\mathfrak{o}\text{-span}\{z^{\lfloor \frac{i}{n} \rfloor}e_1, z^{\lfloor \frac{i+1}{n} \rfloor}e_2, \ldots, z^{\lfloor \frac{i+(n-1)}{n} \rfloor}e_n\}\}_{i\in\mathbb{Z}},$$

where $\{e_j\}_{j=1}^n$ is the standard basis for F^n. The associated filtration is the *standard Iwahori filtration* $\{\mathfrak{i}^i\}_{i\in\mathbb{Z}}$. Note that if $B \subset \mathrm{GL}_n(\mathbb{C})$ is the Borel subgroup of invertible upper triangular matrices, then I is the preimage of B under the homomorphism $\mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathbb{C})$ induced by the "evaluation at zero" map $z \mapsto 0$.

Define $\omega \in \mathfrak{gl}_n(F)$ by $\omega = \sum_{i=1}^{n-1} e_{i,i+1} + ze_{n,1}$; i.e., ω is the matrix with 1's in each entry of the superdiagonal, z in the lower-left entry, and 0's elsewhere. Then $\omega^i \mathfrak{i}^j = \mathfrak{i}^j \omega^i = \mathfrak{i}^{i+j}$ [Bus87, Proposition 1.18] for all $i, j \in \mathbb{Z}$. Note that $\mathfrak{i}^{i+n} = z\mathfrak{i}^i$ for all i. Let T denote the standard diagonal maximal torus in $\mathrm{GL}_n(\mathbb{C})$. Then $\mathfrak{t} = \mathrm{Lie}(T)$ consists of all diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$. The standard Iwahori grading $\{\mathfrak{i}(i)\}_{i\in\mathbb{Z}}$ is then given by

$$\mathfrak{i}(i) = \omega^i \mathfrak{t} = \mathfrak{t}\omega^i. \tag{4}$$

Example 3.2. Let $n = 3$. Then

$$\omega = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad I = \begin{bmatrix} \mathfrak{o}^* & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o}^* & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o}^* \end{bmatrix}.$$

Some steps in the standard Iwahori filtration on $\mathfrak{gl}_n(F)$ are shown below:

$$\mathfrak{i}^{-2} = \begin{bmatrix} \mathfrak{o} & z^{-1}\mathfrak{o} & z^{-1}\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & z^{-1}\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \quad \supsetneq \quad \mathfrak{i}^{-1} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & z^{-1}\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \quad \supsetneq \quad \mathfrak{i}^{0} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} \end{bmatrix}.$$

If $\mathfrak{h} \subset \mathfrak{gl}_n(F)$ is any $\mathbb{C}$-subspace containing some $\mathfrak{p}^i$, we denote the space of continuous linear functionals on $\mathfrak{h}$ by $\mathfrak{h}^\vee$. Every continuous functional on $\mathfrak{h}$ extends to a continuous functional on $\mathfrak{gl}_n(F)$, so $\mathfrak{h}^\vee \cong \mathfrak{gl}_n(F)^\vee/\mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ denotes the annihilator of $\mathfrak{h}$. The space of 1-forms $\Omega^1(\mathfrak{gl}_n(F))$ can be identified with the space of functionals $\mathfrak{gl}_n(F)^\vee$ by associating a 1-form ν with the functional $Y \mapsto \mathrm{Res}(\mathrm{Tr}(Y\nu))$. This identification is well-behaved with respect to lattice chain filtrations [BS18, Proposition 3.6]:

$$(\mathfrak{p}^i)^\vee \cong \mathfrak{gl}_n(F)\tfrac{dz}{z}/\mathfrak{p}^{-i+1}\tfrac{dz}{z}.$$

When the filtration comes from a grading, one can be even more explicit. In particular, we get

$$\mathfrak{gl}_n(\mathfrak{o})^\vee \cong \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\tfrac{dz}{z} \qquad \text{and} \qquad \mathfrak{i}^\vee \cong \mathfrak{t}[\omega^{-1}]\tfrac{dz}{z}. \tag{5}$$

For applications to connections, it is important to consider the relationship between filtrations on Cartan subalgebras of $\mathfrak{gl}_n(F)$ and filtrations on parahoric subalgebras [BS13b, BS13a].

Note that, since F is not algebraically closed, it is not true that all maximal tori (or equivalently, all Cartan subalgebras) are conjugate. In fact, there is a bijection between the set of conjugacy classes of maximal tori in $\mathrm{GL}_n(F)$ and the set of conjugacy classes in the Weyl group for $\mathrm{GL}_n(\mathbb{C})$ (i.e., the symmetric group $\mathfrak{S}_n$) [KL88, Lemma 2]. We also remark that each Cartan subalgebra $\mathfrak{s}$ comes equipped with a natural filtration $\{\mathfrak{s}^i\}_{i\in\mathbb{Z}}$ (see, e.g., [BS13a, Section 3]). In the cases of interest to us in this paper, the filtration will be induced by a grading $\{\mathfrak{s}(i)\}_{i\in\mathbb{Z}}$.

Let S be a maximal torus and let P be a parahoric subgroup, both in $\mathrm{GL}_n(F)$. Let $\mathfrak{s} = \mathrm{Lie}(S)$ be the Cartan subalgebra associated to S. We say that S and P (or $\mathfrak{s}$ and $\mathfrak{p}$) are *compatible* (resp. *graded* *compatible*) if $\mathfrak{s}^i = \mathfrak{p}^i \cap \mathfrak{s}$ (resp. $\mathfrak{s}(i) = \mathfrak{p}(i) \cap \mathfrak{s}$) for all i. For present purposes, it suffices to consider two examples: the diagonal subalgebra $\mathfrak{t}(F)$ and the "standard Coxeter Cartan subalgebra" $\mathfrak{c} = \mathbb{C}(\!(\omega)\!)$. The diagonal Cartan subalgebra $\mathfrak{t}(F)$ (corresponding to the trivial class in the Weyl group $\mathfrak{S}_n$) is endowed with a filtration which comes from the obvious grading $\mathfrak{t}(\mathbb{C}[z, z^{-1}]) = \bigoplus_i z^i\mathfrak{t}$. It is immediate that $T(F)$ is graded compatible with $\mathrm{GL}_n(\mathfrak{o})$.

At the opposite extreme, there is a unique class of maximal tori in $\mathrm{GL}_n(F)$ that are anisotropic modulo the center, meaning that they have no non-central rational cocharacters. Concretely, such tori are as far from being split as possible. This class corresponds to the Coxeter class in $\mathfrak{S}_n$, i.e., the class of n-cycles. A specific representative of this class is the standard Coxeter torus $\mathcal{C} = \mathbb{C}(\!(\omega)\!)^*$ with Lie algebra $\mathfrak{c}$. Note that ω is regular semisimple — its eigenvalues are the n distinct nth roots of z — so its centralizer $\mathbb{C}(\!(\omega)\!)$ is indeed a Cartan subalgebra. The natural grading by powers of ω on $\mathbb{C}[\omega, \omega^{-1}]$ induces a filtration on $\mathfrak{c}$, and it is clear that $\mathcal{C}$ is graded compatible with I.

4. Toral and maximally ramified connections

4.1. **Formal connections.** A *formal connection of rank n* is a connection $\widehat{\nabla}$ on an F-vector bundle V of rank n over the formal punctured disk $\mathrm{Spec}(F)$. Given a trivialization ϕ for V (which is always trivializable), the connection can be written in matrix form as $\widehat{\nabla} = d + [\widehat{\nabla}]_\phi$, where $[\widehat{\nabla}]_\phi \in \Omega^1_F(\mathfrak{gl}_n(F))$. The loop group $\mathrm{GL}_n(F)$ acts simply transitively on the set of trivializations via left multiplication. The corresponding action of $\mathrm{GL}_n(F)$ on the connection matrix is given by the *gauge action*: if $g \in \mathrm{GL}_n(F)$, then $g \cdot [\widehat{\nabla}]_\phi = [\widehat{\nabla}]_{g\cdot\phi} = \mathrm{Ad}(g)([\widehat{\nabla}]_\phi) - (dg)g^{-1}$. Hence, the set of isomorphism classes of rank n formal connections is isomorphic to the orbit space $\mathfrak{gl}_n(F)\frac{dz}{z} / \mathrm{GL}_n(F)$ for the gauge action.

A formal connection $\widehat{\nabla}$ is called *regular singular* if the connection matrix with respect to some trivialization has a simple pole. If the matrix has a higher order pole for every trivialization, $\widehat{\nabla}$ is said to be *irregular singular*. Katz defined an invariant of formal connections called the *slope* which gives one measure of the degree of irregularity of a formal connection [Del70]. The slope is a nonnegative rational number whose denominator in lowest form is at most n. The slope is positive if and only if $\widehat{\nabla}$ is irregular.

4.2. **Fundamental strata.** The classical approach to the study of formal connections involves an analysis of the "leading term" of the connection matrix with respect to the degree filtration on $\mathfrak{gl}_n(F)$ [Was76]. To review, suppose that the matrix for $\widehat{\nabla}$ with respect to ϕ is expanded with respect to the degree filtration on $\mathfrak{gl}_n(F)$; i.e., suppose

$$[\widehat{\nabla}]_\phi = (M_{-r}z^{-r} + M_{-r+1}z^{-r+1} + \cdots)\frac{dz}{z}, \tag{6}$$

where $r \geq 0$ and $M_i \in \mathfrak{gl}_n(\mathbb{C})$ for all i. When the *leading term* M_{-r} is well-behaved, it gives useful information about the connection. For example, if M_{-r} is non-nilpotent, then $\operatorname{slope}(\widehat{\nabla}) = r$. Moreover, if $r > 0$ and M_{-r} is diagonalizable with distinct eigenvalues, then $\widehat{\nabla}$ can be diagonalized into a "$T(F)$-formal type of depth r". This means that there exists $g \in \mathrm{GL}(F)$ such that $[\widehat{\nabla}]_{g\phi}$ is an element of

$$\mathcal{A}(T(F), r) := \{(D_{-r}z^{-r} + \cdots + D_1 z + D_0)\tfrac{dz}{z} \mid \forall i, D_i \in \mathfrak{t} \text{ and } D_{-r} \text{ has distinct eigenvalues}\}$$

[Was76]. Note that many interesting connections have nilpotent leading terms. For example, the leading term of the formal Frenkel–Gross connection $\widehat{\nabla}_{\mathrm{FG}} = d + \omega^{-1}\frac{dz}{z}$ [FG09] is strictly upper triangular (and thus nilpotent). In fact, the leading term is nilpotent no matter what trivialization one chooses for V.

More recently, Bremer and Sage — borrowing well-known tools from representation theory developed by Bushnell [Bus87], Moy–Prasad [MP94], and others — have introduced a more general approach to the study of formal connections, where leading terms are replaced by "strata" [BS13b, BS13a, BS18]. A GL_n*-stratum* is a triple (P, r, β) with $P \subset \mathrm{GL}_n(F)$ a parahoric subgroup, r a nonnegative integer, and β a functional on $\mathfrak{p}^r/\mathfrak{p}^{r+1}$. Consider the special case where P corresponds to a simplex in the standard apartment (see Remark 3.1). Here, a functional $\beta \in (\mathfrak{p}^r/\mathfrak{p}^{r+1})^\vee$ can be written uniquely as $\beta^\flat \frac{dz}{z}$ for $\beta^\flat$ homogeneous (i.e., for $\beta^\flat \in \mathfrak{p}(-r)$). The stratum is called *fundamental* if $\beta^\flat$ is non-nilpotent. A formal connection $\widehat{\nabla}$ *contains the stratum* (P, r, β) (with respect to a fixed trivialization) if $\widehat{\nabla} = d + X\frac{dz}{z}$ with $X \in \mathfrak{p}^{-r}$ and β induced by $X\frac{dz}{z}$. More general definitions of fundamental strata and stratum containment are given in [BS13b, BS18].

Fundamental strata can be viewed as a generalization of the notion of a non-nilpotent leading term. In particular, fundamental strata can be used to compute the slope of any connection, not merely those with integer slopes. Recall that if $P \subset \mathrm{GL}_n(F)$ is a parahoric subgroup, then e_P denotes the period of the lattice chain stabilized by P.

Theorem 4.1 ([BS13b, Theorem 4.10], [Sag17, Theorem 1])**.** Any formal connection $\widehat{\nabla}$ contains a fundamental stratum. If $\widehat{\nabla}$ contains the fundamental stratum (P, r, β), then $\operatorname{slope}(\widehat{\nabla}) = r/e_P$.

We now investigate some examples. The connection in (6) (with $M_i \in \mathfrak{gl}_n(\mathbb{C})$ for all i) contains the stratum $(\mathrm{GL}_n(\mathfrak{o}), r, M_{-r}z^{-r}\frac{dz}{z})$, which is fundamental if and only if M_{-r} is non-nilpotent. The formal Frenkel–Gross connection $\widehat{\nabla}_{\mathrm{FG}}$ contains the fundamental stratum $(I, 1, \omega^{-1}\frac{dz}{z})$. Moreover, any rank n formal connection of the form $\widehat{\nabla} = d + (a\omega^{-r} + X)\frac{dz}{z}$, with $\gcd(r, n) = 1$, $a \in \mathbb{C}^*$, and $X \in \mathfrak{i}^{-r+1}$, contains the fundamental stratum $(I, r, a\omega^{-r}\frac{dz}{z})$, and thus has slope r/n.

4.3. **Toral connections.** The notion of a diagonalizable leading term with distinct eigenvalues is generalized by the notion of a "regular stratum". For simplicity, we only consider the case where P comes from the standard apartment (see Remark 3.1). General definitions can be found in [BS13b] and [BS13a]. For such a P, consider the stratum $(P, r, \beta^\flat \frac{dz}{z})$. If $\beta^\flat$ is regular semisimple, then its centralizer $C(\beta^\flat)$ is regular semisimple, and we say that $(P, r, \beta^\flat \frac{dz}{z})$ is a $C(\beta^\flat)$-regular stratum. A connection that contains an S-regular stratum is called an *S-toral connection*.

It turns out that toral connections do not exist for every maximal torus S. In fact, an S-toral connection of slope s exists if and only if S corresponds to a regular conjugacy class in $\mathfrak{S}_n$ (in the sense of Springer [Spr74]) and if $e^{2\pi i s}$ is a regular eigenvalue of this conjugacy class [BS13a]. The regular classes are parametrized by the partitions $\{b^{n/b}\}$ (for positive divisors b of n) and $\{b^{(n-1)/b}, 1\}$ (for positive divisors b of $n-1$). Representatives for each of the corresponding conjugacy classes of maximal tori are given by the S^b's defined in §1.3. An S^b-toral connection has slope r/b for some $r > 0$ with $\gcd(r, b) = 1$. Note that S^1 is the diagonal torus $T(F)$ while S^n is the standard Coxeter torus $\mathcal{C}$.

Just as for connections whose naive leading term is regular semisimple, there exist "rational canonical forms" for toral connections involving the notion of a formal type. Fix a divisor b of either n or $n-1$. We define the set of *S^b-formal types of depth r* (with $\gcd(r, b) = 1$) by

$$\mathcal{A}(S^b, r) = \{A\tfrac{dz}{z} \mid A = \sum_{i=0}^{r} A_{-i} \in \bigoplus_{i=0}^{r} \mathfrak{s}^b(-i) \text{ with } A_{-r} \text{ regular semisimple}\}.$$

Every toral connection $\widehat{\nabla}$ of slope r/b is formally isomorphic to a connection of the form $d + A\frac{dz}{z}$ with $A\frac{dz}{z} \in \mathcal{A}(S^b, r)$; we view this as a rational canonical form for $\widehat{\nabla}$.

We will need a more precise variation of this statement. As mentioned in §1.3, for each b, there is a standard parahoric subgroup P^b which is compatible (in fact, graded compatible) with S^b.

Theorem 4.2 ([BS13b, Theorem 4.13], [BS13a, Theorem 5.1])**.** Suppose $\widehat{\nabla}$ is a formal connection containing an S^b-regular stratum $(P^b, r, \beta^\flat \frac{dz}{z})$ with $\beta^\flat \in \mathfrak{s}^b(-r)$. Then there exists $p \in P^{b,1}$ such that $p \cdot [\widehat{\nabla}]$ is a formal type in $\mathcal{A}(S^b, r)$ whose component in $\mathfrak{s}^b(-r)\frac{dz}{z}$ is $\beta^\flat \frac{dz}{z}$.

4.4. **Maximally ramified connections.**

Definition 4.3. A formal connection $\widehat{\nabla}$ of rank n is called *maximally ramified* if it has slope r/n with $\gcd(r, n) = 1$.

Thus, a formal connection is maximally ramified if the denominator of the slope (in lowest terms) is as big as possible. Another interpretation involves the slope decomposition of $\widehat{\nabla}$. It is a well-known result of Turrittin [Tur55] and Levelt [Lev75] that after extending scalars to $\mathbb{C}((z^{1/b}))$ for some $b \in \mathbb{Z}_{>0}$, there exists a trivialization in which the matrix of $\widehat{\nabla}$ is block-diagonal:

$$\widehat{\nabla} = d + \operatorname{diag}(p_1(z^{-1/b})\mathrm{id}_{m_1} + R_1, \dots, p_k(z^{-1/b})\mathrm{id}_{m_k} + R_k)\tfrac{dz}{z};$$

here, the p_i's are polynomials and the R_i's are nilpotent matrices. This is the Levelt–Turrittin normal form of $\widehat{\nabla}$. The *slopes* of $\widehat{\nabla}$ are the n rational numbers $\deg(p_i)/b$, each appearing with multiplicity m_i. This collection of invariants gives more detailed information about how irregular $\widehat{\nabla}$ is than the single invariant $\mathrm{slope}(\widehat{\nabla})$. Indeed, one can define $\mathrm{slope}(\widehat{\nabla})$ to be the maximum of the slopes of $\widehat{\nabla}$.

One can show that the slopes are nonnegative rational numbers with denominators at most n. Moreover, if at least one slope is r/n with $\gcd(r,n) = 1$, then all slopes are r/n. Thus, $\widehat{\nabla}$ is maximally ramified if at least one slope has denominator n.

It turns out that maximally ramified connections are the same thing as Coxeter toral connections. If we specialize our results on formal types to the standard Coxeter torus $\mathcal{C}$, we see that the $\mathcal{C}$-formal types of depth r are given by

$$\mathcal{A}(\mathcal{C}, r) = \{p(\omega^{-1})\tfrac{dz}{z} \mid p \in \mathbb{C}[x], \deg(p) = r\}.$$

We thus obtain the following result on rational canonical forms for maximally ramified connections:

Theorem 4.4 ([KS21a])**.** Let $\widehat{\nabla}$ be a maximally ramified connection of slope r/n with $\gcd(r,n) = 1$. Then $\widehat{\nabla}$ is formally gauge equivalent to a connection of the form $d + p(\omega^{-1})\frac{dz}{z}$ with p a polynomial of degree r.

This theorem may be obtained as a direct corollary of Sabbah's refined Levelt–Turrittin decomposition [Sab08, Corollary 3.3]. Indeed, since $\widehat{\nabla}$ has slope r/n, Sabbah's theorem shows that it is formally isomorphic to a connection of the form $[n]_*(d + d\phi + \lambda)$, where $[n] : \mathrm{Spec}(F) \to \mathrm{Spec}(F)$ is the n-fold covering induced by $z \mapsto z^n$, $\phi \in z^{-1}\mathbb{C}[z^{-1}]$ has degree r, and $\lambda \in \mathbb{C}$. It is now easy to conclude that $\widehat{\nabla}$ has the desired rational canonical form, with the coefficients of p determined by λ and the coefficients of ϕ.

However, the theory of toral connections allows one to prove a generalized version of this theorem for G-connections, where G is a reductive group with connected Dynkin diagram [KS21a]. Here, n is replaced by the Coxeter number h, $\mathcal{C}$ is an appropriate fixed Coxeter torus in $G(F)$, and $\widehat{\nabla}$ is formally isomorphic to $d + \mathscr{A}$, where $\mathscr{A}$ is a $\mathcal{C}$-formal type of slope r/h. Below, we provide a concise stratum-theoretic proof for the specific case $G = \mathrm{GL}_n$, which is simpler than the general proof.

Proof. By Theorem 4.1, $\widehat{\nabla}$ contains a fundamental stratum (P, r, β') with respect to some trivialization. Since $\mathrm{slope}(\widehat{\nabla}) = r/e_P$, it follows that $e_P = n$ and P is an Iwahori subgroup. By equivariance of stratum containment and the fact that Iwahori subgroups are all conjugate, we may modify the trivialization so that $\widehat{\nabla}$ contains the fundamental stratum (I, r, β). The functional β is represented by $\beta^\flat \frac{dz}{z}$, where $\beta^\flat \in \mathfrak{i}(-r)$ is non-nilpotent.

By (4), $\beta^\flat = \mathrm{diag}(a_1, \dots, a_n)\omega^{-r}$ for some constants a_i. Let $a = \prod a_i$. Since $\mathrm{char}(\beta^\flat) = x^n - az^{-r}$, it follows that $a \neq 0$. The polynomial thus has distinct roots, and $\beta^\flat$ is regular semisimple. Let

$a^{1/n}$ be a fixed nth root of a. It is easy to see that there exists $t \in T$ such that $\mathrm{Ad}(t)(\beta^\flat) = a^{1/n}\omega^{-r}$, and since t normalizes I, $\widehat{\nabla}$ contains the stratum $(I, r, a^{1/n}\omega^{-r}\frac{dz}{z})$. By Theorem 4.2, $\widehat{\nabla}$ is formally isomorphic to $d + (b_r\omega^{-r} + \cdots + b_0)\frac{dz}{z}$, where $b_r = a^{1/n}$. □

5. The Deligne–Simpson problem for Coxeter connections

5.1. **Moduli spaces of connections with toral singularities.** We now turn our attention to meromorphic connections ∇ on a rank n trivializable vector bundle V over the complex Riemann sphere $\mathbb{P}^1$. To discuss the Deligne–Simpson problem, we need to define what it means for ∇ to be framable at a singularity with respect to a given toral formal type. We will assume that the singular point is 0. The only modification needed if the singularity is at an arbitrary point $a \in \mathbb{P}^1$ is to replace the uniformizer z by $z - a$ if a is finite and by z^{-1} if $a = \infty$.

Fix a trivialization ϕ of V, and write $\nabla = d + [\nabla]_\phi$. The principal part of $[\nabla]_\phi$ is an element of $\mathfrak{gl}_n(\mathbb{C}[z^{-1}])\frac{dz}{z}$, and so may be viewed as a continuous functional on $\mathfrak{gl}_n(\mathfrak{o})$ by (5). Similarly, the restriction of $[\nabla]_\phi$ to $\mathfrak{p}^b$ is uniquely determined by the truncation of $[\nabla]_\phi/\frac{dz}{z}$ to $\bigoplus_{i=0}^\infty \mathfrak{p}^b(-i)$. Thus, if $\mathscr{A} = A\frac{dz}{z}$ is an S^b-formal type, then $\mathscr{A}$ may naturally be viewed as an element of $(\mathfrak{p}^b)^\vee$.

Definition 5.1. Let $\nabla = d + [\nabla]_\phi$ be a global connection on $\mathbb{P}^1$ with a singular point at 0, and let $\mathscr{A} = A\frac{dz}{z}$ be an S^b-formal type of depth r. We say that ∇ is *framable at* 0 *with respect to* $\mathscr{A}$ if

1. there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $[\nabla]_{g\phi} = g \cdot [\nabla]_\phi \in (\mathfrak{p}^b)^{-r}$ and $[\nabla] - A \in (\mathfrak{p}^b)^{1-r}$, and
2. there exists an element $p \in (P^b)^1$ such that $\mathrm{Ad}^*(p)([\nabla]_{g\phi}\frac{dz}{z})|_{\mathfrak{p}^b} = \mathscr{A}$.

It is a consequence of Theorem 4.2 that this definition is equivalent to Definition 1.1.

Fix two disjoint subsets $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_\ell\}$ of $\mathbb{P}^1$ with $m \geq 1$ and $\ell \geq 0$. Let $\mathbf{A} = (\mathscr{A}_1, \dots, \mathscr{A}_m)$ be a collection of toral formal types at the a_i's, and let $\mathbf{O} = (\mathscr{O}_1, \dots, \mathscr{O}_\ell)$ be a collection of adjoint orbits at the b_j's. Assume that all of the orbits $\mathscr{O}_j$ are *nonresonant*, meaning that no two eigenvalues of an orbit differ by a nonzero integer. One can now consider the category $\mathscr{C}(\mathbf{A}, \mathbf{O})$ of meromorphic connections ∇ satisfying the following properties:

1. ∇ has irregular singularities at the a_i's, regular singularities at the b_j's, and no other singular points;
2. for each i, ∇ is framable at a_i with respect to the formal type $\mathscr{A}_i$; and
3. for each j, ∇ has residue at b_j in $\mathscr{O}_j$.

In [BS13b], Bremer and Sage constructed the moduli space $\mathcal{M}(\mathbf{A}, \mathbf{O})$ of this category as a Hamiltonian reduction of a product over the singular points of certain symplectic manifolds, each of which is endowed with a Hamiltonian action of $\mathrm{GL}_n(\mathbb{C})$. At a regular singular point with adjoint orbit $\mathscr{O}$, the manifold is just $\mathscr{O}$, viewed as the coadjoint orbit $\mathscr{O}\frac{dz}{z}$. To define the symplectic manifold $\mathcal{M}_{\mathscr{A}}$ associated to an S^b-toral formal type $\mathscr{A}$, we first remark that the parahoric subgroup P^b is the pullback of a certain standard parabolic subgroup $Q^b \subset \mathrm{GL}_n(\mathbb{C})$ under the map

$\mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathbb{C})$ induced by $z \mapsto 0$. For example, $P^1 = \mathrm{GL}_n(\mathfrak{o})$ is the pullback of $Q^1 = \mathrm{GL}_n(\mathbb{C})$, and $P^n = I$ is the pullback of $Q^n = B$. The "extended orbit" $\mathcal{M}_{\mathscr{A}} \subset (P^b \backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{gl}_n(\mathfrak{o})^\vee$ is defined by

$$\mathcal{M}_{\mathscr{A}} = \{(Q^b g, \alpha) \mid (\mathrm{Ad}^*(g)\alpha)|_{\mathfrak{p}^b} \in \mathrm{Ad}^*(P^b)(\mathscr{A})\}.$$

The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathcal{M}_{\mathscr{A}}$ via $h \cdot (Q^b g, \alpha) = (Q^b g h^{-1}, \mathrm{Ad}^*(h)\alpha)$, with moment map $(Q^b g, \alpha) \mapsto \alpha|_{\mathfrak{gl}_n(\mathbb{C})}$.

Theorem 5.2 ([BS13b, Theorem 5.26])**.** The moduli space $\mathcal{M}(\mathbf{A}, \mathbf{O})$ is given by

$$\mathcal{M}(\mathbf{A}, \mathbf{O}) \cong \left[\left(\prod_i \mathcal{M}_{\mathscr{A}_i} \right) \times \left(\prod_j \mathscr{O}_j \right) \right] /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$

Let $\mathcal{M}_{\mathrm{irr}}(\mathbf{A}, \mathbf{O})$ be the subset of $\mathcal{M}(\mathbf{A}, \mathbf{O})$ consisting of irreducible connections. One can now restate the toral Deligne–Simpson problem as

> *Given the toral formal types* $\mathbf{A}$ *and the nonresonant adjoint orbits* $\mathbf{O}$*, determine whether* $\mathcal{M}_{\mathrm{irr}}(\mathbf{A}, \mathbf{O})$ *is nonempty.*

Note that a $\mathcal{C}$-toral connection is irreducible, so if any $\mathscr{A}_i$ is $\mathcal{C}$-toral, then the Deligne–Simpson problem reduces to the question of whether $\mathcal{M}(\mathbf{A}, \mathbf{O})$ is nonempty.

5.2. **Coxeter connections.** We now specialize to an important special case: connections on $\mathbb{G}_m$ with a maximally ramified singular point at 0 and (possibly) a regular singularity at ∞. Since maximally ramified formal connections are Coxeter toral, we will follow [KS21b] and refer to such connections as *Coxeter connections*.

It is possible to give a simpler expression for moduli spaces of Coxeter connections.

Proposition 5.3. Let $\mathscr{A}$ be a $\mathcal{C}$-formal type, and let $\mathscr{O}$ be a nonresonant adjoint orbit. Then

$$\mathcal{M}(\mathscr{A}, \mathscr{O}) \cong \{(\alpha, Y) \mid \alpha \in \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\tfrac{dz}{z}, Y \in \mathscr{O}, \alpha|_{\mathfrak{i}} \in \mathrm{Ad}^*(I)(\mathscr{A}), \text{ and } \mathrm{Res}(\alpha) + Y = 0\}/B.$$

Proof. Applying Theorem 5.2, we have

$$\begin{aligned}
\mathcal{M}(\mathscr{A}, \mathscr{O}) &\cong (\mathcal{M}_{\mathscr{A}} \times \mathscr{O}) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}) \\
&\cong \{(Bg, \alpha, Y) \mid (Bg, \alpha) \in \mathcal{M}_{\mathscr{A}}, Y \in \mathscr{O}, \text{ and } \mathrm{Res}(\alpha) + Y = 0\}/\, \mathrm{GL}_n(\mathbb{C}) \\
&\cong \{(B, \alpha, Y) \mid \alpha \in \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\tfrac{dz}{z}, Y \in \mathscr{O}, \alpha|_{\mathfrak{i}} \in \mathrm{Ad}^*(I)(\mathscr{A}), \text{ and } \mathrm{Res}(\alpha) + Y = 0\}/B.
\end{aligned}$$

□

We can now state the solution to the Deligne–Simpson problem for Coxeter connections. For a given $\mathcal{C}$-formal type and a monic polynomial q of degree n, let

$$\mathrm{DS}(\mathscr{A}, q) = \{\mathscr{O} \in \pi_q \mid \mathcal{M}(\mathscr{A}, \mathscr{O}) \neq \varnothing\}.$$

Theorem 5.4. Let r and n be positive integers with $\gcd(r,n)=1$, let $\mathscr{A}$ be a maximally ramified formal type of slope r/n, and let $q=\prod_{i=1}^{s}(x-a_i)^{m_i}\in\mathbb{C}[x]$ with $a_1,\dots,a_s\in\mathbb{C}$ distinct modulo $\mathbb{Z}$. Then

$$\mathrm{DS}(\mathscr{A},q)=\begin{cases}\langle\mathscr{O}_q^r\rangle & \text{if } \mathrm{Res}(\mathrm{Tr}(\mathscr{A}))=-\sum_{i=1}^{s}m_ia_i,\\ \varnothing & \text{else.}\end{cases}$$

We prove this theorem in the next subsection.

Remark 5.5. If we write $\mathscr{A}=p(\omega^{-1})\frac{dz}{z}$, then $\mathcal{M}(\mathscr{A},\mathscr{O})$ is nonempty if and only if $np(0)=-\mathrm{Tr}(\mathscr{O})$ and if $\mathscr{O}$ has at most r Jordan blocks for each eigenvalue. This second condition is always satisfied if $r>n$. Note that the solution only depends on the slope and the residue of the formal type.

This theorem immediately gives the corresponding result for SL_n-connections. (In terms of a global trivialization, one may view an SL_n-connection on $\mathbb{P}^1$ as an operator $d+X\frac{dz}{z}$ with $X\in\mathfrak{sl}_n(\mathbb{C}[z,z^{-1}])$.) In this case, maximally ramified formal types are of the form $p(\omega^{-1})\frac{dz}{z}$ with $p(0)=0$ and $\mathrm{Tr}(\mathscr{O})=0$, so the trace condition becomes vacuous. Accordingly, we obtain the following solution to the Deligne–Simpson for Coxeter SL_n-connections.

Corollary 5.6. The moduli space of Coxeter SL_n-connections with formal type of slope r/n and adjoint orbit $\mathscr{O}$ is nonempty if and only if $\mathscr{O}\succeq\mathscr{O}^r_{\mathrm{char}(\mathscr{O})}$.

The notion of a Coxeter G-connection makes sense for any reductive group with connected Dynkin diagram, and there is an analogue of the Deligne–Simpson problem in this context. We give a specific conjecture about this problem (under the additional hypothesis of unipotent monodromy) in the introduction.

5.3. **Proof of Theorem 5.4.** We begin with some preliminaries on I^1-orbits in $\mathfrak{i}^\vee$.

Lemma 5.7. For any $r,s\in\mathbb{Z}$ with r relatively prime to n, the linear map $\phi_{r,s}:\mathfrak{t}\to\mathfrak{gl}_n(\mathbb{C})$ given by $\gamma\mapsto[\gamma\omega^{r-s},\omega^{-r}]\omega^s$ has image $\mathfrak{t}\cap\mathfrak{sl}_n(\mathbb{C})$.

Proof. Since $[\gamma\omega^{r-s},\omega^{-r}]=\gamma\omega^{-s}-\omega^{-r}\gamma\omega^{r-s}=(\gamma-\omega^{-r}\gamma\omega^r)\omega^{-s}$, we have $\phi_{r,s}(\gamma)=\gamma-\omega^{-r}\gamma\omega^r$. Note that $\omega^{-r}\in N(T)$ — indeed, it is a monomial matrix which represents the $-r$th power of a Coxeter element in the Weyl group $\mathfrak{S}_n$ — so the entries in $-\omega^{-r}\gamma\omega^r\in\mathfrak{t}$ are a reordering of the entries in $\gamma\in\mathfrak{t}$. It follows that $\phi_{r,s}(\mathfrak{t})\subset\mathfrak{t}\cap\mathfrak{sl}_n(\mathbb{C})$. To show equality, it suffices to check that $\dim\ker\phi_{r,s}=1$. We have $\gamma\in\ker\phi_{r,s}$ if and only if γ commutes with ω^{-r}. Since $\gcd(r,n)=1$, the centralizers of ω^{-r} and ω coincide and equal $\mathbb{C}(\!(\omega)\!)$. Thus, $\ker(\phi_{r,s})=\mathbb{C}(\!(\omega)\!)\cap\mathfrak{t}=\mathbb{C}\mathrm{id}$. ☐

We can now give a convenient representative of certain coadjoint I^1-orbits in $\mathfrak{i}^\vee$.

Proposition 5.8. Let r and n be positive integers with $\gcd(r,n)=1$. Suppose $\alpha \in \mathfrak{i}^\vee$ is given by $\alpha = (a\omega^{-r}+X)\frac{dz}{z}$ for some $a \in \mathbb{C}^*$ and $X \in \mathfrak{i}^{-r+1}$. Then there exists $g \in I^1$ such that

$$\mathrm{Ad}^*(g)\alpha = (a\omega^{-r} + \sum_{i=0}^{r-1} c_i e_{ii}\omega^{-i})\tfrac{dz}{z}. \tag{7}$$

Proof. All elements of $\mathfrak{i}^1\frac{dz}{z}$ represent the zero functional on $\mathfrak{i}$, so it suffices to show that if $\alpha \in (a\omega^{-r}+\sum_{i=s+1}^{r-1} c_ie_{ii}\omega^{-i}+\beta\omega^{-s}+\mathfrak{i}^{-s+1})\frac{dz}{z}$ for $1 \le s \le r-1$, then there exists $\gamma \in \mathfrak{t}$ such that $\mathrm{Ad}^*(1+\gamma\omega^{r-s})(\alpha) \in (a\omega^{-r}+\sum_{i=s+1}^{r-1} c_ie_{ii}\omega^{-i}+c_se_{ss}\omega^{-s}+\mathfrak{i}^{-s+1})\frac{dz}{z}$.

Write $\beta = (b_1,\dots,b_n)$. By Lemma 5.7, there exists $\gamma \in \mathfrak{t}$ such that

$$\phi_{r,s}(a\gamma) = (-b_1,\dots,-b_{s-1},\sum_{i\neq s} b_i, -b_{s+1},\dots,-b_n).$$

Setting $c_s = \sum_i b_i$ gives

$$\begin{aligned}\mathrm{Ad}^*(1+\gamma\omega^{r-s})(\alpha) &\in (a\omega^{-r} + \sum_{i=s+1}^{r-1} c_ie_{ii}\omega^{-i} + \beta\omega^{-s} + a[\gamma\omega^{r-s},\omega^{-r}] + \mathfrak{i}^{-s+1})\tfrac{dz}{z}\\ &= (a\omega^{-r} + \sum_{i=s+1}^{r-1} c_ie_{ii}\omega^{-i} + (\beta+\phi_{r,s}(a\gamma))\omega^{-s} + \mathfrak{i}^{-s+1})\tfrac{dz}{z}\\ &= (a\omega^{-r} + \sum_{i=s+1}^{r-1} c_ie_{ii}\omega^{-i} + c_se_{ss}\omega^{-s} + \mathfrak{i}^{-s+1})\tfrac{dz}{z}.\end{aligned}$$

□

Corollary 5.9. Given α as in the proposition and $D \in \mathfrak{t}$ with $\mathrm{Tr}(D) = \mathrm{Res}(\mathrm{Tr}(\alpha))$, there exists $h \in I^1$ such that $\mathrm{Res}(\mathrm{Ad}^*(h)\alpha) = \mathrm{Res}(a\omega^{-r}\frac{dz}{z}) + D$.

Proof. Given g as in (7), we claim that

$$\mathrm{Res}(\mathrm{Ad}^*(g)\alpha) \in \mathrm{Res}(a\omega^{-r}\tfrac{dz}{z}) + \mathfrak{t}.$$

To see this, write $s = kn+u$ with $0 < u \le n$. Recall that N_u is the matrix with 1's on the uth subdiagonal and 0's elsewhere. Similarly, let E_u be the matrix whose only nonzero entries are 1's on the $(n-u)$th superdiagonal. (We make the convention that $N_n = 0$ and $E_n = \mathrm{id}$.) It is easy to verify that $\omega^{-s} = z^{-k}(N_u + z^{-1}E_u)$. Since $e_{uu}E_u = e_{un}$, and $e_{uu}N_u = 0$, we have $e_{ss}\omega^{-s} = z^{-(k+1)}e_{sn}$. In particular, $\mathrm{Res}(e_{ss}\omega^{-s}\frac{dz}{z}) = 0$ if $s > 0$ and equals e_{nn} if $s = 0$. Applying this to (7) gives $\mathrm{Res}(\mathrm{Ad}^*(g)\alpha) = \mathrm{Res}(a\omega^{-r}\frac{dz}{z}) + c_0e_{nn}$ as desired.

To complete the proof, it suffices to show that if $\mathrm{Res}(\alpha) = \mathrm{Res}(a\omega^{-r}\frac{dz}{z}) + D'$ for some $D' \in \mathfrak{t}$, and if $D \in \mathfrak{t}$ satisfies $\mathrm{Tr}(D) = \mathrm{Tr}(D')$, then there exists $\gamma \in \mathfrak{t}$ such that $\mathrm{Res}(\mathrm{Ad}^*(1+\gamma\omega^r)\alpha) = \mathrm{Res}(a\omega^{-r}\frac{dz}{z}) + D$. Since $\mathrm{Res}(\mathrm{Ad}^*(1+\gamma\omega^r)\alpha) = \mathrm{Res}(a\omega^{-r}\frac{dz}{z}) + D' + a[\gamma\omega^r,\omega^{-r}]$, it suffices to find

$\gamma \in \mathfrak{t}$ such that $D'+a[\gamma\omega^r, \omega^{-r}] = D$. This follows from Lemma 5.7, since $a^{-1}(D-D') \in \mathfrak{t}\cap\mathfrak{sl}_n(\mathbb{C}) = \mathrm{Image}(\phi_{r,0})$. □

Lemma 5.10. Let r and n be positive integers with $\gcd(r,n) = 1$. Suppose that $\mathscr{A} \in \mathcal{A}(\mathcal{C}, r)$ and $\mathscr{O}$ is a nonresonant adjoint orbit in $\mathfrak{gl}_n(\mathbb{C})$. If $\mathcal{M}(\mathscr{A}, \mathscr{O}) \neq \varnothing$, then the Jordan form of $\mathscr{O}$ has at most r blocks for each eigenvalue.

Proof. This is trivial for $r > n$, so assume that $r < n$. Choose $\alpha \in \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\frac{dz}{z}$ and $Y \in \mathscr{O}$ such that $\alpha|_{\mathfrak{i}} \in \mathrm{Ad}^*(I)(\mathscr{A})$ and $Y = -\operatorname{Res}(\mathrm{Ad}^*(b)\alpha)$. Write $\mathscr{A} = (a\omega^{-r} + X)\frac{dz}{z}$ for some $a \in \mathbb{C}^*$ and for some $X \in \mathfrak{c}^{-r+1}$. Since $I = TI^1$, we may assume without loss of generality that $\alpha|_{\mathfrak{i}} \in \mathrm{Ad}^*(I^1)\mathscr{A}$. This implies that $\alpha = (a\omega^{-r} + X')\frac{dz}{z}$ for some $X' \in \mathfrak{i}^{-r+1}$. It is easy to see that $\operatorname{Res}(X'\frac{dz}{z})$ has 0's in the rth subdiagonal and below. In the notation of §2, this means that $Y := -\operatorname{Res}(\alpha) \in V^r$. By Theorem 2.3, Y has at most r blocks for each eigenvalue. □

Lemma 5.11. Let r and n be positive integers with $\gcd(r,n) = 1$, let $\mathscr{A}$ be a $\mathcal{C}$-formal type of depth r, and let $q = \prod_{i=1}^s (x-a_i)^{m_i} \in \mathbb{C}[x]$ be a degree n polynomial with $a_1, \dots, a_s \in \mathbb{C}$ distinct modulo $\mathbb{Z}$. If $\operatorname{Res}(\operatorname{Tr}(\mathscr{A})) = -\sum_{i=1}^s m_i a_i$, then $\mathscr{O}_q^r \in \mathrm{DS}(\mathscr{A}, q)$.

Proof. Write $\mathscr{A} = (a\omega^{-r} + X)\frac{dz}{z}$ for some $a \in \mathbb{C}^*$ and for some $X \in \mathfrak{c}^{-r+1}$. By Proposition 2.4, there exists D with trace $\sum_{i=1}^s m_i a_i$ such that $-aN_r + D \in \mathscr{O}_q^r$. We now apply Corollary 5.9 to obtain $g \in I^1$ such that $\operatorname{Res}(\mathrm{Ad}^*(g)\mathscr{A}) = aN_r - D$. Let $\alpha \in \mathfrak{gl}_n(\mathfrak{o})^\vee$ be any functional extending $\mathrm{Ad}^*(g)\mathscr{A} \in \mathfrak{i}^\vee$. Then the B-orbit of $(\alpha, -aN_r+D)$ gives a point in $\mathcal{M}(\mathscr{A}, \mathscr{O}_q^r)$, so $\mathscr{O}_q^r \in \mathrm{DS}(\mathscr{A}, q)$. □

Proof of Theorem 5.4. If $\mathcal{M}(\mathscr{A}, \mathscr{O}) \neq \varnothing$, then there exists $\alpha \in \mathfrak{gl}_n(\mathfrak{o})^\vee$ and $Y \in \mathscr{O}$ with $\operatorname{Res}(\alpha)+Y = 0$. Since $\operatorname{Res}(\operatorname{Tr}(\alpha)) = \operatorname{Res}(\operatorname{Tr}(\mathscr{A}))$, we see that $\operatorname{Res}(\operatorname{Tr}(\mathscr{A})) = -\operatorname{Tr}(Y) = -\sum_{i=1}^s m_i a_i$.

By Lemma 5.11, $\mathscr{O}_q^r \in \mathrm{DS}(\mathscr{A}, q)$, and by Lemma 5.10, $\mathrm{DS}(\mathscr{A}, q) \subset \langle \mathscr{O}_q^r \rangle$. To show equality, take $\mathscr{O} \in \langle \mathscr{O}_p^r \rangle$. Since $\mathscr{O}_q^r \in \mathrm{DS}(\mathscr{A}, q)$, there exists some $X \in \mathfrak{gl}_n(\mathbb{C}[z^{-1}])$ and $Y \in \mathscr{O}_q^r$ such that $(X\frac{dz}{z})|_{\mathfrak{i}} \in \mathrm{Ad}^*(I)(\mathscr{A})$ and $\operatorname{Res}(X\frac{dz}{z}) + Y = 0$. By a theorem of Krupnik [Kru97, Theorem 1], there exists a strictly upper triangular matrix $N \in \mathfrak{gl}_n(\mathbb{C})$ such that $Y+N \in \mathscr{O}$. Note that $N \subset \mathfrak{i}^1$, so $(N\frac{dz}{z})|_{\mathfrak{i}} = 0$. Thus, $(X-N) \in \mathfrak{gl}_n(\mathbb{C}[z^{-1}])$, $((X-N)\frac{dz}{z})|_{\mathfrak{i}} \in \mathrm{Ad}^*(I)(\mathscr{A})$, and $\operatorname{Res}((X-N)\frac{dz}{z}) + (Y+N) = 0$. Hence $\mathcal{M}(\mathscr{A}, \mathscr{O}) \neq \varnothing$, and the proof is finished. □

Remark 5.12. At least in the case of unipotent monodromy, it is possible to avoid using Krupnik's Theorem by giving an explicit construction of an element of the moduli space. We discuss this in [KLM+21].

6. Rigidity for Coxeter connections

Let ∇ be a meromorphic G-connection on $\mathbb{P}^1$ which is regular on the Zariski-open set $U = \mathbb{P}^1 \setminus \{x_1, \dots, x_k\}$. Let $\widehat{\nabla}_{x_i}$ denote the induced formal G-connection at x_i. The connection is called *physically rigid* if, for any meromorphic G-connection ∇' which is regular on U and satisfies $\widehat{\nabla}'_{x_i} \cong \widehat{\nabla}_{x_i}$ for all i, we have $\nabla' \cong \nabla$.

In general, it is very difficult to determine whether a connection is physically rigid. A more accessible notion is given by *cohomological rigidity*, which means that $H^1(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla) = 0$, where $j : U \hookrightarrow \mathbb{P}^1$ is the inclusion. If ∇ is irreducible, then ∇ being cohomologically rigid implies that ∇ admits no infinitesimal deformations [Yun14]. For $G = \mathrm{GL}_n(\mathbb{C})$, Bloch and Esnault have shown that cohomological rigidity and physical rigidity are equivalent [BE04].

We call a $\mathcal{C}$-formal type *homogeneous* when it is of the form $a\omega^{-r}\frac{dz}{z}$ for $a \in \mathbb{C}^*$; it gives rise to a "homogeneous Coxeter connection" $d + a\omega^{-r}\frac{dz}{z}$ on $\mathbb{G}_m$. This connection has a toral singularity at 0 and (possibly) a regular singularity at ∞ with unipotent monodromy. This notion also makes sense for any complex simple group G [KS21b]. Again, one can define formal types with respect to a certain maximal torus $\mathcal{C}_G \subset G(F)$ called the Coxeter torus. Moreover, if r is any positive integer relatively prime to the Coxeter number h, there exists an element $\omega_{-r} \in \mathrm{Lie}(\mathcal{C}_G)$ such that $a\omega_{-r}\frac{dz}{z}$ may be viewed as a homogeneous formal type. One can again consider the corresponding Coxeter G-connection on $\mathbb{G}_m$ with a homogeneous $\mathcal{C}_G$-toral irregular singularity of slope r/h at 0 and (possibly) a regular singular point with unipotent monodromy at ∞. The case $r = 1$ is the remarkable rigid connection constructed by Frenkel and Gross [FG09].

In [KS21b], Kamgarpour and Sage determined when these homogeneous Coxeter G-connections are (cohomologically) rigid for any simple G. It turns out $r = 1$ and $r = h+1$ always give rigid connections: the Frenkel–Gross and "Airy G-connection" respectively. For the exceptional groups, there are no other such rigid connections except for $r = 7$ in E_7. However, for the classical groups, one also has rigidity for $1 < r < h$ with r satisfying certain divisibility conditions. For example, in type A, these connections are rigid if and only if $r|(n \pm 1)$.

In this paper, we generalize this result in type A to give a classification of rigid framable Coxeter connections with unipotent monodromy at ∞. (Framable means that we only consider Coxeter connections contained in the relevant framable moduli space $\mathcal{M}(\mathscr{A}, \mathscr{O})$.)

Theorem 6.1. Let $\mathscr{A}$ be a rank n maximally ramified formal type of slope r/n, and let $\mathscr{O}$ be any nilpotent orbit with $\mathscr{O} \succeq \mathscr{O}^r_{x^n}$. Then there exists a rigid connection with the given formal type and unipotent monodromy determined by $\mathscr{O}$ if and only if $\mathscr{O} = \mathscr{O}^r_{x^n}$ and $r|(n \pm 1)$.

If ∇ is a Coxeter connection, let $\mathcal{I}$ denote the global differential Galois group, and let $\mathcal{I}_0$ and $\mathcal{I}_\infty$ denote the local differential Galois groups at 0 and ∞. These differential Galois groups are all algebraic subgroups of $\mathrm{GL}_n(\mathbb{C})$. Also, let $\mathrm{Irr}(\mathrm{ad}_{\widehat{\nabla}_0})$ denote the irregularity of the formal connection $\mathrm{ad}_{\widehat{\nabla}_0}$. This is the sum of all the slopes appearing in the slope decomposition of $\mathrm{ad}_{\widehat{\nabla}_0}$; it is a nonnegative integer.

Let $n(\nabla) = \mathrm{Irr}(\mathrm{ad}_{\widehat{\nabla}_0}) - \dim(\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}_0}) - \dim(\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}_\infty}) + 2\dim(\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}})$, and let $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ be the inclusion. It is shown in [FG09, Proposition 11] that $\dim(H^1(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla)) = n(\nabla)$. Thus, we get the *numerical criterion for rigidity* that ∇ is rigid if and only if $n(\nabla) = 0$.

We now calculate the numerical criterion as in Section 4 of [KS21b]. The local differential Galois group $\mathcal{I}_0$ is given by $\mathcal{I}_0 \cong H \ltimes \langle\theta\rangle$, where H is a certain torus containing a regular semisimple element and θ is an order n element of $N(H)$ [KS21a]. The centralizer of H is thus a maximal torus T', and $\theta \in N(T')$ represents a Coxeter element in the Weyl group. We conclude (as in [KS21b]) that $\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}_0} = (\mathfrak{gl}_n(\mathbb{C})^H)^\theta = \mathrm{Lie}(T')^\theta = \mathbb{C}\mathrm{id}$. Since $\mathcal{I}_0 \subset \mathcal{I}$, we also have $\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}} = \mathbb{C}\mathrm{id}$.

In general, if $\widehat{\nabla}$ is a toral G-connection with slope s, then by Lemma 19 of [KS19], $\mathrm{Irr}(\mathrm{ad}_{\widehat{\nabla}}) = s|\Phi|$, where Φ is the set of roots with respect to the maximal torus T. In our particular case, we obtain $\mathrm{Irr}(\mathrm{ad}_{\widehat{\nabla}_0}) = \frac{r}{n}n(n-1) = r(n-1)$.

Finally, if we fix some element $N_{\mathscr{O}} \in \mathscr{O}$, then $\widehat{\nabla}_\infty$ is regular singular with unipotent monodromy $\exp(2\pi i N_{\mathscr{O}})$. This means that $\mathcal{I}_\infty \cong \langle\exp(2\pi i N_{\mathscr{O}})\rangle$, so $\mathfrak{gl}_n(\mathbb{C})^{\mathcal{I}_\infty} = C(N_{\mathscr{O}})$, the centralizer of $N_{\mathscr{O}}$. Using the fact that $\dim(\mathscr{O}) = n^2 - \dim(C(N_{\mathscr{O}}))$, we obtain

$$n(\nabla) = (r-n-1)(n-1) + \dim(\mathscr{O}).$$

Since $\mathcal{M}(\mathscr{A}, \mathscr{O}^r_{x^n}) \neq \varnothing$, we can take ∇' with this local behavior. Suppose that $\mathscr{O} \supsetneq \mathscr{O}^r_{x^n}$. We then have $n(\nabla) = (r-n-1)(n-1) + \dim(\mathscr{O}) > (r-n-1)(n-1) + \dim(\mathscr{O}^r_{x^n}) = n(\nabla') = 0$. It follows that in this case, ∇ is never rigid.

Finally, take $\mathscr{O} = \mathscr{O}^r_{x^n}$. It was shown in [KS21b] that $\dim(\mathscr{O}^r_{x^n}) = (n+1-r)(n-1)$ if and only if $r|(n\pm 1)$. This finishes the proof of the theorem.

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