On the Best Lattice Quantizers

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***Abstract*—A lattice quantizer approximates an arbitrary real-valued source vector with a vector taken from a specific discrete lattice. The quantization error is the difference between the source vector and the lattice vector. In a classic 1996 paper, Zamir and Feder show that the globally optimal lattice quantizer (which minimizes the mean square error) has white quantization noise: for a uniformly distributed source, the covariance of the error is the identity matrix, multiplied by a positive real factor. We generalize the theorem, showing that the same property holds (i) for any locally optimal lattice quantizer and (ii) for an optimal product lattice, if the component lattices are themselves locally optimal. We derive an upper bound on the normalized second moment (NSM) of the optimal lattice in any dimension, by proving that any lower- or upper-triangular modification to the generator matrix of a product lattice reduces the NSM. Using these tools and employing the best currently known lattice quantizers to build product lattices, we construct improved lattice quantizers in dimensions 13 to 15, 17 to 23, and 25 to 48. In some dimensions, these are the first reported lattices with normalized second moments below the Zador upper bound.**

***Index Terms*—Dither autocorrelation, laminated lattice, lattice theory, mean square error, moment of inertia, normalized second moment, product lattice, quantization constant, quantization noise, vector quantization, Voronoi region, white noise.**

I. Introduction

LATTICES are regular arrays of points in $\mathbb{R}^n$. They are obtained as arbitrary linear combinations of (at most n) linearly independent basis vectors, with integer coefficients. Hence, lattices are a countably infinite set of vectors, closed under addition. The remarkable book by Conway and Sloane [1] provides a comprehensive review of lattices and their properties.

As fundamental geometric structures, lattices have found applications in a variety of disciplines, including digital communications [2], experimental design [3], data analysis [4], and particle physics [5]. In each application, the problem of designing the best lattice for a given purpose arises. Such optimization challenges often reduce to familiar mathematical problems such as sphere-packing, sphere-covering, or quantization [1, Ch. 1–2].

In this paper, we are concerned with the *quantization problem*, which can be defined as follows. Random vectors in $\mathbb{R}^n$ are drawn from some (source) probability distribution, and approximated by their closest lattice points. This approximation (or quantization) process creates a round-off (or quantization) error: the difference between the vector and its closest lattice point. Among all lattices having the same number of lattice points per unit volume, the optimal lattice quantizer is the lattice with the minimum mean square error. This is equivalent to minimizing the *normalized second moment* (NSM), which is a scale-invariant measure of this mean square error.

As in most work on lattice quantization, we assume that the source distribution is smooth on the scale of the lattice spacing, and that its support is much larger than that scale. In this case, the optimal lattice does not depend upon the source probability distribution of the random vectors.

Tables of the NSM, showing the best known lattices for quantization in various dimensions are listed in [6], [7], [1, p. 61], and the quantization performance of some additional lattices is computed in [8]–[13]. Yet, proofs of optimality are known only in dimensions up to three [6], [14].

In a pioneering 1996 paper [15], Zamir and Feder show that the optimal lattice quantizer in any dimension has a *white* quantization error. More precisely, the error defined above (vector difference between a random source vector and its closest lattice vector) has a covariance matrix which is the identity matrix, scaled by a positive real constant.

In this paper, we extend the Zamir and Feder result to *locally optimal* lattices. These are lattices whose NSM cannot be reduced by a small perturbation of the lattice generator matrix. We also consider *product lattices*, which are the Cartesian product of two or more lower-dimensional lattices. The NSM of a product lattice depends on the relative scaling between the component lattices. If each of the lower-dimensional lattices is locally optimal, then we prove that the scaling which minimizes the NSM of the product is the one for which the quantization error is white.

Lastly, we apply these methods to explicitly design some product lattices and analytically optimize their scale factors. These provide constructive upper bounds on the quantization performance of the optimal lattices in their respective dimensions. This simple construction yields better lattice quantizers than previously reported in all dimensions above 12 except for 16 and 24. We also prove that further optimization is possible: the NSM of such product lattices is a saddle point in the space of generator matrices, and can be further reduced by certain perturbations of the generator matrix.

II. Mathematical Preliminaries and Method

Notation: Bold lowercase letters $\boldsymbol{x}$ denote row vectors, while bold uppercase letters $\boldsymbol{X}$ denote either matrices or random vectors. An all-zero vector or matrix of an arbitrary size (inferred from the context) is denoted by $\mathbf{0}$, and identity matrices are denoted by $\boldsymbol{I}$. Sets are denoted by uppercase Greek letters Ω, apart from the integers $\mathbb{Z}$ and real numbers

The work of E. Agrell was supported by a Collaborating Scientist Grant from the Max Planck Institute for Gravitational Physics, Germany, which is gratefully acknowledged.

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$\mathbb{R}$. Arithmetical operations on sets should be understood as operating per element, e.g., $\Omega + \boldsymbol{\lambda} \triangleq \{\boldsymbol{x} + \boldsymbol{\lambda} : \boldsymbol{x} \in \Omega\}$. Definitions are indicated by $\triangleq$.

Without loss of generality, we consider n-dimensional lattices Λ that are generated by square invertible $n \times n$ *generator matrices* $\boldsymbol{B}$. The lattice consists of the set of points $\boldsymbol{uB}$ for all row vectors $\boldsymbol{u}$ with integer components. The all-zero row vector $\boldsymbol{0}$ belongs to all lattices. The cubic lattice $\mathbb{Z}^n$ is the special case for which $\boldsymbol{B}$ is the identity matrix.

Until now, we have used "quantization" to denote the map from a vector in $\mathbb{R}^n$ to the closest lattice point. However, for the proofs in this paper, we consider more general mappings. A *quantization rule* or *quantizer* for a lattice Λ is a function $Q_\Lambda(\boldsymbol{x})$ such that (i) $Q_\Lambda(\boldsymbol{x}) \in \Lambda$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and (ii) $Q_\Lambda(\boldsymbol{x} + \boldsymbol{\lambda}) = Q_\Lambda(\boldsymbol{x}) + \boldsymbol{\lambda}$ for all $\boldsymbol{\lambda} \in \Lambda$ and all $\boldsymbol{x} \in \mathbb{R}^n$. The quantizer's properties are completely determined by its behavior in the *fundamental decision region*

$$\Omega(Q_\Lambda) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : Q_\Lambda(\boldsymbol{x}) = \boldsymbol{0}\}, \tag{1}$$

since condition (ii) may then be used to determine the action anywhere. The translate $\Omega(Q_\Lambda) + \boldsymbol{\lambda}$ of the fundamental decision region is called the decision region of the lattice point $\boldsymbol{\lambda}$. All have the same volume [2, Prop. 2.2.1]

$$V_\Lambda \triangleq \int_{\Omega(Q_\Lambda)} \mathrm{d}^n\boldsymbol{x} = |\det \boldsymbol{B}|. \tag{2}$$

As indicated by the notation V_Λ, this volume depends upon the lattice Λ, but is independent of the quantization rule. Taken together, the decision regions of all lattice points cover $\mathbb{R}^n$ without overlap.

As mentioned in the Introduction, the performance of lattice quantizers does not depend upon the source distribution if this distribution is smooth on the scale of the lattice. To prove this, consider a source probability density function (pdf) $p_{\boldsymbol{X}}(\boldsymbol{x})$, normalized by $\int p_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = 1$. The mean square quantization error of the quantization rule is

$$\mathbb{E}[\|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2] = \int_{\mathbb{R}^n} p_{\boldsymbol{X}}(\boldsymbol{x}) \|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2 \,\mathrm{d}^n\boldsymbol{x}. \tag{3}$$

Since the translates $\Omega(Q_\Lambda) + \boldsymbol{\lambda}, \forall \boldsymbol{\lambda} \in \Lambda$ cover $\mathbb{R}^n$ without overlap, the mean square error can be written as

$$\mathbb{E}[\|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2] = \int_{\Omega(Q_\Lambda)} \sum_{\boldsymbol{\lambda} \in \Lambda} p_{\boldsymbol{X}}(\boldsymbol{\lambda} + \boldsymbol{x}) \|\boldsymbol{x}\|^2 \,\mathrm{d}^n\boldsymbol{x}, \tag{4}$$

where we use condition (ii) to write this as an integral over the fundamental decision region and condition (i) to set $Q_\Lambda(\boldsymbol{x}) = \boldsymbol{0}$ inside that region. If now $p_{\boldsymbol{X}}$ varies slowly over the scale of the lattice, then $\sum_{\boldsymbol{\lambda} \in \Lambda} p_{\boldsymbol{X}}(\boldsymbol{\lambda} + \boldsymbol{\xi})$ is approximately constant, independent of $\boldsymbol{\xi}$. Such a probability distribution can be obtained by rescaling a smooth base pdf $\tilde{p}$ of compact support, for example as $p_{\boldsymbol{X}}(\boldsymbol{x}) = \alpha^n \tilde{p}(\alpha \boldsymbol{x})$ in the limit as $\alpha \to 0$. In such a limit, $\sum_{\boldsymbol{\lambda} \in \Lambda} p_{\boldsymbol{X}}(\boldsymbol{\lambda} + \boldsymbol{\xi}) = 1/V_\Lambda$ for any $\boldsymbol{\xi}$,[1] and the mean square error (4) approaches [16]

$$E(Q_\Lambda) \triangleq \lim_{\alpha \to 0} \mathbb{E}[\|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2] \tag{5}$$

$$= \frac{1}{V_\Lambda} \int_{\Omega(Q_\Lambda)} \|\boldsymbol{x}\|^2 \,\mathrm{d}^n\boldsymbol{x}. \tag{6}$$

Hence, in what follows, the pdf of the source does not appear.

Important quantities that are closely related to the mean square error are the NSM or quantizer constant $G(Q_\Lambda)$ and the *correlation matrix* $\boldsymbol{R}(Q_\Lambda)$, which are [6], [2, pp. 48, 71]

$$G(Q_\Lambda) \triangleq \frac{E(Q_\Lambda)}{n V_\Lambda^{2/n}}, \tag{7}$$

$$\boldsymbol{R}(Q_\Lambda) \triangleq \frac{1}{V_\Lambda} \int_{\Omega(Q_\Lambda)} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} \,\mathrm{d}^n\boldsymbol{x}. \tag{8}$$

Note that the NSM $G(Q_\Lambda)$ is "dimensionless" in the sense that it is invariant under uniform rescaling of the lattice. From (6) and (8), it follows that

$$E(Q_\Lambda) = \mathrm{tr}\boldsymbol{R}(Q_\Lambda), \tag{9}$$

so the trace of the correlation matrix gives the mean square error.

It follows immediately from the definition (8) that the correlation matrix $\boldsymbol{R}$ is real, symmetric, and positive definite. If the quantization noise is not white, then $\boldsymbol{R}$ provides "preferred directions" in the space, for example corresponding to the eigenvector with the largest or the smallest eigenvalue. In the case of white quantization noise, however, $\boldsymbol{R}$ is proportional to the identity, and does not generate preferred directions, since every vector is an eigenvector with the same positive real eigenvalue.

For a given lattice Λ, the most important quantization rule is the *minimum-distance quantization rule*, denoted by a hat:

$$Q_\Lambda(\boldsymbol{x}) = \hat{Q}_\Lambda(\boldsymbol{x}) \triangleq \arg\min_{\boldsymbol{\lambda} \in \Lambda} \|\boldsymbol{x} - \boldsymbol{\lambda}\|^2. \tag{10}$$

For any vector $\boldsymbol{x}$, it returns the closest vector in the lattice (with ties broken arbitrarily). This quantization rule is special because, for a given lattice Λ, it minimizes $E(Q_\Lambda)$ and $G(Q_\Lambda)$. This follows immediately from (5), because the expectation $\mathbb{E}[\|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2]$ is minimal if $\|\boldsymbol{x} - Q_\Lambda(\boldsymbol{x})\|^2$ is minimized for every $\boldsymbol{x}$. Hence, $\hat{Q}_\Lambda$ is the *optimal decision rule* for a given lattice.

For this rule, the fundamental decision region (1) is the *Voronoi region*

$$\Omega(\hat{Q}_\Lambda) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|^2 \le \|\boldsymbol{x} - \boldsymbol{\lambda}\|^2, \ \forall \boldsymbol{\lambda} \in \Lambda\}, \tag{11}$$

which geometrically consists of all points in $\mathbb{R}^n$ whose closest lattice point is the origin.[2] An important property of the Voronoi region of any lattice is that it is symmetric about $\boldsymbol{0}$: the center of gravity $\int_{\Omega(\hat{Q}_\Lambda)} \boldsymbol{x}\,\mathrm{d}^n\boldsymbol{x} = \boldsymbol{0}$. Hence, the correlation matrix $\boldsymbol{R}(Q_\Lambda)$ is equal to the *covariance matrix* whenever $Q_\Lambda = \hat{Q}_\Lambda$ (but not for arbitrary quantization rules Q_Λ).

Throughout this paper, the word "optimal" is used in several senses. For a given lattice, the optimal decision rule is the

[1] This can be proved by writing $\int \tilde{p}(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = 1$ as a Riemann sum over a shifted and scaled lattice $\boldsymbol{x} \in \alpha(\Lambda + \boldsymbol{\xi})$.

[2] More precisely, the interiors of (1) and (11) are equal under the rule (10). With a slight abuse of notation, we disregard their boundaries, which have zero n-volume and do not contribute to any integral over a finite integrand.

one which minimizes the NSM, i.e., (10). Among all lattices of given dimension, the optimal lattice is the one with the smallest NSM. The optimal product of given lattices is the one that minimizes the NSM among all Cartesian products of those lattices, by varying the relative scales between them.

Our main theorem-proving technique is, as in [2, Sec. 4.3], to construct different decision rules for a given lattice, exploiting the fact that their NSMs are equal to or greater than the NSM of the optimal decision rule $\hat{Q}$. For example, in Sec. III, if the quantization noise of a lattice Λ is not white, then $\boldsymbol{R}$ provides preferred directions (say, the eigenvectors with the largest eigenvalue)[3]. With these, we construct a family of lattices $\tilde{\Lambda}$ with nonoptimal decision rules, whose NSM is smaller than that of the original lattice Λ. Since the NSM of $\tilde{\Lambda}$ with an optimal decision rule cannot be larger, we have thus shown that the original lattice is not optimal. A similar proof technique is applied in Sec. V. Starting with a product lattice, we generate a new non-product lattice, with a nonoptimal decision rule, but whose NSM is equal to that of the original starting lattice. Hence, the optimal decision rule on the new non-product lattice must yield a smaller NSM than that of the original product.

III. Locally Optimal Lattices

Our starting point is the following theorem, which states that the globally optimal quantizer lattice has a white quantization error: a covariance matrix proportional to the identity.

Theorem 1 (Zamir–Feder [15], [2, Sec. 4.3]): For the optimal lattice Λ in any dimension n,

$$\boldsymbol{R}(\hat{Q}_\Lambda) = \frac{E(\hat{Q}_\Lambda)}{n}\boldsymbol{I}. \tag{12}$$

We now generalize this to the locally optimal case.

A *locally optimal lattice* is a lattice Λ whose NSM $G(\hat{Q}_\Lambda)$ cannot be decreased by an infinitesimal perturbation of the generator matrix $\boldsymbol{B}$ [14]. The extension to Theorem 1 is:

Theorem 2: Any locally optimal lattice satisfies (12).

Proof: Our proof is constructive. If the covariance matrix of Λ is not proportional to the identity matrix, we use it to build a nearby lattice $\tilde{\Lambda}$ with a smaller NSM than the NSM of Λ.

Let $\tilde{\Lambda} = \Lambda\boldsymbol{A}_\beta$, where $\boldsymbol{A}_\beta$ is an invertible $n \times n$ matrix and β is a real parameter to be defined later. As in [15], we consider the minimum-distance quantizer $\hat{Q}_\Lambda(\boldsymbol{x})$ on Λ and the suboptimal quantizer $\tilde{Q}_{\tilde{\Lambda}}(\boldsymbol{x}) \triangleq \hat{Q}_\Lambda(\boldsymbol{x}\boldsymbol{A}_\beta^{-1})\boldsymbol{A}_\beta$ on $\tilde{\Lambda}$.

It is straightforward to show that $\tilde{Q}_{\tilde{\Lambda}}(\boldsymbol{x})$ satisfies the two conditions in Sec. II and has a fundamental decision region $\Omega(\tilde{Q}_{\tilde{\Lambda}}) = \Omega(\hat{Q}_\Lambda)\boldsymbol{A}_\beta$. Note that while $\Omega(\hat{Q}_\Lambda)$ is the Voronoi region of Λ, the fundamental decision region $\Omega(\tilde{Q}_{\tilde{\Lambda}})$ is generally *not* the Voronoi region of $\tilde{\Lambda}$. By (2), it has volume $V_{\tilde{\Lambda}} = V_\Lambda\,|\det\boldsymbol{A}_\beta|$.

The covariance matrices of the two quantization rules are easily related using the change of variables (mapping) provided by $\boldsymbol{A}_\beta$. From (8), the covariance matrix of $\tilde{Q}_{\tilde{\Lambda}}$ is $\boldsymbol{R}(\tilde{Q}_{\tilde{\Lambda}}) = \boldsymbol{A}_\beta^{\mathrm{T}}\boldsymbol{R}(\hat{Q}_\Lambda)\boldsymbol{A}_\beta$ [15, Eq. (15)]. Hence, the NSM is

$$G(\tilde{Q}_{\tilde{\Lambda}}) = \frac{E(\tilde{Q}_{\tilde{\Lambda}})}{nV_{\tilde{\Lambda}}^{2/n}} = \frac{\mathrm{tr}\,\boldsymbol{A}_\beta\boldsymbol{A}_\beta^T\boldsymbol{R}(\hat{Q}_\Lambda)}{n(V_\Lambda\,|\det\boldsymbol{A}_\beta|)^{2/n}}, \tag{13}$$

where we have used the cyclic property of the trace.

To select $\boldsymbol{A}_\beta$, we follow the approach described earlier, using $\boldsymbol{R}(\hat{Q}_\Lambda)$ to obtain preferred directions.[4] Let $\bar{\boldsymbol{R}}$ denote the traceless part, which by assumption is nonzero:

$$\bar{\boldsymbol{R}} \triangleq \boldsymbol{R}(\hat{Q}_\Lambda) - \frac{\mathrm{tr}\boldsymbol{R}(\hat{Q}_\Lambda)}{n}\boldsymbol{I}, \tag{14}$$

and let $\boldsymbol{A}_\beta \triangleq \exp(\beta\bar{\boldsymbol{R}})$. This choice of mapping is volume-preserving, since for any square matrix $\boldsymbol{M}$, $\det(\exp(\boldsymbol{M})) = \exp(\mathrm{tr}\,\boldsymbol{M})$ [18, p. 16]. Thus, $\det\boldsymbol{A}_\beta = \exp(\beta\,\mathrm{tr}\,\bar{\boldsymbol{R}}) = 1$. Note that because the covariance matrix is symmetric and real, both $\bar{\boldsymbol{R}}$ and $\boldsymbol{A}_\beta$ are symmetric and real.

For the proof, we only need $\boldsymbol{A}_\beta$ for small values of β:

$$\boldsymbol{A}_\beta = \boldsymbol{I} + \beta\bar{\boldsymbol{R}} + O(\beta^2). \tag{15}$$

Substituting $\boldsymbol{A}_\beta$ from (15) and $\boldsymbol{R}(\hat{Q}_\Lambda) = \boldsymbol{I}\,\mathrm{tr}\boldsymbol{R}(\hat{Q}_\Lambda)/n + \bar{\boldsymbol{R}}$ from (14) into (13), the NSM becomes

$$\begin{aligned} G(\tilde{Q}_{\tilde{\Lambda}}) &= \frac{1}{nV_\Lambda^{2/n}}\,\mathrm{tr}\left((\boldsymbol{I}+\beta\bar{\boldsymbol{R}})^2\left[\frac{\mathrm{tr}\boldsymbol{R}(\hat{Q}_\Lambda)}{n}\boldsymbol{I}+\bar{\boldsymbol{R}}\right]\right) \\ &= G(\hat{Q}_\Lambda) + 2\beta\frac{\mathrm{tr}\,\bar{\boldsymbol{R}}^2}{nV_\Lambda^{2/n}}, \end{aligned} \tag{16}$$

where we have distributed the trace over additions, used $\mathrm{tr}\bar{\boldsymbol{R}} = 0$, and dropped terms of order β^2 or higher.

It is clear from (16) that for β negative, we have $G(\tilde{Q}_{\tilde{\Lambda}}) < G(\hat{Q}_\Lambda)$. This follows because, since $\bar{\boldsymbol{R}}$ is a non-vanishing real symmetric matrix, $\mathrm{tr}\,\bar{\boldsymbol{R}}^2$ must be positive[5]. Since the NSM of the minimum distance quantizer on $\tilde{\Lambda}$ satisfies[6] $G(\hat{Q}_{\tilde{\Lambda}}) \leq G(\tilde{Q}_{\tilde{\Lambda}})$, we have established that for β small and negative, $G(\hat{Q}_{\tilde{\Lambda}}) < G(\hat{Q}_\Lambda)$. □

To test Theorem 2, we examine a large number of numerically optimized lattice quantizers. These were designed in 1996 using an iterative algorithm, which converges to different locally optimal lattices [11]. A total of 90 locally optimal lattices are available as online supplementary material to the 1998 article [19]; we estimate the covariance matrices $\boldsymbol{R}(\hat{Q}_\Lambda)$ of their quantization errors using Monte Carlo integration. In all cases, consistent with the theorem, the obtained covariance matrices are proportional to the identity matrix, apart from minor round-off errors.

IV. Product Lattices

In this section, we study lattices that are formed as the Cartesian products of two lower-dimensional lattices.[7] Gersho

[3] A similar argument leads to [17, Eq. 5.1.1].

[4] From here on, our proof deviates from the corresponding proof in [15] for *globally* optimal lattices.

[5] To prove this, write $\bar{\boldsymbol{R}} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$ where $\boldsymbol{U}$ is orthogonal and $\boldsymbol{D}$ is real and diagonal, then use the cyclic property of the trace.

[6] We have no way to directly analyze the performance of $\hat{Q}_{\tilde{\Lambda}}$ because we have no simple expression for the Voronoi region $\Omega(\hat{Q}_{\tilde{\Lambda}})$.

[7] The extension to products of more than two component lattices is straightforward.

applied this technique to obtain upper bounds on the optimal NSM for $n=5$ and $n=100$, without formalizing the expressions [6, Sec. VII].

Let two lattices in dimensions n_1 and n_2 be denoted by Λ_1 and Λ_2, and consider their product $\Lambda_\mathrm{p} = \Lambda_1 \times \Lambda_2$, whose dimension is $n = n_1 + n_2$. A generator matrix for Λ_p is

$$\boldsymbol{B}_\mathrm{p} = \begin{bmatrix} \boldsymbol{B}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B}_2 \end{bmatrix}, \tag{17}$$

where $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are generator matrices of Λ_1 and Λ_2. Relevant quantities associated with the two component lattices, and with their product, are denoted by

$$\begin{aligned} &\Omega_1 \triangleq \Omega(\hat{Q}_{\Lambda_1}), && \Omega_2 \triangleq \Omega(\hat{Q}_{\Lambda_2}), && \Omega \triangleq \Omega(\hat{Q}_{\Lambda_\mathrm{p}}), \\ &V_1 \triangleq V_{\Lambda_1}, && V_2 \triangleq V_{\Lambda_2}, && V \triangleq V_{\Lambda_\mathrm{p}}, \\ &E_1 \triangleq E(\hat{Q}_{\Lambda_1}), && E_2 \triangleq E(\hat{Q}_{\Lambda_2}), && E \triangleq E(\hat{Q}_{\Lambda_\mathrm{p}}), \\ &G_1 \triangleq G(\hat{Q}_{\Lambda_1}), && G_2 \triangleq G(\hat{Q}_{\Lambda_2}), && G \triangleq G(\hat{Q}_{\Lambda_\mathrm{p}}), \\ &\boldsymbol{R}_1 \triangleq \boldsymbol{R}(\hat{Q}_{\Lambda_1}), && \boldsymbol{R}_2 \triangleq \boldsymbol{R}(\hat{Q}_{\Lambda_2}), && \boldsymbol{R} \triangleq \boldsymbol{R}(\hat{Q}_{\Lambda_\mathrm{p}}) \end{aligned}$$

and are related as follows.

Theorem 3: For the product lattice $\Lambda_\mathrm{p} = \Lambda_1 \times \Lambda_2$,

$$\Omega = \Omega_1 \times \Omega_2, \tag{18}$$

$$V = V_1 V_2, \tag{19}$$

$$E = E_1 + E_2, \tag{20}$$

$$G = \frac{n_1}{n} V_1^{\frac{2n_2}{nn_1}} V_2^{-\frac{2}{n}} G_1 + \frac{n_2}{n} V_1^{-\frac{2}{n}} V_2^{\frac{2n_1}{nn_2}} G_2, \tag{21}$$

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}_2 \end{bmatrix}. \tag{22}$$

An example of a 3-dimensional Voronoi region Ω, constructed according to Theorem 3 as the Cartesian product of two lower-dimensional Voronoi regions Ω_1 and Ω_2, is illustrated in Fig. 1.

Proof: To prove (18), let $\boldsymbol{x} = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2]$ and $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1 \;\; \boldsymbol{\lambda}_2]$. By definition, Ω in (11) is formed by all vectors $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$ and $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ such that

$$\|\boldsymbol{x}_1\|^2 + \|\boldsymbol{x}_2\|^2 \le \|\boldsymbol{x}_1 - \boldsymbol{\lambda}_1\|^2 + \|\boldsymbol{x}_2 - \boldsymbol{\lambda}_2\|^2, \tag{23}$$

for all $\boldsymbol{\lambda}_1 \in \Lambda_1$ and $\boldsymbol{\lambda}_2 \in \Lambda_2$.

If $\boldsymbol{x}_1 \in \Omega_1$ and $\boldsymbol{x}_2 \in \Omega_2$, then $\|\boldsymbol{x}_1\|^2 \le \|\boldsymbol{x}_1 - \boldsymbol{\lambda}_1\|^2$ and $\|\boldsymbol{x}_2\|^2 \le \|\boldsymbol{x}_2 - \boldsymbol{\lambda}_2\|^2$. Summing these implies (23), so $\Omega_1 \times \Omega_2 \subseteq \Omega$.

Conversely, if $\boldsymbol{x} \in \Omega$, then setting $\boldsymbol{\lambda}_2 = \mathbf{0}$ in (23) implies $\|\boldsymbol{x}_1\|^2 \le \|\boldsymbol{x}_1 - \boldsymbol{\lambda}_1\|^2$ for all $\boldsymbol{\lambda}_1 \in \Lambda_1$, so that $\boldsymbol{x}_1 \in \Omega_1$. Similarly, setting $\boldsymbol{\lambda}_1 = \mathbf{0}$ in (23) implies that $\boldsymbol{x}_2 \in \Omega_2$. This shows that $\Omega \subseteq \Omega_1 \times \Omega_2$.

Taken together, $\Omega_1 \times \Omega_2 \subseteq \Omega$ and $\Omega \subseteq \Omega_1 \times \Omega_2$ prove (18), which in turn proves (19).

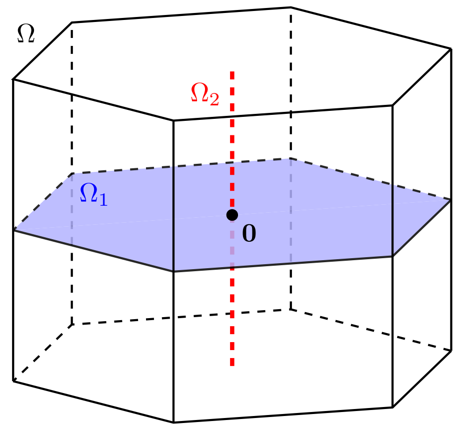


Fig. 1: The Voronoi region Ω of the product lattice $\Lambda_\mathrm{p} = \Lambda_1 \times \Lambda_2$, where Λ_1 is the two-dimensional hexagonal lattice A_2 and Λ_2 is the one-dimensional integer lattice $\mathbb{Z}$. The origin $\mathbf{0}$ belongs to all three lattices and is the centroid of all three Voronoi regions. The top and bottom facets of Ω are shifted copies of Ω_1, and the six vertical edges are shifted copies of Ω_2.

The definition (6), applied to a product lattice using (18) and (19), implies that

$$\begin{aligned} E &= \frac{1}{V_1 V_2} \int_{\Omega_1 \times \Omega_2} \left(\|\boldsymbol{x}_1\|^2 + \|\boldsymbol{x}_2\|^2 \right) \mathrm{d}^{n_1}\boldsymbol{x}_1 \, \mathrm{d}^{n_2}\boldsymbol{x}_2 \\ &= \frac{1}{V_1 V_2} \left(\int_{\Omega_1} \|\boldsymbol{x}_1\|^2 \, \mathrm{d}^{n_1}\boldsymbol{x}_1 \int_{\Omega_2} \mathrm{d}^{n_2}\boldsymbol{x}_2 \right. \\ &\qquad \left. + \int_{\Omega_1} \mathrm{d}^{n_1}\boldsymbol{x}_1 \int_{\Omega_2} \|\boldsymbol{x}_2\|^2 \, \mathrm{d}^{n_2}\boldsymbol{x}_2 \right) \\ &= \frac{1}{V_1 V_2} \left(V_1 E_1 V_2 + V_1 V_2 E_2 \right). \end{aligned} \tag{24}$$

This proves (20).

Equation (21) follows by substituting $E = G/(nV^{2/n})$ and the corresponding expressions for E_1 and E_2 into (20), and simplifying using (19) and $n = n_1 + n_2$.

Lastly, to prove (22), we use (18) and (19) in (8) to obtain

$$\begin{aligned} \boldsymbol{R} &= \frac{1}{V_1 V_2} \int_{\Omega_1} \int_{\Omega_2} \begin{bmatrix} \boldsymbol{x}_1^\mathrm{T}\boldsymbol{x}_1 & \boldsymbol{x}_1^\mathrm{T}\boldsymbol{x}_2 \\ \boldsymbol{x}_2^\mathrm{T}\boldsymbol{x}_1 & \boldsymbol{x}_2^\mathrm{T}\boldsymbol{x}_2 \end{bmatrix} \mathrm{d}^{n_1}\boldsymbol{x}_1 \, \mathrm{d}^{n_2}\boldsymbol{x}_2 \\ &= \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^\mathrm{T} & \boldsymbol{R}_2 \end{bmatrix}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \boldsymbol{R}_{12} &\triangleq \frac{1}{V_1 V_2} \int_{\Omega_1} \int_{\Omega_2} \boldsymbol{x}_1^\mathrm{T} \boldsymbol{x}_2 \, \mathrm{d}^{n_1}\boldsymbol{x}_1 \, \mathrm{d}^{n_2}\boldsymbol{x}_2 \\ &= \frac{1}{V_1 V_2} \left(\int_{\Omega_1} \boldsymbol{x}_1 \, \mathrm{d}^{n_1}\boldsymbol{x}_1 \right)^\mathrm{T} \int_{\Omega_2} \boldsymbol{x}_2 \, \mathrm{d}^{n_2}\boldsymbol{x}_2 \\ &= \mathbf{0}. \end{aligned} \tag{26}$$

These vanish because (as pointed out after (11)) the Voronoi regions Ω_1 and Ω_2 are symmetric about zero and thus have their center of gravity at the origin. □

We now generalize the product construction by introducing an explicit real scale factor $a > 0$ to build a one-parameter family of product lattices $\Lambda(a) = \Lambda_1 \times a\Lambda_2$. A generator matrix for $\Lambda(a)$ is

$$\boldsymbol{B}(a) = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{0} & a\boldsymbol{B}_2 \end{bmatrix}. \tag{27}$$

Denoting the parameters of $\Lambda(a)$ by $\Omega(a)$, $V(a)$, $E(a)$, $G(a)$, and $\boldsymbol{R}(a)$, Theorem 3 generalizes as follows.

Corollary 4: For the product lattice $\Lambda(a) = \Lambda_1 \times a\Lambda_2$,

$$\Omega(a) = \Omega_1 \times a\Omega_2, \tag{28}$$

$$V(a) = a^{n_2} V_1 V_2, \tag{29}$$

$$E(a) = E_1 + a^2 E_2, \tag{30}$$

$$G(a) = \frac{n_1}{n} a^{-\frac{2n_2}{n}} V_1^{\frac{2n_2}{nn_1}} V_2^{-\frac{2}{n}} G_1 + \frac{n_2}{n} a^{\frac{2n_1}{n}} V_1^{-\frac{2}{n}} V_2^{\frac{2n_1}{nn_2}} G_2, \tag{31}$$

$$\boldsymbol{R}(a) = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & a^2\boldsymbol{R}_2 \end{bmatrix}. \tag{32}$$

Proof: In Theorem 3, replace Ω_2 by $a\Omega_2$, V_2 by $a^{n_2}V_2$, E_2 by a^2E_2, and $\boldsymbol{R}_2$ by $a^2\boldsymbol{R}_2$, while the scale-invariant G_2 remains unchanged. □

For given Λ_1 and Λ_2, what scale factor $a = a_{\text{opt}}$ produces the *optimal product lattice* $\Lambda(a)$, in the sense of minimizing $G(a)$? Since (31) diverges in the positive direction as $a \to 0$ and as $a \to \infty$, it has at least one minimum for a finite $a > 0$. This minimum is unique, and has a closed form as follows.

Corollary 5: For given lattices Λ_1 and Λ_2, the optimal product lattice $\Lambda(a_{\text{opt}}) = \Lambda_1 \times a_{\text{opt}}\Lambda_2$ is obtained for

$$a_{\text{opt}} = \frac{V_1^{\frac{1}{n_1}}}{V_2^{\frac{1}{n_2}}} \sqrt{\frac{G_1}{G_2}} \tag{33}$$

$$= \sqrt{\frac{n_2 E_1}{n_1 E_2}}. \tag{34}$$

The optimal NSM is

$$G(a_{\text{opt}}) = G_1^{\frac{n_1}{n}} G_2^{\frac{n_2}{n}}. \tag{35}$$

Proof: Let

$$C_1 = V_1^{\frac{2n_2}{nn_1}} V_2^{-\frac{2}{n}} G_1 \text{ and } C_2 = V_1^{-\frac{2}{n}} V_2^{\frac{2n_1}{nn_2}} G_2. \tag{36}$$

The NSM $G(a)$ in (31) may be written as

$$G(a) = \frac{n_1}{n} a^{-\frac{2n_2}{n}} C_1 + \frac{n_2}{n} a^{\frac{2n_1}{n}} C_2. \tag{37}$$

Its derivative with respect to a is

$$G'(a) = -\frac{2n_1n_2}{n^2} a^{-\frac{2n_2}{n}-1} C_1 + \frac{2n_1n_2}{n^2} a^{\frac{2n_1}{n}-1} C_2 = \frac{2n_1n_2}{n^2} a^{\frac{2n_1}{n}-3} \left(a^2 C_2 - C_1\right), \tag{38}$$

where to obtain the final line we use $-\frac{2n_2}{n} - 1 = \frac{2n_1}{n} - 3$ for the C_1 term. The unique positive solution to $G'(a_{\text{opt}}) = 0$ is

$$a_{\text{opt}} = \sqrt{\frac{C_1}{C_2}}. \tag{39}$$

Substituting this back into (37) and again using $n_1 + n_2 = n$ gives the optimal product NSM

$$G(a_{\text{opt}}) = C_1^{\frac{n_1}{n}} C_2^{\frac{n_2}{n}}. \tag{40}$$

Substituting (36) into (39) and (40) now yields (33) and (35), respectively. Finally, (34) follows from (33) and (7). □

The minimum $G(a)$ in (35) depends only on G_1, G_2, and the lattice dimensions, but not on V_1 or V_2. It satisfies the elegant relationship

$$G^n(a_{\text{opt}}) = G_1^{n_1} G_2^{n_2}, \tag{41}$$

which is useful and easy to recall.

We are now ready to calculate the covariance matrix. Consider the case where $\boldsymbol{R}_1 = \rho_1\boldsymbol{I}$ and $\boldsymbol{R}_2 = \rho_2\boldsymbol{I}$ are each proportional to the identity matrix, with different constants of proportionality. Because of (9) and (34), the optimal scale factor satisfies ${a_{\text{opt}}}^2\rho_2 = \rho_1$, ensuring that $\boldsymbol{R}(a_{\text{opt}}) = \rho_1\boldsymbol{I}$ is proportional to the identity matrix. We state this in a way similar to Theorems 1 and 2, but with different conditions.

Corollary 6: If Λ_1 and Λ_2 are locally optimal lattices and $\Lambda(a_{\text{opt}}) = \Lambda_1 \times a_{\text{opt}}\Lambda_2$, then

$$\boldsymbol{R}(a_{\text{opt}}) = \frac{E(a_{\text{opt}})}{n}\boldsymbol{I}. \tag{42}$$

Proof: From Theorem 2, $\boldsymbol{R}_1 = (E_1/n_1)\boldsymbol{I}$ and $\boldsymbol{R}_2 = (E_2/n_2)\boldsymbol{I}$. Using these together with (34) in (32) yields

$$\boldsymbol{R}(a_{\text{opt}}) = \begin{bmatrix} \frac{E_1}{n_1}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{n_2E_1}{n_1E_2}\frac{E_2}{n_2}\boldsymbol{I} \end{bmatrix} = \frac{E_1}{n_1}\boldsymbol{I}. \tag{43}$$

To complete the proof of (42), we conclude from (30) and (34) that

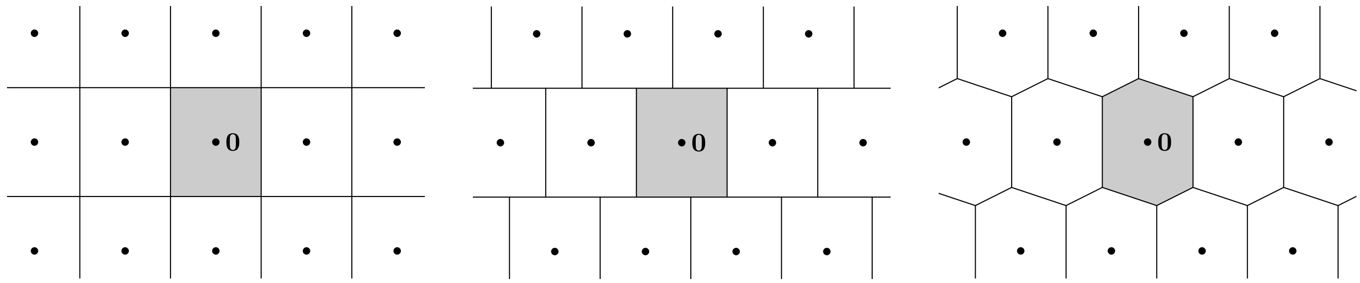
$$E(a_{\text{opt}}) = E_1 + \frac{n_2E_1}{n_1E_2}E_2 = \frac{n}{n_1}E_1, \tag{44}$$

or equivalently $E(a_{\text{opt}})/n = E_1/n_1$. □

Theorem 2 and Corollary 6 are curiously related to each other: both give sufficient (but not necessary) conditions for white quantization error. Is Corollary 6 perhaps a special case of Theorem 2? In other words, is the optimal product lattice $\Lambda(a_{\text{opt}})$, to which Corollaries 5 and 6 apply, also a locally optimal lattice, as defined in Sec. III? We will see that the answer is "no". To say this more precisely, consider the generator matrix $\boldsymbol{B}(a)$ in (27) for $a = a_{\text{opt}}$. By the assumptions in Corollary 6, the NSM $G(a)$ is locally minimal with respect to small variations in $\boldsymbol{B}_1$ or $\boldsymbol{B}_2$, and also minimal with respect to $a = a_{\text{opt}}$. However, the behavior of $G(a)$ as a function of the bottom-left or top-right submatrices of $\boldsymbol{B}(a)$, which vanish in (27), remains to be studied. In the next section, we show that setting either of these submatrices nonzero will *decrease* the NSM.

V. Upper Bound

In this section, we show that the NSM of any lattice is upper-bounded by that of a product lattice, which is given by Theorem 3 or Corollaries 4 or 5.



(a) $\hat{Q}_{\Lambda_\mathrm{p}}$ (b) $\tilde{Q}_{\Lambda}$ (c) $\hat{Q}_{\Lambda}$

Fig. 2: An example of Theorem 7 with $n_1 = n_2 = 1$. Each cell is the decision region of the lattice point it contains, and the shaded cells are the fundamental decision regions. Comparing (a) and (b) shows that the NSMs of $\hat{Q}_{\Lambda_\mathrm{p}}$ and $\tilde{Q}_{\Lambda}$ are equal, because their fundamental decision regions are identical. Comparing (b) and (c) shows that $\tilde{Q}_{\Lambda}$ cannot have a smaller NSM than $\hat{Q}_{\Lambda}$, because the lattices are identical and $\hat{Q}_{\Lambda}(\boldsymbol{x})$ minimizes the quantization error for every $\boldsymbol{x}$.

Consider a square generator matrix of the form

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{B}_2 \end{bmatrix}, \tag{45}$$

where $\boldsymbol{B}_1$ is $n_1 \times n_1$ and $\boldsymbol{B}_2$ is $n_2 \times n_2$. Let Λ_1, Λ_2, and Λ be the lattices generated by $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, and $\boldsymbol{B}$, respectively, and let $\Lambda_\mathrm{p} = \Lambda_1 \times \Lambda_2$ as in Sec. IV.

Theorem 7: For any $\boldsymbol{H}$, $G(\hat{Q}_{\Lambda}) \le G(\hat{Q}_{\Lambda_\mathrm{p}})$, with equality if $\boldsymbol{H} = \boldsymbol{0}$.

Proof: The method of proof is to define a suboptimal quantizer $\tilde{Q}_{\Lambda}$ for which $G(\tilde{Q}_{\Lambda}) = G(\hat{Q}_{\Lambda_\mathrm{p}})$. The theorem then follows, because by the definition of $\hat{Q}_{\Lambda}$, $G(\hat{Q}_{\Lambda}) \le G(\tilde{Q}_{\Lambda})$. The decision regions for the three different quantizers are illustrated in Fig. 2.

We construct the suboptimal quantization rule from the optimal quantization rules for Λ_1 and Λ_2. As earlier, write the source vector as $\boldsymbol{x} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2]$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ have respective dimensions n_1 and n_2. Our suboptimal quantization rule is defined by the following four-step algorithm:

$$\hat{\boldsymbol{x}}_2 \triangleq \hat{Q}_{\Lambda_2}(\boldsymbol{x}_2), \tag{46}$$

$$\boldsymbol{z}_1 \triangleq \hat{\boldsymbol{x}}_2 \boldsymbol{B}_2^{-1} \boldsymbol{H}, \tag{47}$$

$$\hat{\boldsymbol{x}}_1 \triangleq \hat{Q}_{\Lambda_1}(\boldsymbol{x}_1 - \boldsymbol{z}_1), \tag{48}$$

$$\tilde{Q}_{\Lambda}(\boldsymbol{x}) \triangleq [\hat{\boldsymbol{x}}_1 + \boldsymbol{z}_1 \quad \hat{\boldsymbol{x}}_2]. \tag{49}$$

Quantities with subscript "1" lie in the subspace spanned by the first n_1 coordinates (horizontal in Fig. 2) and the quantities with subscript "2" lie in the subspace spanned by the final n_2 coordinates (vertical in Fig. 2). We first show that this is a quantization rule: it satisfies the two conditions given at the start of Sec. II.

Condition (i): Since $\hat{Q}_{\Lambda_1}(\boldsymbol{x}_1 - \boldsymbol{z}_1) \in \Lambda_1$ and $\hat{Q}_{\Lambda_2}(\boldsymbol{x}_2) \in \Lambda_2$, there exist integer vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that $\hat{\boldsymbol{x}}_1 = \boldsymbol{u}_1 \boldsymbol{B}_1$ and $\hat{\boldsymbol{x}}_2 = \boldsymbol{u}_2 \boldsymbol{B}_2$. Then by (47), $\boldsymbol{z}_1 = \boldsymbol{u}_2 \boldsymbol{H}$ and by (49), $\tilde{Q}_{\Lambda}(\boldsymbol{x}) = [\boldsymbol{u}_1 \boldsymbol{B}_1 + \boldsymbol{u}_2 \boldsymbol{H} \quad \boldsymbol{u}_2 \boldsymbol{B}_2] = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2] \boldsymbol{B}$, which shows that $\tilde{Q}_{\Lambda}(\boldsymbol{x}) \in \Lambda$.

Condition (ii): Consider $\tilde{Q}_{\Lambda}(\boldsymbol{x} + \boldsymbol{\lambda})$ for an arbitrary $\boldsymbol{\lambda} \in \Lambda$. Let $\boldsymbol{\lambda} \triangleq [\boldsymbol{v}_1 \quad \boldsymbol{v}_2] \boldsymbol{B} = [\boldsymbol{v}_1 \boldsymbol{B}_1 + \boldsymbol{v}_2 \boldsymbol{H} \quad \boldsymbol{v}_2 \boldsymbol{B}_2]$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are integer vectors. Let $\boldsymbol{x}' \triangleq [\boldsymbol{x}'_1 \quad \boldsymbol{x}'_2] \triangleq \boldsymbol{x} + \boldsymbol{\lambda}$, and let $\hat{\boldsymbol{x}}'_2$, $\boldsymbol{z}'_1$, $\hat{\boldsymbol{x}}'_1$, and $\tilde{Q}_{\Lambda}(\boldsymbol{x}')$ denote the quantities obtained via the algorithm (46)–(49) when $\boldsymbol{x}$ is replaced by $\boldsymbol{x}'$. Since condition (ii) applies to $\hat{Q}_{\Lambda_1}$ and $\hat{Q}_{\Lambda_2}$, one finds

$$\begin{aligned} \hat{\boldsymbol{x}}'_2 &= \hat{Q}_{\Lambda_2}(\boldsymbol{x}'_2) \\ &= \hat{Q}_{\Lambda_2}(\boldsymbol{x}_2 + \boldsymbol{v}_2 \boldsymbol{B}_2) \\ &= \hat{Q}_{\Lambda_2}(\boldsymbol{x}_2) + \boldsymbol{v}_2 \boldsymbol{B}_2 \\ &= \hat{\boldsymbol{x}}_2 + \boldsymbol{v}_2 \boldsymbol{B}_2, \end{aligned} \tag{50}$$

$$\begin{aligned} \boldsymbol{z}'_1 &= \hat{\boldsymbol{x}}'_2 \boldsymbol{B}_2^{-1} \boldsymbol{H} \\ &= (\hat{\boldsymbol{x}}_2 + \boldsymbol{v}_2 \boldsymbol{B}_2) \boldsymbol{B}_2^{-1} \boldsymbol{H} \\ &= \hat{\boldsymbol{x}}_2 \boldsymbol{B}_2^{-1} \boldsymbol{H} + \boldsymbol{v}_2 \boldsymbol{H} \\ &= \boldsymbol{z}_1 + \boldsymbol{v}_2 \boldsymbol{H}, \end{aligned} \tag{51}$$

$$\begin{aligned} \hat{\boldsymbol{x}}'_1 &= \hat{Q}_{\Lambda_1}(\boldsymbol{x}'_1 - \boldsymbol{z}'_1) \\ &= \hat{Q}_{\Lambda_1}(\boldsymbol{x}_1 + \boldsymbol{v}_1 \boldsymbol{B}_1 + \boldsymbol{v}_2 \boldsymbol{H} - \boldsymbol{z}_1 - \boldsymbol{v}_2 \boldsymbol{H}) \\ &= \hat{Q}_{\Lambda_1}(\boldsymbol{x}_1 - \boldsymbol{z}_1) + \boldsymbol{v}_1 \boldsymbol{B}_1 \\ &= \hat{\boldsymbol{x}}_1 + \boldsymbol{v}_1 \boldsymbol{B}_1, \end{aligned} \tag{52}$$

$$\begin{aligned} \tilde{Q}_{\Lambda}(\boldsymbol{x}') &= [\hat{\boldsymbol{x}}'_1 + \boldsymbol{z}'_1 \quad \hat{\boldsymbol{x}}'_2] \\ &= [\hat{\boldsymbol{x}}_1 + \boldsymbol{v}_1 \boldsymbol{B}_1 + \boldsymbol{z}_1 + \boldsymbol{v}_2 \boldsymbol{H} \quad \hat{\boldsymbol{x}}_2 + \boldsymbol{v}_2 \boldsymbol{B}_2] \\ &= [\hat{\boldsymbol{x}}_1 + \boldsymbol{z}_1 \quad \hat{\boldsymbol{x}}_2] + [\boldsymbol{v}_1 \boldsymbol{B}_1 + \boldsymbol{v}_2 \boldsymbol{H} \quad \boldsymbol{v}_2 \boldsymbol{B}_2] \\ &= \tilde{Q}_{\Lambda}(\boldsymbol{x}) + \boldsymbol{\lambda}. \end{aligned} \tag{53}$$

Thus, since it satisfies both conditions, $\tilde{Q}_{\Lambda}$ is a valid quantization rule.

The NSM of $\tilde{Q}_{\Lambda}$ is determined by its fundamental decision region, defined by (1). From (49) and (47), $\tilde{Q}_{\Lambda}(\boldsymbol{x}) = \boldsymbol{0}$ if and only if $\hat{\boldsymbol{x}}_1 = \boldsymbol{0}$ and $\hat{\boldsymbol{x}}_2 = \boldsymbol{0}$. Hence, (1) yields

$$\begin{aligned} \Omega(\tilde{Q}_{\Lambda}) &= \{\boldsymbol{x} \in \mathbb{R}^n : \hat{Q}_{\Lambda_1}(\boldsymbol{x}_1) = \boldsymbol{0} \text{ and } \hat{Q}_{\Lambda_2}(\boldsymbol{x}_2) = \boldsymbol{0}\} \\ &= \Omega(\hat{Q}_{\Lambda_1}) \times \Omega(\hat{Q}_{\Lambda_2}) \\ &= \Omega(\hat{Q}_{\Lambda_\mathrm{p}}). \end{aligned} \tag{54}$$

Thus, the fundamental decision region of the (suboptimal) quantization rule for Λ is identical to the fundamental decision region of the optimal quantization rule for Λ_p. This can be intuitively understood by comparing Figs. 2(a) and 2(b).

Since the fundamental decision regions of $\tilde{Q}_{\Lambda}$ and $\hat{Q}_{\Lambda_\mathrm{p}}$ are identical, so are all parameters derived from these regions, e.g., $E(\tilde{Q}_{\Lambda}) = E(\hat{Q}_{\Lambda_\mathrm{p}})$, $\boldsymbol{R}(\tilde{Q}_{\Lambda}) = \boldsymbol{R}(\hat{Q}_{\Lambda_\mathrm{p}})$, and $G(\tilde{Q}_{\Lambda}) = G(\hat{Q}_{\Lambda_\mathrm{p}})$. But since the optimal decision rule for Λ satisfies

$G(\hat{Q}_\Lambda) \le G(\tilde{Q}_\Lambda)$, our proof is complete: $G(\hat{Q}_\Lambda) \le G(\hat{Q}_{\Lambda_\mathrm{p}})$. Equality if and only if $\boldsymbol{H} = \mathbf{0}$ follows, since $\boldsymbol{B}$ in (45) and $\boldsymbol{B}_\mathrm{p}$ in (17) are equal if and only if $\boldsymbol{H} = \mathbf{0}$. □

Like Theorem 3, Theorem 7 can also be extended by a scale factor a. Specifically, if $\boldsymbol{B}_2$ in (45) is multiplied by a scale factor a, then the NSM of the resulting lattice is upper-bounded by $G(a)$ in Corollary 4 or for $a = a_\mathrm{opt}$ by $G(a_\mathrm{opt})$ in Corollary 5.

We now return to the question of whether the lattice generated by $\boldsymbol{B}(a)$ in (27) is locally optimal at $a = a_\mathrm{opt}$. It was observed at the end of Sec. IV that under the assumptions of Corollary 6, $G(a_\mathrm{opt})$ is locally *minimal* with respect to variations in $\boldsymbol{B}_1$ and $a_\mathrm{opt}\boldsymbol{B}_2$, which are the upper-left and bottom-right submatrices of $\boldsymbol{B}(a_\mathrm{opt})$. On the other hand, it follows from Theorem 7 that $G(a_\mathrm{opt})$ is locally *maximal* with respect to variations about $\mathbf{0}$ in the bottom-left submatrix of $\boldsymbol{B}(a_\mathrm{opt})$. By analogy, it can also be shown that $G(a_\mathrm{opt})$ is locally maximal with respect to variations in the top-right submatrix of $\boldsymbol{B}(a_\mathrm{opt})$.

Since the NSM must decrease for any variations of either the top-right or bottom-left blocks of $\boldsymbol{B}$, the first derivative of G with respect to these entries must vanish. If the product lattice Λ_p is locally or globally optimal, then the NSM must increase for any variations of $\boldsymbol{B}_1$ or $\boldsymbol{B}_2$, and hence the first derivative of G with respect to those entries must also vanish. Since all first derivatives vanish, and the NSM increases in some directions and decreases in others, we conclude that the NSM has a *saddle point* at $\boldsymbol{B}(a_\mathrm{opt})$.

VI. Laminated Lattices

One way to construct lattices is by lamination of a lower-dimensional lattice. To build an n-dimensional laminated lattice Λ, take a generator matrix $\boldsymbol{B}_1$ for an $(n-1)$-dimensional lattice Λ_1 and an arbitrary $(n-1)$-dimensional vector $\boldsymbol{h}$. Then construct the $n \times n$ generator matrix

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \mathbf{0} \\ \boldsymbol{h} & a \end{bmatrix}. \tag{55}$$

Here $a > 0$ is a real number, which is the distance between the shifted lattice copies in the direction orthogonal to their subspace, and the vector $\boldsymbol{h}$ is the stacking offset in the $(n-1)$-plane. Fig. 1 illustrates the Voronoi region of a laminated lattice with $n = 3$ and $\boldsymbol{h} = \mathbf{0}$.

In the classical lattice literature, "laminated lattices" are built recursively in this way, to maximize the packing density, starting from $\mathbb{Z}$. So $\boldsymbol{B}_1$ is itself the generator for a laminated lattice, and in each recursive iteration, $\boldsymbol{h}$ and a in (55) are selected to maximize the packing density. This construction gives rise to the well-studied Λ and K series in [20], [21], [1, Sec. 4 of Ch. 5 and Ch. 6].

In this paper, our focus is the NSM rather than the packing density, so we use "laminated lattice" more broadly for *any* lattice generated via (55). This broader meaning is consistent with [13], [22]. Note that to maximize the packing density, the optimal choice of $\boldsymbol{h}$ is a "deep hole" in Λ_1 (a vertex of the Voronoi region most distant from the origin). This is also (intuitively) a good choice to minimize the NSM, although it may not be optimal.

An upper bound on the quantization performance of a laminated lattice follows directly from the results of Secs. IV and V, as follows.

Corollary 8: An n-dimensional lattice Λ obtained by lamination of an $(n-1)$-dimensional lattice Λ_1 satisfies, for an arbitrary offset $\boldsymbol{h}$ and the optimal layer separation a,

$$G(\hat{Q}_\Lambda) \le \frac{G(\hat{Q}_{\Lambda_1})^{1-\frac{1}{n}}}{12^{\frac{1}{n}}}. \tag{56}$$

Proof: Setting $n_2 = 1$, $\Lambda_2 = a\mathbb{Z}$, and $\boldsymbol{H} = \boldsymbol{h}$ in Theorem 7 yields

$$G(\hat{Q}_\Lambda) \le G(\hat{Q}_{\Lambda_\mathrm{p}}), \tag{57}$$

where $\Lambda_\mathrm{p} = \Lambda_1 \times a\mathbb{Z}$. Furthermore, setting $n_2 = 1$, $\Lambda_2 = a\mathbb{Z}$, $G_1 = G(\hat{Q}_{\Lambda_1})$, and $G_2 = G(\hat{Q}_\mathbb{Z}) = 1/12$ in Corollary 5 yields

$$G(\hat{Q}_{\Lambda_\mathrm{p}}) \le \frac{G(\hat{Q}_{\Lambda_1})^{1-\frac{1}{n}}}{12^{\frac{1}{n}}}. \tag{58}$$

Combining (57) and (58) completes the proof. □

As a curiosity, we observe that (56) can be elegantly cast as

$$\left(12G(\hat{Q}_\Lambda)\right)^n \le \left(12G(\hat{Q}_{\Lambda_1})\right)^{n-1}. \tag{59}$$

The relation would have been even more appealing if G had been defined a factor of 12 larger than the standard definition (7). With that alternative definition, the factors of 12 disappear from (59), and $\mathbb{Z}^n$ would have an NSM of 1 in any dimension.

VII. Best Known Lattice Quantizers

The upper bound in Corollary 8 has interesting implications. These call to mind an observation made by Cohn in the context of sphere packing [24]. Referring to a plot of sphere-packing density as a function of dimension, he comments that "*Certain dimensions, most notably 24, have packings so good that they seem to pull the entire curve in their direction. The fact that this occurs is not so surprising, since one expects cross sections and stackings of great packings to be at least good, but the effect is surprisingly large.*" Here, in the context of lattice quantizers, Corollary 8 does precisely this for NSMs. A lattice with particularly small NSM G in dimension $n-1$ makes it possible to also obtain a small NSM in dimension n, and hence "pulls down the NSM curve" for larger dimensions. More generally, Theorem 7 can pull down the curve over intervals of more than one dimension.

We designed product lattices in dimension n by applying Corollary 5 to the best known lattices in dimensions n_1 and $n - n_1$, constructing product lattices for n_1 ranging from 1 to $n-1$. The minimal NSM obtained in each dimension provides a constructive upper bound on the optimal NSM. It follows from Theorem 7 that better lattice quantizers can be found among lattices of the form (45), where $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are the best known lattices in dimensions n_1 and $n - n_1$, respectively, for example by lamination if $n_1 = 1$ or $n_1 = n - 1$.

Tab. I summarizes the best known lattice quantizers in dimensions $n \le 48$. The first such list was compiled in

TABLE I: The best known lattice quantizers. Columns 2–3 list the smallest normalized second moments (NSM) G previously reported in dimension n, while columns 4–5 list the conjectured lower and upper bounds, respectively. Columns 6–7 list our best product lattices. Columns 8–9 indicate if the product lattice in Column 7 provides a smaller NSM than previously reported in Column 2 (<G) and/or is below the upper bound of Column 5 (<U). NSMs that are are known exactly are listed with nine decimals, whereas NSMs with five decimals are derived from numerical estimates in [8].

	Best previously reported		Generic bounds		Best product			
n	NSM	Lattice	Lower [23]	Upper [16]	NSM	Lattice	Better?	
1	0.083333333	$\mathbb{Z}$	0.083333333	0.500000000				
2	0.080187537	A_2	0.080187537	0.159154943	0.083333333	$\mathbb{Z} \otimes \mathbb{Z}$		<U
3	0.078543281	A_3^*	0.077874985	0.115802581	0.081222715	$A_2 \otimes \mathbb{Z}$		<U
4	0.076603235	D_4	0.076087080	0.099735570	0.079714343	$A_3^* \otimes \mathbb{Z}$		<U
5	0.075625443	D_5^*	0.074654327	0.091319469	0.077904301[8]	$D_4 \otimes \mathbb{Z}$		<U
6	0.074243697	E_6^*	0.073474906	0.086084334	0.076858706	$D_5^* \otimes \mathbb{Z}$		<U
7	0.073116493	E_7^*	0.072483503	0.082478806	0.075478834	$E_6^* \otimes \mathbb{Z}$		<U
8	0.071682099	E_8	0.071636064	0.079824101	0.074321725	$E_7^* \otimes \mathbb{Z}$		<U
9	0.071622594	AE_9	0.070901661	0.077775626	0.072891732	$E_8 \otimes \mathbb{Z}$		<U
10	0.070813818	D_{10}^+	0.070257874	0.076139300	0.072715487	$AE_9 \otimes \mathbb{Z}$		<U
11	0.070426259	A_{11}^3	0.069688002	0.074797093	0.071869620	$D_{10}^+ \otimes \mathbb{Z}$		<U
12	0.070095600	K_{12}	0.069179323	0.073672867	0.071420842	$A_{11}^3 \otimes \mathbb{Z}$		<U
13	0.074873919	D_{13}^*	0.068721956	0.072715163	0.071034583	$K_{12} \otimes \mathbb{Z}$	<G	<U
14	0.074954492	D_{14}^*	0.068308096	0.071887858	0.071455542	$K_{12} \otimes A_2$	<G	<U
15	0.075039738	D_{15}^*	0.067931488	0.071164794	0.071709124	$K_{12} \otimes A_3^*$	<G	
16	0.06830	Λ_{16}	0.067587055	0.070526523	0.071668753	$K_{12} \otimes D_4$		
17	0.075216213	D_{17}^*	0.067270625	0.069958259	0.06910	$\Lambda_{16} \otimes \mathbb{Z}$	<G	<U
18	0.075304924	D_{18}^*	0.066978741	0.069448546	0.06953	$\Lambda_{16} \otimes A_2$	<G	
19	0.075392902	D_{19}^*	0.066708503	0.068988355	0.06982	$\Lambda_{16} \otimes A_3^*$	<G	
20	0.075479665	D_{20}^*	0.066457468	0.068570467	0.06988	$\Lambda_{16} \otimes D_4$	<G	
21	0.075554858	A_{21}^*	0.066223553	0.068189035	0.06998	$\Lambda_{16} \otimes D_5^*$	<G	
22	0.075577414	A_{22}^*	0.066004976	0.067839266	0.06987	$\Lambda_{16} \otimes E_6^*$	<G	
23	0.075601888	A_{23}^*	0.065800200	0.067517194	0.06973	$\Lambda_{16} \otimes E_7^*$	<G	
24	0.06577	Λ_{24}	0.065607893	0.067219503	0.06941	$\Lambda_{16} \otimes E_8$		
25	0.075655156	A_{25}^*	0.065426891	0.066943400	0.06640	$\Lambda_{24} \otimes \mathbb{Z}$	<G	<U
26	0.075683386	A_{26}^*	0.065256179	0.066686513	0.06678	$\Lambda_{24} \otimes A_2$	<G	
27	0.075712385	A_{27}^*	0.065094858	0.066446812	0.06708	$\Lambda_{24} \otimes A_3^*$	<G	
28	0.075741975	A_{28}^*	0.064942137	0.066222551	0.06722	$\Lambda_{24} \otimes D_4$	<G	
29	0.075772009	A_{29}^*	0.064797312	0.066012219	0.06737	$\Lambda_{24} \otimes D_5^*$	<G	
30	0.075802366	A_{30}^*	0.064659756	0.065814499	0.06738	$\Lambda_{24} \otimes E_6^*$	<G	
31	0.075832940	A_{31}^*	0.064528911	0.065628241	0.06736	$\Lambda_{24} \otimes E_7^*$	<G	
32	0.075863646	A_{32}^*	0.064404271	0.065452432	0.06720	$\Lambda_{24} \otimes E_8$	<G	
33	0.075894409	A_{33}^*	0.064285386	0.065286178	0.06732	$\Lambda_{24} \otimes AE_9$	<G	
34	0.075925169	A_{34}^*	0.064171846	0.065128688	0.06722	$\Lambda_{24} \otimes D_{10}^+$	<G	
35	0.075955874	A_{35}^*	0.064063282	0.064979257	0.06720	$\Lambda_{24} \otimes A_{11}^3$	<G	
36	0.075986480	A_{36}^*	0.063959359	0.064837254	0.06718	$\Lambda_{24} \otimes K_{12}$	<G	
37	0.076016949	A_{37}^*	0.063859771	0.064702116	0.06757	$\Lambda_{24} \otimes K_{12} \otimes \mathbb{Z}$	<G	
38	0.076047252	A_{38}^*	0.063764240	0.064573336	0.06781	$\Lambda_{24} \otimes K_{12} \otimes A_2$	<G	
39	0.076077363	A_{39}^*	0.063672511	0.064450456	0.06799	$\Lambda_{24} \otimes K_{12} \otimes A_3^*$	<G	
40	0.076107259	A_{40}^*	0.063584352	0.064333062	0.06677	$\Lambda_{24} \otimes \Lambda_{16}$	<G	
41	0.076136923	A_{41}^*	0.063499548	0.064220781	0.06713	$\Lambda_{24} \otimes \Lambda_{16} \otimes \mathbb{Z}$	<G	
42	0.076166341	A_{42}^*	0.063417902	0.064113272	0.06736	$\Lambda_{24} \otimes \Lambda_{16} \otimes A_2$	<G	
43	0.076195500	A_{43}^*	0.063339234	0.064010223	0.06753	$\Lambda_{24} \otimes \Lambda_{16} \otimes A_3^*$	<G	
44	0.076224390	A_{44}^*	0.063263376	0.063911352	0.06761	$\Lambda_{24} \otimes \Lambda_{16} \otimes D_4$	<G	
45	0.076253004	A_{45}^*	0.063190174	0.063816399	0.06770	$\Lambda_{24} \otimes \Lambda_{16} \otimes D_5^*$	<G	
46	0.076281336	A_{46}^*	0.063119483	0.063725126	0.06770	$\Lambda_{24} \otimes \Lambda_{16} \otimes E_6^*$	<G	
47	0.076309381	A_{47}^*	0.063051171	0.063637315	0.06768	$\Lambda_{24} \otimes \Lambda_{16} \otimes E_7^*$	<G	
48	0.076337136	A_{48}^*	0.062985115	0.063552764	0.06577	$\Lambda_{24} \otimes \Lambda_{24}$	<G	

[8] Proposed as an upper bound in [6, Tab. I].

1979 for $n = 1$ to 5 [6]. It was extended to $n \leq 10$ in 1982 [7], which also analytically calculated the NSMs of the classical lattices A_n, A_n^*, D_n, and D_n^* for any n. Since then, progress has been much slower. Better lattices were reported for $n = 6$ and 7 in [8] and for $n = 9$ and 10 in [11], although the NSMs of these lattices were only computed numerically. Reference [8] also gave the best known lattice quantizers for $n = 12$, 16, and 24 with numerically computed NSMs. Later, the corresponding exact NSMs were calculated for $n = 6$ [9], $n = 7$ [10], $n = 9$ [13], and $n = 10$ and 12 [12]. In the latter reference, a new best known lattice quantizer was identified for $n = 11$. NSM results for $n \leq 15$ were summarized in [4].

The upper bound of Zador [16, Lemma 5] and the conjectured lower bound from Conway and Sloane [23], which we evaluated numerically by high-resolution trapezoidal integration, are also shown in Tab. I. The Zador upper bound is for arbitrary quantizers; we conjecture that there is always at least one lattice quantizer satisfying this bound. Taken together, these two results provide conjectured lower and upper bounds on the NSM of optimal lattice quantizers. While intuitively plausible, these remain unproven.

For each dimension, the last four columns of Tab. I list the best product lattice that can be constructed from the lattices given on the preceding rows of the table. While Theorem 7 establishes that these lattices are not even locally optimal, they nevertheless improve significantly on the previously lowest reported NSMs. For brevity, we use $\otimes$ to denote the Cartesian product of two lattices *with the optimal choice of relative scale*. More precisely, $\Lambda_1 \otimes \Lambda_2 \triangleq \Lambda_1 \times a_{\text{opt}} \Lambda_2$ with a_{opt} given by Corollary 5. Thus, $\Lambda_1 \otimes \Lambda_2$ and $\Lambda_2 \otimes \Lambda_1$ are equivalent up to scaling, and have the same NSM.

The first dimension in which this construction provides a better lattice quantizer than previously reported is $n = 13$, where our best product lattice is $K_{12} \otimes \mathbb{Z}$. With an NSM of 0.0710, it is the first reported 13-dimensional lattice whose NSM falls below the generic upper bound 0.0727. It is significantly better than the currently best known 13-dimensional lattice quantizer D_{13}^*, whose NSM is 0.0749. Our optimized product lattices are also the first reported lattices which lie below the Zador upper bound in dimensions $n = 14$, 17, and 25.

VIII. Conclusion

As far as we know, Table I is currently the most extensive published table of record NSM lattices. Previous tables have not gone beyond $n = 16$, apart from a numerical estimate for $n = 24$. It is disconcerting that in many dimensions the smallest known NSMs are for product lattices, since we have proven that these cannot be optimal. This reflects the complexity of designing good lattices and of evaluating their NSMs. The theoretical results in this paper provide some properties of optimal lattice quantizers, which in combination with the tabulated product lattices may hopefully provide guidance and benchmarks for further progress in lattice quantization.

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