Cyclic framed little disks algebras, Grothendieck-Verdier duality and handlebody group representations

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Abstract

We characterize cyclic algebras over the associative and the framed little 2-disks operad in any symmetric monoidal bicategory. The cyclicity is appropriately treated in a coherent way, i.e. up to coherent isomorphism. When the symmetric monoidal bicategory is specified to be a certain symmetric monoidal bicategory of linear categories subject to finiteness conditions, we prove that cyclic associative and cyclic framed little 2-disks algebras, respectively, are equivalent to pivotal Grothendieck-Verdier categories and ribbon Grothendieck-Verdier categories, a type of category that was introduced by Boyarchenko-Drinfeld based on Barr's notion of a ⋆-autonomous category. We use these results and Costello's modular envelope construction to obtain two applications to quantum topology: I) We extract a consistent system of handlebody group representations from any ribbon Grothendieck-Verdier category inside a certain symmetric monoidal bicategory of linear categories and show that this generalizes the handlebody part of Lyubashenko's mapping class group representations. II) We establish a Grothendieck-Verdier duality for the category extracted from a modular functor by evaluation on the circle (without any assumption on semisimplicity), thereby generalizing results of Tillmann and Bakalov-Kirillov.

Contents

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1   Introduction and summary

Algebras over the associative and the framed little 2-disks operad with values in the symmetric monoidal bicategory of (linear) categories are well-known to be equivalent to (linear) monoidal and balanced braided monoidal categories, respectively. It is equally well-known that both operads naturally have the structure of a cyclic operad, as introduced by Getzler and Kapranov [GK95]. This means that they come with a specific way to cyclically permute inputs of operations with the output. Given a cyclic operad, one may, of course, forget the cyclic structure and consider *ordinary* algebras over it, but one can also consider *cyclic* algebras which are defined to be compatible with the cyclic structure in the appropriate sense. This raises the immediate question how *cyclic* algebras (with values in a symmetric monoidal bicategory such as suitable bicategories of linear categories) over the associative and the framed little 2-disks operad (hereafter just referred to as framed little disks operad) can be characterized. It is implicitly understood here that we will consider these algebraic structures *up to coherent isomorphism* in the appropriate sense. In this article, we give an explicit characterization of these cyclic algebras in terms of *Grothendieck-Verdier duality*; the precise statements appear as Theorems 4.12 and 5.12 below and will also be momentarily discussed in course of this introduction. Grothendieck-Verdier duality is a notion proposed and investigated by Boyarchenko and Drinfeld in [BD13] as a weakening of rigidity. It is based on earlier notions due to Barr [Bar79]. Afterwards, we present applications of our results in quantum topology: We give a new class of explicitly computable handlebody groups representations that satisfy excision. These representations generalize the handlebody part of Lyubashenko's mapping class group representations [Lyu95a, Lyu95b, Lyu96]. Moreover, we prove a duality result for the category obtained from a modular functor by evaluation on the circle.

    Let us give the precise statements and elaborate on the structure of the article: The sections 2 to 5 are of purely operadic nature and devoted to the characterization of cyclic associative and framed little disks algebras in an arbitrary symmetric monoidal bicategory $\mathcal{M}$. The fact that we consider cyclic algebras in a symmetric monoidal *bi*category has two reasons. Firstly, both the associative and the framed little disks operad are aspherical, i.e. they may be seen as category-valued (in fact, groupoid-valued) operads. This makes it possible and natural to consider algebras over these operads in a symmetric monoidal bicategory because any bicategory is naturally enriched over categories. Secondly, this choice perfectly matches with our motivation coming from quantum topology as our applications will show.

    We use Costello's description of cyclic (and modular) operads [Cos04], i.e. we describe a category-valued cyclic operad as a symmetric monoidal functor $\mathcal{O} : \mathsf{Forests} \longrightarrow \mathsf{Cat}$ from the *forest category* to the category of categories (symmetric monoidal functors will here be automatically understood in the weak sense). The objects of $\mathsf{Forests}$ are the graphs with one vertex and n legs for $n \geq 1$ (the so-called *corollas*) and finite disjoint unions

thereof. Morphisms are given by forests; we recall the necessary details in Section 2.1. Disjoint union provides a symmetric monoidal structure. In order to define cyclic $\mathcal{O}$-algebras in a symmetric monoidal bicategory $\mathcal{M}$ in Section 2.4, we define the cyclic endomorphism operad $\mathrm{End}_{\kappa}^{X} : \mathsf{Forests} \longrightarrow \mathsf{Cat}$ for any object X of $\mathcal{M}$ equipped with a symmetric non-degenerate pairing $\kappa : X \otimes X \longrightarrow I$, i.e. a morphism from $X \otimes X$ to the monoidal unit I of $\mathcal{M}$ that is symmetric up to coherent isomorphism and exhibits X as its own dual in the homotopy category of $\mathcal{M}$. Explicitly, $\mathrm{End}_{\kappa}^{X}$ will send a corolla T with set $\mathsf{Legs}(T)$ of legs to the morphism category $\mathcal{M}(X^{\otimes \mathsf{Legs}(T)}, I)$, where $X^{\otimes \mathsf{Legs}(T)}$ is the unordered monoidal product of X over $\mathsf{Legs}(T)$ that we define in Section 2.3. The structure of a cyclic $\mathcal{O}$-algebra on (X, κ) is a symmetric monoidal transformation $A : \mathcal{O} \longrightarrow \mathrm{End}_{\kappa}^{X}$.

Cyclic associative algebras can be characterized in more concrete terms if $\mathcal{M}$ is the symmetric monoidal bicategory $\mathsf{Lex}^{\mathsf{f}}$ of finite linear categories over an algebraically closed field k that we fix throughout the article (this symmetric monoidal bicategory is frequently used in quantum algebra). Its objects are finite k-linear categories (the full definition is given on page 11), its 1-morphisms are left exact functors, and its 2-morphisms are natural transformations. For this target category, we characterize cyclic associative algebras by means of Grothendieck-Verdier duality [BD13]: A *Grothendieck-Verdier category* (Definition 4.3) is

- a monoidal category $\mathcal{C}$ together with an object $K \in \mathcal{C}$ (the *dualizing object*) such that the functor $\mathcal{C}(K, X \otimes -)$ is representable for every $X \in \mathcal{C}$

- subject to the condition that the functor $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ sending X to a representing object DX for $\mathcal{C}(K, X \otimes -)$ is an equivalence.

One should understand DX as the *dual* of X. The functor D is referred to as the *duality functor*.

If $\mathcal{C}$ is rigid, one obtains such a structure with the monoidal unit as the dualizing object, but the notion of a Grothendieck-Verdier category is strictly weaker (see e.g. [BD13, Example 2.3]). A *pivotal structure* on a Grothendieck-Verdier category (Definition 4.9) consists of natural isomorphisms $\psi_{X,Y} : \mathcal{C}(K, X \otimes Y) \longrightarrow \mathcal{C}(K, Y \otimes X)$ subject to two coherence conditions. We should remark that our conventions are dual to the ones in [BD13] for convenience. Pivotal Grothendieck-Verdier categories can also be defined inside $\mathsf{Lex}^{\mathsf{f}}$ (meaning that all structure consists of (higher) morphisms in $\mathsf{Lex}^{\mathsf{f}}$ instead of Cat), which allows us to formulate our first main result:

Theorem 4.12. *The structure of a cyclic associative algebra in $\mathsf{Lex}^{\mathsf{f}}$ amounts precisely to a pivotal Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$.*

Theorem 4.12 can be further combined with a result of Street who proves in [Str04, Proposition 3.2] that any Grothendieck-Verdier category aka $\star$-autonomous category can be equivalently described as a Frobenius pseudomonoid, see also Remark 4.14. Then Theorem 4.12 tells us that a cyclic associative algebra in $\mathsf{Lex}^{\mathsf{f}}$ will in particular inherit the structure of a Frobenius pseudomonoid.

The strategy for the proof of Theorem 4.12 is as follows: When considering cyclic algebras e.g. in vector spaces, it is standard that a cyclic $\mathcal{O}$-algebra is an ordinary $\mathcal{O}$-algebra plus an invariant pairing [GK95, MSS02]. The same remains true in a higher categorical context (and in particular the bicategorical context considered in the paper) with the subtle difference that the invariance of the pairing becomes *structure* instead of just a *property* and leads to two types of coherence conditions that we identify in the *Lifting Theorem* 3.1 (that gives us the structure to *lift* a non-cyclic algebra to a cyclic one). We formulate the Lifting Theorem in Corollary 3.3 more concretely for an operad given in terms of generators and relations. On this basis, Theorem 4.12 can be obtained through relatively tedious algebraic manipulations and several facts on finite linear categories extracted from [FSS20]. It is important to note that we can explicitly characterize cyclic associative algebras in *any* symmetric monoidal bicategory, but prove the relation to Grothendieck-Verdier duality only for $\mathsf{Lex}^{\mathsf{f}}$ or similar categories (Remark 4.13). For example, the description of cyclic associativity through Grothendieck-Verdier duality is not possible in the bicategory Cat of categories.

In order to treat the cyclic framed little disks operad $\mathfrak{f}E_2$ in a similar way by means of the Lifting Theorem, we use the presentation of this operad through ribbon braids [SW03]. On the operad of ribbon braids, we establish a cyclic structure (Proposition 5.2) which under the equivalence to the framed little disks operad corresponds to the cyclic structure coming from the identification of $\mathfrak{f}E_2$ with the cyclic operad of genus zero

surfaces (or also the cyclic structure from [Bud08]), see Proposition 5.3. We may then prove the second main result:

Theorem 5.13. *The structure of a cyclic framed little disks algebra in $\mathsf{Lex}^{\mathsf{f}}$ is equivalent to a ribbon Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$.*

The precise definition of a ribbon structure on a (pivotal) Grothendieck-Verdier category $\mathcal{C}$ is given in Section 5.4: Roughly, it amounts to a braiding, i.e. natural isomorphisms $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ subject to the usual hexagon relations, and a balancing, i.e. a natural automorphism $\theta_X : X \longrightarrow X$ with $\theta_I = \mathrm{id}_I$, $\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$; moreover, one requires $D\theta_X = \theta_{DX}$. Together, the braiding and the balancing give rise to a pivotal structure. Therefore, a ribbon Grothendieck-Verdier category may be equivalently described as a pivotal ribbon Grothendieck-Verdier category with a compatibility condition on the pivotal structure, braiding and balancing (Definition 5.9) as we explain in Lemma 5.10.

In order to profit from the results in concrete applications, we make use of the following construction: A modular operad [GK98] is, roughly speaking, a cyclic operad additionally admitting self-compositions of operations. To any cyclic operad, one may assign the 'smallest' modular operad containing it, namely the modular envelope [Cos04]. For a given cyclic operad $\mathcal{O}$, it is obtained by freely completing it to a modular operad (in a homotopically correct way). Moreover, any cyclic algebra over this cyclic operad extends to a modular algebra over its modular envelope by a purely abstract argument. The reason why this becomes particularly interesting for the associative and framed little disks operad is that their modular envelopes have been computed by Costello [Cos04] and Giansiracusa [Gia11] in terms of interesting and well-studied objects in low-dimensional topology. In light of these results, Theorem 4.12 and 5.12 have important consequences in quantum topology that are treated in Section 7 and summarized now.

Application I: Handlebody group representation from ribbon Grothendieck-Verdier structures.
By a result of Giansiracusa [Gia11, Theorem A] there is a canonical map from the modular envelope of $\mathsf{f}E_2$ to the modular operad $\mathsf{Hbdy}^{\mathsf{a}}$ of handlebodies (the subscript 'a' indicates the restriction to certain allowed handlebodies used in [Gia11]: the three-dimensional ball and the three-dimensional ball with one embedded disk are excluded). On the level of topological modular operads, this map is an isomorphism between connected components and a homotopy equivalence on all connected components except for the one of the solid closed torus. By means of Theorem 5.12 we may now prove that a ribbon Grothendieck-Verdier category leads to a consistent system of handlebody group representations:

Theorem 7.9 (combined with Theorem 7.8). *Let $\mathcal{C}$ be a ribbon Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$ with dualizing object K. For integers $g, n \geq 0$ and any family $X_1, \ldots, X_n \in \mathcal{C}$ of objects in $\mathcal{C}$, the finite-dimensional morphism space*

$$V_{g,n}(X_1, \ldots, X_n) := \mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g}) \tag{1.1}$$

defined using the canonical coend $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes DX$ (D is the duality functor of $\mathcal{C}$) comes naturally with an action of the handlebody group, i.e. the mapping class group of the handlebody of genus g and n embedded disks in the boundary, whenever $(g,n) \neq (1,0)$. The vector spaces (1.1) behave locally under the sewing of handlebodies. More explicitly, there are canonical isomorphisms

$$\oint^{Y \in \mathcal{C}} V_{g,n+2}(-, Y, DY) \cong V_{g+1,n}(-) \ ,$$

$$\oint^{Y \in \mathcal{C}} V_{g_1,n+1}(-, Y) \otimes V_{g_2,m+1}(-, DY) \cong V_{g_1+g_2,n+m}(-)$$

of left exact functors $\mathcal{C}^{\boxtimes n} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ and $\mathcal{C}^{\boxtimes (n+m)} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, respectively, where $\oint$ is the left exact coend. These isomorphisms are compatible with the handlebody group actions.

For the precise definition of $\mathbb{F}$, we refer to the statement of Theorem 7.8 in the main text. We refer to Remark 7.11 for a comment on the case $(g, n) = (1, 0)$.

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Strictly speaking, the objects $X_1, \ldots, X_n \in \mathcal{C}$ should not be thought of as ordered, but rather attached to the embedded disks in the boundary of the handlebody. In order to obtain the vector space (1.1), an order is *chosen*. These subtleties are suppressed in the above Theorem for presentation purposes, but explained in detail in Remark 7.10. A graphical presentation summarizing the Theorems 5.12 and 7.9 is given in Figure 1.

Theorem 7.9 applies in particular to finite ribbon categories in the sense of [EGNO15]. For a finite ribbon category that arises as the category of finite-dimensional modules over a ribbon Hopf algebra, Theorem 7.9 specializes to the following statement:

Corollary 7.13. *Let A be a finite-dimensional ribbon Hopf algebra and denote by A^*_{coadj} the dual of A with coadjoint action. Then for any non-negative integers g and n with $(g,n) \neq (1,0)$ and any finite-dimensional A-modules $X_1, \ldots, X_n$, the vector space*

$$\mathrm{Hom}_A\left(k, X_1 \otimes \cdots \otimes X_n \otimes \left(A^*_{coadj}\right)^{\otimes g}\right)$$

of A-invariants of the module $X_1 \otimes \cdots \otimes X_n \otimes \left(A^*_{coadj}\right)^{\otimes g}$ comes canonically with an action of the mapping class group of the handlebody with genus g and n disks embedded in the boundary of the handlebody.

Note that A is not assumed to be factorizable. In Example 5.14, we consider the handlebody group representations for a ribbon Grothendieck-Verdier category whose Grothendieck-Verdier duality does not come from rigidity.

The basic ingredient for the proof of Theorem 7.9 is clearly Theorem 5.12 because it abstractly leads to the desired handlebody representation using Giansiracusa's Theorem. To derive the explicit description from above, however, one needs further tools, namely a locality property for the handlebody group representations. This is afforded by an excision result (Theorem 6.4) that states, roughly, that modular algebras with values in Lex^f behave 'locally' under Lyubashenko's left exact coend $\oint$ [Lyu96]. As a by-product, this opens a new perspective on Lyubashenko's left exact coend as the composition in the cyclic (actually, even modular) endomorphism operad.
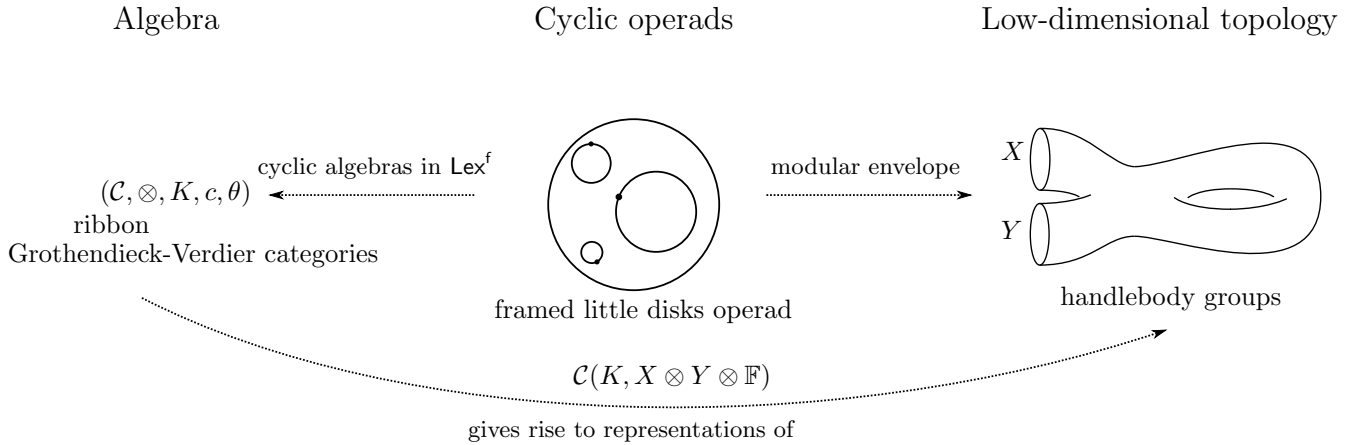
Figure 1: A visualization of the different mathematical structures involved in our main result and their relations.

In the special case that $\mathcal{C}$ is actually a modular category (meaning that the Grothendieck-Verdier duality actually comes from rigidity and that the braiding is non-degenerate; we review the terminology in Section 7.2), Theorem 7.9 relates to classical constructions as follows: Under the far stronger assumption of modularity, $\mathcal{C}$ yields a modular functor, i.e. a consistent system of projective mapping class group representations by the Lyubashenko construction [Lyu95a, Lyu95b, Lyu96]. The restriction to handlebody group representations will

then coincide with the handlebody group representations from Theorem 7.9. Note however that Theorem 7.9 can be applied to way more general situations: Neither rigidity nor non-degeneracy of the braiding are needed. As a price to pay, one just finds handlebody group representations, but not mapping class group representations.

The relation to Lyubashenko's mapping class group representations is quite appealing on a conceptual level: In the original papers [Lyu95a, Lyu95b, Lyu96], the mapping class group actions are established through tedious computations relying on a presentation of mapping class groups in terms of generators and relations; in [FS17] a description of these mapping class group actions through the combinatorial Lego-Teichmüller game of Bakalov and Kirillov [BK00] is given. Our approach using the modular envelope and our characterization of cyclic fE_2-algebras puts at least the handlebody part of Lyubashenko's construction on purely topological grounds. In particular, it allows us to obtain from a topological construction the conformal blocks of a modular category (meaning the spaces (1.1)) — at least as vector spaces with handlebody group representations — without making any algebraic ad-hoc ansatz.

Application II: Grothendieck-Verdier duality for the evaluation of a modular functor on the circle. For many of the representation categories appearing in conformal and topological field theory, *duality results* have been established:

- In [Hua08] Huang gives a class of vertex operator algebras whose category of modules is rigid.

- In [BDSPV15] Bartlett, Douglas, Schommer-Pries and Vicary prove that the value of an extended three-dimensional topological field theory on the circle is rigid.

- In [BK01] Bakalov and Kirillov prove under the assumption of semisimplicity, simplicity of the monoidal unit and a normalization axiom for the sphere that the category $\mathcal{C}$ obtained by evaluation of a modular functor on the circle is *weakly rigid*, i.e. they conclude that there is an anti-equivalence $* : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ with natural isomorphisms

$$\mathcal{C}(X, Y \otimes Z) \cong \mathcal{C}(Y^* \otimes X, Z) \cong \mathcal{C}(X \otimes Z^*, Y) \quad \text{for all} \quad X, Y, Z \in \mathcal{C} \ . \tag{1.2}$$

A similar result is given by Turaev in [Tur94, V.] for so-called *rational* modular functors. In [Tur94, BK01] semisimplicity is directly imposed. Tillmann [Til98, Section 3] presents related duality results by working in a different category of linear categories in which semisimplicity for the category on the circle is obtained as a consequence. We should emphasize here again that, in contrast to [Tur94, Til98, BK01], we will not build in semisimplicity (neither directly or indirectly) into our definitions; in fact, going beyond the semisimple case is one of our main motivations.

We refer to [BDSPV15, Section 1.3] for an excellent discussion of these (and more) duality results including more references to the literature.

Our characterization of cyclic framed little disks algebras allows us to improve on the third result. Without the assumption of semisimplicity, simplicity of the monoidal unit and a normalization axiom for the sphere, we prove:

Theorem 7.17. *The linear category extracted from a* $\mathsf{Lex}^{\mathsf{f}}$*-valued modular functor inherits a ribbon Grothendieck-Verdier structure. In particular, it is a pivotal Grothendieck-Verdier category.*

For the notion of a modular functor, various slightly different definitions exists. We fix our notion in Definition 7.15 following essentially [FS17, SW21a].

It might seem a little confusing that we find a Grothendieck-Verdier structure *without* the requirement that the dualizing object actually coincides with the monoidal unit as one would assume after seeing (1.2) (in language of [BD13], such a Grothendieck-Verdier category would be called an *r-category*; this is still strictly weaker than rigidity). However, we explain in Corollary 7.18 that once one imposes the (in fact very strong) requirements of semisimplicity, simplicity of the monoidal unit and a normalization axiom, the dualizing object will be forced to coincide with the monoidal unit. Therefore, Theorem 7.17 actually recovers the results of Tillmann and Bakalov-Kirillov as a special case.

2 Cyclic and modular operads and algebras over them

In this section, we recall the notions of cyclic and modular operads introduced by Getzler and Kapranov in [GK95, GK98]. The definition of *algebras* over cyclic and modular operads requires the notion of an endomorphism operad that is defined by means of non-degenerate pairings.

2.1 Preliminaries on the definition of cyclic and modular operads via graphs

In [Cos04] Costello gives a very efficient description of operads, cyclic operads and modular operads based on different categories of graphs. We will adopt this description and therefore briefly recall the most important definitions: A *graph* consists of a set H of *half edges* and a set V of *vertices* together with a map $H \longrightarrow V$ and an involution $\iota : H \longrightarrow H$ specifying how half edges are glued together. The orbits of the involution ι are the *edges* of the graph. Fixed points of ι are called *external legs* (*legs*, for short). We denote by $\mathsf{Legs}(\Gamma)$ the set of external legs of a graph Γ. We may realize a graph Γ as a topological space $|\Gamma|$ with the vertices of Γ as the 0-cells and the edges of Γ as the 1-cells. A *corolla* is a graph with one vertex and only external legs. Often we will denote a graph as a pair $\Gamma = (V, H)$ of the set of vertices and the set of half edges suppressing all other parts of the structure in the notation.

Let $\Gamma = (V, H)$ and $\Gamma' = (V', H')$ be graphs. A *morphism* of graphs consists of maps $V \longrightarrow V'$ and $H \longrightarrow H'$ which are compatible with the graph structure in the obvious way.
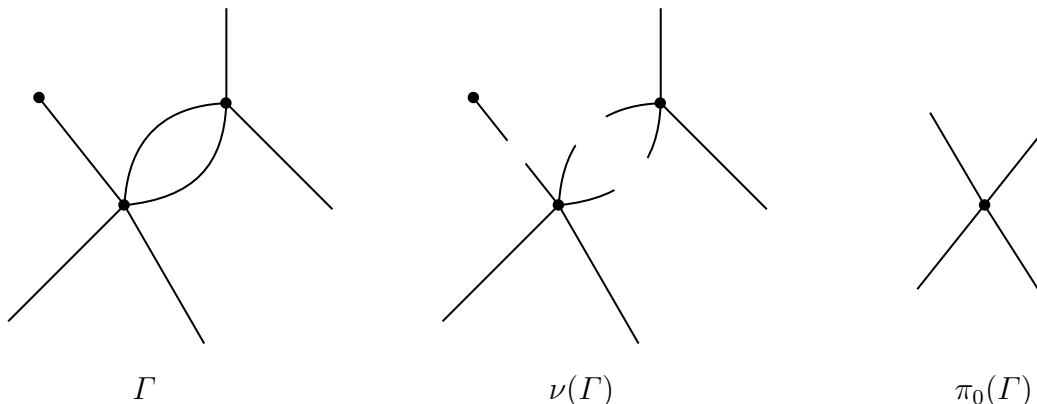

Figure 2: A sketch for the graphs $\nu(\Gamma)$ and $\pi_0(\Gamma)$.

Given a graph Γ we can form a new graph $\nu(\Gamma)$ by cutting open all internal edges. Formally, this replaces

the involution on the half edges by the identity map. We can also form a graph $\pi_0(\Gamma)$ by contracting all internal edges. In Figure 2 a pictorial presentation of these operations is given.

The category Graphs has as objects graphs which are finite disjoint unions of corollas. A morphism $\gamma_1 \longrightarrow \gamma_2$ is given by an equivalence class of a graph Γ together with isomorphisms $\varphi_1 : \gamma_1 \longrightarrow \nu(\Gamma)$ and $\varphi_2 : \gamma_2 \longrightarrow \pi_0(\Gamma)$; note that here φ_1 and φ_2 are morphisms of graphs in the above sense, but not morphisms in the category Graphs (which is named afters its morphisms). Two such triples $(\Gamma, \varphi_1, \varphi_2)$ and $(\Gamma', \varphi_1', \varphi_2')$ are equivalent if there exists an isomorphism $\psi : \Gamma \longrightarrow \Gamma'$ satisfying $\varphi_1' = \nu(\psi) \circ \varphi_1$ and $\varphi_2' = \pi_0(\psi) \circ \varphi_2$. The composition $\Gamma_2 \circ \Gamma_1$ is defined by replacing the vertices of Γ_2 by the graph Γ_1, see Figure 3. We denote by Forests the subcategory of Graphs whose objects are those objects of Graphs which do not contain a corolla with zero legs and whose morphisms are forests, i.e. disjoint unions of contractible graphs.
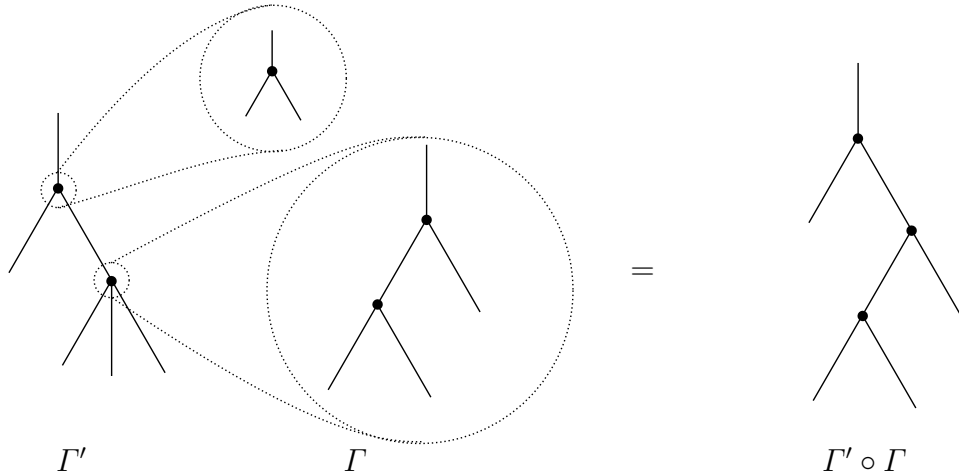
Figure 3: A sketch for the composition $\Gamma' \circ \Gamma$ of two morphisms Γ, Γ' in Graphs.

Finally, we define the category RForests of *rooted forests*: A *rooted graph* is a graph Γ equipped with a section $s : V(\pi_0(\Gamma)) \longrightarrow \mathsf{Legs}(\Gamma)$ of the obvious map $\mathsf{Legs}(\Gamma) \longrightarrow V(\pi_0(\Gamma))$, i.e. in each component of Γ we distinguish an external leg that we refer to as the *root*. Morphisms of rooted graphs are morphisms of the underlying graph which are compatible with the specified sections. Note that a rooted forest Γ induces the structure of a rooted graph on $\nu(\Gamma)$ by declaring for every vertex the edge in the direction of the root of Γ as the root of the vertex in $\nu(\Gamma)$. The category RForests is defined just like Forests with objects and morphisms replaced by their rooted version. There is a functor RForests $\longrightarrow$ Forests which forgets the root.

Remark 2.1. Our definitions slightly differ from the definitions used in [Cos04] where only at least trivalent corollas are allowed as objects of Forests and RForests. In other words, in [Cos04] operads without arity zero and arity one operations are considered.

The categories RForests, Forests and Graphs are used in [Cos04] to give a definition of operads, cyclic operads and modular operads with values in a symmetric monoidal (higher) category $\mathcal{S}$, see also [Gia13, Section 2]. Before making this precise in the special case that we are interested in, let us give the idea: An ordinary/cyclic/modular operad in $\mathcal{S}$ is a symmetric monoidal functor

$$\mathcal{O} : \mathsf{RForests}/\mathsf{Forests}/\mathsf{Graphs} \longrightarrow \mathcal{S} \ .$$

Here 'symmetric monoidal' has to be understood in the appropriate weak sense (to be made precise momentarily in the bicategorical case). This very nicely allows us to define ordinary/cyclic/modular operad for which the associativity of operadic composition is relaxed up to coherent homotopy.

In this paper, the emphasis lies on aspherical topological operads, i.e. topological operads whose spaces of operations are aspherical, where aspherical means that all homotopy groups of degree two and higher are trivial.

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Famous examples include the associative operad and the (framed) little disks operad that will be treated in Section 4 and 5 of this article, respectively. Such operads can naturally be regarded as groupoid-valued or, more generally, category-valued. We will see below in Section 2.4 that for category-valued operads, one can naturally consider algebras in any symmetric monoidal bicategory. Therefore, we will develop the theory of ordinary/cyclic/modular operads and their algebras with values in a symmetric monoidal *bi*category, which is also in line with our motivations coming from quantum topology.

We assume some familiarity with the theory of symmetric monoidal bicategories; we refer e.g. to [SP09, Chapter 2] for a detailed discussion and to [Lei98] for a short introduction (however without the treatment of monoidal structures). In particular, we rely on the following notions: By a *bicategory* we mean a three-layered categorical structure with objects, 1-morphisms and 2-morphisms (sometimes also called 0-cells, 1-cells and 2-cells) in the weak sense. A morphism between bicategories (sometimes also referred to as (weak) 2-functor) will just be called *functor*. *Symmetric monoidal functors* between *symmetric monoidal bicategories* are to be understood in a strong (not in any kind of lax) sense unless otherwise stated. In particular, a symmetric monoidal functor between symmetric monoidal bicategories comprises various sorts of coherence data subject to coherence conditions. We will briefly illustrate this after the next definition.

Definition 2.2. Let $\mathcal{M}$ be a symmetric monoidal bicategory. An *operad* in $\mathcal{M}$ is a symmetric monoidal functor $\mathcal{O} : \mathsf{RForests} \longrightarrow \mathcal{M}$, where we consider $\mathsf{RForests}$ as a symmetric monoidal bicategory with only trivial 2-morphisms. We define *cyclic* and *modular operads* by replacing $\mathsf{RForests}$ with $\mathsf{Forests}$ and Graphs, respectively.

As explained before the definition, we understand the symmetric monoidal functors $\mathcal{O}$ out of $\mathsf{RForests}$, $\mathsf{Forests}$ and Graphs to $\mathcal{M}$ to be weak and not strict. This means in particular that for composable morphisms $T \xrightarrow{\Gamma} T' \xrightarrow{\Gamma'} T''$ the 1-morphisms $\mathcal{O}(\Gamma' \circ \Gamma)$ and $\mathcal{O}(\Gamma') \circ \mathcal{O}(\Gamma)$ are not necessarily equal, but related by a specific 2-isomorphism

$$\mathcal{O}(\Gamma' \circ \Gamma) \cong \mathcal{O}(\Gamma') \circ \mathcal{O}(\Gamma) \tag{2.1}$$

that is part of the data. Furthermore, there are 1-equivalences $\mathcal{O}(T) \otimes \mathcal{O}(T') \xrightarrow{\simeq} \mathcal{O}(T \sqcup T')$.

Let us compare Definition 2.2 to the 'usual' definition of a (symmetric) operad $\mathcal{O}$ in $\mathcal{M}$ which is typically given in terms of the following data:

- Objects $(\mathcal{O}(n))_{n \geq 0}$ in $\mathcal{M}$ carrying a right action of the permutation group Σ_n on n letters. We interpret $\mathcal{O}(n)$ as the object of n-ary operations.

- Maps for all $1 \leq j \leq n$ and $m \geq 0$

$$\circ_j : \mathcal{O}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{O}(n - 1 + m) .$$

We think of these maps as partial composition maps.

Note that we consider a definition of an operad *without* operadic identity.

The description of an operad just given in terms of objects of operations can be extracted from a symmetric monoidal functor $F : \mathsf{RForests} \longrightarrow \mathcal{M}$. To see this, evaluate F on the corolla T_n with $n + 1$ legs which we identify with the set $\{0, \ldots, n\}$ with 0 marked as the root. We then obtain the objects $\mathcal{O}(n) := F(T_n)$. From a permutation $\sigma \in \Sigma_n$, we can construct an automorphism of T_n with T_n as underlying graph. The isomorphism φ_2 permutes the edges $\{1, \ldots, n\}$ according to σ and φ_1 is defined to be the identity of T_n. This gives us a Σ_n-action on $\mathcal{O}(n)$ up to coherent isomorphism. The coherence isomorphisms for the Σ_n-action are a special case of the isomorphisms (2.1) applied to permutation morphisms. By evaluation of F on morphisms in $\mathsf{RForests}$ and monoidality of F we obtain the composition map for the objects $\mathcal{O}(n)$. The equivariance and associativity axioms follow from the functoriality of F (again, these holds up to coherent isomorphism thanks to (2.1)). This can be seen to provide an equivalence between the two descriptions which holds analogously for cyclic and modular operads, see also [Cos04]. In the case of cyclic and modular operads, the cyclic permutation of legs induces an isomorphism $\tau_n : T_n \cong T_n$ in $\mathsf{Forests}$ (or Graphs) with $\tau_n^{n+1} = \mathrm{id}_{T_n}$. As above, the underlying graph

of τ_n is T_n, φ_1 is the identity and φ_2 the map sending the edge i to $i+1$ modulo n (note that τ_n is *not* a morphism in RForests). These morphisms give rise to an action of the cyclic group $\mathbb{Z}_{n+1}$ on $\mathcal{O}(T_n)$ which is part of the more traditional definition of cyclic (and modular) operads.

Again, let us emphasize that this defines operads *without* an operadic identity. However, all operads appearing in this paper have operadic unit, and it will be important to correctly keep track of those. To this end, we make the following definition:

Definition 2.3. Let $\mathcal{O}$ be an $\mathcal{M}$-valued modular operad. An *operadic identity* for $\mathcal{O}$ consists of the following data:

- A 1-morphism $1_{\mathcal{O}} : I \longrightarrow \mathcal{O}(T_1)$ which is a homotopy fixed point for the $\mathbb{Z}_2$-action on $\mathcal{O}(T_1)$, i.e. it comes with a 2-isomorphism

$$
\begin{array}{ccc}
 & & \mathcal{O}(T_1) \\
 & \nearrow{\scriptstyle 1_{\mathcal{O}}} & \downarrow{\scriptstyle \mathcal{O}(\tau_1)} \\
I & \underset{\cong}{\Longrightarrow} & \\
 & \searrow{\scriptstyle 1_{\mathcal{O}}} & \\
 & & \mathcal{O}(T_1)
\end{array}
$$

squaring to the identity,

- and for all $n \in \mathbb{N}$ and all morphisms $\Gamma : T_1 \sqcup T_n \longrightarrow T_n$ in Graphs, 2-isomorphisms

$$
\begin{array}{ccc}
I \otimes \mathcal{O}(T_n) & \xrightarrow{1_{\mathcal{O}} \otimes \mathrm{id}_{\mathcal{O}(T_n)}} \mathcal{O}(T_1) \otimes \mathcal{O}(T_n) \xrightarrow{\simeq} \mathcal{O}(T_1 \sqcup T_n) \\
\downarrow{\scriptstyle \simeq} & \\
\mathcal{O}(T_n) & \xleftarrow{\ \mathcal{O}(\Gamma)\ }
\end{array}
$$

with 2-isomorphism 1_{Γ},

such that the application of the unitality morphisms $(1_{\mathcal{O}}, 1_{\Gamma})$ is compatible with composition (applying the unitality morphisms *before* and *after* composing is required to yield the same result) and cyclic symmetry. In more detail: For all morphisms $\Omega : \sqcup_{i=1}^{m} T_{n_i} \longrightarrow T_{n'}$, $\Gamma : T_1 \sqcup T_{n'} \longrightarrow T_{n'}$ and $\Gamma' : T_1 \sqcup T_{n_j} \longrightarrow T_{n_j}$ in Graphs such that $\Gamma \circ (\tau_1^{\varepsilon} \sqcup \Omega) = \Omega \circ (\mathrm{id}_{\sqcup_{i \neq j} T_{n_i}} \sqcup \Gamma')$, where ε is either 0 or 1,

$$
\begin{array}{ccccc}
I \otimes \mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) & \xrightarrow{\mathrm{id}_I \otimes \mathcal{O}(\Omega)} & I \otimes \mathcal{O}(T_{n'}) & \xrightarrow{1_{\mathcal{O}} \otimes \mathrm{id}_{\mathcal{O}(T_{n'})}} & \mathcal{O}(T_1) \otimes \mathcal{O}(T_{n'}) \xrightarrow{\simeq} \mathcal{O}(T_1 \sqcup T_{n'}) \\
\downarrow{\scriptstyle \simeq} & {\scriptstyle \cong} & \downarrow{\scriptstyle \simeq} & {\scriptstyle 1_{\Gamma}} & \\
\mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) & \xrightarrow{\mathcal{O}(\Omega)} & \mathcal{O}(T_{n'}) & \xleftarrow{\ \mathcal{O}(\Gamma)\ } &
\end{array}
$$

$$
\begin{array}{c}
I \otimes \mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) \xrightarrow{\mathrm{id}_I \otimes \mathcal{O}(\Omega)} I \otimes \mathcal{O}(T_{n'}) \xrightarrow{1_{\mathcal{O}} \otimes \mathrm{id}_{\mathcal{O}(T_{n'})}} \mathcal{O}(T_1) \otimes \mathcal{O}(T_{n'}) \xrightarrow{\simeq} \mathcal{O}(T_1 \sqcup T_{n'}) \\[2ex]
{\scriptstyle 1_{\mathcal{O}} \otimes \mathrm{id}_{\mathcal{O}(\sqcup_{i=1}^{m} T_{n_i})}} \quad \| {\scriptstyle \cong} \\[1ex]
{\scriptstyle \cong} \quad \mathcal{O}(T_1) \otimes \mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) \qquad {\scriptstyle \mathrm{id}_{\mathcal{O}(T_1)} \otimes \mathcal{O}(\Omega)} \\[1ex]
{\scriptstyle 1_{\mathcal{O}} \otimes \mathrm{id}_{\mathcal{O}(\sqcup_{i=1}^{m} T_{n_i})}} \quad \downarrow{\scriptstyle \mathcal{O}(\tau_1^{\varepsilon}) \otimes \mathrm{id}_{\mathcal{O}(\sqcup_{i=1}^{m} T_{n_i})}} \\[1ex]
\mathcal{O}(T_1) \otimes \mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) \\[1ex]
{\scriptstyle 1_{\Gamma'}} \quad \downarrow{\scriptstyle \simeq} \\[1ex]
\mathcal{O}(T_1 \sqcup T_{n_j}) \otimes \mathcal{O}(\sqcup_{i \neq j} T_{n_i}) \xrightarrow{\cong} \mathcal{O}(\Gamma) \\[1ex]
= \quad {\scriptstyle \simeq} \qquad {\scriptstyle \mathcal{O}(\Gamma') \otimes \mathrm{id}} \\[1ex]
\mathcal{O}(\sqcup_{i=1}^{m} T_{n_i}) \xrightarrow{\mathcal{O}(\Omega)} \mathcal{O}(T_{n'})
\end{array}
$$

holds, where the 1-equivalences labeled with $\simeq$ are the coherences of $\mathcal{M}$ and $\mathcal{O}$ and where the unlabeled 2-morphisms come from the homotopy fixed point structure for the operadic unit in the case $\varepsilon = 1$ and trivial for $\varepsilon = 0$. Note that we suppress some coherence isomorphisms for readability in the diagram. We say that $\mathcal{O}$ is a *unital operad* if there exists a unit in the sense just defined (we do not fix the unit).

The distinction of the two cases $\varepsilon = 0, 1$ is important to ensure the right compatibility with the morphisms τ_n.

Units for cyclic and ordinary operads are defined analogously, where in the ordinary case one does not require it to be a homotopy fixed point.

Remark 2.4. One might expect the choice of a unit to be structure on a modular (or cyclic or ordinary) operad $\mathcal{O}$. However, if we have two choices of units $(1_\mathcal{O}, 1_\Gamma)$ and $(1'_\mathcal{O}, 1'_\Gamma)$, there exists a unique 2-isomorphism between $1_\mathcal{O}$ and $1'_\mathcal{O}$ compatible with the 2-isomorphisms 1_Γ and $1'_\Gamma$. It is constructed by using the morphism $c : T_1 \sqcup T_1 \longrightarrow T_1$ that realizes the composition of arity one operations:

$$
\begin{array}{ccc}
 & & \mathcal{O}(T_1) \\
 & \overset{1_\mathcal{O}}{\nearrow} \quad \overset{1'_c}{\Rightarrow} \quad & \uparrow \mathcal{O}(c) \\
I \quad \xrightarrow{1_\mathcal{O} \otimes 1'_\mathcal{O}} \quad & \mathcal{O}(T_1) \otimes \mathcal{O}(T_1) \\
 & \overset{1_c^{-1}}{\Leftarrow} \quad \overset{1'_\mathcal{O}}{\searrow} \quad & \downarrow \mathcal{O}(c) \\
 & & \mathcal{O}(T_1)
\end{array}
$$

In this sense, the existence of an operadic unit is a property of $\mathcal{O}$ and not a structure.

Example 2.5. In this article, the following examples of symmetric monoidal bicategories will be relevant:

	objects	1-morphisms	2-morphisms	monoidal structure
Cat	categories	functors	natural transformations	Cartesian product
Lex$^{\text{f}}$	finite categories over our fixed algebraically closed field k (linear Abelian categories with finite-dimensional morphism spaces, enough projective objects, finitely many isomorphism classes of simple objects such that every object has finite length)	left exact functors	natural transformations	Kelly product (coincides here with the Deligne product)

The symmetric monoidal bicategory Lex$^{\text{f}}$ is defined dually to a finite version of Rex in [BZBJ18], see also [FSS20, Theorem 3.2]. It is frequently used in many areas of representation theory, in particular in quantum algebra. Recall that a k-linear category is finite if and only if it is k-linearly equivalent to the category of finite-dimensional modules over some finite-dimensional k-algebra; we refer e.g. to [DSPS19, Proposition 1.4] for this well-known statement. Note that describing a finite category as the finite-dimensional modules over a finite-dimensional algebra is rarely useful because we will mostly consider finite categories with additional structure like a monoidal product, and this additional structure cannot necessarily be described on the level of the algebra.

Remark 2.6 (Graphical calculus in symmetric monoidal bicategories)**.** Symmetric monoidal bicategories admit a graphical calculus that we briefly recall now: Objects correspond to lines, and the monoidal product corresponds to juxtaposition. The juxtaposition of the empty collection of lines (i.e. no line) represents the monoidal

unit. A 1-morphism $F : \bigotimes_{i=1}^{m} X_i \longrightarrow \bigotimes_{j=1}^{n} Y_j$ will be written as a box

$$Y_1 \mid \ldots \mid Y_n$$
$$\boxed{F}$$
$$X_1 \mid \cdots \mid X_m$$

with m ingoing legs labeled with the objects $X_1, \ldots, X_m$ and n outgoing legs labeled with $Y_1, \ldots, Y_n$. The legs corresponding to the source objects and target objects will be attached to the bottom and the top of the box representing F, respectively. The symmetric braiding will be represented by a crossing of lines (thanks to symmetry, overcrossing and undercrossing need not be distinguished). A 2-morphism α between 1-morphisms F and G with coinciding source and target object will be represented by an arrow

$$Y_1 \mid \ldots \mid Y_n \qquad\qquad Y_1 \mid \ldots \mid Y_n$$
$$\boxed{F} \quad \xrightarrow{\ \alpha\ } \quad \boxed{G}$$
$$X_1 \mid \cdots \mid X_m \qquad\qquad X_1 \mid \cdots \mid X_m$$

allowing us to efficiently write commuting diagrams for 2-morphisms. We suppress coherence morphisms in the graphical calculus because these can be inserted in an essentially unique way (up to canonical higher isomorphism).

2.2 Non-degenerate pairings

For an operad $\mathcal{O}$ in Cat, one can define an algebra over $\mathcal{O}$ in a symmetric monoidal bicategory $\mathcal{M}$ as an operad map from $\mathcal{O}$ to the Cat-valued endomorphism operad built from some object in $\mathcal{M}$ that is supposed to carry the algebra structure; we refer to [MSS02, LV12, Fre17] for an introduction to the theory of operads and the algebras over them. To generalize the notion of an algebra over an operad to cyclic and modular operads, we need a cyclic and modular version of an endomorphism operad. This is accomplished by considering (non-degenerate and symmetric) pairings on objects in $\mathcal{M}$ [GK95]. If we are dealing with vector spaces as our target category, then it is clear what is meant by a non-degenerate pairing. For higher categories, a little more care is required.

Definition 2.7. Let $\mathcal{M}$ be a symmetric monoidal bicategory. A *pairing* on $X \in \mathcal{M}$ is defined to be a morphism $\kappa : X \otimes X \longrightarrow I$, where $I \in \mathcal{M}$ is the monoidal unit of $\mathcal{M}$. A pairing $\kappa : X \otimes X \longrightarrow I$ is called *non-degenerate* if κ exhibits X as its own dual in the homotopy category of $\mathcal{M}$ (by symmetry left and right dual coincide). The pairing κ exhibiting X as its own dual means the following: There is a map $\Delta : I \longrightarrow X \otimes X$, called *coevaluation*, such that κ and Δ satisfy the usual snake relations

$$\boxed{\kappa} \quad \cong \quad \Big| \qquad \text{and} \qquad \boxed{\kappa} \quad \cong \quad \Big| \tag{2.2}$$
$$\boxed{\Delta} \qquad\qquad\qquad\qquad\qquad \boxed{\Delta}$$

up to isomorphism, i.e. they lead to *snake isomorphisms* instead of snake relations.

Remark 2.8. The isomorphisms in (2.2) can always be chosen to be coherent [Pst14, Section 2], and our convention will be to always choose them that way, see also Remark 2.11 for more details.

Remark 2.9. If $\kappa : X \otimes X \longrightarrow I$ is a non-degenerate pairing on an object X in a symmetric monoidal bicategory, we obtain adjunctions $- \otimes X \dashv - \otimes X$ and $X \otimes - \dashv X \otimes -$, i.e. natural equivalences

$$\mathcal{M}(Y \otimes X, Z) \simeq \mathcal{M}(Y, Z \otimes X) \, , \tag{2.3}$$
$$\mathcal{M}(X \otimes Y, Z) \simeq \mathcal{M}(Y, X \otimes Z)$$

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for $Y, Z \in \mathcal{M}$. These equivalences can be expressed explicitly in terms of the pairing and its coevaluation Δ. For example, (2.3) sends $f : Y \otimes X \longrightarrow Z$ to

$$Y \xrightarrow{\mathrm{id}_Y \otimes \Delta} Y \otimes X \otimes X \xrightarrow{f \otimes \mathrm{id}_X} Z \otimes X \ .$$

As motivated above, we can use the notion of a pairing to define cyclic and modular endomorphism operads. It will be crucial to consider symmetric pairings:

Definition 2.10. Let $\mathcal{M}$ be a symmetric monoidal bicategory. For $X \in \mathcal{M}$, consider the $\mathbb{Z}_2$-action on $X \otimes X$ coming from the symmetric braiding on $\mathcal{M}$ and the induced $\mathbb{Z}_2$-action on the morphism category $\mathcal{M}(X \otimes X, I)$. We define a *symmetric pairing* on X as a homotopy fixed point of the $\mathbb{Z}_2$-action on $\mathcal{M}(X \otimes X, I)$. We call a symmetric pairing *non-degenerate* if the underlying pairing is non-degenerate.

Remark 2.11. Concretely, the structure of a homotopy fixed point on a pairing $\kappa : X \otimes X \longrightarrow I$ consists of a natural isomorphism $\Sigma_\kappa : \kappa \cong \kappa\tau$, where τ is the braiding $X \otimes X \longrightarrow X \otimes X$, such that the composition $\kappa \cong \kappa\tau \cong \kappa\tau^2 = \kappa$ is the identity transformation. The symmetry can dually be described by an isomorphism $\Sigma_\Delta : \Delta \cong \tau\Delta$ for the coevaluation $\Delta : I \longrightarrow X \otimes X$ via
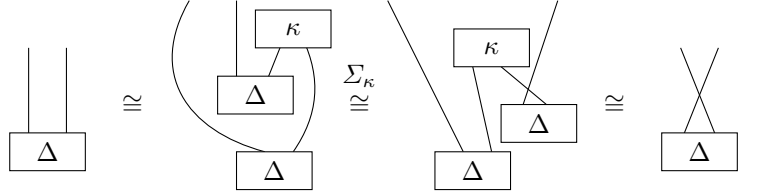
where the unlabeled isomorphisms are the snake isomorphisms. Again, we find that the composition $\Delta \cong \tau\Delta \cong \tau^2\Delta = \Delta$ is the identity. The coherence relations for the snake isomorphism imply that various diagrams one can form using the symmetry isomorphisms Σ_κ and Σ_Δ commute. For example, the diagram
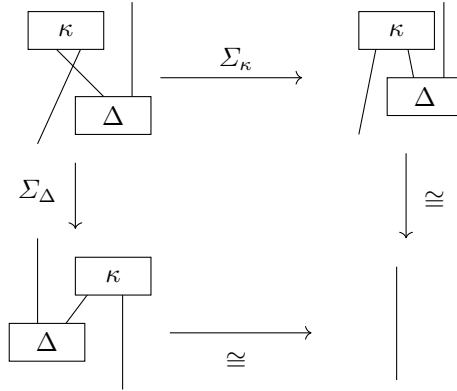
commutes.

2.3 Cyclic and modular endomorphism operads

Having defined non-degenerate symmetric pairings, we are now in a position to define cyclic and modular endomorphism operads: As a first step, let us define for a corolla $T \in \mathsf{Graphs}$ and an object $X \in \mathcal{M}$ in a symmetric monoidal bicategory $\mathcal{M}$ the *unordered monoidal product* $X^{\otimes \mathsf{Legs}(T)}$ (we could make this definition for any finite set instead of the set $\mathsf{Legs}(T)$ of legs of T). The definition uses that $\mathcal{M}$ is symmetric monoidal and can be given as follows: Let $\mathcal{L}$ be the category whose objects are orders of the set $\mathsf{Legs}(T)$ and whose morphisms are bijections compatible with the orders. Consider the natural functor $\mathcal{L} \longrightarrow \mathcal{M}$ sending an object $\{\ell_1 < \ell_2 < \cdots < \ell_n\}$ to $\otimes_{j=1}^n X$ and a morphism to the corresponding permutation of tensor factors. We define $X^{\otimes \mathsf{Legs}(T)}$ as the 2-colimit of this functor. After choosing an order for $\mathsf{Legs}(T)$, the unordered monoidal

product is equivalent to $X^{\otimes|\mathsf{Legs}(T)|}$, but the unordered monoidal product has the advantage that it can be defined *without choosing an order*.

Now for a symmetric pairing $\kappa : X \otimes X \longrightarrow I$ on an object X in a symmetric monoidal bicategory $\mathcal{M}$ and a corolla T, we set

$$\mathrm{End}_\kappa^X(T) := \mathcal{M}\left(X^{\otimes\mathsf{Legs}(T)}, I\right)$$

and extend monoidally. For a morphism $(\Gamma, \varphi_1, \varphi_2) : \sqcup_{i=1}^{k_1} T^{(i)} \longrightarrow \sqcup_{j=1}^{k_2} T'^{(j)}$ in Graphs, we need to construct a functor

$$\mathrm{End}_\kappa^X(\Gamma, \varphi_1, \varphi_2) : \prod_{i=1}^{k_1} \mathcal{M}\left(X^{\otimes\mathsf{Legs}(T^{(i)})}, I\right) \longrightarrow \prod_{j=1}^{k_2} \mathcal{M}\left(X^{\otimes\mathsf{Legs}(T'^{(j)})}, I\right) \tag{2.4}$$

(later we will again suppress φ_1 and φ_2 in the notation and write $\mathrm{End}_\kappa^X(\Gamma)$ instead of $\mathrm{End}_\kappa^X(\Gamma, \varphi_1, \varphi_2)$). For the definition of (2.4), we can concentrate on the case that Γ is connected. Afterwards, End_κ^X can be extended monoidally to non-connected graphs. In the connected case, $k_2 = 1$ holds, and we write $T' = T'^{(1)}$. As a first step, we define maps

$$X^{\otimes|\mathsf{Legs}(T')|} \longrightarrow \bigotimes_{i=1}^{k_1} X^{\otimes|\mathsf{Legs}(T^{(i)})|} \tag{2.5}$$

for every ordering of $\mathsf{Legs}(T^{(1)}), \dots, \mathsf{Legs}(T^{(k_1)})$ and $\mathsf{Legs}(T')$ (an order is chosen for *each* of these sets separately). These are defined by the commutativity of the square

Here the right vertical arrow uses the coevaluation, and the lower horizontal arrow uses the identifications induced by φ_1 and φ_2. In order to construct the latter map, recall that by definition φ_1 and φ_2 tell us how to identify the legs of T' and the pairs of edges obtained by cutting at the internal edges of Γ with the legs of $\sqcup_{i=1}^{k_1} T^{(i)}$.

The symmetry of Δ ensures that this family (2.5) of functors is compatible in the sense that they descend to the 2-colimit used to define unordered monoidal products. Hence, they provide maps

$$X^{\otimes\mathsf{Legs}(T')} \longrightarrow \bigotimes_{i=1}^{k_1} X^{\otimes\mathsf{Legs}(T^{(i)})} . \tag{2.6}$$

We now define the functor $\mathrm{End}_\kappa^X(\Gamma, \varphi_1, \varphi_2)$ by

$$\mathrm{End}_\kappa^X(\Gamma, \varphi_1, \varphi_2) : \prod_{i=1}^{k_1} \mathcal{M}\left(X^{\otimes\mathsf{Legs}(T^{(i)})}, I\right) \longrightarrow \mathcal{M}\left(\bigotimes_{i=1}^{k_1} X^{\otimes\mathsf{Legs}(T^{(i)})}, I\right) \longrightarrow \mathcal{M}(X^{\otimes\mathsf{Legs}(T')}, I) ,$$

where the first map is induced by the tensor product in $\mathcal{M}$, and the second is the precomposition with (2.6).

As one verifies directly, these assignments define the desired modular endomorphism operad in the sense that the following statement holds:

Proposition 2.12. *Let $\mathcal{M}$ be a symmetric monoidal category and κ be a non-degenerate symmetric pairing on $X \in \mathcal{M}$. Then $\mathrm{End}_\kappa^X : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ is a symmetric monoidal functor, i.e. a modular operad in Cat. We call this modular operad the modular endomorphism operad of (X, κ). Furthermore, End_κ^X admits an operadic unit given by the pairing $\kappa \in \mathrm{End}_\kappa^X(T_1)$.*

By restricting End^X_κ to Forests we get the *cyclic endomorphism operad* of (X, κ). By further pulling back to RForests we find the endomorphism operad of (X, κ). It is important to note that for End^X_κ to be a symmetric monoidal functor, non-degeneracy of κ is not needed. It is needed, however, to identify the restriction of End^X_κ with the endomorphism operad in the 'traditional sense'. Indeed, the natural map

$$\boxed{f} \quad \longmapsto \quad \raisebox{-1em}{\begin{array}{c}\boxed{f}\\ \boxed{\Delta}\end{array}}\ ,$$
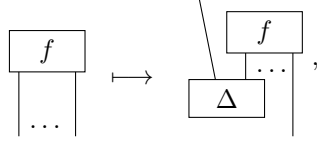
induced by a choice of root provides an equivalence between $\mathrm{End}^X_\kappa(T_n)$ and $\mathcal{M}(X^{\otimes n}, X)$. Therefore, we insist on non-degeneracy in Proposition 2.12.

2.4 Cyclic and modular algebras

Using the cyclic and modular endomorphism operad, one defines cyclic and modular algebras, respectively:

Definition 2.13. Let $\mathcal{O}$ be a modular operad in Cat and $\mathcal{M}$ any symmetric monoidal bicategory. A *modular $\mathcal{O}$-algebra A* in $\mathcal{M}$ is an object $X \in \mathcal{M}$ together with the choice of a non-degenerate symmetric pairing κ on X and a symmetric monoidal transformation $A : \mathcal{O} \longrightarrow \mathrm{End}^X_\kappa$ of symmetric monoidal functors Graphs $\longrightarrow$ Cat. We denote the components of a modular algebra A by $A_T : \mathcal{O}(T) \longrightarrow \mathrm{End}^X_\kappa(T)$ for $T \in$ Graphs and its naturality isomorphisms for every morphism $\Gamma : T \longrightarrow T'$ by

$$
\begin{array}{ccc}
\mathcal{O}(T) & \xrightarrow{\ A_T\ } & \mathrm{End}^X_\kappa(T) \\
{\scriptstyle \mathcal{O}(\Gamma)}\big\downarrow & \underset{A_\Gamma}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathrm{End}^X_\kappa(\Gamma)} \\
\mathcal{O}(T') & \xrightarrow[\ A_{T'}\ ]{} & \mathrm{End}^X_\kappa(T')\ .
\end{array}
$$

If $\mathcal{O}$ admits a unit $(1_\mathcal{O}, 1_\Gamma)$, we additionally require A to be equipped with compatibility data in the form of a natural isomorphism

$$
\begin{array}{c}
 & & \mathcal{O}(T_1) \\
 & \overset{1_\mathcal{O}}{\nearrow} & \big\downarrow{\scriptstyle A_{T_1}} \\
\star & \underset{1_A}{\Longrightarrow} & \\
 & \underset{1_{\mathrm{End}^X_\kappa}=\kappa}{\searrow} & \mathrm{End}^X_\kappa(T_1)
\end{array}
$$

such that

$$
\begin{array}{c}
\text{(left diagram)}
\end{array}
\quad = \quad
\begin{array}{c}
\text{(right diagram)}
\end{array}
$$

where the left diagram involves $1_\mathcal{O}$, $\mathcal{O}(T_1) \xrightarrow{\mathcal{O}(\tau_1)} \mathcal{O}(T_1)$, A_{T_1}, A_{τ_1}, $\star$, 1_A, $1_{\mathrm{End}^X_\kappa}$, $\mathrm{End}^X_\kappa(T_1) \xrightarrow{\mathrm{End}^X_\kappa(\tau_1)} \mathrm{End}^X_\kappa(T_1)$, and the right diagram involves $1_\mathcal{O}$, $\mathcal{O}(1)$, A_{T_1}, $\star$, $1_{\mathrm{End}^X_\kappa}$, $\mathrm{End}^X_\kappa(1)$, $\mathrm{End}^X_\kappa(\tau_1)$.

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and for all morphisms $\Gamma : T_1 \sqcup T_n \longrightarrow T_n$ in Graphs

$$
\begin{array}{ccccc}
\star \times \mathcal{O}(T_n) & \xrightarrow{\ 1_{\mathcal{O}}\ } & \mathcal{O}(T_1) \times \mathcal{O}(T_n) & \xrightarrow{\ \mathcal{O}(\Gamma)\ } & \mathcal{O}(T_n) \\
\downarrow{\scriptstyle \star \times A_{T_n}} & {\scriptstyle 1_A \times A} & \downarrow{\scriptstyle A_{T_1} \times A_{T_n}} & {\scriptstyle A_\Gamma} & \downarrow{\scriptstyle A_{T_n}} \\
\star \times \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ 1_{\mathrm{End}^X_\kappa}\ ]{} & \mathrm{End}^X_\kappa(T_1) \times \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ \mathrm{End}^X_\kappa(\Gamma)\ ]{} & \mathrm{End}^X_\kappa(T_n)
\end{array}
$$

$$
\begin{array}{ccc}
\star \times \mathcal{O}(T_n) & \xrightarrow{\ \simeq\ } & \mathcal{O}(T_n) \\
\downarrow{\scriptstyle 1 \times A_{T_n}} & {\scriptstyle \cong} & \downarrow{\scriptstyle A_{T_n}} \\
\star \times \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ \simeq\ ]{} & \mathrm{End}^X_\kappa(T_n)\ .
\end{array}
$$

(with 1_Γ, $\simeq$, and $(1_\Gamma^{\mathrm{End}^X_\kappa})^{-1}$, $\simeq$ labels)

Again, cyclic algebras over cyclic operads are defined analogously by replacing Graphs with $\mathsf{Forests}$.

Remark 2.14. The compatibility condition in Definition 2.13 with respect to operadic units might seem to depend on a specific choice of a unit for $\mathcal{O}$ and to consist of additional structure. However, note that for another choice of unitality data $(1'_{\mathcal{O}}, 1'_\Gamma, 1'_A)$, we have

$$
\begin{array}{ccc}
 & \mathcal{O}(T_1) & \\
\nearrow{\scriptstyle 1'_{\mathcal{O}}} & & \\
\star & \overset{1_A}{\swarrow} \quad \downarrow{\scriptstyle A_{T_1}} & \\
\searrow{\scriptstyle 1_{\mathrm{End}^X_\kappa}} & & \\
 & \mathrm{End}^X_\kappa(T_1) &
\end{array}
\quad = \quad
\begin{array}{ccc}
 & \mathcal{O}(T_1) & \\
\nearrow{\scriptstyle 1'_{\mathcal{O}}} & & \\
\star & \overset{1'_A}{\swarrow} \quad \downarrow{\scriptstyle A_{T_1}} & \\
\searrow{\scriptstyle 1_{\mathrm{End}^X_\kappa}} & & \\
 & \mathrm{End}^X_\kappa(T_1)\ , &
\end{array}
$$

where the unlabeled 2-isomorphism is the one mentioned in Remark 2.4. In this sense, being compatible with operadic units is a property of a symmetric monoidal natural transformation $A : \mathcal{O} \longrightarrow \mathrm{End}^X_\kappa$.

Remark 2.15. The symmetric monoidal bicategory $\mathcal{M}$ in Definition 2.13 can be arbitrary. The case $\mathcal{M} = \mathsf{Cat}$ (with the 'usual' symmetric monoidal structure from Example 2.5) can, of course, be considered, but it is not interesting because a non-degenerate symmetric pairing $\mathcal{C} \times \mathcal{C} \longrightarrow \star$ exists if and only if $\mathcal{C} \simeq \star$. The situation will be significantly more interesting in the example $\mathcal{M} = \mathsf{Lex}^{\mathsf{f}}$, see Section 2.5.

In the approach to operads chosen in this paper, the definition of morphisms between algebras requires some care. Naïvely, one might try to define them as symmetric monoidal modifications. However, this does not relate algebra structures on *different* underlying objects in $\mathcal{M}$. We sketch a construction of the bicategory $\mathsf{ModAlg}\,(\mathcal{O})$ for a modular operad $\mathcal{O}$ using the (symmetric monoidal) arrow category $\mathrm{ar}(\mathcal{M}) := [0 \longrightarrow 1, \mathcal{M}]$, i.e. the category of functors from the interval category $0 \longrightarrow 1$ with two objects 0 and 1 and one non-identity morphism to $\mathcal{M}$. Since the bicategory of algebras will not play an essential role throughout the paper, we will limit the exposition to the very essential points and omit some details. The construction we give will just be spelled out for modular operads, but can be easily transferred to cyclic and ordinary operads.

Definition 2.16. Let (X, κ) and (X', κ') be objects of a symmetric monoidal bicategory $\mathcal{M}$ equipped with non-degenerate symmetric pairings. A *morphism compatible with the pairings* $f : (X, \kappa) \longrightarrow (X', \kappa')$ is a morphism $f : X \longrightarrow X'$ plus the structure required to make

$$
\begin{array}{ccc}
X \otimes X & \xrightarrow{\ f \otimes f\ } & X' \otimes X' \\
\downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa'} \\
I & \xrightarrow[\ \mathrm{id}_I\ ]{} & I
\end{array}
\tag{2.7}
$$

into a symmetric pairing in $\mathrm{ar}(\mathcal{M})$ on f.

Now let $f : (X, \kappa) \longrightarrow (X', \kappa')$ be a morphism compatible with non-degenerate symmetric pairings κ and κ' on X and X', respectively. Then we can associate the endomorphism operad $\mathrm{End}^f : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ to the pairing (2.7) in the arrow category. It fits into a span

$$
\begin{array}{ccc}
 & \mathrm{End}^f & \\
 \swarrow & & \searrow \\
\mathrm{End}^X_\kappa & & \mathrm{End}^{X'}_{\kappa'}
\end{array}
$$

of Cat-valued modular operads.

Definition 2.17. For a Cat-valued modular operad $\mathcal{O}$, the bicategory $\mathsf{ModAlg}\,(\mathcal{O})$ of modular $\mathcal{O}$-algebras in a symmetric monoidal bicategory $\mathcal{M}$ is defined as follows: Objects are modular $\mathcal{O}$-algebras in $\mathcal{M}$ in the sense of Definition 2.13. A *1-morphism of modular algebras* $f : A \longrightarrow B$ consists of a morphism $f : (A, \kappa_A) \longrightarrow (B, \kappa_B)$ together with a symmetric monoidal transformation $\mathcal{O} \longrightarrow \mathrm{End}^f$ and a filling of the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{O} & & \\
 & \overset{A}{\swarrow} & \downarrow & \overset{B}{\searrow} & \\
 & & \mathrm{End}^f & & \\
 & \swarrow & & \searrow & \\
\mathrm{End}^A_{\kappa_A} & & & & \mathrm{End}^B_{\kappa_B}
\end{array}
$$

with natural symmetric monoidal isomorphisms. We define 2-morphisms of modular algebras similarly using spans of spans of modular operads constructed from pairing preserving 2-morphisms in $\mathcal{M}$ and the endomorphism operad in $\mathrm{ar}(\mathrm{ar}(\mathcal{M}))$.

Morphisms between algebras over cyclic and modular operads are always equivalences:

Proposition 2.18. *Let $\mathcal{O}$ be a modular operad. The bicategory $\mathsf{ModAlg}\,\mathcal{O}$ is a 2-groupoid. Similarly, the bicategory of cyclic algebras over a cyclic operad is a 2-groupoid.*

Proof. We give the proof for the modular case. The cyclic case works completely analogously. Let $f : (A, \kappa_A) \longrightarrow (B, \kappa_B)$ be a morphism of modular $\mathcal{O}$-algebras. We define an inverse to f by

It is straightforward to verify that this defines a weak inverse for f in $\mathcal{M}$ using the snake isomorphisms and the 2-isomorphisms $\kappa_B \circ (f \otimes f) \cong \kappa_A$ and $(f \otimes f) \circ \Delta_A \cong \Delta_B$. These isomorphisms exist by definition because f is compatible with the pairings. The inverse can be equipped with the additional structure needed to make it into a morphism of modular algebras. This shows that every 1-morphism is weakly invertible.

Similarly, for a 2-morphism $\eta : f \longrightarrow g$ between 1-morphisms $f, g : (A, \kappa_A) \longrightarrow (B, \kappa_B)$ of modular algebras, we can construct an inverse as follows:

$\square$

We call a morphism of modular operads an *equivalence* if it is an equivalence arity-wise (the same definition is made for cyclic and ordinary operads). It is not clear that equivalent modular operads give rise to equivalent categories of modular algebras. For ordinary operads, such results are proven in [BM07, Theorem 4.1] under the name *Comparison Theorem*. We will establish such a result for modular algebras over modular operads with values in bicategories. To this end, we will need the following, probably well-known result:

Proposition 2.19 (Whitehead's Theorem for symmetric monoidal functors between symmetric monoidal bicategories)**.** *Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be symmetric monoidal functors between symmetric monoidal bicategories. Then every symmetric monoidal equivalence α from F to G has a weak inverse, i.e. there is a symmetric monoidal transformation β from G to F such that $\alpha\beta$ and $\beta\alpha$ are the identity transformation of G and F, respectively, up to invertible monoidal modification.*

In lack of a reference, we at least sketch the proof:

Sketch of the proof. The symmetric monoidal equivalence α consists of components $\alpha_c : F(c) \longrightarrow G(c)$ for $c \in \mathcal{C}$ plus coherence data:

(i) An invertible 2-morphism $\alpha_f : G(f)\alpha_c \longrightarrow \alpha_{c'} F(f)$ for all 1-morphisms $f : c \longrightarrow c'$ in $\mathcal{C}$.

(ii) A 2-isomorphism

where $I_\mathcal{C}$ and $I_\mathcal{D}$ is the monoidal unit of $\mathcal{C}$ and $\mathcal{D}$, respectively, and the unlabeled 1-equivalences are part of the coherence data for F and G.

(iii) 2-Isomorphisms

for all objects $c, c' \in \mathcal{C}$. The unlabeled arrows are again the coherence morphisms for F and G.

This data is subject to the coherence conditions that are given in [SP09, Definition 2.7].

By assumption α_c is an equivalence for every $c \in \mathcal{C}$. Hence, we find a weak inverse $\beta_c : G(c) \longrightarrow F(c)$ that we may choose to be additionally adjoint to α. The unit and counit of this adjunction for varying $c \in \mathcal{C}$ can be used to equip β with the necessary coherence data (of the type (i)-(iii) as above for α):

(i) For a morphism $f : c \longrightarrow c'$, the needed invertible 2-morphism $\beta_f : F(f)\beta_c \longrightarrow \beta_{c'}G(f)$ is given by

$$
\begin{array}{c}
G(c) \xrightarrow{\ \beta_c\ } F(c) \\
\text{id} \quad \alpha_c \quad F(f) \\
G(c) \xLeftarrow{\ \alpha_f^{-1}\ } F(c') \\
G(f) \quad G(f) \quad \alpha_{c'} \quad F(f) \\
G(c') \xLeftarrow{\ } \mathrm{id}_{F(c)} \\
\text{id} \quad \beta_{c'} \\
G(c') \xrightarrow[\ \beta_{c'}\ ]{} F(c') \ ,
\end{array}
$$

where the unlabeled 2-morphisms are all 2-isomorphisms and arise as coherence data or through the unit and counit of the pair $(\alpha_{I_{\mathcal{C}}}, \beta_{I_{\mathcal{D}}})$ of adjoint equivalences (all 2-morphisms in this proof will be invertible, but we will omit the symbol '$\cong$' in the diagrams to save space).

(ii) The coherence data of type (ii) is given by

$$
\begin{array}{c}
G(I_{\mathcal{C}}) \\
\simeq \quad \alpha_{I_{\mathcal{D}}} \quad \alpha_{I_{\mathcal{C}}} \quad \beta_{1_{\mathcal{C}}} \\
I_{\mathcal{D}} \xLeftarrow{\quad} \\
\simeq \\
F(I_{\mathcal{C}}) \xrightarrow{\ \text{id}\ } F(I_{\mathcal{C}}) \ , \\
\simeq
\end{array}
$$

where the unlabeled 1-equivalences are part of the coherence data for F and G and the unlabeled 2-isomorphisms are the counit of the adjoint equivalence and the coherence isomorphisms $f \circ \mathrm{id} \cong f$ for the composition with identities in the bicategory $\mathcal{C}$.

(iii) The 2-isomorphisms of type (iii) are given by (we suppress tensoring with identities in the notation for 1-morphisms and 2-morphisms to ensure readability)

$$
\begin{array}{c}
G(c \otimes_{\mathcal{C}} c') \\
\simeq \quad \simeq \quad \beta_{c \otimes c'} \\
G(c) \otimes_{\mathcal{D}} G(c') \qquad \alpha_{c \otimes c'} \qquad F(c \otimes_{\mathcal{C}} c') \\
\text{id} \quad \alpha_{c,c'}^{-1} \quad \text{id} \\
\beta_c \quad \beta_{c'} \quad G(c) \otimes_{\mathcal{D}} G(c') \quad F(c \otimes_{\mathcal{C}} c') \\
\alpha_{c'} \quad \alpha_{c'} \\
G(c) \otimes_{\mathcal{D}} F(c') \xrightarrow{\ \text{id}\ } G(c) \otimes_{\mathcal{D}} F(c') \quad \simeq \\
\beta_c \quad \alpha_c \\
F(c) \otimes_{\mathcal{D}} G(c') \xrightarrow[\ \beta_{c'}\ ]{} F(c) \otimes_{\mathcal{D}} F(c') \ ,
\end{array}
$$

where the unlabeled 2-isomorphisms are coherence data of F and G or the unit and counit of the adjoint equivalences α and β evaluated at various objects.

Now we consider the composition $\beta \circ \alpha$. The adjunction data combines into an invertible symmetric monoidal modification (see [SP09, Definition 2.8] for a definition) $\beta \circ \alpha \longrightarrow \mathrm{id}_F$, i.e. coherent isomorphisms $\beta_c \circ \alpha_c \longrightarrow \mathrm{id}_c$ for all $c \in \mathcal{C}$. Similarly, we get an invertible symmetric monoidal modification $\mathrm{id}_G \longrightarrow \alpha \circ \beta$. $\qquad\square$

If we are given a map $\Phi : \mathcal{O} \longrightarrow \mathcal{P}$ of Cat-valued modular operads, we obtain a functor $\Phi^* : \mathsf{ModAlg}\,\mathcal{P} \longrightarrow \mathsf{ModAlg}\,\mathcal{O}$ for the category of modular algebras with values in a symmetric monoidal bicategory $\mathcal{M}$, namely by precomposition with Φ. If Φ is an equivalence, we may find thanks to Proposition 2.19 a weak inverse $\Psi : \mathcal{P} \longrightarrow \mathcal{O}$. This gives us a functor $\Psi^* : \mathsf{ModAlg}\,\mathcal{O} \longrightarrow \mathsf{ModAlg}\,\mathcal{P}$ — again by precomposition — that can easily be seen to be a weak inverse for Φ^*. This leads to the desired *Comparison Theorem* for modular and cyclic operads (in a bicategorical context):

Theorem 2.20 (Comparison Theorem). *Any equivalence $\Phi : \mathcal{O} \longrightarrow \mathcal{P}$ of modular Cat-valued operads induces by precomposition an equivalence $\Phi^* : \mathsf{ModAlg}\,\mathcal{P} \longrightarrow \mathsf{ModAlg}\,\mathcal{O}$ between the 2-groupoids of modular algebras with values in any symmetric monoidal bicategory $\mathcal{M}$. An analogous statement holds for cyclic operads and ordinary operads.*

2.5 Non-degenerate pairings in $\mathsf{Lex}^{\mathsf{f}}$

In Section 2.4 we have defined cyclic and modular algebras over a Cat-valued operad. The algebras take values in an arbitrary symmetric monoidal bicategory. When characterizing associative and framed little disks algebras, we will also allow general symmetric monoidal bicategories for the algebras to take values in. Only afterwards, we will specialize to a specific example of a symmetric monoidal bicategory that allows us to study applications of our results in quantum algebra, namely the symmetric monoidal bicategory $\mathsf{Lex}^{\mathsf{f}}$ formed by finite categories, left exact functors and natural transformations (Example 2.5). In this situation, there is an intimate relation between non-degenerate symmetric pairings and the morphism spaces that we will exploit.

A key tool for the investigation of pairings on finite categories will be the *categorical Eilenberg-Watts Theorem* stated in terms of coends (we refer to [FSS20] for an account on coends in finite categories). For the formulation of the result, we use that $\mathsf{Lex}^{\mathsf{f}}$ is enriched over itself; we denote the internal hom by $\mathsf{Lex}^{\mathsf{f}}[-,-]$.

Theorem 2.21 (Fuchs-Schaumann-Schweigert [FSS20, Theorem 3.2]). *For finite categories $\mathcal{C}$ and $\mathcal{D}$, the functors*

$$\Psi : \mathsf{Lex}^{\mathsf{f}}[\mathcal{C}, \mathcal{D}] \longrightarrow \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{D}$$

$$F \longmapsto \int^{X \in \mathcal{C}^{\mathrm{opp}}} X \boxtimes F(X) \ ,$$

$$\Phi : \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{D} \longrightarrow \mathsf{Lex}^{\mathsf{f}}[\mathcal{C}, \mathcal{D}]$$

$$X \boxtimes Y \longmapsto \mathcal{C}(X, -) \otimes Y \ ,$$

where we denote by a slight abuse of notation the $\mathsf{Vect}^{\mathsf{f}}$-tensoring of $\mathcal{C}$ by $\otimes$, provide a pair of adjoint equivalences

$$\Psi : \mathsf{Lex}^{\mathsf{f}}[\mathcal{C}, \mathcal{D}] \xleftrightarrow{\ \sim\ } \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{D} : \Phi \ .$$

Definition 2.22. For any pairing κ on a finite category $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$, i.e. a left exact functor $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, we define the functor $-^{\kappa} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ as the composition

$$-^{\kappa} : \mathcal{C} \xrightarrow{\ X \longmapsto \kappa(X, -)\ } \widehat{\mathcal{C}} := \mathsf{Lex}^{\mathsf{f}}[\mathcal{C}, \mathsf{Vect}^{\mathsf{f}}] \xrightarrow{\ \Psi\ } \mathcal{C}^{\mathrm{opp}} \ ,$$

i.e. we set

$$X^{\kappa} := \int^{Y \in \mathcal{C}^{\mathrm{opp}}} Y \otimes \kappa(X, Y) \quad \text{for} \quad X \in \mathcal{C} \ .$$

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By Theorem 2.21 there is a canonical natural isomorphism $\mathcal{C}(X^\kappa, -) = \Phi\Psi(\kappa(X, -)) \cong \kappa(X, -)$, namely the inverse of the unit of the adjunction $\Psi \dashv \Phi$. This implies directly the following basic, but important fact:

Lemma 2.23. *Let κ be a pairing on a finite category $\mathcal{C} \in \mathsf{Lex}^f$. Then there is a canonical natural isomorphism $\Lambda_{X,Y} : \mathcal{C}(X^\kappa, Y) \longrightarrow \kappa(X, Y)$ for $X, Y \in \mathcal{C}$*

Remark 2.24. Recall from Definition 2.10 that a pairing $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^f$ is symmetric if it is equipped with a homotopy fixed point structure with respect to the natural $\mathbb{Z}_2$-action, i.e. a natural isomorphism $\Sigma_{X,Y} : \kappa(X, Y) \longrightarrow \kappa(Y, X)$ squaring to the identity. From Lemma 2.23, we now get a natural isomorphism

$$\mathcal{C}(X^\kappa, Y) \xrightarrow{\Lambda_{X,Y}} \kappa(X, Y) \xrightarrow{\Sigma_{X,Y}} \kappa(Y, X) \xrightarrow{\Lambda_{Y,X}^{-1}} \mathcal{C}(Y^\kappa, X) \xrightarrow{-^\kappa} \mathcal{C}(X^\kappa, Y^{2\kappa}) , \text{ where } Y^{2\kappa} := (Y^\kappa)^\kappa .$$

Provided that $-^\kappa$ is an equivalence, the Yoneda Lemma leads to a natural isomorphism $-^{2\kappa} \cong \mathrm{id}_\mathcal{C}$.

Remark 2.25. According to Definition 2.7, the fact that the pairing κ exhibits $\mathcal{C}$ as its own dual means that there is a functor $\Delta : \mathsf{Vect}^f \longrightarrow \mathcal{C} \boxtimes \mathcal{C}$, called coevaluation, such that κ and Δ satisfy the usual snake relations up to natural isomorphism. Since we are working in Lex^f, the functor Δ is determined by its value on the ground field k that we also denote by $\Delta \in \mathcal{C} \boxtimes \mathcal{C}$ and will refer to as the *coevaluation object*. It will be convenient to write $\Delta = \Delta' \boxtimes \Delta''$. This notation is inspired by the Sweedler notation in the theory of Hopf algebras [Kas15, Notation 1.6] and should not be understood in the sense that Δ is actually a 'pure tensor'. The Δ' and the Δ'' are merely placeholders for the different factors of $\mathcal{C}$. These may help to write the snake isomorphisms. For example, the composition

$$\mathcal{C} \boxtimes \mathcal{C} \xrightarrow{\Delta \boxtimes \mathrm{id}_\mathcal{C} \boxtimes \mathrm{id}_\mathcal{C}} \mathcal{C}^{\boxtimes 4} \xrightarrow{\mathrm{id}_\mathcal{C} \boxtimes \kappa \boxtimes \mathrm{id}_\mathcal{C}} \mathcal{C} \boxtimes \mathcal{C} \xrightarrow{\kappa} \mathsf{Vect}^f$$

is naturally isomorphic to κ by a snake isomorphism. The component of this natural isomorphism at $X \boxtimes Y \in \mathcal{C} \boxtimes \mathcal{C}$ can now be written as

$$\kappa(\Delta', Y) \otimes \kappa(\Delta'', X) \cong \kappa(X, Y) .$$

Proposition 2.26. *A pairing κ on a finite category $\mathcal{C} \in \mathsf{Lex}^f$ is non-degenerate if and only if $-^\kappa : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ from Definition 2.22 is an equivalence. In that case, the coevaluation object is given by the coend $\Delta = \int^{X \in \mathcal{C}} X \boxtimes X^{-\kappa} \in \mathcal{C} \boxtimes \mathcal{C}$, where $X^{-\kappa}$ is the image of X under the weak inverse of $-^\kappa$. If κ is symmetric, then $-^{-\kappa} \cong -^\kappa$ by a canonical isomorphism, so that the coevaluation object is canonically isomorphic to the coend $\int^{X \in \mathcal{C}} X \boxtimes X^\kappa \in \mathcal{C} \boxtimes \mathcal{C}$.*

Proof. Suppose κ is non-degenerate and denote by Δ the coevaluation object. Sending $\alpha \in \widehat{\mathcal{C}} := \mathsf{Lex}^f[\mathcal{C}, \mathsf{Vect}^f]$ to $(\alpha \boxtimes \mathrm{id}_\mathcal{C})(\Delta)$ yields a weak inverse for the functor $\mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ sending X to $\kappa(X, -)$. By definition this implies that $-^\kappa$ is also an equivalence.

Conversely, let $-^\kappa : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ be an equivalence. Since by [FSS20, Section 3.7] $\mathcal{C}^{\mathrm{opp}}$ is dual to $\mathcal{C}$ with duality pairing $\mathcal{C}(-, -) : \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^f$, we see that $\mathcal{C}$ is self-dual with duality pairing

$$\mathcal{C} \boxtimes \mathcal{C} \xrightarrow{-^\kappa \boxtimes \mathrm{id}_\mathcal{C}} \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} \mathsf{Vect}^f .$$

This composition is canonically isomorphic to κ by Lemma 2.23. As we can also extract from [FSS20, Section 3.7], the coevaluation object for the duality of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{opp}}$ is the coend $\int^{X \in \mathcal{C}} X \boxtimes X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}}$. Hence, $\int^{X \in \mathcal{C}} X \boxtimes X^{-\kappa}$ is the coevaluation object for the self-duality of $\mathcal{C}$.

The additional statement on the symmetric case follows from Remark 2.24. $\qquad\square$

In order to prove excision results later on in Section 6, we need to relate the composition in the endomorphism operad to left exact coends studied in [Lyu96], see also the treatment in [FSS20] and additionally [SW21b] for the relation to homotopy coends and derived traces. The notion of a left exact coend is relevant in the following

situation: If we are given a left exact functor $G : \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}} \boxtimes \mathcal{D} \longrightarrow \mathcal{A}$, where $\mathcal{A}, \mathcal{C}$ and $\mathcal{D}$ are finite categories, the functor $\mathcal{D} \ni Y \longmapsto \int^{X \in \mathcal{C}} F(X \boxtimes X \boxtimes Y)$ will be linear, but not necessarily left exact. Hence, it does not belong to $\mathsf{Lex}^{\mathsf{f}}$. As a remedy, we see G as a left exact functor $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathsf{Lex}^{\mathsf{f}}[\mathcal{D}, \mathcal{A}]$. The coend of this functor exists; it will by construction give us a left exact functor $\mathcal{D} \longrightarrow \mathcal{A}$ that one refers to as left exact coend and denotes by $\oint^{X \in \mathcal{C}} G(X \boxtimes X \boxtimes -)$. To make the connection to the endomorphism operad, we first establish a relation between the left exact coend and dualizability in $\mathsf{Lex}^{\mathsf{f}}$ by proving that the left exact coend can be described through evaluation on the coevaluation object:

Lemma 2.27. *Let $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$ have a non-degenerate symmetric pairing $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$. Then for any finite category $\mathcal{D}$ and any left exact functor $F : \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{D} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, there is a canonical isomorphism*

$$\oint^{X \in \mathcal{C}} F(X \boxtimes X^{\kappa} \boxtimes -) \cong F\left(\left(\int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right) \boxtimes - \right) ,$$

where $-^{\kappa} : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathcal{C}$ is the equivalence induced by κ.

Proof. By [DSPS19, Proposition 1.7 & Corollary 1.10] F is representable, i.e. there is an object $L \in \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{D}$ such that F can be written as the hom functor $(\mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{D})(L, -)$. Using for $Y \in \mathcal{C} \boxtimes \mathcal{C}$ the contraction

$$\langle L, Y \rangle := (\mathcal{C} \boxtimes \mathcal{C})(L', Y) \otimes L'' \in \mathcal{D} \quad \text{with Sweedler notation} \quad L = L' \boxtimes L'' \in (\mathcal{C} \boxtimes \mathcal{C}) \boxtimes \mathcal{D}$$

of L and Y via the morphism spaces, we find

$$F(X \boxtimes X^{\kappa} \boxtimes -) = \mathcal{D}(\langle L, X \boxtimes X^{\kappa} \rangle, -) ,$$

$$F\left(\left(\int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right) \boxtimes - \right) = \mathcal{D}\left(\left\langle L, \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right\rangle, - \right) . \tag{2.8}$$

It remains to prove that the dinatural family

$$\mathcal{D}(\langle L, X \boxtimes X^{\kappa} \rangle, -) \longrightarrow \mathcal{D}\left(\left\langle L, \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right\rangle, - \right)$$

induced by the dinatural family $X \boxtimes X^{\kappa} \longrightarrow \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa}$ is universal because this exhibits (2.8) as the coend $\oint^{X \in \mathcal{C}} F(X \boxtimes X^{\kappa} \boxtimes -)$ and hence proves the assertion. For this, we need to show that any dinatural family $\mathcal{D}(\langle L, X \boxtimes X^{\kappa} \rangle, -) \longrightarrow G$ for some left exact functor $G : \mathcal{D} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ descends uniquely to $\mathcal{D}\left(\left\langle L, \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right\rangle, - \right)$. But by invoking again the representability statement from [DSPS19, Proposition 1.7 & Corollary 1.10], we may write $G = \mathcal{D}(M, -)$ for some $M \in \mathcal{D}$, which by the Yoneda Lemma implies that it suffices to prove that the dinatural family

$$\langle L, X \boxtimes X^{\kappa} \rangle \longrightarrow \left\langle L, \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right\rangle$$

is universal or, in other words, that there is a canonical isomorphism

$$\int^{X \in \mathcal{C}} \langle L, X \boxtimes X^{\kappa} \rangle \cong \left\langle L, \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa} \right\rangle ,$$

which in fact exists by [FSS20, Proposition 3.4]. $\qquad\qquad\square$

As a consequence of this Lemma, we find that the composition operation in the endomorphism operad corresponding to a graph is a left exact coend with one dummy variable for each internal edge of the graph:

Proposition 2.28. *Let $\Gamma : T \longrightarrow T'$ be a morphism in Graphs between objects T and T'. We assume that T' is connected and denote by $T = \sqcup_{i=1}^{m} T^{(i)}$ the decomposition of T into connected components. Then for any $\mathcal{C} \in$*

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$\mathsf{Lex}^{\mathsf{f}}$ *with non-degenerate symmetric pairing* $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ *and* $F = (F_i)_{1 \leq i \leq m} \in \prod_{i=1}^{m} \mathsf{Lex}^{\mathsf{f}}(\mathcal{C}^{\boxtimes \mathsf{Legs}(T^{(i)})}, \mathsf{Vect}^{\mathsf{f}})$ *in* $\mathsf{Lex}^{\mathsf{f}}$, *we have a canonical isomorphism*

$$\mathrm{End}_{\kappa}^{\mathcal{C}}(\Gamma)F \cong \oint^{X_1, \dots, X_r \in \mathcal{C}} F^{\boxtimes}(\dots, X_j, \dots, X_j^{\kappa}, \dots) \quad with \quad F^{\boxtimes} = F_1 \boxtimes \cdots \boxtimes F_m : \mathcal{C}^{\boxtimes \mathsf{Legs}(T)} \longrightarrow \mathsf{Vect}^{\mathsf{f}} \ .$$

The left exact coends $\oint$ *runs over the variables* $X_1, \dots, X_r$ *corresponding to internal edges of* Γ. *(The integrand of the coend is just a mnemonic notation for the insertion of the variables corresponding to internal edges into the correct arguments of* $F^{\boxtimes}$ *according to the identification of* $\nu(\Gamma)$ *with* T.)*

Proof. Since we can decompose Γ into morphisms contracting one internal edge at a time, we may assume that Γ has just one internal edge. Then

$$\mathrm{End}_{\kappa}^{\mathcal{C}}(\Gamma)F \cong F^{\boxtimes}(\dots, \Delta', \dots, \Delta'', \dots)$$

with the Sweedler notation for the coevaluation object $\Delta \in \mathcal{C} \boxtimes \mathcal{C}$ discussed in Remark 2.25. This coevaluation object is given by $\Delta = \int^{X \in \mathcal{C}} X \boxtimes X^{\kappa}$ by Proposition 2.26. Now the assertion follows from Lemma 2.27. $\qquad \square$

3 The Lifting Theorem

Pulling back a cyclic operad $\mathcal{O}$ along the functor $\mathsf{RForests} \longrightarrow \mathsf{Forests}$ yields an ordinary operad $\overline{\mathcal{O}}$, the underlying ordinary operad of $\mathcal{O}$. It is a natural question whether the structure of an algebra over $\overline{\mathcal{O}}$ on some object X (in some (higher) symmetric monoidal category — depending on the context that one is working in) can be lifted to a *cyclic* algebra over $\mathcal{O}$ on (X, κ). More precisely: What kind of additional data and/or properties are needed?

When the target category is given by the category of vector spaces, there is a very classical answer [GK95, MSS02]: The structure of a cyclic $\mathcal{O}$-algebra on a vector space V with a pairing κ is equivalent to the structure of an $\overline{\mathcal{O}}$-algebra on V such that the pairing satisfies an invariance property.

In this section we provide an answer in the context of symmetric monoidal bicategories. In this framework, we have to take further coherence data into account making the situation slightly more subtle. Still the principle that an $\mathcal{O}$-algebra is an $\overline{\mathcal{O}}$-algebra plus an invariant pairing can be generalized to this setting. However, the invariance of the pairing will amount to additional *structure* and will not be just a property. More precisely, for each operation, there will be a cyclic invariance isomorphism. The isomorphisms for different operations will generally not be independent, but related in a way prescribed by the operad. This leads to a significantly richer algebraic structure.

3.1 General version

In order to state the Lifting Theorem, we need to make some preliminary observations: Consider the corollas T_n (these have $H = \{0, \dots, n\}$ as the set of edges and one vertex) and their disjoint unions. Whenever needed, we may regard T_n as a rooted corolla with 0 as the root. Recall from Section 2.1 that the cyclic permutation of legs induces an isomorphism $\tau_n : T_n \cong T_n$ in $\mathsf{Forests}$ with $\tau_n^{n+1} = \mathrm{id}_{T_n}$.

The graph underlying a morphism $\Gamma : \gamma_1 = \sqcup_i T_{n_i} \longrightarrow \sqcup_j T_{n_j} = \gamma_2$ in $\mathsf{Forests}$ between finite disjoint unions of corollas can be equipped with the structure of a rooted forest using the roots for the T_{n_j} (by convention the root for any of the T_{n_j} is the zeroth leg). Hence, we can interpret it as a morphism $\Gamma' : \gamma_1 \longrightarrow \gamma_2$ in $\mathsf{RForests}$, where the identification $\varphi_1' : \gamma_1 \longrightarrow \nu(\Gamma')$ differs from the original one $\varphi_1 : \gamma_1 \longrightarrow \nu(\Gamma)$ by a cyclic permutation of the corollas. This allows us to write $\Gamma = \Gamma' \circ \sqcup_i \tau_{n_i}^{k_i}$ with unique $k_i \in \mathbb{Z}_{n_i}$, see Figure 4 for an illustration. We will refer to this factorization of Γ as the *standard factorization* and will denote it (as above) by Γ'.

Now let $\Omega : \sqcup_i T_{n_i} \longrightarrow T_n$ be a morphism in $\mathsf{RForests}$. Then $\tau_n \circ \Omega$ can be seen as morphism in $\mathsf{Forests}$, and hence, we can consider its standard factorization

$$\tau_n \circ \Omega = \Omega^{(n)} \circ \sqcup_i \tau_{n_i}^{k_i} \ , \quad \text{where} \quad \Omega^{(n)} := (\tau_n \circ \Omega)' \ . \tag{3.1}$$
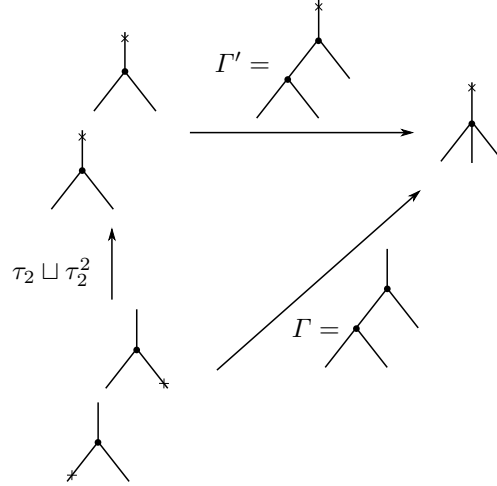
Figure 4: A sketch for the factorization of a morphism $\Gamma \in$ Forests into a cyclic permutation and $\Gamma' \in$ RForests The roots are marked by $\times$.

With this notation, we can now formulate the coherent version of the principle that a cyclic algebra over a cyclic operad is an non-cyclic algebra over the underlying operad plus an invariant pairing.

Theorem 3.1 (Lifting Theorem). *Let $\mathcal{M}$ be a symmetric monoidal bicategory, κ a non-degenerate symmetric pairing on $X \in \mathcal{M}$ and $\mathcal{O}$ a* Cat*-valued operad. Then the structure of an $\mathcal{M}$-valued cyclic algebra over $\mathcal{O}$ on (X, κ) can equivalently be described as an algebra A over the underlying non-cyclic operad $\overline{\mathcal{O}}$ together with natural isomorphisms ϕ_n*

$$
\begin{array}{ccc}
\mathcal{O}(T_n) & \xrightarrow{\mathcal{O}(\tau_n)} & \mathcal{O}(T_n) \\
{\scriptstyle A_{T_n}} \downarrow & \overset{\phi_n}{\Longleftarrow} & \downarrow {\scriptstyle A_{T_n}} \\
\mathrm{End}^X_\kappa(T_n) & \xrightarrow[\mathrm{End}^X_\kappa(\tau_n)]{} & \mathrm{End}^X_\kappa(T_n)
\end{array}
\tag{3.2}
$$

for $n \geq 1$ subject to the following coherence conditions:

(C1) *The $n+1$-fold composition of ϕ_n is the identity, i.e. we have the following equality of natural isomorphisms:*

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$$= \quad \begin{array}{ccc} \mathcal{O}(T_n) & \xrightarrow{\ \mathcal{O}(\mathrm{id}_{T_n})\ } & \mathcal{O}(T_n) \\ {\scriptstyle A_{T_n}}\big\downarrow & {\scriptstyle A_{\mathrm{id}_{T_n}}} & \big\downarrow{\scriptstyle A_{T_n}} \\ \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ \mathrm{End}^X_\kappa(\mathrm{id}_{T_n})\ ]{} & \mathrm{End}^X_\kappa(T_n) \ . \end{array}$$

(C2) *The isomorphisms ϕ intertwine with the $\mathcal{O}$-action, i.e. for all rooted morphisms $\Omega : \sqcup_i T_{n_i} \longrightarrow T_n$, the equality*

$$\begin{array}{ccccc} & & \mathcal{O}(\tau_n \circ \Omega) & & \\ \mathcal{O}(\sqcup_i T_{n_i}) & \xrightarrow{\ \mathcal{O}(\Omega)\ } & \mathcal{O}(T_n) & \xrightarrow{\ \mathcal{O}(\tau_n)\ } & \mathcal{O}(T_n) \\ {\scriptstyle A_{\sqcup_i T_{n_i}}}\big\downarrow & {\scriptstyle A_\Omega} & \big\downarrow{\scriptstyle A_{T_n}} & {\scriptstyle \phi_n} & \big\downarrow{\scriptstyle A_{T_n}} \\ \mathrm{End}^X_\kappa(\sqcup_i T_{n_i}) & \xrightarrow[\ \mathrm{End}^X_\kappa(\Omega)\ ]{} & \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ \mathrm{End}^X_\kappa(\tau_n)\ ]{} & \mathrm{End}^X_\kappa(T_n) \\ & & \mathrm{End}^X_\kappa(\tau_n \circ \Omega) & & \end{array}$$

$$= \quad \begin{array}{ccccc} & & \mathcal{O}(\tau_n \circ \Omega) & & \\ \mathcal{O}(\sqcup_i T_{n_i}) & \xrightarrow{\ \mathcal{O}(\sqcup \tau_{n_i}^{k_i})\ } & \mathcal{O}(\sqcup_i T_{n_i}) & \xrightarrow{\ \mathcal{O}(\Omega^{(n)})\ } & \mathcal{O}(T_n) \\ {\scriptstyle A_{\sqcup_i T_{n_i}}}\big\downarrow & {\scriptstyle \sqcup \phi_{n_i}^{k_i}} & \big\downarrow{\scriptstyle A_{T_n}} & {\scriptstyle A_{\Omega^{(n)}}} & \big\downarrow{\scriptstyle A_{T_n}} \\ \mathrm{End}^X_\kappa(\sqcup_i T_{n_i}) & \xrightarrow[\ \mathrm{End}^X_\kappa(\sqcup \tau_{n_i}^{k_i})\ ]{} & \mathrm{End}^X_\kappa(T_n) & \xrightarrow[\ \mathrm{End}^X_\kappa(\Omega^{(n)})\ ]{} & \mathrm{End}^X_\kappa(T_n) \\ & & \mathrm{End}^X_\kappa(\tau_n \circ \Omega) & & \end{array}$$

of natural isomorphisms holds, where $\tau_n \circ \Omega = \Omega^{(n)} \circ \sqcup_i \tau_{n_i}^{k_i}$ is the standard factorization of the morphism $\tau_n \circ \Omega$ in Forests *given in (3.1).*

Proof. It is clear that we can extract from the structure of a cyclic algebra the natural isomorphisms (3.2) through $\phi_n := A_{\tau_n}$. By definition these have to satisfy the coherence conditions (C1) and (C2) given above.

Conversely, suppose we are given an $\overline{\mathcal{O}}$-algebra A. To extend the structure of A to a cyclic algebra, we have to define natural transformations

$$\begin{array}{ccc} \mathcal{O}(\sqcup_i T_{n_i}) & \xrightarrow{\ \mathcal{O}(\Gamma)\ } & \mathcal{O}(T_n) \\ {\scriptstyle A_{\sqcup_i T_{n_i}}}\big\downarrow & {\scriptstyle A_\Gamma} & \big\downarrow{\scriptstyle A_{T_n}} \\ \mathrm{End}^X_\kappa(\sqcup_i T_{n_i}) & \xrightarrow[\ \mathrm{End}^X_\kappa(\Gamma)\ ]{} & \mathrm{End}^X_\kappa(T_n) \end{array}$$

for all morphisms $\Gamma : \sqcup_i T_{n_i} \longrightarrow T_n$ in Forests. We define them to agree with the ones provided by A on rooted morphisms and by ϕ_n on the morphisms τ_n. Finally, for an arbitrary morphism Γ in Forests, we consider its

factorization $\Gamma = \Gamma' \circ \sqcup_i \tau_{n_i}^{k_i}$ from (3.1) and define A_Γ by

$$
\begin{array}{ccc}
\mathcal{O}(\sqcup_i T_{n_i}) \xrightarrow{\mathcal{O}(\Gamma)} \mathcal{O}(T_n) \\
\Big\downarrow{}^{A_{\sqcup_i T_{n_i}}} \quad \swarrow^{A_\Gamma} \quad \Big\downarrow{}^{A_{T_n}} \\
\mathrm{End}^X_\kappa(\sqcup_i T_{n_i}) \xrightarrow{\mathrm{End}^X_\kappa(\Gamma)} \mathrm{End}^X_\kappa(T_n)
\end{array}
\quad := \quad
\begin{array}{c}
\mathcal{O}(\Gamma) \\
\mathcal{O}(\sqcup_i T_{n_i}) \xrightarrow{\mathcal{O}(\sqcup \tau_{n_i}^{k_i})} \mathcal{O}(\sqcup_i T_{n_i}) \xrightarrow{\mathcal{O}(\Gamma')} \mathcal{O}(T_n) \\
A_{\sqcup_i T_{n_i}} \quad \sqcup \phi_{n_i}^{k_i} \quad A_{T_n} \quad A_{\Gamma'} \quad A_{T_n} \\
\mathrm{End}^X_\kappa(\sqcup_i T_{n_i}) \xrightarrow{\mathrm{End}^X_\kappa(\sqcup \tau_{n_i}^{k_i})} \mathrm{End}^X_\kappa(T_n) \xrightarrow{\mathrm{End}^X_\kappa(\Gamma')} \mathrm{End}^X_\kappa(T_n) \, . \\
\mathrm{End}^X_\kappa(\Gamma)
\end{array}
$$

Now the conditions (C1) and (C2) imply that this defines a cyclic algebra.

Both constructions are inverse to each other. $\qquad\square$

Remark 3.2. By the definition of the endomorphism operad, any operation $o \in \mathcal{O}(T_n)$ acts as a 1-morphism $o : A^{\otimes(n+1)} \longrightarrow I$, and the component ϕ_n^o of the natural isomorphism ϕ_n at o is an isomorphism in $\mathcal{M}(A^{\otimes(n+1)}, I)$ which in the graphical calculus (Remark 2.6) we write as

Here $\tau.o$ is the image of o under the action with τ. The coherence condition (C1) implies that applying this isomorphism $n + 1$ times yields the identity. The coherence condition (C2) ensures that these isomorphism are compatible with the composition of operations.

3.2   Adaption to a presentation in terms of generators and relations

Many operads, including the ones appearing in this paper, have a description in terms of generators and relations. For us, the case of Cat-valued operads is most important. Although Cat is a symmetric monoidal bicategory, we will treat it as an ordinary symmetric monoidal category by forgetting the 2-morphisms in order to build operads from generators and relations. In this case, the operadic composition is strictly associative. This has the practical advantage that we can rely on the well-developed theory of generators and relations in this setting, see e.g. [Fre17, Section 1.2.5]. Specifically for the category-valued case, we make a few conventional adjustments following [MW20, Section 4.1] that we now also briefly recall: The generators are a sequence $(G_{\mathrm{Ob}}(n))_{n \in \mathbb{N}}$ of sets of generating objects and another sequence $(G_{\mathrm{Mor}}(n))_{n \in \mathbb{N}}$ of sets of generating morphisms. They come with maps $s_n, t_n : G_{\mathrm{Mor}}(n) \longrightarrow G_{\mathrm{Ob}}(n)$ specifying the source and target for generating morphisms. When listing generators for a Cat-valued operad, we only list them as a set and complete them freely to a symmetric sequence in categories. This means for example if we specify a generating object μ in arity two, we also add freely an operation $\sigma_{1,2}\mu$ corresponding to the application of the permutation $\sigma_{1,2}$ to μ. Furthermore, we freely add the composition of generating morphisms.

If $\mathcal{O}$ is a Cat-valued cyclic operad and if the underlying operad $\overline{\mathcal{O}}$ is presented by generating objects and generating morphisms subject to relations in the above-explained sense, we can give the following more explicit version of the Lifting Theorem that we will need in the next section:

Corollary 3.3 (Lifting Theorem in terms of generators and relations)**.** *Let $\mathcal{O}$ be a Cat-valued cyclic operad with a fixed presentation in terms of generators and relations for the underlying operad $\overline{\mathcal{O}}$. For a given $\overline{\mathcal{O}}$-algebra in a symmetric monoidal bicategory $\mathcal{M}$, a lift of this $\overline{\mathcal{O}}$-algebra structure to a cyclic $\mathcal{O}$-algebra structure amounts*

precisely to the choice of a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$ on the underling object X of the $\overline{\mathcal{O}}$-algebra and isomorphisms

$$(3.3)$$

for every generating object $o \in \mathcal{O}(T_n)$ with $n \neq 0$ subject to the following relations:

(C) *The $n+1$-fold composition of ϕ_n^o with itself is the identity.*

(R) *Suppose we have a relation $\mathcal{O}(\Omega)(o_1, \ldots, o_\ell) = o = \mathcal{O}(\widetilde{\Omega})(\widetilde{o}_1, \ldots, \widetilde{o}_k)$ between generating objects of $\mathcal{O}$ (here Ω and $\widetilde{\Omega}$ are morphisms in $\mathsf{RForests}$, and o is an operation in arity n). Then we obtain two natural isomorphisms*

induced by (3.3) and the standard decomposition of $\tau_n \circ \Omega$ and $\tau_n \circ \widetilde{\Omega}$, respectively. We need to impose the relation that these two isomorphisms agree, thereby allowing us to extend the definition of the isomorphisms (3.3) consistently from generating operations to all operations.

(M) *For every generating morphism, $r : o \longrightarrow o'$ the square*

commutes. Here the vertical arrows are induced by $\tau.r$ and r, respectively.

Proof. Assume we are given a cyclic algebra A over $\mathcal{O}$. Then the evaluation of the isomorphisms (3.2) on a generating operation o gives rise to the 2-isomorphism ϕ_n^o. Their naturality with respect to generating morphisms amounts exactly to (M). Condition (C1) and (C2) in Theorem 3.1 above imply (C) and (R), respectively.

Conversely, let us assume that we are given a non-cyclic algebra A together with isomorphism ϕ_n^o of the form (3.3) for all generating objects subject to (C), (R) and (M). We need to define ϕ_n

at an arbitrary object $O \in \mathcal{O}(T_n)$. We can write O non-uniquely as $O = \mathcal{O}(\Omega)(o_1, \ldots, o_m)$ for a rooted tree $\Omega : \sqcup_{i=1}^m T^{(i)} \longrightarrow T_n$ and generating objects $o_i \in \mathcal{O}(T^{(i)})$. Now the standard factorization $\tau_n \circ \Omega = \Omega' \circ (\sqcup \tau_i^{k_i})$ allows us to define ϕ_n^O in terms of the $\tau_n^{o_i}$ through the right hand side of point (C2) in Theorem 3.1. Condition (R) is equivalent to this being well-defined. The collection of the ϕ_n^O assemble into a natural transformation ϕ_n by condition (M) since every morphism can be written as a combination of generating morphisms. We need to verify that ϕ_n satisfies condition (C1) and (C2). Condition (C1) follows directly from (C). Moreover, the extension of ϕ_n^o (for generating objects) to ϕ_n^O (for arbitrary objects) is precisely constructed in such a way that (C2) holds as soon as it is well-defined. $\qquad\square$

4 Cyclic associative algebras in a symmetric monoidal bicategory and Grothendieck-Verdier structures

The cyclic associative operad $\mathsf{As} : \mathsf{Forests} \longrightarrow \mathsf{Set}$ sends a corolla with edges E to the set of cyclic orders on E. By means of the symmetric monoidal functor $\mathsf{Set} \longrightarrow \mathsf{Cat}$ forming the discrete category for a given set, we consider As as a category-valued cyclic operad. In this section, we characterize cyclic associative algebras in an arbitrary symmetric monoidal bicategory and afterwards specialize to the symmetric monoidal bicategory $\mathsf{Lex}^{\mathsf{f}}$ to find a connection to Grothendieck-Verdier duality in the sense of Boyarchenko-Drinfeld [BD13].

4.1 A characterization of cyclic associative algebras in a symmetric monoidal bicategory

First we present the characterization of cyclic associative algebra in an *arbitrary* symmetric monoidal bicategory. In order to do this in a compact way, we introduce the following notion:
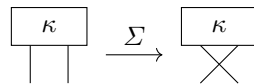
Definition 4.1. A *self-dual algebra* in a symmetric monoidal bicategory $\mathcal{M}$ is an object X with the following structure:

(M) The object X is endowed with the structure of an associative algebra up to coherent isomorphism whose product, unit, associator and unitors we denote as follows:
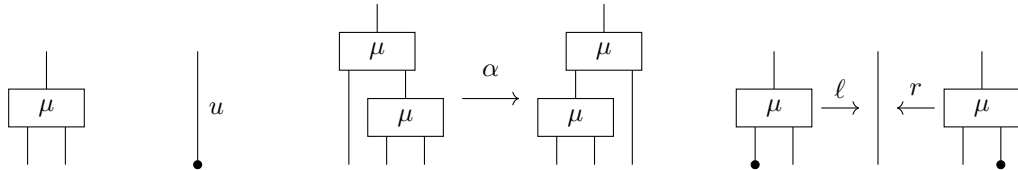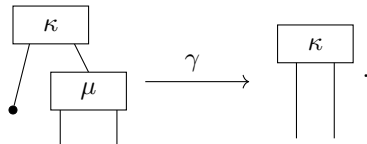

(The term 'up to coherent isomorphism' includes the standard coherence conditions on the associators and unitors [ML98, Section VII.1], in particular the pentagon axiom.)

(P) The object X is endowed with a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$ whose coevaluation we denote by $\Delta : I \longrightarrow X \otimes X$. We denote by

the symmetry isomorphisms.

(Z) The product μ and the pairing κ come with an isomorphism

.

This data is subject to the following relations:

(H1) The isomorphism

$$\psi: \quad \xrightarrow{\gamma} \quad \xrightarrow{\Sigma} \quad \xrightarrow{\gamma^{-1}} \quad \tag{4.1}$$

makes the hexagon

$$\xrightarrow{\psi} \xrightarrow{\alpha} \quad \downarrow \psi$$
$$\alpha \uparrow \qquad \xleftarrow{\psi} \xleftarrow{\alpha}$$

commute.

(H2) The isomorphism

$$\xrightarrow{\gamma_{u,-}}$$

induced by γ agrees with the one induced by the left unitor ℓ.

Theorem 4.2. *The structure of a cyclic associative algebra on an object X in a symmetric monoidal bicategory $\mathcal{M}$ amounts precisely to the structure of a self-dual algebra on X.*

Proof. By Definition 2.13 a cyclic associative algebra in a symmetric monoidal bicategory $\mathcal{M}$ is an object $X \in \mathcal{M}$ and a morphism $A : \mathsf{As} \longrightarrow \mathrm{End}_\kappa^X$ of cyclic operads from the cyclic associative operad to the cyclic endomorphism operad formed by a non-degenerate symmetric pairing κ on X. This already gives us the non-degenerate symmetric pairing mentioned in point (P).

We conclude now from the Lifting Theorem 3.1 that a cyclic algebra $A : \mathsf{As} \longrightarrow \mathrm{End}_\kappa^X$ on (X, κ) precisely amounts to the following:

(∗) A *non-cyclic* $\mathcal{M}$-valued algebra over As, i.e. an associative algebra in $\mathcal{M}$ up to coherent isomorphism, thereby giving us precisely part (M) in Definition 4.1. The fact that an associative algebra up to coherent isomorphism yields the algebraic structure listed under point (M) with the coherence conditions described in [ML98, Section VII.1], most notably the pentagon axiom, is well-known, but a concrete reference to a proof that, without further non-trivial comparisons, is valid in our framework is not easy to give. We can, however, extract it from the literature as follows: Denote by $\mathsf{As}^\otimes$ the symmetric monoidal enveloping category of As [Hor17, Definition 1.7]. Then associative algebras in $\mathcal{M}$ up to coherent isomorphism

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are symmetric monoidal functors $\mathsf{As}^{\otimes} \longrightarrow \mathcal{M}$ up to coherent isomorphism. In this picture, we can see that they are exactly algebras over the ∞-operad As, or equivalently E_1, in the sense of Lurie [Lur17, Chapter 3]. The needed statement is then given in [Lur17, Example 5.1.2.4]. A detailed derivation of the coherence conditions can be extracted by restricting the statements of [Wee19, Section 4] from E_2 to E_1. (It might be helpful to mention that [Wee19, Section 6] contains also a detailed unpacking of the notion of ∞-operads and their algebras from [Lur17] with a special focus on the bicategorical situation.)

($**$) Natural isomorphisms as given by the Lifting Theorem 3.1 subject to the coherence conditions (C1) and (C2) also given there.

The proof will proceed in two steps: In step (i), we use the Lifting Theorem in terms of generators and relations (Corollary 3.3) to explicitly give the natural isomorphisms from ($**$) and their coherence conditions. In step (ii), we prove that these are equivalent to (Z), (H1) and (H2) in the statement of the Theorem.

(i) In order to explicitly describe the natural isomorphisms appearing in ($**$), we use Corollary 3.3 where we have spelled out the Lifting Theorem for the situation that the underlying operad is given in terms of generators and relations.

For the associative operad, a very easy presentation in terms of generators and relations is available: There are three generating operations

$$\mathrm{id} \; = \; \Big| \qquad u \; = \; \Big|_{\bullet} \qquad \mu \; = \; \bigwedge$$

corresponding to the operadic identity (our operads do not have an operadic identity by default, so it has to be included into the list of generators), the monoidal unit and the monoidal product, respectively, subject to the relations

$$\bigwedge \!\!\! \bigwedge \; = \; \bigwedge \!\!\! \bigwedge \quad (\mathrm{As}) \qquad \bigvee_{\bullet} \; = \; \Big| \quad (\mathrm{LU}) \qquad \bigvee_{\bullet} \; = \; \Big| \quad (\mathrm{RU})$$

(corresponding to associativity and unitality, respectively) and the rather trivial relations for the operadic identity which we do not include here. All these relations hold strictly, but we still always obtain algebras up to coherent isomorphism because by the conventions set in Section 2 all functors are weak. Let us recall that a cyclic permutation acts trivially on the generating operations. Note that this does not imply that it acts trivially on all composed operations.

We now spell out Corollary 3.3 for the associative operad: From (3.3), we obtain an isomorphism for each non-nullary generator:

- The isomorphism that the operadic identity gives rise to agrees with the symmetry isomorphism of the pairing. Point (C) in Corollary 3.3 tells us that the square of the first of these isomorphism is the identity, but this already holds because of the symmetry of the pairing.

- The product generator μ gives an isomorphism

$$\boxed{\kappa} \; \mspace \overset{\Omega}{\longrightarrow} \; \boxed{\kappa} \tag{4.2}$$

 whose threefold composition is the identity.

Point (M) in Corollary 3.3 is not relevant for the associative operad because the latter is discrete; it has no morphisms between operations. According to point (R) in Corollary 3.3, there is a relation for each of the three non-trivial relations (As), (LU) and (RU). The relations corresponding to the operadic unit do

not induce any additional conditions due to our conventions laid out in Definition 2.13 which allow us to assume without loss of generality that the operadic unit acts as the identity of X.

Let us first derive the relation coming from associativity (As). For notational convenience, we introduce the operation $m : X \otimes X \otimes X \longrightarrow I$ encoding the combination of κ and μ:
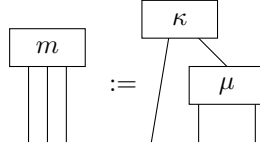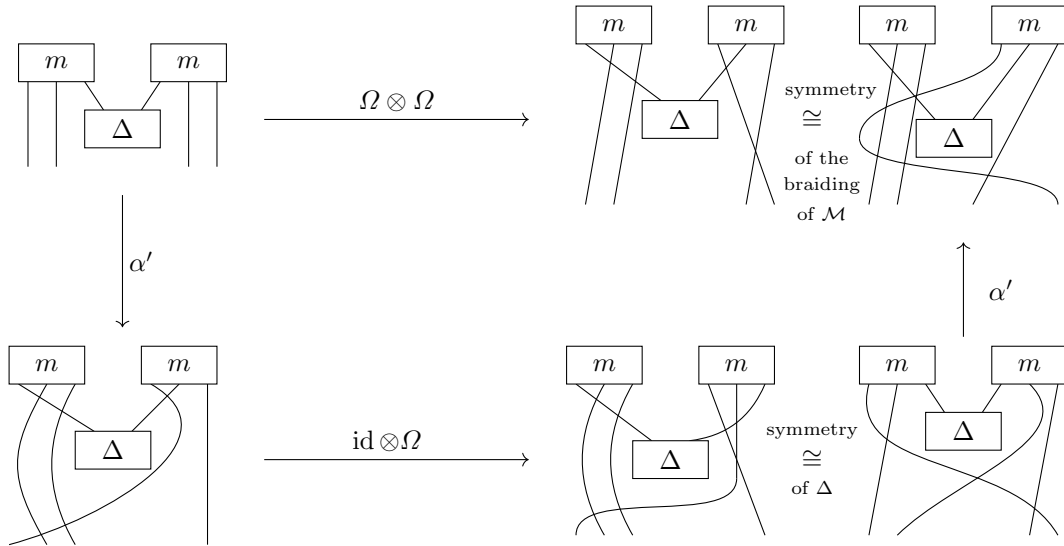

Then the associativity relation induces an isomorphism
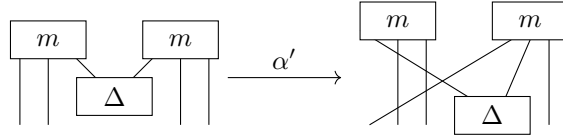
build from the associator α and the snake isomorphism. The relation coming from associativity (As) now corresponds to the equality of the two natural isomorphisms
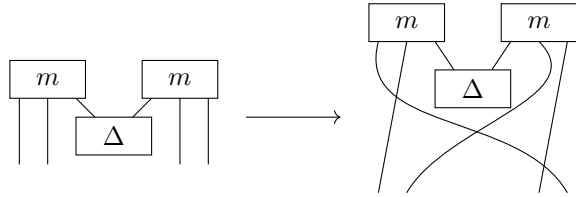
we can build from α' and Ω. Concretely, this condition can be rewritten in terms of the commutativity of the diagram

.

When reformulated in terms of μ and α, this amounts to the commutativity of

$$(4.3)$$

Similarly, the left and right unitality relation (LU) and (RU) give us the following commuting diagrams:

$$(4.4)$$

$$(4.5)$$

In the relation corresponding to (LU), Ω^2 appears since acting with the generating cyclic permutations on the composed operation on the left corresponds to acting twice with the generator of the cyclic permutations on μ. This concludes the derivation of all the structure and conditions coming from the Lifting Theorem.

(ii) Let us summarize step (i): A cyclic associative algebra on (X, κ) amounts precisely to the structure of an associative algebra up to coherent isomorphism on X together with an isomorphism Ω from (4.2) whose three-fold composition is the identity and which makes (4.3), (4.4) and (4.5) commute. It remains to prove that there is the following 1:1 correspondence of structure and relations:

$$\Omega \text{ as in (4.2) subject to } \Omega^3 = \text{id and (4.3), (4.4) and (4.5)} \xleftrightarrow{\ 1:1\ } \gamma \text{ as in (Z) subject to (H1) and (H2) .}$$

We establish the two directions of this correspondence separately:

$(\longrightarrow)$ We send Ω to the isomorphism

$$(4.6)$$

In order to prove that γ actually satisfies (H1), we need the following preliminary consideration: We insert in (4.3) the monoidal unit into the first argument from the left and obtain the following diagram in which the outer diagram commutes thanks to (4.3) (we suppress the unitors):

Now we observe: The upper triangle commutes thanks to the symmetry requirements on Σ, the left middle square commutes since the horizontal composition of 2-morphisms applied to *different* 1-morphisms is commutative, and the right middle square commutes since the snake isomorphisms are chosen to be coherent. Therefore, the outer diagram and all inner diagrams except for the lower pentagon commute. From this we conclude that the lower pentagon commutes as well, and this tells us that Ω can be written as the following composition:

$$\Omega = \quad \xrightarrow{\gamma^{-1}} \quad \xrightarrow{\alpha} \quad \xrightarrow{\gamma} \quad \xrightarrow{\Sigma} \quad \qquad (4.7)$$

One can now directly verify that $\Omega^3 = \mathrm{id}$ implies the property (H1) for γ (or rather for ψ that is used to state (H1)). Property (H2) follows the diagram

$$\xrightarrow{\quad \Sigma \quad}$$
$$\ell \uparrow \qquad \qquad r \uparrow \qquad \qquad =$$
$$\xrightarrow{\quad \Omega \quad} \qquad \xrightarrow{\quad \ell \quad} \qquad \xrightarrow{\quad \Sigma \quad}$$

in which the square commutes by (4.5) and the triangle is one of the standard coherence conditions for any non-cyclic associative algebra in a symmetric monoidal bicategory. By definition the composition of the lower horizontal arrows is $\gamma_{u,-}$. Since Σ squares to the identity, (H2) follows.

($\longleftarrow$) Starting from the isomorphism γ subject to (H1) and (H2), we can define Ω via

$$\Omega := \quad \xrightarrow{\gamma^{-1}} \quad \xrightarrow{\alpha} \quad \xrightarrow{\gamma} \quad \xrightarrow{\Sigma} \quad \qquad (4.8)$$

After all, we already know that Ω must be of this form. It remains to prove that Ω — if defined that way — satisfies $\Omega^3 = \mathrm{id}$ and (4.3), (4.4) and (4.5). This boils down to writing out the corresponding diagrams and filling them in with smaller squares and triangles using the fact that the vertical composition of 2-morphisms applied at different 1-morphisms is commutative and the fact that the snake isomorphisms are chosen to be coherent. Let us give the details:

– For the proof of (4.4), consider the following diagram:
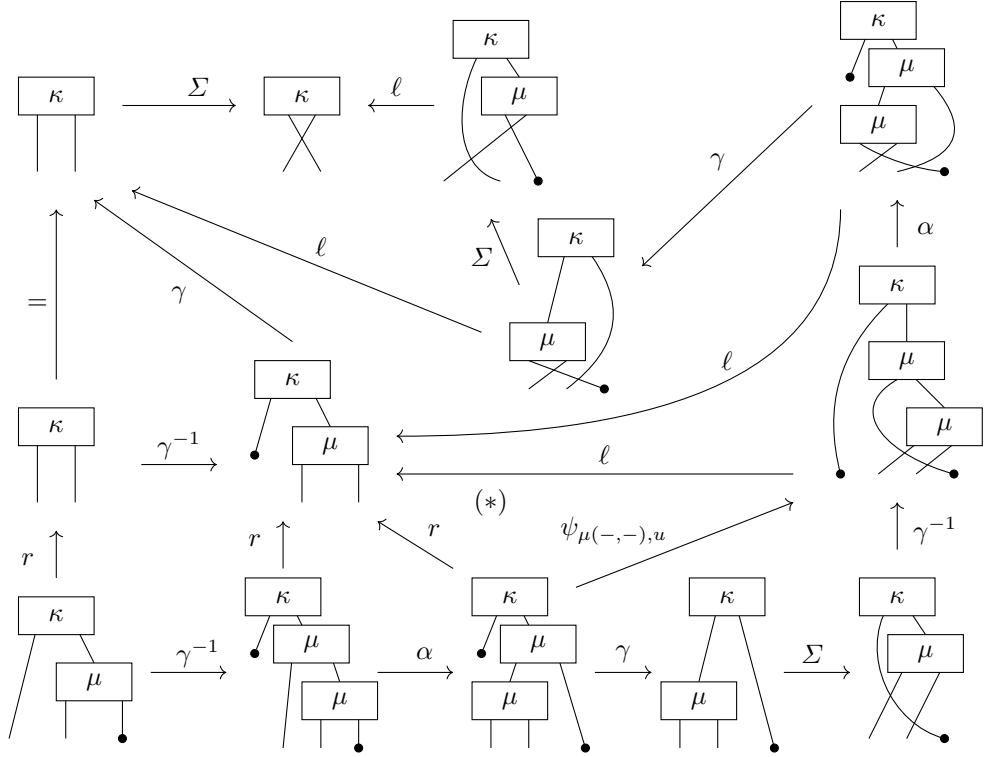
The outer diagram is really (4.4) by definition. The commutativity follows from the commutativity of all the subdiagrams. The commutativity of the subdiagrams is clear (or holds by definition) for all except $(*)$. The commutativity of $(*)$ follows from (H1) and the symmetry property that ψ inherits from Σ.

– Similarly to (4.4), the commutativity of (4.5) follows from the commutativity of the following diagram:
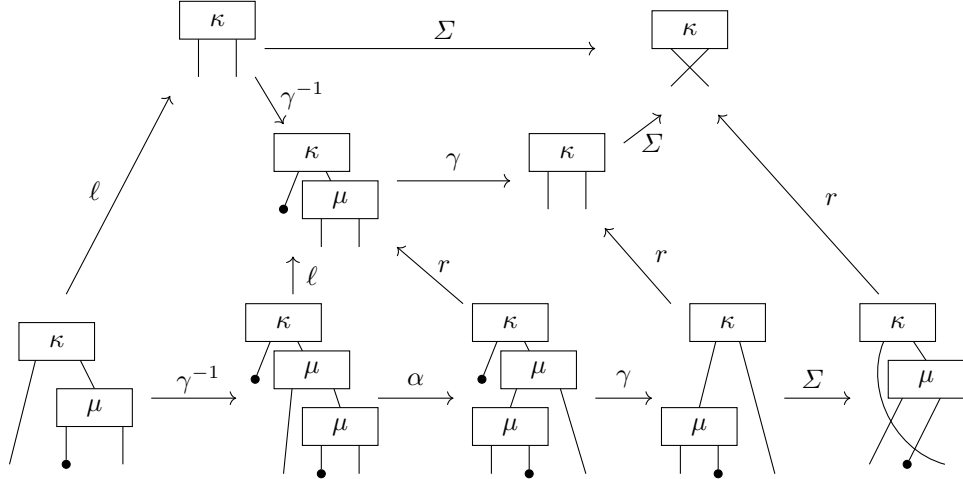
– The relation $\Omega^3 = \mathrm{id}$ follows from (H1).
– The proof of (4.3) follows from a computation similar to the ones leading to (4.4) and (4.5).

The assignments $(\longrightarrow)$ and $(\longleftarrow)$ are inverse to each other: Suppose we start with Ω and define γ via (4.6) and define Ω' using γ via (4.8). We need to show $\Omega' = \Omega$. But this is a consequence of (4.7). Next suppose

we start with γ and define Ω via (4.7) and use the so-defined Ω to define γ' via (4.6). By definition γ' is then composition in clockwise direction in the following diagram:
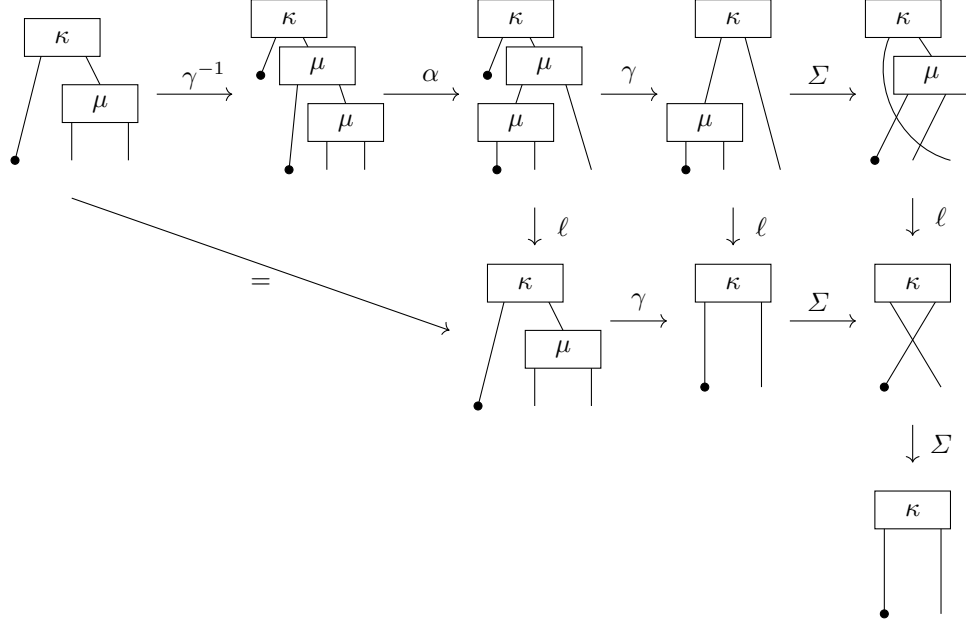
The triangle on the left commutes by (H2). The two squares commute because the horizontal composition of 2-morphisms applied to different 1-morphisms is commutative. Since Σ squares to the identity, $\gamma' = \gamma$. This proves that the assignments ($\longrightarrow$) and ($\longleftarrow$) are inverse to each other and completes step (ii) and hence the proof.

$\square$

4.2 Relation between cyclic associative algebras and Grothendieck-Verdier categories

In order to phrase Theorem 4.2 in terms of Grothendieck-Verdier duality, we recall the definition of the latter from [BD13] (Grothendieck-Verdier categories have been considered earlier in [Bar79] under the name $\star$-autonomous categories). We will use conventions dual to the ones in [BD13]. This is merely for convenience and does not make an essential difference, see Remark 4.4.

Definition 4.3. A *Grothendieck-Verdier category* is a monoidal category $\mathcal{C}$ together with an object $K \in \mathcal{C}$ such that $\mathcal{C}(K, X \otimes -)$ is representable for every $X \in \mathcal{C}$ and such that the functor $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ sending X to a representing object DX for $\mathcal{C}(K, X \otimes -)$ is an equivalence. The object K is referred to as *dualizing object*. The functor D is referred to as *duality functor*.

In more detail, the functor D is defined by *choosing* an object $DX \in \mathcal{C}$ and a natural isomorphism $\mathcal{C}(K, X \otimes -) \cong \mathcal{C}(DX, -)$ for every $X \in \mathcal{C}$. The assignment $X \longmapsto DX$ extends to a functor by the Yoneda Lemma. Note that representability of $\mathcal{C}(K, X \otimes -)$ is a property; the pair of the choice of DX as representing object and the isomorphism $\mathcal{C}(K, X \otimes -) \cong \mathcal{C}(DX, -)$, however, is only unique up to canonical isomorphism. While D is only essentially unique in the sense just explained, the requirement that D is an equivalence does not depend on the choice involved in the definition of D.

Remark 4.4. Definition 4.3 means that for some distinguished object $K \in \mathcal{C}$ we have fixed isomorphisms

$$\mathcal{C}(K, X \otimes Y) \cong \mathcal{C}(DX, Y)$$

natural in X and Y. Since $I \otimes -$ is isomorphic to the identity functor, we obtain

$$K \cong DI \tag{4.9}$$

by a canonical isomorphism. In [BD13] natural isomorphisms $\mathcal{C}(X \otimes Y, K) \cong \mathcal{C}(X, DY)$ are used instead. Both definitions are equivalent via categorical duality in the following sense: If a category $\mathcal{C}$ with monoidal product $\otimes$ has a Grothendieck-Verdier structure with duality D according to Definition 4.3, then $\mathcal{C}^{\mathrm{opp}}$ equipped with $\otimes^{\mathrm{opp}}$ (here, the 'opp' on the monoidal product means that we pass to the opposite category *and* flip the tensor factors) is a Grothendieck-Verdier category with duality $D^{\mathrm{opp}} : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathcal{C}$ in the sense of [BD13].

Remark 4.5 (Normalized duality)**.** We have seen in (4.9) that, regardless of *how* the duality functor is defined, there is always a canonical isomorphism $K \cong DI$. In fact, it is most convenient to have $\mathcal{C}(K, I \otimes -)$ represented through $\mathcal{C}(K, I \otimes -) \overset{\ell}{\cong} \mathcal{C}(K, -)$, where ℓ is the left unitor. Then $DI = K$ strictly on object level. We will refer to any duality functor that extends these choices made for the monoidal unit as *normalized*. By the essential uniqueness of duality functors, it can always be assumed without loss of generality that the duality functor is normalized.

Example 4.6. Every (right) rigid monoidal category is an example of a Grothendieck-Verdier category. Recall that a monoidal category $(\mathcal{C}, \otimes, I)$ is *(right) rigid* if every object $X \in \mathcal{C}$ admits a right dual $X^{\vee}$. This is an object $X^{\vee} \in \mathcal{C}$ together with an evaluation map $\mathrm{ev}_X : X^{\vee} \otimes X \longrightarrow I$ and coevaluation map $\mathrm{coev}_X : I \longrightarrow X \otimes X^{\vee}$ which satisfy the usual snake relations. We can define a Grothendieck-Verdier structure on the monoidal category $\mathcal{C}$ that consists of the object $K = I$ and the natural isomorphisms

$$\mathcal{C}(I, X \otimes Y) \longrightarrow \mathcal{C}(X^{\vee}, Y)$$
$$(f : I \to X \otimes Y) \longmapsto \left(X^{\vee} \xrightarrow{\mathrm{id}_{X^{\vee}} \otimes f} X^{\vee} \otimes X \otimes Y \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_Y} Y \right)$$

for all $X, Y \in \mathcal{C}$.

Example 4.7. The following example is well-known: For a set X, denote by $\wp(X)$ the category of subsets of X with inclusions as morphisms. The union provides a monoidal structure on $\wp(X)$ with monoidal unit $\emptyset \in \wp(X)$. For $U \in \wp(X)$, denote by $\mathsf{C}(U) \in \wp(X)$ the complement. The canonical isomorphisms

$$\wp(X)(X, U \cup -) \cong \wp(X)(\mathsf{C}(U), -)$$

endow $(\wp(X), \cup)$ with a Grothendieck-Verdier structure with dualizing object X and duality C. If X is not the empty set, this provides us with an example of a Grothendieck-Verdier category which does not come from a rigid monoidal category in the sense of Example 4.6.

Definition 4.8. A Grothendieck-Verdier category whose dualizing object coincides up to isomorphism with the monoidal unit is called an *r-category*.

By Example 4.6 every rigid category can be seen as an r-category, but by [BD13, Example 0.9] it is false that conversely every r-category comes from a rigid monoidal structure.

Since a Grothendieck-Verdier structure is a weakening of rigidity, one might hope that there is also a notion of a pivotal structure. Boyarchenko and Drinfeld [BD13, Definition 5.1] propose the following (again, we present the dualized version, see Remark 4.4):

Definition 4.9. A *pivotal structure* on a Grothendieck-Verdier category $\mathcal{C}$ with dualizing object K and duality D is the choice of an isomorphism

$$\psi_{X,Y} : \mathcal{C}(K, X \otimes Y) \longrightarrow \mathcal{C}(K, Y \otimes X)$$

natural in $X, Y \in \mathcal{C}$ satisfying

$$\psi_{X,Y} = \psi_{Y,X}^{-1} \tag{4.10}$$

and making the diagram

$$
\begin{array}{ccc}
\mathcal{C}(K,(X\otimes Y)\otimes Z) & \xrightarrow{\;\psi_{X\otimes Y,Z}\;} & \mathcal{C}(K,Z\otimes(X\otimes Y)) & \xrightarrow{\;\mathcal{C}(K,\alpha_{Z,X,Y})\;} & \mathcal{C}(K,(Z\otimes X)\otimes Y)
\end{array}
$$

$$
\tag{4.11}
$$

commute for $X,Y,Z \in \mathcal{C}$. Here α is the associator of the monoidal category $\mathcal{C}$.

Remark 4.10. By [BD13, Proposition 5.7], a pivotal structure amounts precisely to a natural monoidal isomorphism $D^2 \cong \mathrm{id}_{\mathcal{C}}$ whose component at the unit I is the canonical isomorphism $D^2 I \cong I$.

Example 4.11. The notion of a pivotal structure on a Grothendieck-Verdier category generalizes the notion of a pivotal structure on a rigid monoidal category: Recall that a *pivotal structure* on a rigid monoidal category $(\mathcal{C}, \otimes, I)$ is a monoidal natural isomorphism $\omega : -^{\vee\vee} \Longrightarrow \mathrm{id}_{\mathcal{C}}$. This induces a pivotal structure for the corresponding rigid Grothendieck-Verdier structure (see Example 4.6) by sending a morphism $f : I \longrightarrow X \otimes Y$ to

$$
I \xrightarrow{\;\mathrm{coev}_{X^\vee}\;} X^\vee \otimes X^{\vee\vee} \cong X^\vee \otimes (I \otimes X^{\vee\vee})
$$

$$
\xrightarrow{\;\mathrm{id}_{X^\vee}\otimes(f\otimes\mathrm{id}_{X^{\vee\vee}})\;} X^\vee \otimes ((X \otimes Y) \otimes X^{\vee\vee}) \cong ((X^\vee \otimes X) \otimes Y) \otimes X^{\vee\vee}
$$

$$
\xrightarrow{\;(\mathrm{ev}_X \otimes \mathrm{id}_Y)\otimes\omega_X\;} (I \otimes Y) \otimes X \cong Y \otimes X \ .
$$

By a Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$ we mean an object $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$ together not only with a Grothendieck-Verdier structure on the underlying category, but actually a lift of all the structure to structure living inside $\mathsf{Lex}^{\mathsf{f}}$. This means in particular that the monoidal product will be left exact by construction.

Non-degenerate symmetric pairings in $\mathsf{Lex}^{\mathsf{f}}$ are intimately related to the morphism spaces by Lemma 2.23 (this was a consequence of the Eilenberg-Watts calculus for finite categories). Now a careful comparison of the characterization of cyclic associative algebras in Theorem 4.2 and the axioms of a pivotal Grothendieck-Verdier category leads to the following specialization of Theorem 4.2:

Theorem 4.12. *The structure of a cyclic associative algebra in $\mathsf{Lex}^{\mathsf{f}}$ amounts precisely to a pivotal Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$.*

Proof. The result is a straightforward specialization of Theorem 4.2 once we take the description of non-degenerate symmetric pairings in $\mathsf{Lex}^{\mathsf{f}}$ given in Lemma 2.23 into account. More precisely, a non-degenerate symmetric pairing $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ can be equivalently written as $\kappa(X,Y) = \mathcal{C}(DX,Y)$, where $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ is an equivalence.

Taking this into consideration, Theorem 4.2 allows us to conclude that a cyclic associative algebra $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$ is precisely the following structure:

- By point (M) we obtain a monoidal structure on $\mathcal{C}$.

- By point (P) — without exploiting the symmetry yet — $\mathcal{C}$ comes with a non-degenerate pairing κ which can equivalently be described by an equivalence $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$.

- By point (Z) we obtain natural isomorphisms $\mathcal{C}(K, X \otimes Y) \cong \mathcal{C}(DX, Y)$ with $K = DI$. For $X = I$, this isomorphism is induced by the left unitor thanks to (H2).

- The structure in the three preceding points describes precisely a Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$ with normalized duality (which one can always assume without loss of generality by Remark 4.5). The

symmetry part of (P) and condition (H1) now amount precisely to a pivotal structure on this Grothendieck-Verdier category. In more detail, the fact that the symmetry isomorphisms of κ square to the identity gives us (4.10), and the commutativity of the diagram in (H1) gives us (4.11).

$\square$

Remark 4.13. The characterization of cyclic associative algebras in Theorem 4.2 works in an *arbitrary* symmetric monoidal bicategory. But the translation to pivotal Grothendieck-Verdier categories (Theorem 4.12) makes use of the specialization to the symmetric monoidal bicategory $\mathsf{Lex}^{\mathsf{f}}$. For example, Theorem 4.12 fails in Cat because for any cyclic associative algebra in Cat the underlying category has to be dualizable in Cat. But by Remark 2.15 this actually implies that this category is equivalent to the one-point category. Clearly, this is not the case for all Grothendieck-Verdier structures in Cat.

Remark 4.14. Cyclic associative algebras in the symmetric monoidal category of vector spaces are symmetric Frobenius algebras (if one, as in this article, considers *symmetric* pairings). Motivated by this fact, one might refer to the algebraic structure that in Theorem 4.2 (and the specialization in Theorem 4.12) we prove to be equivalent to a cyclic algebra over the associative operad (in a symmetric monoidal bicategory) as a *symmetric Frobenius algebra up to coherent isomorphism*. The pairing $\kappa : X \otimes X \longrightarrow I$ for the underlying *non-symmetric* Frobenius structure (i.e. the structure in Theorem 4.2 without the isomorphism Σ and (H1) and (H2)) can be described in terms of a trace $\varepsilon : X \longrightarrow I$. This follows from arguments given by Street in [Str04, Proposition 3.2]. While this allows us to describe parts of the algebraic structure found in Theorem 4.2 in an equivalent way, it does, as far as we see, not provide a shortcut in the proof of the relation to *operadically* defined cyclic associative algebras — and it is the latter that we need for applications in low-dimensional topology.

5 Categorical framed little disks algebras and ribbon Grothendieck-Verdier structures

The *operad E_2 of little disks* is the topological operad whose space $E_2(r)$ of r-ary operations is given by the space of embeddings $\left(\mathbb{D}^2\right)^{\sqcup r} \longrightarrow \mathbb{D}^2$ of the disjoint union of r disks into another disk that are composed of a translation and rescaling, see [BV68, BV73] and additionally [Fre17] for a textbook introduction. There is a well-known extension of the little disks operad, the *operad $\mathsf{f}E_2$ of framed little disks* which allows, in addition to translations and rescalings, also rotations, i.e.

$$\mathsf{f}E_2(r) = E_2(r) \times (\mathbb{S}^1)^{\times r} \tag{5.1}$$

as spaces, where the r factors of $\mathbb{S}^1$ encode the rotation parameter. The operadic composition is given by composition of maps and can be obtained from the semidirect product construction in [SW03]. In [Bud08] it is proven that $\mathsf{f}E_2$ is equivalent to the operad of conformal balls, and that the latter comes with a cyclic structure. Therefore, $\mathsf{f}E_2$ is equivalent to a cyclic operad.

This section is devoted to the characterization of cyclic algebras over $\mathsf{f}E_2$ with values in an arbitrary symmetric monoidal bicategory.

## 5.1 A groupoid model for the cyclic operad of framed little disks

From (5.1) it follows that the framed little disks operad is aspherical. A groupoid model can be given in terms of ribbon braids [SW03]. The purpose of this subsection is to also describe the cyclic structure on the level of this groupoid model. We do this by defining a cyclic structure on the ribbon braid operad. In [CIW19] Campos, Idrissi and Willwacher give a cyclic action on a parenthesized version of RBr. The action defined in this section is the analogue of this cyclic structure on RBr. In Section 5.2, we prove that it corresponds to the *geometric* cyclic structure.

We first recall the operad of ribbon braids: Denote by $\pi : RB_n \longrightarrow \Sigma_n$ the canonical map from the ribbon braid group on n strands to the symmetric group on n letters. This map defines an RB_n-action on Σ_n. The

corresponding action groupoids $\Sigma_n /\!/ RB_n$ for varying $n \geq 0$ provide a groupoid model RBr for the framed little disks operad, see [Wah01] and [SW03, Section 7] for details. Concretely, the groupoid of arity n-operations is given by $\mathsf{RBr}(n) \coloneqq \Sigma_n /\!/ B_n \times (\star /\!/ \mathbb{Z})^n$, where $\star /\!/ \mathbb{Z}$ is the groupoid with one object with automorphism group $\mathbb{Z}$. We refer to [Wah01, Section 1.2] for the details on the operad structure; we will also momentarily give a presentation in terms of generators and relations. The operad RBr being a groupoid model for $\mathsf{f}E_2$ means that there is an equivalence

$$\mathsf{RBr} \xrightarrow{\simeq} \Pi \mathsf{f}E_2 \tag{5.2}$$

of operads. On the level of objects, the category-valued operad RBr coincides with the associative operad As meaning that it has a binary associative and unital operation μ (the generators and relations were listed explicitly on page 30). As a consequence, there is a canonical operad map $\mathsf{As} \longrightarrow \mathsf{RBr}$. However, RBr has non-trivial morphisms in the groupoids of operations that are generated under operadic composition by the braiding $c : \mu \longrightarrow \mu^{\mathrm{opp}}$ and the balancing $\theta \in \mathsf{RBr}(1)$. In order to formulate the relations for these morphisms, we will use a graphical notation: We denote the braiding by an overcrossing and the balancing by a filled disk (or ellipse depending on the number of strands it has to be stretched over). The inverse braiding $\mu^{\mathrm{opp}} \longrightarrow \mu$ is denoted by an undercrossing and the inverse balancing by a white disk with boundary. This graphical calculus for the *morphisms* in the groupoids of operations is different from the graphical calculus for the *objects* of operations that we have already used. In order to avoid confusion, we will print the graphical computations for the morphisms with fat lines (this is merely for convenience; it is logically not needed). The generators that RBr has in addition to those of As and their relations are now as follows (when listing generating morphisms for groupoid-valued operads, we do not list their inverses explicitly):
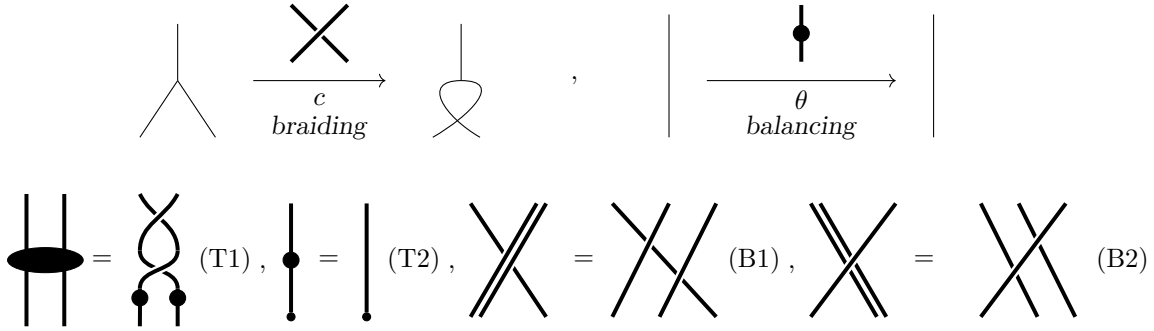
The relations (B1) and (B2) require some explanation and are only well-defined after the relations from the associative operad are imposed. For example, the equation (B1) has to be understood as the commutativity of the diagram


in $\mathsf{RBr}(3)$. We have used here the usual notation $\circ_i$ for the partial composition of operations. The relation (B2) has to be interpreted analogously. Clearly, these relations just encode the usual hexagon relations for braided monoidal categories.

We need to elaborate on a notational subtlety: In the graphical calculus, we have the braiding $c : \mu \longrightarrow \mu^{\mathrm{opp}}$ and the inverse braiding $c^{-1} : \mu^{\mathrm{opp}} \longrightarrow \mu$. But by symmetry, there is also a braiding $\mu^{\mathrm{opp}} \longrightarrow \mu$ and its
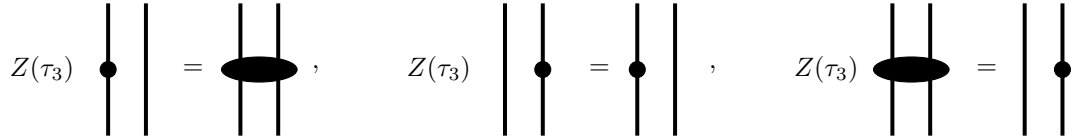
inverse $\mu \longrightarrow \mu^{\mathrm{opp}}$. In other words, we have four types of braidings if we take into account not only the type of crossing, but also the source and target operation. Very often this distinction is suppressed in the notation. In the sequel, this distinction will be relevant. For this reason, we will (whenever needed) indicate the source and target operation with numbers as follows:
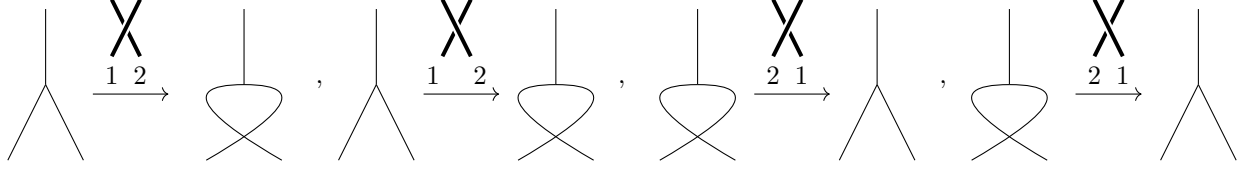
In order to equip RBr with a cyclic structure, we first exhibit the cyclic action on the groupoids of operations $\mathsf{RBr}(n)$ for $n = 1, 2$:

- The $\mathbb{Z}_2$-action on $\mathsf{RBr}(1)$ is defined to be trivial.

- The generator τ_3 of $\mathbb{Z}_3$ acts by the functor $Z(\tau_3) : \mathsf{RBr}(2) \longrightarrow \mathsf{RBr}(2)$ which is defined to be the identity on objects. On morphisms it is given as follows:

  - The action $Z(\tau_3)$ on the twist on one of two strands or both strands is actually already fixed by the definition on $\mathsf{RBr}(1)$ and the requirements for a cyclic structure:

 - The definition $Z(\tau_3)$ on the braiding is given by:

  (5.3)

  Note that one of these assignments fixes the remaining three.

  - The cyclic action on the other generators is trivial.

Lemma 5.1. $Z(\tau_2) : \mathsf{RBr}(1) \longrightarrow \mathsf{RBr}(1)$ and $Z(\tau_3) : \mathsf{RBr}(2) \longrightarrow \mathsf{RBr}(2)$ are functors that extend the permutation actions of the operad structure on RBr. More precisely, $Z(\tau_2)$ yields a Σ_2-action on $\mathsf{RBr}(1)$, and $Z(\tau_3)$ extends the Σ_2-action on $\mathsf{RBr}(2)$ to a Σ_3-action.

Proof. For $Z(\tau_2)$, the statement is clear.

In order to prove that $Z(\tau_3)$ is a functor, we need to verify that it preserves the relation (T1). This is indeed the case:
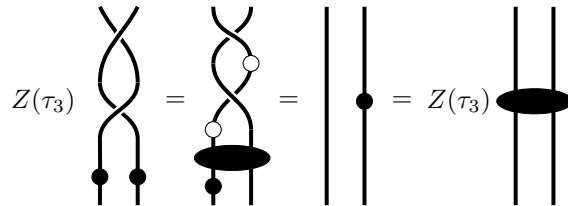
In order to see that $Z(\tau_3)$ extends the Σ_2-action on $\mathsf{RBr}(2)$ to a Σ_3-action, we need to verify two things:

- The relation $\tau_3\sigma_{1,2} = \sigma_{1,2}\tau_3^2$ with the transposition $\sigma_{1,2}$ is respected. Indeed:

$$Z(\tau_3)Z(\sigma_{1,2})\ \Big|\!\times\!\Big| \ = Z(\tau_3)\ \Big|\!\times\!\Big| \ = \ \Big|\!\times\!\Big| \ ,$$
$$\quad\quad\quad\quad\quad 1\ \ 2 \quad\quad\quad\quad 2\ \ 1 \quad\quad 2\ \ 1$$

$$Z(\sigma_{1,2})Z(\tau_3)^2\ \Big|\!\times\!\Big| \ = \ Z(\sigma_{1,2})Z(\tau_3)\ \Big|\!\times\!\Big| \ = Z(\sigma_{1,2})\ \Big|\!\times\!\Big| \ = \ \Big|\!\times\!\Big|$$
$$\quad\quad\quad\quad\quad\quad 1\ \ 2 \quad\quad\quad\quad\quad 1\ \ 2 \quad\quad\quad\quad\quad 1\ \ \ 2 \quad\quad 2\ \ 1$$

- The functor $Z(\tau_3)$ triples to the identity. For this, it suffices to check the following:

$$Z(\tau_3)^3\ \Big|\!\times\!\Big| \ = Z(\tau_3)^2\ \Big|\!\times\!\Big| \ = \ Z(\tau_3)\ \Big|\!\times\!\Big| \ = \ \Big|\!\circ\!\Big| \ = \ \Big|\!\times\!\Big|$$
$$\quad\quad\quad\quad 1\ \ 2 \quad\quad\quad\quad 1\ \ 2 \quad\quad\quad\quad 1\quad\ 2 \quad\quad 1\ \ 2 \quad\quad 1\ \ 2$$

$$\square$$

Proposition 5.2. *The cyclic action from Lemma 5.1 naturally extends to the structure of a* Cat*-valued cyclic operad on* RBr.

Proof. We can extend the cyclic action from from Lemma 5.1 to all of RBr by using the following general formula for the cyclic action on composed operations [MSS02, Remark 5.3]:

$$\tau(a \circ_1 b) = \tau(b) \circ_m \tau(a) \tag{5.4}$$
$$\tau(a \circ_i b) = \tau(a) \circ_{i-1} b \tag{5.5}$$

Here a and b are arbitrary operations of arity n and m larger than 0, respectively, and $1 < i \le n$. When $m = 0$ and $i \neq 1$, we can use the same formula. However, if $i = 1$, we need to use $\tau(a \circ_1 b) = \tau(a)^2 \circ_{m-1} b$ instead. For this to be well-defined, we need to verify the compatibility with the relations (B1) and (B2) (the compatibility with (T1) and (T2) is already a part of Lemma 5.1): We use (5.4) and (5.5) to apply the cyclic permutation to the diagram (B1) and get

$$\tag{5.6}$$

For the upper horizontal map, no cyclic permutation is applied to c in accordance with (5.5). When evaluating τ on the operation which is the result of applying σ_{12}, we use the relation $\tau \circ \sigma_{23} = \sigma_{12} \circ \tau$ to evaluate one of

the actions. We now need to verify that (5.6) commutes. This amounts precisely to the identity
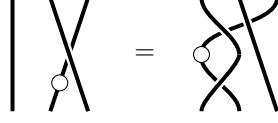
which can be easily seen to hold. A similar computation shows that (B2) is respected. This completes the proof. □

5.2 Equivalence of the cyclic structure on RBr to the one on fE_2

In this section, we show that the cyclic structure defined on RBr above agrees with the cyclic structure on fE_2 induced by identifying it with the operad of genus zero surfaces (here, surfaces will always be compact and oriented) and hence also with the cyclic structure found by Budney in [Bud08]. To this end, we will, similarly to [Bud08], exhibit an equivalence from fE_2 to a topological operad S_0 built from disk configurations in the 2-sphere (it is essentially the operad of genus zero surfaces). The operad S_0 has an obvious cyclic structure, and we prove that there is an equivalence $RBr \simeq \Pi S_0$ of cyclic operads, where the cyclic structure on RBr is the one given above.

Before defining S_0 we need to recall some basic properties of the stereographic projection: For every point $p \in \mathbb{S}^2$, the stereographic projection is a diffeomorphism $\mathrm{st} : \mathbb{S}^2 \setminus \{p\} \longrightarrow T_{-p}\mathbb{S}^2$ from $\mathbb{S}^2 \setminus \{p\}$ to the tangent space to $\mathbb{S}^2$ at $-p$. We consider $\mathbb{S}^2$ here as a submanifold of $\mathbb{R}^3$ and hence can identify the tangent space $T_{-p}\mathbb{S}^2$ with an affine plane in $\mathbb{R}^3$. The canonical scalar product on $\mathbb{R}^3$ induces a natural scalar product on $T_{-p}\mathbb{S}^2$. A vector $x \in T_{-p}\mathbb{S}^2$ on the unit circle in $T_{-p}\mathbb{S}^2$ can be uniquely extended to a positively oriented orthonormal basis and thereby induces a linear isomorphism $T_{-p}\mathbb{S}^2 \cong \mathbb{R}^2$.

Let us now define the operad S_0: For $n \geq 0$, a map $\sqcup_{j=0}^n \mathbb{D}_j^2 \longrightarrow \mathbb{S}^2$ is called *admissible* if

- the images of the interiors are pairwise disjoint,

- for every map $f_j : \mathbb{D}_j^2 \longrightarrow \mathbb{S}^2$, the point $-f_j(0) \in \mathbb{S}^2$ is not contained in the image of f_j,

- and the composition $\mathbb{D}_j^2 \xrightarrow{f_j} \mathbb{S}^2 \setminus \{-f_j(0)\} \xrightarrow{\mathrm{st}} T_{f_j(0)}\mathbb{S}^2 \cong \mathbb{R}^2$ is given by a rescaling, where the last isomorphism is induced, as explained above, by the choice of a point on the unit circle of $T_{f_j(0)}\mathbb{S}^2$, namely the intersection of the unit circle of $T_{f_j(0)}\mathbb{S}^2$ with the line in $T_{f_j(0)}\mathbb{S}^2$ from 0 to the point $\mathrm{st} \circ f_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We equip the space of admissible maps with the topology induced by the compact-open topology on the mapping space. There is an SO(3)-action on the space of all admissible maps $\sqcup_{i=0}^n \mathbb{D}_i^2 \longrightarrow \mathbb{S}^2$ coming from the SO(3)-action on $\mathbb{S}^2$. An operation in $S_0(n)$ is an SO(3)-orbit of admissible maps $\sqcup_{j=0}^n \mathbb{D}_j^2 \longrightarrow \mathbb{S}^2$. The partial composition $\underline{f} \circ_\ell \underline{f'}$ of two operations $\underline{f} : \sqcup_{i=0}^n \mathbb{D}_i^2 \xrightarrow{\sqcup f_i} \mathbb{S}^2$ and $\underline{f'} : \sqcup_{j=0}^{n'} \mathbb{D}_j^2 \xrightarrow{\sqcup f'_j} \mathbb{S}^2$ can be defined as follows: For $\underline{f'}$, perform a stereographic projection from the center of the zeroth disk. This yields a configuration of n' disks inside a bigger disk in the plane. We insert these n' disks into the ℓ-th disk in $\underline{f}$ to get the partial composition $\underline{f} \circ_\ell \underline{f'}$.

The permutation group on $n + 1$ letters acts on $S_0(n)$ and thereby endows S_0 with the structure of a cyclic operad. There is a canonical equivalence of operads

$$fE_2 \xrightarrow{\simeq} S_0 \tag{5.7}$$

which sends a disk configuration in the standard disk to a disk configuration in the lower hemisphere via (the inverse of) the stereographic projection and adds the upper hemisphere as the zeroth disk.

The next result shows that the cyclic structure on S_0 can be combinatorially described by the cyclic structure on RBr from Proposition 5.2.

Proposition 5.3. *There is an equivalence* $RBr \longrightarrow \Pi S_0$ *of groupoid-valued cyclic operads.*

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Proof. We start by constructing an equivalence $\Phi : \mathsf{RBr} \longrightarrow \mathit{\Pi}\mathsf{S}_0$ of operads. As recalled in (5.2), there is an equivalence of operads $\mathsf{RBr} \longrightarrow \mathit{\Pi}\mathsf{f}E_2$ sending the generator μ to the disk configuration in the lower left corner of Figure 5b, the braiding to the homotopy which slides disk one over disk two (see lower part of Figure 5b) and the twist to the rotation of a disk by 360 degrees against its orientation. This map is not a strict map of operads (associativity is not respected strictly), but only up to coherent isomorphism (which in the bicategorical framework is allowed). We can obtain the map as follows: There is a zigzag of strict equivalences (where 'strict' means that the coherence data is trivial) between Cat-valued operads $\mathsf{RBr} \longleftarrow \mathsf{PaRB} \longrightarrow \mathit{\Pi}(\mathsf{f}E_2)$, where PaRB is the operad of parenthesized braids [CIW19, Section 3.1], and the first map forgets the parentheses. Now by Proposition 2.19 we can invert the first map with a non-strict morphism of Cat-valued operads. The composition of this inverse with the second map gives us the desired morphism. The equivalence $\Phi : \mathsf{RBr} \longrightarrow \mathit{\Pi}\mathsf{S}_0$ is the composition of the equivalence from (5.2) with the one from (5.7).

It remains to show that the equivalence Φ is compatible with the cyclic structure — possibly up to coherent isomorphism (again, let us emphasize that our notion of cyclic operads and their morphisms naturally accounts for that). For an object $o \in \mathsf{RBr}(n)$, this compatibility means that the cyclic action on $\Phi(o)$ is trivial up to coherent isomorphism because the cyclic action on RBr is trivial on the object level. Indeed, acting with a cyclic permutation on $\Phi(o)$ changes only the sizes of the disks (since S_0-operations are $SO(3)$-orbits, the configuration stays the same otherwise). We denote by h_{n+1} the corresponding rescaling homotopy. The rescaling homotopies are coherent in the sense that they are compatible with composition. The reason for this is that the rescalings are unique up to higher homotopy, i.e. unique in $\mathit{\Pi}\mathsf{S}_0$.

It remains to show that these homotopies form *natural transformations*

$$
\begin{array}{ccc}
\mathsf{RBr}(n) & \xrightarrow{\ \mathsf{RBr}(\tau_{n+1})\ } & \mathsf{RBr}(n) \\[2pt]
\Big\downarrow{\scriptstyle \Phi} & \quad{\scriptstyle h_{n+1}}\ \nearrow & \Big\downarrow{\scriptstyle \Phi} \\[2pt]
\mathit{\Pi}\mathsf{S}_0(n) & \xrightarrow[\ \mathit{\Pi}(\mathsf{S}_0)(\tau_{n+1})\ ]{} & \mathit{\Pi}\mathsf{S}_0(n)\ ,
\end{array}
$$

i.e. are compatible with the morphisms in the groupoids of operations. It is enough to verify this on the generating morphisms c in arity two and θ in arity one (because, as argued above, the homotopies are compatible with composition):

- The generator θ is mapped to the automorphism of the configuration in Figure 5a which rotates the first disk $\mathbb{D}_1^2$ by -360 degrees. Applying the cyclic structure maps this to the automorphism of the lower configuration in Figure 5a which rotates zeroth disk $\mathbb{D}_0^2$ by -360 degrees. After applying an element in $SO(3)$, this agrees with the automorphism of the original configuration rotating $\mathbb{D}_0^2$ by 360 degrees. But this is equivalent to rotating $\mathbb{D}_1^2$ by -360 degrees. This shows that h_2 (which is actually the identity) is natural with respect to θ.

- The image of the braiding under the equivalence Φ has the following description (using the freedom of choosing representatives in the $SO(3)$-orbits): We rotate $\mathbb{D}_0^2$ by -180 degrees, $\mathbb{D}_1^2$ by 180 degrees and $\mathbb{D}_2^2$ by 180 degrees, see Figure 5b for a sketch. Up to a rescaling, which is absorbed by the homotopy h_3, the cyclic action produces the morphism rotating $\mathbb{D}_0^2$ by 180 degrees, rotating $\mathbb{D}_1^2$ by 180 degrees and rotating $\mathbb{D}_2^2$ by -180 degrees. Figure 5c explains that this agrees (after rescaling) with the image of $\bar{c} \circ_1 \theta^{-1}$ under Φ — as it should be by the definition of the cyclic action on RBr.

This shows that Φ is compatible with the cyclic structure and finishes the proof. $\qquad\square$

5.3 A characterization of cyclic RBr-algebras in a symmetric monoidal bicategory

A (non-cyclic) RBr-algebra in a symmetric monoidal bicategory $\mathcal{M}$ is a homotopy coherent associative algebra (i.e. an object $X \in \mathcal{M}$ with multiplication μ, associator α, unit u and unitors ℓ and r) plus two isomorphisms,
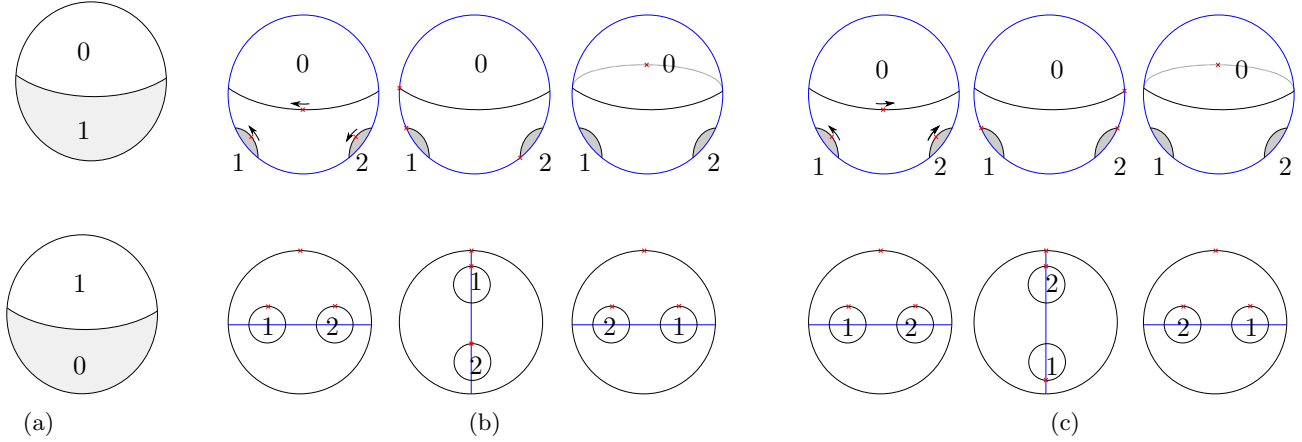
Figure 5: (a) The top figure shows the image of the operadic identity in RBr under the equivalence from Theorem 5.3. The bottom figure shows the same, but *after* the action with a cyclic permutation. The two configurations can be transformed into each other by acting with an element of SO(3).
(b) The top figure shows the path in $\mathsf{S}_0(2)$ corresponding to the braiding. The bottom figure is its stereographic projection from the center of $\mathbb{D}_0^2$. We marked the base point of every disk by a cross. The missing crosses on the right are on the backside of the sphere. The blue line above is always mapped to the blue line below by the stereographic projection.
(c) The action of τ_3 on the image of c in $\mathsf{S}_0(2)$. We use the same drawing conventions as in (b).

namely a braiding and a balancing
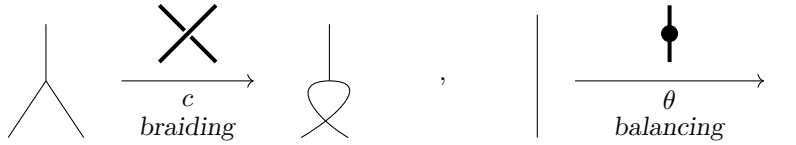
subject to the analogue of the relations (T1), (T2), (B1) and (B2), but with the coherence isomorphisms α, ℓ and r inserted in the necessary places, see [JS91] and also [Wee19, Section 4] and [Woi20, Section 5.4.2]. We call $X \in \mathcal{M}$ endowed with this structure a *balanced braided algebra* in $\mathcal{M}$. Taking this notion as a starting point, we can define the following algebraic structure:

Definition 5.4. A *self-dual balanced braided algebra* in a symmetric monoidal bicategory $\mathcal{M}$ is a balanced braided algebra A with a non-degenerate pairing $\kappa : A \otimes A \longrightarrow I$ with an isomorphism γ
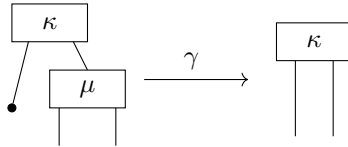
subject to the following conditions:

(U) The isomorphism
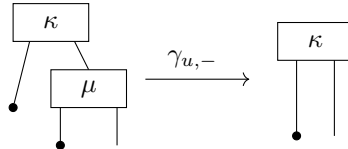
induced by γ agrees with the one induced by the left unitor ℓ (this is condition (H2) from Theorem 4.2).

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(PB) The automorphisms of κ induced by $\theta \otimes \mathrm{id}$ and $\mathrm{id} \otimes \theta$ agree:

$$\kappa \xrightarrow{\;\theta \otimes \mathrm{id} \,=\, \mathrm{id} \otimes \theta\;} \kappa$$
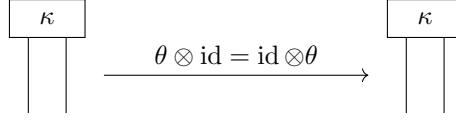
The above definition suggests that a self-dual balanced braided algebra is in particular a self-dual algebra in the sense of Definition 4.1. A priori, this is not obvious, but it follows from the following Lemma:

Lemma 5.5. *The structure of a self-dual balanced braided algebra on an object X in a symmetric monoidal bicategory amounts precisely to the following:*

(BB) *The object X carries the structure of a balanced braided algebra in $\mathcal{M}$ with product μ, associator α, unit u, unitors ℓ and r, braiding c and balancing θ.*

(PZH) *The object X is endowed with a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$ with symmetry isomorphism Σ in the sense of Theorem 4.2 (P) and an isomorphism γ*

$$\overset{\kappa}{\underset{\mu}{\bullet}} \xrightarrow{\;\gamma\;} \kappa$$

which satisfy the compatibility conditions (H1) and (H2) from Theorem 4.2.

(RM) *The following relations are satisfied (the first one is formulated in terms of ψ which was defined using γ in (4.1)):*

$$\overset{\kappa}{\underset{\mu}{\bullet}} \xrightarrow{\;\psi \;=\;} \overset{\kappa}{\underset{\mu}{\bullet}} \quad (RB)$$

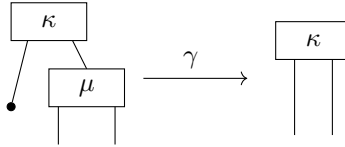$$\begin{array}{ccc} \kappa & \xrightarrow{\;\Sigma\;} & \kappa \\ {\scriptstyle \theta \otimes \mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id} \otimes \theta} \\ \kappa & \xrightarrow{\;\Sigma\;} & \kappa \end{array} \quad (RT)$$
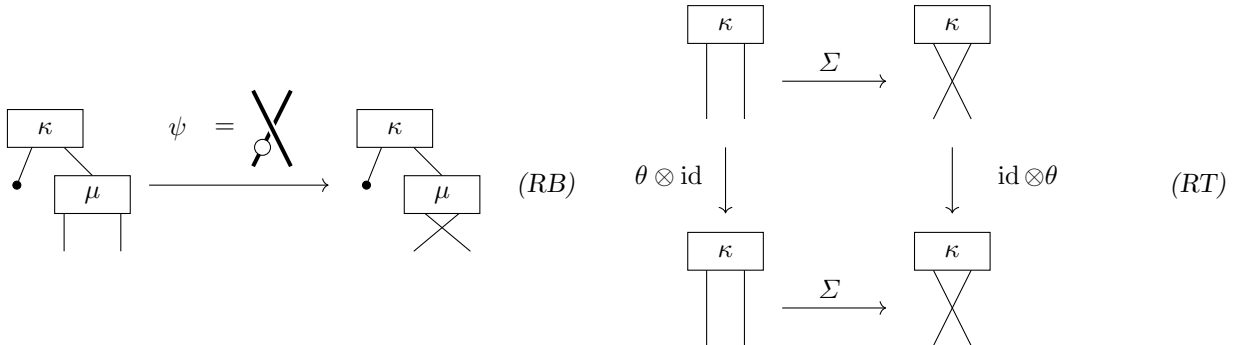
Proof. We observe that if we are given the structure from the statement of the Lemma, we can forget the symmetry isomorphism. Then we have a self-dual balanced braided algebra. In fact, all axioms hold by definition except for (PB) which, however, easily follows from (RT).

The non-trivial part is the converse: Suppose we are given a self-dual balanced braided algebra and want to obtain the algebraic structure described in the Lemma. The structure and conditions that are part of the structure described in the above Lemma, but are *not* part of the definition of a self-dual balanced braided algebra are the symmetry data for κ, (H1), (RB) and (RT). And in fact, we can show that these four pieces of information are redundant. The symmetry isomorphism Σ is redundant information because, if (RB) is supposed to hold, then Σ must be given by

$$\Sigma := \kappa \xrightarrow{\;\gamma^{-1}\;} \overset{\kappa}{\underset{\mu}{\bullet}} \xrightarrow{\quad} \overset{\kappa}{\underset{\mu}{\bullet}} \xrightarrow{\;\gamma\;} \kappa \qquad (5.8)$$

and in fact, this way of defining Σ automatically ensures that it squares to the identity (this follows from (T1), (PB) and (T2)). At the same time, any symmetry isomorphism defined using (5.8) automatically satisfies (RB), thereby making (RB) redundant. Moreover, we note that (H1), when spelled out explicitly, means that the isomorphism
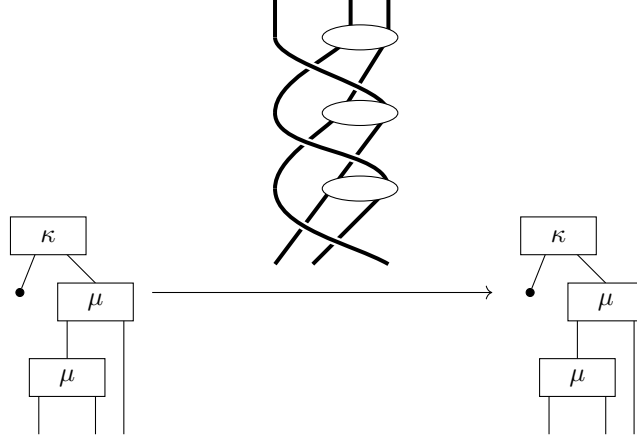
is the identity. This indeed follows from a direct computation using (T1), (PB) and (T2), which makes (H1) redundant. Finally, we observe that (RB) implies (RT). □

Theorem 5.6. *The structure of a cyclic* RBr*-algebra on an object X in a symmetric monoidal bicategory $\mathcal{M}$ amounts precisely to the structure of a self-dual balanced braided algebra in $\mathcal{M}$.*

The proof of the Theorem follows from Lemma 5.5 and the following Proposition:

Proposition 5.7. *The structure of a cyclic* RBr*-algebra on an object X in a symmetric monoidal bicategory $\mathcal{M}$ amounts precisely to the structure described in Lemma 5.5.*

Proof. By the Lifting Theorem 3.1 and its version in terms of generators and relations (Corollary 3.3) a cyclic RBr-algebra amounts precisely to the following:

(i) A non-cyclic RBr-algebra, which gives us precisely the structure of a balanced algebra in point (BB).

(ii) Isomorphisms from generating objects and the relations between them. But the generating *objects* and their relations are identical for As and RBr (the two operads just differ on the level of morphisms), therefore this gives us precisely the isomorphisms and relations found in Theorem 4.2, i.e. (PZH).

(iii) Finally, we get according to Corollary 3.3 (M) one relation for each generating morphism:

- One relation for the balancing, namely (RT) — because the cyclic action preserves the balancing on one strand by definition.
- One relation for the braiding, namely (as follows from (5.3)) the commutativity of the following

square formulated in terms of Ω (which is related to γ by (4.6)):

$$(5.9)$$

where the lower arrow is given by Ω^{-1} which is the value of the natural isomorphism for the cyclic action at μ^{opp}.

In order to complete the proof, it remains to show (5.9) $\Longleftrightarrow$ (RB):

($\Longrightarrow$) If we insert in (5.9) the unit into the first argument from the left, we find after a short calculation (RB) by using the definition of γ in terms of Ω in (4.6), the definition of ψ in terms of γ in (4.1), relation (4.4), and the fact that braiding with the unit is trivial.

($\Longleftarrow$) If we express ψ entirely in terms of Ω and Σ (using (4.1) and (4.6)), we can deduce from a lengthy, but straightforward computation that the commutativity of (5.9) is equivalent to the commutativity of the hexagon

$$(5.10)$$

In order to prove that this hexagon really commutes, we replace ψ by means of (RB). The isomorphism obtained by composing in *clockwise* direction in (5.10) is now (we suppress the associator in the notation):

The isomorphism that we arrive at on the right hand side of this equation is the isomorphism obtained by composing in *counterclockwise* direction in (5.10). Hence, (5.10) and therefore (5.9) commutes.

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$\square$

5.4 Relation between cyclic framed little disks algebras and ribbon Grothendieck-Verdier categories

Theorem 5.6 can be expressed in terms of ribbon Grothendieck-Verdier structures defined in [BD13]:

Definition 5.8. A *braided Grothendieck-Verdier category* is a Grothendieck-Verdier category whose underlying monoidal category is equipped with a braiding c. A *balancing* on $\mathcal{C}$ is a natural automorphism of the identity functor $\mathrm{id}_{\mathcal{C}}$ whose components $\theta_X : X \longrightarrow X$ satisfy

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \quad \text{for} \quad X, Y \in \mathcal{C} , \tag{5.11}$$
$$\theta_I = \mathrm{id}_I .$$

A braided Grothendieck-Verdier structure with a balancing satisfying

$$D\theta_X = \theta_{DX} \quad \text{for} \quad X \in \mathcal{C} \tag{5.12}$$

will be referred to as a *ribbon Grothendieck-Verdier structure*. In that case, the balancing is also called a *ribbon twist*.

Definition 5.9. Let $\mathcal{C}$ be a braided Grothendieck-Verdier category with pivotal structure ψ (Definition 4.9) and balancing θ. We call ψ and θ *compatible* if for all $X, Y \in \mathcal{C}$ the triangle

$$\begin{array}{ccc} \mathcal{C}(K, X \otimes Y) & \xrightarrow{\psi_{X,Y}} & \mathcal{C}(K, Y \otimes X) \\ {\scriptstyle \mathcal{C}(K, c_{X,Y}^{-1})} \Big\downarrow & \nearrow {\scriptstyle \mathcal{C}(K, \mathrm{id}_Y \otimes \theta_X^{-1})} & \\ \mathcal{C}(K, Y \otimes X) & & \end{array} \tag{5.13}$$

commutes.

Lemma 5.10. *For any braided Grothendieck-Verdier category $\mathcal{C}$, a ribbon twist on $\mathcal{C}$ gives rise to a unique pivotal structure compatible with the ribbon twist. Hence, a ribbon twist on $\mathcal{C}$ can be equivalently described as a pivotal structure and a ribbon twist which are compatible.*

Proof. The statement can be reduced to the following: If we are given a balancing and a braiding and *define* ψ by (5.13), the so-defined ψ is a pivotal structure. Indeed, the needed conditions (4.10) and (4.11) follow from a direct computation using (5.11)-(5.12) (it is helpful to first deduce $\theta_K = \mathrm{id}_K$). $\square$

Remark 5.11. As discussed in Remark 4.14, we may describe the pairing $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ in terms of a trace $\varepsilon : \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ related to κ by $\varepsilon(X) = \kappa(I, X)$. Reformulating Definition 5.8 in terms of ε gives rise to a structure related to the notion of a *contraction* on a balanced braided monoidal category defined in [Enr10, Definition 12]. The key differences, however, are that the pairing defined by $\kappa(X, Y) := \varepsilon(X \otimes Y)$ is not required to be non-degenerate and that various coherence isomorphisms (including the pivotal structure) are assumed to be the identity in [Enr10].

As for the associative operad, we can express Theorem 5.6 in terms of Grothendieck-Verdier duality.

Theorem 5.12. *The structure of a cyclic* RBr*-algebra in* $\mathsf{Lex}^{\mathsf{f}}$ *amounts precisely to a ribbon Grothendieck-Verdier category in* $\mathsf{Lex}^{\mathsf{f}}$.

Proof. Thanks to Proposition 5.7, a cyclic RBr-algebra in $\mathsf{Lex}^{\mathsf{f}}$ amounts to the algebraic structure described in Lemma 5.5, i.e. a balanced braided monoidal category in $\mathsf{Lex}^{\mathsf{f}}$ (this is (BB)) and a pivotal Grothendieck-Verdier structure for the underlying monoidal structure (this is (PZH) as follows from our treatment of cyclic As-algebras in Theorem 4.2 and 4.12). Therefore, by Lemma 5.10, it suffices to prove that (RB) $\Longleftrightarrow$ (5.13) and that (RT) $\Longleftrightarrow$ (5.12). While the former follows from the definitions, the latter requires a proof: Relation (RT) is the commutativity of

$$
\begin{array}{ccc}
\kappa(X,Y) & \xrightarrow{\ \Sigma_{X,Y}\ } & \kappa(Y,X) \\
{\scriptstyle \kappa(X,\theta_Y)}\Big\downarrow & & \Big\downarrow{\scriptstyle \kappa(Y,\theta_X)} \\
\kappa(X,Y) & \xrightarrow{\ \Sigma_{Y,X}\ } & \kappa(Y,X) \ ,
\end{array}
$$

where Σ is the symmetry isomorphism of κ. By recalling the description of the pivotal Grothendieck-Verdier structure in terms of the symmetric pairing κ (Theorem 4.12), we see that (RT) is equivalent to the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{C}(K,X\otimes Y) & \xrightarrow{\ \psi_{X,Y}\ } & \mathcal{C}(K,Y\otimes X) \\
{\scriptstyle \mathcal{C}(K,X\otimes\theta_Y)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{C}(K,Y\otimes\theta_X)} \\
\mathcal{C}(K,X\otimes Y) & \xrightarrow{\ \psi_{X,Y}\ } & \mathcal{C}(K,Y\otimes X) \ ,
\end{array}
$$

where ψ is the pivotal structure. By the naturality of ψ this is equivalent to the equality

$$
\mathcal{C}(K,X\otimes Y) \xrightarrow{\ \mathcal{C}(K,\theta_X\otimes Y)=\mathcal{C}(K,X\otimes\theta_Y)\ } \mathcal{C}(K,X\otimes Y)
$$

of maps. Under the natural isomorphism $\mathcal{C}(DX,Y)\cong\mathcal{C}(K,X\otimes Y)$ from the Grothendieck-Verdier structure, this is equivalent to the equality

$$
\mathcal{C}(DX,Y) \xrightarrow{\ \mathcal{C}(D\theta_X,Y)=\mathcal{C}(DX,\theta_Y)\ } \mathcal{C}(DX,Y) \ .
$$

The equivalence of (RT) and $D\theta_X=\theta_{DX}$ for all $X\in\mathcal{C}$ is thereby reduced to the equivalence

$$
\mathcal{C}(D\theta_X,Y)=\mathcal{C}(DX,\theta_Y) \text{ for all } X,Y\in\mathcal{C} \quad\Longleftrightarrow\quad D\theta_X=\theta_{DX} \text{ for all } X\in\mathcal{C} \ ,
$$

which we will prove now:

($\Longrightarrow$) This follows by setting $Y=DX$ and evaluating the left hand side on id_{DX}.

($\Longleftarrow$) For this, we take any morphism $f:DX\longrightarrow Y$ and observe

$$
\begin{aligned}
\theta_Y f &= f\theta_{DX} \quad \text{(by naturality of } \theta) \\
&= fD\theta_X \quad \text{(since } D\theta_X=\theta_{DX}) \ .
\end{aligned}
$$

$\square$

Theorem 5.13. *The 2-groupoid of cyclic* $\mathsf{f}E_2$*-algebras in an arbitrary symmetric monoidal bicategory $\mathcal{M}$ is equivalent to the 2-groupoid of self-dual balanced braided algebras in $\mathcal{M}$. For $\mathcal{M}=\mathsf{Lex}^{\mathsf{f}}$, this 2-groupoid is equivalent to the 2-groupoid of ribbon Grothendieck-Verdier categories in* $\mathsf{Lex}^{\mathsf{f}}$.

Proof. The Theorems 5.6 and 5.12 characterize cyclic RBr-algebras. Using the Comparison Theorem 2.20, we can transfer these results to the cyclic $\mathsf{f}E_2$-operad which is equivalent to RBr by Proposition 5.3. $\square$

Example 5.14. In order to discuss a class of ribbon Grothendieck-Verdier categories, let us recall the construction of pointed braided fusion categories from Abelian group cocycles, see [EML53] and [EGNO15, Section 8.4]: For a finite Abelian group G, denote by $\mathsf{Vect}^{\mathsf{f}}_G$ the category of finite-dimensional G-graded vector spaces over the complex numbers. For G-graded vector spaces V and W, one can define a monoidal product $V \otimes W$ by

$$(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b \quad \text{for all} \quad g \in G \ .$$

In order to specify the associator, we denote by $\mathbb{C}_g$ the ground field $\mathbb{C}$ seen as G-graded vector space supported in degree g. The associator is determined by its values on the simple objects $\mathbb{C}_g$ and given on the simple objects by

$$\alpha_{\mathbb{C}_{g_1}, \mathbb{C}_{g_2}, \mathbb{C}_{g_3}} : (\mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2}) \otimes \mathbb{C}_{g_3} \longrightarrow \mathbb{C}_{g_1} \otimes (\mathbb{C}_{g_2} \otimes \mathbb{C}_{g_3})$$
$$(v_1 \otimes v_2) \otimes v_3 \longmapsto \lambda(g_1, g_2, g_3) v_1 \otimes (v_2 \otimes v_3) \ ,$$

where the numbers $\lambda(g_1, g_2, g_3) \in \mathbb{C}^\times$ form a 3-cocycle $\lambda \in Z^3(G; \mathbb{C}^\times)$. In order to construct a braiding for this monoidal product, we need to complete λ to an Abelian 3-cocycle $\omega = (\lambda, \tau) \in Z^3_{\mathrm{ab}}(G; \mathbb{C}^\times)$, i.e. we additionally need a 2-cochain τ on G such that

$$\lambda(g_2, g_3, g_1)\tau(g_1, g_2 g_3)\lambda(g_1, g_2, g_3) = \tau(g_1, g_3)\lambda(g_2, g_1, g_3)\tau(g_1, g_2) \ ,$$
$$\lambda(g_3, g_1, g_2)^{-1}\tau(g_1 g_2, g_3)\lambda(g_1, g_2, g_3)^{-1} = \tau(g_1, g_3)\lambda(g_1, g_3, g_2)^{-1}\tau(g_2, g_3) \quad \text{for all} \quad g_1, g_2, g_3 \in G \ .$$

Now a braiding is given by

$$c_{\mathbb{C}_{g_1}, \mathbb{C}_{g_2}} : \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2} \longrightarrow \mathbb{C}_{g_2} \otimes \mathbb{C}_{g_1}$$
$$v_1 \otimes v_2 \longmapsto \tau(g_1, g_2) v_2 \otimes v_1$$

This monoidal category is rigid with the left and right dual V^* of $V \in \mathsf{Vect}^{\mathsf{f}}_G$ given by $(V^*)_g = V^*_{g^{-1}}$. Therefore, finite-dimensional G-graded vector spaces with the structure specified above by means of the Abelian 3-cocycle $\omega = (\lambda, \tau)$ give us a braided fusion category that we denote by $\mathsf{Vect}^{\mathsf{f}\,\omega}_G$. It is pointed in the sense that all simple objects are invertible, and in fact, all pointed braided fusion categories are of this form.

The Abelian 3-cocycle $\omega = (\lambda, \tau)$ can be equivalently described by a quadratic form: A *quadratic form* on a finite Abelian group G is a map $q : G \longrightarrow \mathbb{C}^\times$ of sets with $q\left(g^{-1}\right) = q(g)$ for all $g \in G$ such that the symmetric function $b_q : G \times G \longrightarrow k^\times$ defined by

$$b_q(g, h) := \frac{q(gh)}{q(g)q(h)} \quad \text{for} \quad g, h \in G$$

is a bicharacter in the sense that $b(g_1 g_2, h) = b(g_1, h)b(g_2, h)$ for $g_1, g_2, h \in G$. Then by [EML53] the canonical map

$$H^3_{\mathrm{ab}}(G; \mathbb{C}^\times) \longrightarrow \mathrm{Quad}(G; \mathbb{C}^\times) \ , \quad (\lambda, \tau) \longmapsto (g \longmapsto \tau(g, g))$$

is an isomorphism.

Although $\mathsf{Vect}^{\mathsf{f}\,\omega}_G$ is rigid, it can still have Grothendieck-Verdier structures that do not come from rigidity: Since any Grothendieck-Verdier duality has to be an anti-equivalence which maps the simple unit $\mathbb{C}_e$ to the dualizing object K, the dualizing object must be simple and hence given by $K = \mathbb{C}_{g_0}$ for some fixed $g_0 \in G$. It is easy to observe that for each such choice, we can find a canonical Grothendieck-Verdier structure with duality functor $D_{g_0} = \mathbb{C}_{g_0} \otimes (-)^*$. Note that $D_e = (-)^*$ coincides with the usual (rigid) duality.

From [Zet18, Theorem 4.2.2], we may now deduce the following statement: Suppose $g_0 = h_0^{-2}$ for some $h_0 \in G$, and denote by $q : G \longrightarrow \mathbb{C}^\times$ the quadratic form associated to the Abelian cocycle ω and by $b_q : G \times G \longrightarrow \mathbb{C}^\times$ the symmetric function corresponding to q. We define the group morphism $\eta : G \longrightarrow \mathbb{C}^\times$ by $\eta(g) := b_q(g, h_0)$ for $g \in G$. Then $\mathsf{Vect}^{\mathsf{f}\,\omega}_G$ together with duality $D_{g_0} = D_{h_0^{-2}}$ and balancing

$$\theta_{\mathbb{C}_g} : \mathbb{C}_g \longrightarrow \mathbb{C}_g$$
$$v \longmapsto q(g)\eta(g)v = q(g)b_q(g, h_0)v = \frac{q(gh_0)}{q(h_0)}v$$

is a pivotal braided Grothendieck-Verdier category with compatible balancing.

6   The calculus construction

Consider a Cat-valued modular operad $\mathcal{O} : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ and a modular $\mathcal{O}$-algebra structure $A : \mathcal{O} \longrightarrow \mathrm{End}_\kappa^X$ on an object X in a symmetric monoidal bicategory $\mathcal{M}$ equipped with a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$ (see Definition 2.13). In this section, we provide a general construction of a calculus induced by the algebra structure. This calculus will formalize and simplify computations related to the algebra A. It should be seen as an auxiliary construction needed for the applications in quantum topology that we will present in the last section. In particular, it will allow us to concisely formulate and prove gluing properties for modular algebras.

As a preparation, denote by Ω the symmetric monoidal category of finite sets and surjections (the monoidal product is disjoint union). For $T \in \mathsf{Graphs}$, we denote by $\mathsf{C}(T)$ the set of components of T. In other words, $T = \sqcup_{c \in \mathsf{C}(T)} K_c$, where K_c is a corolla. By abuse of notation, we will just write c for the corolla K_c. A morphism $\Gamma : T \longrightarrow T'$ induces a surjective map $\mathsf{C}(\Gamma) : \mathsf{C}(T) \longrightarrow \mathsf{C}(T')$. This turns $\mathsf{C} : \mathsf{Graphs} \longrightarrow \Omega$ into a symmetric monoidal functor. Now we define for $X \in \mathcal{M}$ and a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$, a category $\mathcal{E}_\kappa^X$ whose objects are pairs $(T, \underline{\varphi})$, where $T \in \mathsf{Graphs}$ and $\underline{\varphi} = (\varphi_c)_{c \in \mathsf{C}(T)}$ with $\varphi_c \in \mathcal{M}(I, X^{\otimes \mathsf{Legs}(c)})$. A morphism $(S, \underline{\varphi} = (\varphi_c)_{c \in \mathsf{C}(S)}) \longrightarrow (T, \underline{\psi} = (\psi_d)_{d \in \mathsf{C}(T)})$ is a morphism $\Gamma : S \longrightarrow T$ in Graphs and for any $d \in \mathsf{C}(T)$ a morphism $\bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} \varphi_c \longrightarrow \psi_d^\Gamma$ in $\mathcal{M}(I, \bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} X^{\otimes \mathsf{Legs}(c)})$, where ψ_d^Γ is given by

$$\psi_d^\Gamma : I \xrightarrow{\ \psi_d\ } X^{\otimes \mathsf{Legs}(d)} \xrightarrow[\text{of the restriction } \Gamma_d : \sqcup_{c \in \mathsf{C}(\Gamma)^{-1}(d)} c \longrightarrow d \text{ of } \Gamma \text{ to } d]{\overset{\Delta \text{ applied for each internal edge}}{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} X^{\otimes \mathsf{Legs}(c)} \ .$$

The category $\mathcal{E}_\kappa^X$ comes with a projection $\mathcal{E}_\kappa^X \longrightarrow \mathsf{Graphs}$. Moreover, disjoint union and juxtaposition of families of morphisms provide a symmetric monoidal structure on $\mathcal{E}_\kappa^X$ such that $\mathcal{E}_\kappa^X \longrightarrow \mathsf{Graphs}$ becomes a symmetric monoidal functor.

In the next step, let us recall a familiar construction: For a functor $F : \mathcal{C} \longrightarrow \mathsf{Cat}$, we denote by $\int F$ its *Grothendieck construction*, see e.g. [MLM92, Section I.5]. By definition this is the category whose objects are pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$. A morphism $(c, x) \longrightarrow (c', x')$ is a pair (f, α), where $f : c \longrightarrow c'$ is a morphism in $\mathcal{C}$ and $\alpha : F(f)x \longrightarrow x'$ is a morphism in $F(c')$. The Grothendieck construction comes with a projection functor $\int F \longrightarrow \mathcal{C}$.

The Grothendieck construction applied to $\mathcal{O} : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ and $\mathrm{End}_\kappa^X : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ gives us a functor $\int \mathcal{O} \longrightarrow \mathsf{Graphs}$, and we may define the category $\mathcal{O} \star_\kappa X$ as the pullback

$$
\begin{array}{ccc}
\mathcal{O} \star_\kappa X & \longrightarrow & \int \mathcal{O} \\
\downarrow & & \downarrow \\
\int \mathcal{E}_\kappa^X & \longrightarrow & \mathsf{Graphs}
\end{array}
$$

in Cat (it is also a homotopy pullback because $\int \mathcal{O} \longrightarrow \mathsf{Graphs}$ is a Grothendieck fibration). The category $\mathcal{O} \star_\kappa X$ comes with a projection $\pi_\mathcal{O}^X : \mathcal{O} \star_\kappa X \longrightarrow \mathsf{Graphs}$ (by composition of either pair of composable arrows in the above square) and is naturally a symmetric monoidal category such that $\pi_\mathcal{O}^X$ is a symmetric monoidal functor.

By virtue of the monoidal product in the symmetric monoidal bicategory $\mathcal{M}$, the category $\mathcal{M}(I, I)$ is symmetric monoidal (the monoidal product can also be described through the composition as an Eckmann-Hilton argument shows; this will then also establish the symmetry). The symmetric monoidal structure on $\mathcal{M}(I, I)$ can be used to define a symmetric monoidal functor $\underline{\mathcal{M}(I, I)} : \Omega \longrightarrow \mathsf{Cat}$ from the category of finite sets and surjections to Cat sending a finite set S to $\mathcal{M}(I, I)^{\times S}$ and a surjective map $f : S \longrightarrow S'$ to the functor $\mathcal{M}(I, I)^{\times S} \longrightarrow \mathcal{M}(I, I)^{\times S'}$ whose component for $s' \in S$ is

$$\mathcal{M}(I, I)^{\times S} = \prod_{s' \in S'} \mathcal{M}(I, I)^{\times f^{-1}(s')} \xrightarrow{\ \text{projection}\ } \mathcal{M}(I, I)^{\times f^{-1}(s')} \xrightarrow{\ \otimes\ } \mathcal{M}(I, I)$$

Lemma 6.1. *Let $\mathcal{O}$ be a Cat-valued modular operad and $A : \mathcal{O} \longrightarrow \mathrm{End}_\kappa^X$ a modular algebra on an object X in a symmetric monoidal bicategory $\mathcal{M}$ with non-degenerate symmetric pairing κ. Then A induces a lax (in the sense that the coherence morphisms are not necessarily isomorphisms) symmetric monoidal transformation between symmetric monoidal functors between symmetric monoidal bicategories (where the involved 1-categories are seen as bicategories without non-trivial 2-morphisms):*

$$
\begin{array}{ccc}
 & \overset{\star}{\longrightarrow} & \\
\mathcal{O} \star_\kappa X & \Big\Downarrow \gamma^A & \mathsf{Cat} \ . \\
 & \underset{\underline{\mathcal{M}}(I,I) \circ \mathsf{C}}{\longrightarrow} &
\end{array}
$$

Here $\star : \mathcal{O} \star_\kappa X \longrightarrow \mathsf{Cat}$ is the constant functor whose single value is the one-point category.

Proof. The component of γ^A at $(T, o, \underline{\varphi})$ with $\underline{\varphi} = (\varphi_c)_{c \in \mathsf{C}(T)}$ is given at follows: We have

$$
A_o \in \prod_{c \in \mathsf{C}(T)} \mathcal{M}(X^{\otimes \mathsf{Legs}(c)}, I)
$$

(this uses the $\mathcal{O}$-algebra structure on X) and denote the c-component by $A_o^c \in \mathcal{M}(X^{\otimes \mathsf{Legs}(c)}, I)$. With this notation, we define

$$
\gamma_{(T,o,\underline{\varphi})}^A := \left(I \xrightarrow{\varphi_c} X^{\otimes \mathsf{Legs}(c)} \xrightarrow{A_o^c} I \right)_{c \in \mathsf{C}(T)} \ .
$$

The value at a morphism given by the data $\Gamma : S \longrightarrow T$, $\omega : \mathcal{O}(\Gamma)(o) \longrightarrow o'$ and $\theta_d : \bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} \varphi_c \longrightarrow \psi_d^\Gamma$ is the natural transformation

$$
\begin{array}{ccc}
* & \xrightarrow{\hspace{3cm}} & * \\
{\scriptstyle \gamma_{(S,o,\underline{\varphi})}^A} \Big\downarrow & \overset{\gamma_{(\Gamma,\omega,\theta)}}{\nearrow} & \Big\downarrow {\scriptstyle \gamma_{(T,o',\underline{\psi})}^A} \\
\mathcal{M}(I,I)^{|\mathsf{C}(S)|} & \underset{\underline{\mathcal{M}}(I,I) \circ \mathsf{C}(\Gamma)}{\longrightarrow} & \mathcal{M}(I,I)^{|\mathsf{C}(T)|}
\end{array}
$$

given by the collection of morphisms $\gamma_{(\Gamma,\omega,\theta)}^d$ for $d \in \mathsf{C}(T)$

$$
\gamma_{(\Gamma,\omega,\theta)}^d : \left(\bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} A_o^c \right) \circ \varphi_c \xrightarrow{\theta_d} \left(\bigotimes_{c \in \mathsf{C}(\Gamma)^{-1}(d)} A_o^c \right) \circ \psi_d^\Gamma \cong A_{\mathcal{O}(\Gamma_d)(\times_{c \in C(\Gamma)^{-1}(d)} o_c)} \circ \psi_d \xrightarrow{A_\omega} A_{o'} \circ \psi_d \ .
$$

A direct computation shows that this is a symmetric monoidal transformation. $\qquad \square$

Proposition 6.2 (Calculus functor)**.** *Let $\mathcal{M}$ be a symmetric monoidal bicategory. For an $\mathcal{M}$-valued modular algebra A on (X, κ) over a Cat-valued modular operad $\mathcal{O}$,*

$$
\mathsf{Calc}_A : \mathcal{O} \star_\kappa X \longrightarrow \mathcal{M}(I,I) \ , \quad (T, o, \underline{\varphi}) \longmapsto \left(\gamma_{(T,o,\underline{\varphi})}^A \right)^\otimes \ .
$$

is a symmetric monoidal functor that we refer to as the calculus (functor) for A.

Remark 6.3. The calculus construction cannot only be performed for modular algebras, but — by replacing Graphs with $\mathsf{RForests}$ or $\mathsf{Forests}$ — also for algebras over ordinary operads or cyclic operads.

Proof of Proposition 6.2. A morphism $f : (T, o, \underline{\varphi}) \longrightarrow (T', o', \underline{\varphi'})$ in $\mathcal{O} \star_\kappa X$ gives rise to a natural transformation in the left square of the following diagram:

$$
\begin{array}{ccc}
\star & \xrightarrow{\gamma_{(T,o,\underline{\varphi})}} & \mathcal{M}(I,I)^{\times \pi_0(T)} \\
\Big\| = & & \Big\downarrow {\underline{\mathcal{M}(I,I)}(\mathsf{C}(f))} \\
\star & \xrightarrow[\gamma_{(T',o',\underline{\varphi'})}]{} & \mathcal{M}(I,I)^{\times \pi_0(T')}
\end{array}
\qquad
\begin{array}{c}
\xrightarrow{\otimes} \\
\mathcal{M}(I,I) \; . \\
\xrightarrow[\otimes]{}
\end{array}
$$

The square commutes up to a canonical natural transformation. The triangle on the right commutes up to a canonical natural isomorphism. As a consequence, we obtain a morphism $\mathsf{Calc}_A(T, o, \underline{\varphi}) = \left(\gamma^A_{(T,o,\underline{\varphi})} \right)^{\otimes} \longrightarrow$ $\mathsf{Calc}_A(T', o', \underline{\varphi'}) = \left(\gamma^A_{(T',o',\underline{\varphi'})} \right)^{\otimes}$ in $\mathcal{M}(I,I)$ which we define to be $\mathsf{Calc}_A(f)$. This way, Calc_A extends to a functor $\mathsf{Calc}_A : \mathcal{O} \star_\kappa X \longrightarrow \mathcal{M}(I,I)$.

Since γ^A from (6.1) is a symmetric monoidal transformation, one concludes that Calc_A is a symmetric monoidal functor: For a disjoint union $(T, o, \underline{\varphi}) = (T^{(0)}, o_0, \underline{\varphi_0}) \sqcup (T^{(1)}, o_1, \underline{\varphi_1})$, where without loss of generality $T^{(0)}$ and $T^{(1)}$ are connected, we find $\gamma^A_{(T,o,\underline{\varphi})} \cong \gamma^A_{(T^{(0)},o_0,\underline{\varphi_0})} \times \gamma^A_{(T^{(0)},o_0,\underline{\varphi_0})}$ in $\mathcal{M}(I,I)^{\times 2}$ by a canonical isomorphism because γ^A is symmetric monoidal. As a consequence,

$$
\mathsf{Calc}_A(T, o, \underline{\varphi}) \cong \gamma^A_{(T^{(0)},o_0,\underline{\varphi_0})} \otimes \gamma^A_{(T^{(1)},o_1,\underline{\varphi_1})} = \mathsf{Calc}_A(T^{(0)}, o_0, \underline{\varphi_0}) \otimes \mathsf{Calc}_A(T^{(1)}, o_1, \underline{\varphi_1})
$$

by a canonical isomorphism. $\qquad\qquad\square$

Given an $\mathcal{M}$-valued modular $\mathcal{O}$-algebra A, the calculus construction assigns to an operation $o \in \mathcal{O}(T_n)$ and $\varphi \in \mathcal{M}(I, X^{\otimes(n+1)})$ the object in $\mathcal{M}(I,I)$ obtained by evaluation of A_o on φ.

In the case $\mathcal{M} = \mathsf{Lex}^{\mathsf{f}}$, we will use the following conventions:

- The category $\mathcal{M}(I,I) = \mathsf{Lex}^{\mathsf{f}}(\mathsf{Vect}^{\mathsf{f}}, \mathsf{Vect}^{\mathsf{f}})$ is canonically equivalent to $\mathsf{Vect}^{\mathsf{f}}$ and we will therefore identify $\mathcal{M}(I,I)$ in this case with $\mathsf{Vect}^{\mathsf{f}}$. Therefore, the calculus functor will be seen as a $\mathsf{Vect}^{\mathsf{f}}$-valued functor.

- Let $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$ be the underlying object for our algebra. For $o \in \mathcal{O}(T_n)$, the evaluation $\mathsf{Calc}_A(T_n, o, -)$ of the calculus on $(T_n, o, -)$ is a functor $\mathsf{Lex}^{\mathsf{f}}(\mathsf{Vect}^{\mathsf{f}}, \mathcal{C}^{\boxtimes(n+1)}) \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, but through the identification $\mathsf{Lex}^{\mathsf{f}}(\mathsf{Vect}^{\mathsf{f}}, \mathcal{C}^{\boxtimes(n+1)}) \simeq \mathcal{C}^{\boxtimes(n+1)}$, we agree to see it as functor

$$
\mathsf{Calc}_A(T_n, o, -) : \mathcal{C}^{\boxtimes(n+1)} \longrightarrow \mathsf{Vect}^{\mathsf{f}}
$$

which after these identifications is just A_o. In particular, $\mathsf{Calc}_A(T_n, o, -)$ can be naturally seen as a left exact functor.

One of the key properties of the calculus construction that we will need later is its 'locality' that we formulate in the following excision result. It crucially relies on the relation between the composition in the endomorphism operad and Lyubashenko's left exact coend that we established earlier in Proposition 2.28.

Theorem 6.4 (Excision)**.** *Let A be a $\mathsf{Lex}^{\mathsf{f}}$-valued modular algebra on $(\mathcal{C}, \kappa)$ over a Cat-valued modular operad $\mathcal{O}$, moreover $\Gamma : T \longrightarrow T'$ a morphism in Graphs, where T' is a connected. Then for $o \in \mathcal{O}(T)$ and $o' := \mathcal{O}(\Gamma)o$, we have a canonical natural isomorphism*

$$
\mathsf{Calc}_A(T', o', -) \cong \oint^{X_1,\dots,X_r \in \mathcal{C}} \mathsf{Calc}_A(T, o, \dots, X_j^\kappa, \dots, X_j, \dots)
$$

of functors $\mathcal{C}^{\boxtimes \mathsf{Legs}(T')} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, where $\oint^{X_1,\dots,X_r \in \mathcal{C}}$ is the left exact coend running over r variables, each one corresponding to an internal edges of Γ.

The notation on the right hand side was explained in Proposition 2.28.

Proof. By definition and the conventions above we have $\mathsf{Calc}_A(T', o', -) = A_{\mathcal{O}(\Gamma)o}$ as functors $\mathcal{C}^{\boxtimes \mathsf{Legs}(T')} \longrightarrow$ $\mathsf{Vect}^{\mathsf{f}}$. Thanks to naturality of A up to coherent isomorphism, we find

$$A_{\mathcal{O}(\Gamma)o} \cong \mathrm{End}^{\mathcal{C}}_{\kappa}(\Gamma)A_o \ ,$$

where A_o is the evaluation of A on o. By Proposition 2.28 we have a canonical isomorphism $\mathrm{End}^{\mathcal{C}}_{\kappa}(\Gamma)A_o \cong \oint^{X_1,\ldots,X_r \in \mathcal{C}} A_o^{\boxtimes}(\ldots, X_j^{\kappa}, \ldots, X_j, \ldots)$. By combining these facts we obtain the assertion. $\qquad\square$

7 Applications to quantum topology

In this section, we present two applications of our characterization of cyclic algebras. Besides the calculus construction from Section 6, a key ingredient will be the modular envelope of a cyclic operad [Cos04], a concept recalled in Section 7.1.

7.1 A reminder on the modular envelope

Any modular operad may be seen as a cyclic operad, i.e. there is a forgetful functor from modular operads to cyclic operads. Conversely, one can assign to any cyclic operad the 'smallest' modular operad containing it, the the so-called *modular envelope* [Cos04]. Formally, the modular envelope of a cyclic operad $\mathcal{O}$ is obtained via left Kan extension along the inclusion $\ell : \mathsf{Forests} \longrightarrow \mathsf{Graphs}$. This construction can be performed for arbitrary cocomplete target categories. Depending on the target category and the extent to which the axioms of a cyclic and modular operad are relaxed, one needs to work with the 'homotopically correct' version of the modular envelope. In our framework of Cat-valued cyclic operads, the modular envelope of a cyclic operad $\mathcal{O}$ evaluated at $T \in \mathsf{Graphs}$ will be given by a relaxed version of a colimit in categories (more precisely, an *oplax colimit* in the most common terminology) for the functor $\ell/T \longrightarrow \mathsf{Forests} \xrightarrow{\mathcal{O}} \mathsf{Cat}$. This type of colimit may be modeled by the Grothendieck construction recalled on page 52, see [HGN17] and [JY20, Theorem 10.2.3]. Hence,

$$(\mathsf{U}_\int \mathcal{O})(T) = \int \left(\ell/T \longrightarrow \mathsf{Forests} \xrightarrow{\mathcal{O}} \mathsf{Cat} \right) .$$

The subscript '$\int$' is supposed to remind us that, for Cat-valued cyclic operads, we consider a version of the modular envelope defined via the Grothendieck construction. With this definition, it follows from Thomason's Theorem [Tho79, Theorem 1.2] that the topological modular operad $|B\mathsf{U}_\int \mathcal{O}|$ obtained by applying arity-wise the nerve and the geometric realization of $\mathsf{U}_\int \mathcal{O}$ is characterized by the homotopy equivalence

$$\underset{(T_0, \Gamma) \in \ell/T}{\mathrm{hocolim}} |B\mathcal{O}(T_0)| \xrightarrow{\simeq} |B\mathsf{U}_\int \mathcal{O}|(T) = |B\mathsf{U}_\int \mathcal{O}(T)| \ , \tag{7.1}$$

i.e. it is given by the appropriately derived version of the modular envelope of the topological cyclic operad $|B\mathcal{O}|$, as considered in [Gia11]. If we denote the derived modular envelope of topological cyclic operads via $\mathbb{L}\mathsf{U}$, (7.1) gives us a homotopy equivalence

$$\mathbb{L}\mathsf{U} |B\mathcal{O}| \xrightarrow{\simeq} |B\mathsf{U}_\int \mathcal{O}| \tag{7.2}$$

of topological modular operads.

Proposition 7.1 (Modular extension). *Let $\mathcal{O}$ be a cyclic operad in Cat and $A : \mathcal{O} \longrightarrow \mathrm{End}^X_{\kappa}$ a cyclic $\mathcal{O}$-algebra on an object X in a symmetric monoidal bicategory $\mathcal{M}$ with non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$. Then A naturally gives rise to a modular algebra $\mathsf{U}_\int \mathcal{O} \longrightarrow \mathrm{End}^X_{\kappa}$ over the modular envelope $\mathsf{U}_\int \mathcal{O}$ which we denote by $\widehat{A}$ and refer to as the modular extension of the cyclic $\mathcal{O}$-algebra to a modular $\mathsf{U}_\int \mathcal{O}$-algebra.*

Proof. For $T \in$ Graphs and $(T^{(0)}, \Gamma) \in \ell/T$, the cyclic $\mathcal{O}$-algebra A provides us with maps

$$\mathcal{O}(T^{(0)}) \xrightarrow{A_{T^{(0)}}} (\mathrm{End}_\kappa^X)(T^{(0)}) \xrightarrow{(\mathrm{End}_\kappa^X)(\Gamma)} (\mathrm{End}_\kappa^X)(T) \tag{7.3}$$

that, by virtue of A being a symmetric monoidal transformation $\mathcal{O} \longrightarrow \mathrm{End}_\kappa^X$, form a co-cone up to natural transformation to the ℓ/T-shaped diagram sending $(T^{(0)}, \Gamma)$ to $\mathcal{O}(T^{(0)})$. In more detail, for any morphism $\Omega : (T^{(0)}, \Gamma_0) \longrightarrow (T^{(1)}, \Gamma_1)$ in ℓ/T, the diagram

$$
\begin{array}{ccc}
\mathcal{O}(T^{(0)}) & \xrightarrow{A_{T^{(0)}}} & (\mathrm{End}_\kappa^X)(T^{(0)}) \\
\downarrow{\scriptstyle \mathcal{O}(\Omega)} & & \downarrow{\scriptstyle (\mathrm{End}_\kappa^X)(\Omega)} \quad \searrow^{(\mathrm{End}_\kappa^X)(\Gamma_0)} \\
& & \qquad\qquad (\mathrm{End}_\kappa^X)(T) \\
\mathcal{O}(T^{(1)}) & \xrightarrow{A_{T^{(1)}}} & (\mathrm{End}_\kappa^X)(T^{(1)}) \quad \nearrow_{(\mathrm{End}_\kappa^X)(\Gamma_1)}
\end{array}
\tag{7.4}
$$

commutes up to canonical natural isomorphism: The natural isomorphism for the square is part of data of A, the natural isomorphism for the triangle comes from the functoriality of the endomorphism operad.

As a consequence, the maps (7.3) induce a map

$$(\mathsf{U}_{\textstyle\int} \mathcal{O})(T) = \int \left(\ell/T \longrightarrow \mathsf{Forests} \xrightarrow{\mathcal{O}} \mathsf{Cat} \right) \longrightarrow \mathrm{End}_\kappa^X(T)$$

providing us with a modular $\mathcal{O}$-algebra structure on (X, κ). $\qquad\qquad\square$

Remark 7.2. For the maps (7.3) to descend to the Grothendieck construction, it is not needed that the transformations exhibited in (7.4) are actually isomorphisms (a transformation running from top to bottom would have sufficed). Nonetheless, the fact that they actually are isomorphisms will become relevant in the following situation: Suppose that in Proposition 7.1 the operad $\mathcal{O}$ is actually groupoid-valued. Then the proof above tells us that the modular extension $\widehat{A}$ of any cyclic $\mathcal{O}$-algebra A will consist of functors $\mathsf{U}_{\textstyle\int} \mathcal{O}(T) \longrightarrow \mathrm{End}_\kappa^X(T)$ for all $T \in$ Graphs which send *all* morphisms to isomorphisms. In other words, $\widehat{A}$ descends to a modular $\Pi|B\mathsf{U}_{\textstyle\int} \mathcal{O}|$-algebra that by abuse of notation we also denote by $\widehat{A}$. Any $\Pi|B\mathsf{U}_{\textstyle\int} \mathcal{O}|$-algebra B can be restricted to a cyclic $\mathcal{O}$-algebra (this would not work for an arbitrary $\mathsf{U}_{\textstyle\int} \mathcal{O}$-algebra) that we denote by B_0. One can then confirm that by construction we find $\widehat{B_0} \simeq B$.

Example 7.3. The modular envelope of the cyclic associative operad has a description in terms of ribbon graphs as noted by Costello [Cos04]: For a corolla $T \in$ Graphs, we define the category $\mathsf{RGraphs}(T)$ as the category of connected ribbon graphs whose legs are identified with those of T. More precisely, the objects of $\mathsf{RGraphs}(T)$ are connected graphs Γ with an identification $\pi_0(\Gamma) \cong T$ and a cyclic order of the half edges incident to each vertex. Note that we may see Γ as a morphism $\Gamma : \nu(\Gamma) \longrightarrow T$ in Graphs. Then the cyclic order for Γ amounts to a cyclic order of the legs of the disjoint union $\nu(\Gamma)$ of corollas, i.e. to an element in the discrete category $\mathsf{As}(\nu(\Gamma))$. Morphisms in $\mathsf{RGraphs}(T)$ are contractions of disjoint unions of trees. If we denote the category of connected graphs Γ with identification $\pi_0(\Gamma) \cong T$ by $\mathsf{Gr}_{\mathrm{conn}}(T)$, then we can describe $\mathsf{RGraphs}(T)$ formally as the Grothendieck construction

$$\mathsf{RGraphs}(T) = \int \left(\mathsf{Gr}_{\mathrm{conn}}(T) \xrightarrow{\Gamma \longmapsto \mathsf{As}(\nu(\Gamma))} \mathsf{Cat} \right) . \tag{7.5}$$

It is straightforward to see that the assignment $T \longmapsto \mathsf{RGraphs}(T)$ extends to a modular operad $\mathsf{RGraphs} :$ Graphs $\longrightarrow$ Cat, the *modular operad of ribbon graphs*. There is a canonical functor $\ell/T \longrightarrow \mathsf{Gr}_{\mathrm{conn}}(T)$ from the slice of the inclusion $\ell :$ Forests $\longrightarrow$ Graphs over T to $\mathsf{Gr}_{\mathrm{conn}}(T)$ to sending $\Gamma : T' \longrightarrow T$ in ℓ/T to Γ. This gives us actually an equivalence

$$\ell/T \xrightarrow{\simeq} \mathsf{Gr}_{\mathrm{conn}}(T) \tag{7.6}$$

of categories. This follows in a straightforward way from the definition of $\mathsf{Gr}_{\mathrm{conn}}(T)$ and is also noted in [Gia11, Section 2.1]. If we restrict the Grothendieck construction (7.5) along the equivalence (7.6), we obtain by definition $\mathsf{U}_\int \mathsf{As}(T)$. This is natural in T, and hence the equivalences (7.6) induce an equivalence

$$\mathsf{U}_\int \mathsf{As} \xrightarrow{\simeq} \mathsf{RGraphs}$$

from the modular envelope of the associative operad As to the modular operad of ribbon graphs $\mathsf{RGraphs}$. In combination with the calculus construction (Section 6), the modular extension (Proposition 7.1) and Theorem 4.12, this yields for any pivotal Grothendieck-Verdier category $\mathcal{C}$ in Lex^f with associated non-degenerate symmetric pairing $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^\mathsf{f}$ a symmetric monoidal functor

$$\mathsf{Calc}_{\widehat{\mathcal{C}}} : \mathsf{RGraphs} \star_\kappa \widehat{\mathcal{C}} \longrightarrow \mathsf{Vect}^\mathsf{f} \ ,$$

i.e. $\mathcal{C}$ gives rise to a symmetric monoidal functor from the category of ribbon graphs with $\mathcal{C}$-labeled legs to vector spaces. To any object $(T, \Gamma, X) \in \mathsf{RGraphs} \star_\kappa \widehat{\mathcal{C}}$, i.e.

- an object $T \in \mathsf{Graphs}$ (without loss of generality, we assume that T is connected, i.e. a corolla),

- a ribbon graph $\Gamma \in \mathsf{RGraphs}(T)$, which we can see as a morphism $\Gamma : T' \longrightarrow T$ in Graphs,

- and an object $X \in \mathcal{C}^{\boxtimes \mathsf{Legs}(T)}$ (to be thought of as a $\mathcal{C}$-label for each of the legs of $\pi_0(\Gamma) \cong T$),

the calculus construction assigns a vector space $\mathsf{Calc}_{\widehat{\mathcal{C}}}(T, \Gamma, X)$. In order to discuss an example, let us assume that Γ is a bouquet of r circles with $n+1$ legs attached to the base vertex. For any order of $\mathsf{Legs}(T)$, i.e. an identification of T with T_n, and the associated order $X_0 \boxtimes \cdots \boxtimes X_n \in \mathcal{C}^{\boxtimes(n+1)}$ of X (which without loss of generality is treated as a pure tensor here), there is a canonical isomorphism

$$\mathsf{Calc}_{\widehat{\mathcal{C}}}(T, \Gamma, X) \cong \mathcal{C}\left(K, X_0 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes r}\right) \quad \text{with the coend} \quad \mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\kappa \ . \tag{7.7}$$

When it comes to how exactly the coend $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\kappa$ has to be understood, a little care is required: It is defined as the image of the coend $\Delta = \int^{X \in \mathcal{C}} X \boxtimes X^\kappa$ under the monoidal product. The isomorphism (7.7) follows from the definition of the modular extension in Proposition 7.1 (in particular (7.3)) and excision (Theorem 6.4). Note that this isomorphism is canonical only after the choice of the order. The left hand side $\mathsf{Calc}_{\widehat{\mathcal{C}}}(T, \Gamma, X)$ is defined independently of this choice (this is a strength of the calculus construction).

7.2 Application I: Handlebody group representation from ribbon Grothendieck-Verdier structures

As a first application, we will prove that a ribbon Grothendieck-Verdier structure gives rise to explicitly computable handlebody group representations. The idea is to combine our characterization of cyclic framed little disks algebras in Theorem 5.12 and Theorem 5.13 with the relation between the derived modular envelope of $\mathsf{f}E_2$ and the modular operad of handlebodies found by Giansiracusa [Gia11] (this statement refers to topological operads).

The *(groupoid-valued) modular handlebody operad* is the symmetric monoidal functor $\mathsf{Hbdy} : \mathsf{Graphs} \longrightarrow \mathsf{Grpd}$ which assigns to a corolla T the groupoid $\mathsf{Hbdy}(T)$ defined as follows: Objects are compact connected oriented handlebodies H (hereafter just referred to as handlebodies for brevity) with $|\mathsf{Legs}(T)|$ many disks embedded in ∂H with a parametrization of the disks, i.e. an orientation-preserving embedding $\psi : \sqcup_{\mathsf{Legs}(T)} \mathbb{D}^2 \longrightarrow H$ which is an orientation-preserving diffeomorphism to the disks embedded in ∂H. Morphisms in $\mathsf{Hbdy}(T)$ are isotopy classes of orientation-preserving diffeomorphisms that respect the disk parametrizations (hence, the automorphism groups are precisely the handlebody groups). Operadic composition is by gluing of handlebodies along their boundaries, see [Gia11, Section 4.3] for the details. Note however that our handlebody operad allows all handlebodies while Giansiracusa only considers those handlebodies $H_{g,n}$ with genus g and n embedded disks for which $(g, n) \neq (0, 0), (0, 1)$. In order to distinguish both versions, we denote Giansiracusa's (groupoid-valued)

handlebody operad by $\mathsf{Hbdy}^{\mathrm{a}}$, where the 'a' indicates that we restrict to the handlebodies allowed in [Gia11]. It comes with an inclusion $\mathsf{Hbdy}^{\mathrm{a}} \subset \mathsf{Hbdy}$ of modular operads. By taking entry-wise the classifying spaces, we may see Hbdy and $\mathsf{Hbdy}^{\mathrm{a}}$ as topological operads that we denote by $\mathbf{Hbdy}$ and $\mathbf{Hbdy}^{\mathrm{a}}$. We may identify $\Pi\mathbf{Hbdy}$ and $\Pi\mathbf{Hbdy}^{\mathrm{a}}$ with Hbdy and $\mathsf{Hbdy}^{\mathrm{a}}$, respectively; in formulae

$$\Pi\mathbf{Hbdy} \simeq \mathsf{Hbdy} , \quad \Pi\mathbf{Hbdy}^{\mathrm{a}} \simeq \mathsf{Hbdy}^{\mathrm{a}} .$$

With a subscript '0', we will indicate the restriction to handlebodies of genus zero. It is straightforward to observe that the restriction Hbdy_0 of Hbdy to genus zero is a cyclic operad. Similarly, $\mathbf{Hbdy}_0$, $\mathsf{Hbdy}_0^{\mathrm{a}}$ and $\mathbf{Hbdy}_0^{\mathrm{a}}$ are cyclic operads. We will use the following crucial result on the modular envelope of $\mathsf{Hbdy}_0^{\mathrm{a}}$. It follows from the results of [Gia11] and (7.2):

Theorem 7.4 (Giansiracusa [Gia11, Theorem A])**.** *There is a canonical map of* Cat*-valued operads*

$$\mathsf{U}_{\int} \mathsf{Hbdy}_0^{\mathrm{a}} \longrightarrow \mathsf{Hbdy}^{\mathrm{a}}$$

which is arity-wise an isomorphism on π_0 *and which when evaluated on* $T \in \mathsf{Graphs}$ *induces a homotopy equivalence after taking nerve and geometric realization if* T *has at least one leg. If* T *has no legs, this remains true except for the path component corresponding to the solid closed torus.*

We refer to Remark 7.11 for a comment on the exception occurring on the component for the solid closed torus.

In [Gia11] the Theorem is proven using a version of cyclic operads without arity zero operations. When applying this result in our context we hence have to ignore the arity zero operations in $\mathsf{f}E_2$. This is also the reason why the solid three-dimensional ball has to be excluded.

Remark 7.5. By Theorem 5.3 we obtain a decomposition

$$\mathsf{U}_{\int} \mathsf{Hbdy}_0^{\mathrm{a}}(T_{n-1}) = \mathsf{U}_{\int} \Pi\mathbf{Hbdy}_0^{\mathrm{a}}(T_{n-1}) = \bigsqcup_{g \in \mathbb{N}_0} M(g,n) \quad \text{for} \quad n \geq 0 ,$$

where $M(g,n)$ is a connected category. The reason why T_{n-1} appears on the left hand side while we use the index n on the right hand side is that T_{n-1} by definition has n legs (with the convention that T_{-1} is the corolla without legs). If $(g,n) \neq (1,0)$, $|BM(g,n)|$ is equivalent to the classifying space of the mapping class group $\mathsf{Map}(H_{g,n})$ of the handlebody $H_{g,n}$ with genus g and n embedded disks. In the sequel, we will denote a base point of $M(g,n)$ by $o_{g,n}$.

If we replace in the definition of Hbdy the handlebodies with surfaces and, consequently, isotopy classes of orientation-preserving diffeomorphisms of handlebodies by isotopy classes of orientation-preserving diffeomorphisms of surfaces, we obtain the *(groupoid-valued) modular operad of surfaces* $\mathsf{Surf} : \mathsf{Graphs} \longrightarrow \mathsf{Grpd}$ and, by taking classifying spaces, its topological counterpart $\mathbf{Surf}$. The automorphism groups of Surf are precisely the mapping class groups of oriented surfaces (with boundary). Since the handlebody group $\mathsf{Map}(H)$ of a handlebody H may be identified with the subgroup of the mapping class group $\mathsf{Map}(\partial H)$ of those isotopy classes of orientation-preserving diffeomorphisms of ∂H that extend to all of H, there are maps $\mathsf{Hbdy} \subset \mathsf{Surf}$ and $\mathbf{Hbdy} \subset \mathbf{Surf}$ of modular operads that induce equivalences of cyclic operads $\mathsf{Hbdy}_0 \simeq \mathsf{Surf}_0$ and $\mathbf{Hbdy}_0 \simeq \mathbf{Surf}_0$ after restriction to genus zero. This follows from the well-known statement that the handlebody subgroups agree with the entire mapping class group for genus zero surfaces, see e.g. [HH12, Proposition 2.1].

The topological cyclic operad S_0 from Section 5.2 can be identified in a straightforward way with $\mathbf{Surf}_0$. We may therefore conclude from Proposition 5.3:

Lemma 7.6. *There are equivalences of cyclic* Grpd*-valued operads* $\mathsf{Hbdy}_0 \simeq \mathsf{Surf}_0 \simeq \Pi\mathsf{f}E_2 \simeq \mathsf{RBr}$.

Together with Theorem 5.12 this observation implies:

Corollary 7.7. *A* $\mathsf{Lex}^{\mathsf{f}}$*-valued cyclic algebra over* Hbdy_0 *or* Surf_0 *can be equivalently described as a ribbon Grothendieck-Verdier category in* $\mathsf{Lex}^{\mathsf{f}}$*.*

If we are given a ribbon Grothendieck-Verdier category $\mathcal{C}$ in $\mathsf{Lex}^{\mathsf{f}}$, we may — by this result — see it as a cyclic Hbdy_0-algebra and restrict it to a cyclic $\mathsf{Hbdy}_0^{\mathrm{a}}$-algebra that we denote by $\mathcal{C}^{\mathrm{a}}$. Its modular extension $\widehat{\mathcal{C}^{\mathrm{a}}}$ in the sense of Proposition 7.1 is a modular algebra over the modular envelope $\mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}}$. Recall from Proposition 6.2 that this modular algebra comes with a calculus functor, i.e. a symmetric monoidal functor $\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}} : \mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}} \star_\kappa \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$, where $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ is the pairing that $\mathcal{C}$ comes equipped with. For this calculus functor, we can prove the following statement that we rephrase afterwards in more concrete terms:

Theorem 7.8. *Let* $\mathcal{C}$ *be a ribbon Grothendieck-Verdier category in* $\mathsf{Lex}^{\mathsf{f}}$ *and* $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ *the associated non-degenerate symmetric pairing. Then the calculus functor of the modular* $\mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}}$*-algebra* $\widehat{\mathcal{C}^{\mathrm{a}}}$ *is a symmetric monoidal functor*

$$\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}} : \mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}} \star_\kappa \mathcal{C} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$$

that we may explicitly describe as follows: After the choice of an order for the n *legs of* T_{n-1}*, there is an isomorphism*

$$\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}}(T_{n-1}, o_{g,n}, X_1, \ldots, X_n) \cong \mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g}) \quad \text{for all} \quad X_1, \ldots, X_n \in \mathcal{C} \ , \qquad (7.8)$$

where

- *the* $o_{g,n}$ *for non-negative integers* g *and* n *are the chosen base points in the components of* $\mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}}(T_{n-1})$*, see Remark 7.5,*

- *and* $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\kappa \in \mathcal{C}$ *is defined by applying the monoidal product to the coevaluation object* $\Delta = \int^{X \in \mathcal{C}} X \otimes X^\kappa$*, i.e.* $\mathbb{F} = \otimes(\Delta)$*.*

Proof. As explained before the statement of the result, we obtain the symmetric monoidal functor as a direct consequence of our characterization of cyclic RBr-algebras and the calculus construction. It remains to prove (7.8): For $g, n \geq 0$, consider the graph $\Gamma_{g,n}$ with one vertex, n legs and g internal edges. Then $\Gamma_{g,n} : T_{n-1+2g} \longrightarrow T_{n-1}$ is a morphism in Graphs with $\mathsf{U}_\int \mathsf{Hbdy}_0^{\mathrm{a}}(\Gamma_{g,n}) o_{0,n+2g} \cong o_{g,n}$ by Theorem 7.4 (on level of π_0) and Remark 7.5. Since by definition

$$\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}}(T_{n-1+2g}, o_{0,n+2g}, X_1, \ldots, X_n, Y_1, \ldots, Y_{2g}) \cong \mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_{2g}) \ ,$$

we can conclude from the Excision Theorem 6.4

$$\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}}(T_{n-1}, o_{g,n}, X_1, \ldots, X_n) \cong \oint^{Y_1, \ldots, Y_g \in \mathcal{C}} \mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes Y_1^\kappa \otimes \cdots \otimes Y_g \otimes Y_g^\kappa) \ .$$

Now (7.8) follows from Lemma 2.27 which allows us to express the left exact coend by means of $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\kappa \in \mathcal{C}$. $\qquad \square$

A less concise, but more explicit version of this result reads as follows:

Theorem 7.9. *Let* $\mathcal{C}$ *be a ribbon Grothendieck-Verdier category in* $\mathsf{Lex}^{\mathsf{f}}$*. Then for* $(g, n) \in \mathbb{N}_0^2$ *and* $X_1, \ldots, X_n \in \mathcal{C}$*, the morphism space* $\mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ *comes with a canonical action of the handlebody group* $\mathsf{Map}(H_{g,n})$ *for the handlebody* $H_{g,n}$ *of genus* g *and* n *boundary disks if* $(g, n) \neq (1, 0)$*.*

Proof. Since $\mathsf{Map}(H_{g,n}) = 1$ for $(g, n) = (0, 0)$ and $(g, n) = (0, 1)$, these two cases are trivial.

Let now $(g, n) \neq (1, 0), (0, 1), (0, 0)$. Theorem 7.8, when combined with the definition of the calculus construction, provides for us functors $M(g, n) \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ sending $o_{g,n}$ to the vector space $\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}}(T_{n-1}, o_{g,n}, X_1, \ldots, X_n)$. The category $M(g, n)$ was defined in Remark 7.5 and has the property that $|BM(G, n)|$ is a classifying space

of $\mathsf{Map}(H_{g,n})$. From the fact that $\mathsf{Hbdy}^{\mathrm{a}}$ is groupoid-valued and Remark 7.2, it follows that this functor $M(g,n) \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ sends *all* morphisms to isomorphisms. As a consequence, the functor $M(g,n) \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ descends to the localization at all morphisms. Hence, $\mathsf{Calc}_{\widehat{\mathcal{C}^{\mathrm{a}}}}(T_{n-1}, o_{g,n}, X_1, \ldots, X_n)$ inherits an action of $\mathsf{Map}(H_{g,n})$. Now the assertion follows from (7.8). $\qquad\square$

Remark 7.10. As a small caveat, we should mention that the statement that $\mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ carries an action of the handlebody group contains a small abuse of language and must be interpreted correctly: In the first place, the value of the calculus functor carries the handlebody group representation. It is then transferred to the morphism space $\mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ by means of (7.8) after *ordering* the boundary components (this is the same choice discussed in Example 7.3). When working with the calculus functor, we do not have to make this choice (there the labels are really attached to the boundaries and never numbered). We still use in Theorem 7.9 the formulation through the morphism spaces and a numbering even if it contains this abuse of notation. This is in order to make contact to the literature where this abuse of notation seems to be standard (after all, it is not very problematic if one is aware of the subtlety).

Remark 7.11. The exception made in Theorem 7.9 for the solid closed torus is a consequence of the corresponding exception appearing in Giansiracusa's result. This issue arises from the non-contractibility of the disk complex for the torus as explained in [Gia11, Section 6.2]. Nonetheless, the calculus functor for a ribbon Grothendieck-Verdier category $\mathcal{C}$ actually can be evaluated on the solid closed torus, where it yields the vector space $\mathcal{C}(K, \mathbb{F})$, see Theorem 7.8. This vector space comes with additional structure from the component of the modular envelope attached to the solid closed torus. The investigation of this structure is beyond the scope of this article, see however the statements that can be made in the modular case below in Proposition 7.14.

Example 7.12. Let G be an Abelian group and $\mathsf{Vect}_G^{\mathsf{f}\ \omega, g_0}$ the ribbon Grothendieck-Verdier category associated to an Abelian 3-cocycle ω on G with coefficients in $\mathbb{C}^\times$ and duality $D_{g_0} = \mathbb{C}_{g_0} \otimes (-)^*$ with $g_0 = h_0^{-2}$ for some $h_0 \in G$, see Example 5.14. For this *semisimple* category, we can conclude by arguments similar to those for [KL01, Corollary 5.1.9] that the coend $\mathbb{F}$ (as defined in Theorem 7.8) is given by

$$\mathbb{F} \cong \bigoplus_{g \in G} \mathbb{C}_g \otimes \mathbb{C}_{g_0 g^{-1}} \cong \bigoplus_{g \in G} \mathbb{C}_{g_0} = \mathbb{C}[G] \otimes \mathbb{C}_{g_0} \ ,$$

where $\mathbb{C}[G] \otimes \mathbb{C}_{g_0}$ is the tensoring of $\mathbb{C}_{g_0}$ with the free $\mathbb{C}$-vector space on the set G. This implies $\mathbb{F}^{\otimes \ell} \cong \mathbb{C}[G]^{\otimes \ell} \otimes \mathbb{C}_{g_0^\ell}$ for $\ell \geq 0$. As a consequence, the vector space associated by Theorem 7.8 to a (for simplicity closed) surface of genus $\ell \geq 1$ is

$$\mathsf{Vect}_G^{\mathsf{f}\ \omega, g_0}(\mathbb{C}_{g_0}, (\mathbb{C}[G] \otimes \mathbb{C}_{g_0})^{\otimes \ell}) \cong \mathbb{C}[G]^{\otimes \ell} \delta_{g_0, g_0^\ell} \ .$$

For $\ell \geq 2$, the handlebody group representation on this vector space can be computed explicitly using excision: For example, it follows from the definition of the balancing in Example 5.14 that the Dehn twist around the m-th handle, $1 \leq m \leq \ell$ acts as the linear automorphism $\mathbb{C}[G]^{\otimes \ell} \delta_{g_0, g_0^\ell}$ which acts as the identity map on all tensor factors $\mathbb{C}[G]$ except for the m-th one where it is given by the map

$$\mathbb{C}[G] \longrightarrow \mathbb{C}[G] \ , \quad g \longmapsto \frac{q(g h_0)}{q(h_0)} \cdot g \ ;$$

here $q : G \longrightarrow \mathbb{C}^\times$ is the quadratic form associated to ω.

In Example 7.12, a category whose Grothendieck-Verdier structure does not come from actual rigidity was covered. But of course, Theorem 7.9 applies in particular in the rigid case. In order to exploit this, let us recall some more terminology: A *finite tensor category* [EO04] is a finite category (as defined in Example 2.5) with a rigid monoidal structure and simple unit. A *finite ribbon category* is a finite braided tensor category equipped with a balancing compatible with the duality. By Theorem 5.12 this is precisely a $\mathsf{Lex}^{\mathsf{f}}$-valued cyclic RBr-algebra with simple unit whose Grothendieck-Verdier structure comes from rigidity.

One way to obtain a finite ribbon category is by taking categories of finite-dimensional modules over a finite-dimensional ribbon Hopf algebra A, see e.g. [Kas15, XIV.6]. In this case, the coend $\mathbb{F}$ is isomorphic to A^*_{coadj} [KL01, Theorem 7.4.13], i.e. the dual A^* of A with the coadjoint action of A given by

$$A \otimes A^* \longrightarrow A^*, \quad a \otimes \alpha \longrightarrow (b \longmapsto \alpha(S(a_{(1)})ba_{(2)})) \ ,$$

where S is the antipode of A and $\Delta a = a_{(1)} \otimes a_{(2)}$ the Sweedler notation for the coproduct. Now Theorem 7.9 specializes to:

Corollary 7.13. *Let A be a finite-dimensional ribbon Hopf algebra. Then for any non-negative integers g and n with $(g, n) \neq (1, 0)$ and any finite-dimensional A-modules $X_1, \ldots, X_n$, the vector space*

$$\operatorname{Hom}_A \left(k, X_1 \otimes \cdots \otimes X_n \otimes \left(A^*_{coadj} \right)^{\otimes g} \right)$$

of A-invariants of the module $X_1 \otimes \cdots \otimes X_n \otimes \left(A^*_{coadj} \right)^{\otimes g}$ comes canonically with an action of the mapping class group of the handlebody with genus g and n boundary components.

The handlebody group representations from Theorem 7.9 (and in particular Corollary 7.13) when given in this generality (note in particular that no non-degeneracy of the braiding is assumed) are new to the best of our knowledge. However, under much stronger assumptions on the category $\mathcal{C}$, namely *modularity* (to be defined momentarily), they relate to the Lyubashenko construction [Lyu95a, Lyu95b, Lyu96] as we will explain now: For a finite braided tensor category $\mathcal{C}$, we may define the *Müger center* [Mü03], i.e. the full subcategory of $\mathcal{C}$ spanned by all objects $X \in \mathcal{C}$ such that $c_{Y,X} c_{X,Y} = \operatorname{id}_{X \otimes Y}$ for all $Y \in \mathcal{C}$ (these objects are called *transparent*). A finite braided tensor category whose Müger center is trivial in the sense that it is generated under finite direct sums by the monoidal unit is called *non-degenerate*. Recently, there has been significant progress in the understanding of non-degeneracy through the equivalent characterizations given in [Shi19] and the factorization homology approach in [BJSS21]. A *modular category* is a finite ribbon category whose underlying finite braided category is non-degenerate (it is important to remark that this definition does *not* include semisimplicity). Modular categories are absolutely central objects living at the intersection of conformal field theory, topological field theory and representation theory. A thorough discussion of modular categories is beyond the scope of this article; a biased and by no means exhaustive list of references is [RT90, BK01, Hua08, KL01, Tur94, BDSPV15, FS17].

By the Lyubashenko construction [Lyu95a, Lyu95b, Lyu96] a modular category $\mathcal{C}$ gives rise to a consistent system of projective mapping class group representations on the morphism spaces $\mathcal{C}(I, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$, see also [FS17] for a perspective on these representations through the Lego-Teichmüller game and [SW21a] for a homotopy coherent perspective. The vector spaces $\mathcal{C}(I, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ together with their mapping class group actions are often referred to as *conformal blocks*. These mapping class group representations can be restricted to the handlebody part (then they will be non-projective, but actually linear). On the other hand, any modular category is in particular a ribbon Grothendieck-Verdier category. Hence, we also have handlebody group representations by Theorem 7.9. The following result is a comparison:

Proposition 7.14. *Let $\mathcal{C}$ be a modular category. Then the following two actions of $\mathsf{Map}(H_{g,n})$ on $\mathcal{C}(I, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ are equivalent:*

- *The action from Theorem 7.9 (it extends in this case also to the solid closed torus).*

- *The restriction of the projective Lyubashenko mapping class group action to the handlebody part.*

We could give the proof already here, but it can be formulated much more concisely using the terminology of the next section. Therefore, we defer the proof to page 63.

Proposition 7.14 has the following significance: The Lyubashenko construction is to a large extent an algebraic construction which starts from the vector spaces $\mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ and establishes the corresponding mapping class group actions by a presentation of mapping class groups in terms of generators and relations. Proposition 7.14 now tells us that the vector spaces $\mathcal{C}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ and at least the handlebody part of the actions do not only have an intrinsically topological description, but in fact also a universal property coming from the modular envelope of the cyclic operad of genus zero surfaces.

7.3 Application II: Grothendieck-Verdier duality for the evaluation of a modular functor on the circle

In the preceding subsection, it was already mentioned that the Lyubashenko construction does not only yield handlebody group representations, but actually projective mapping class group representations — they form a structure that is commonly referred to as a *modular functor*. While modular functors are certainly not the main object of study for the present article, we may still use our results to prove a duality statement for modular functors.

A modular functor [Tur94, Til98, BK01] is, roughly speaking, a consistent system of projective mapping class group representations. Many variants of this notion exist; we will momentarily present the version used in this article and briefly comment on the relation to other definitions.

As already mentioned, modular functors feature certain *projective* mapping class representations, and this projectivity is described by considering certain central extensions of mapping class groups. The relevant central extensions are obtained by means of 2-cocycles on the mapping class groups arising from the framing anomaly [Ati90, GM13]. In the language of modular operads, this can be formulated as follows: For a corolla T, we define the groupoid $\mathsf{Surf}^{\mathsf{c}}(T)$. Objects are compact oriented surfaces with boundary, a parametrization $\sqcup_{\mathsf{Legs}(T)}\mathbb{S}^1 \longrightarrow \Sigma$ of the boundary $\partial\Sigma$ and a maximal isotropic subspace λ of the presymplectic vector space $H_1(\Sigma;\mathbb{Q})$ with respect to the intersection pairing $H_1(\Sigma;\mathbb{Q}) \otimes H_1(\Sigma;\mathbb{Q}) \longrightarrow \mathbb{Q}$ (if Σ is closed, $H_1(\Sigma;\mathbb{Q})$ is symplectic, and a maximal isotropic subspace is precisely a Lagrangian subspace). A morphism $(\Sigma,\lambda) \longrightarrow (\Sigma',\lambda')$ is a pair (ϕ,n) of an isotopy class of orientation-preserving diffeomorphisms $\phi : \Sigma \longrightarrow \Sigma'$ compatible with the boundary parametrizations, and a weight $n \in \mathbb{Z}$. The composition of composable morphisms $(\Sigma,\lambda) \xrightarrow{(\phi_0,n_0)} (\Sigma',\lambda') \xrightarrow{(\phi_1,n_1)} (\Sigma'',\lambda'')$ is the pair $(\phi_1\phi_0, n_0 + n_1 + \mu(\lambda'',\phi_{1*}\lambda',\phi_1\phi_{0*}\lambda))$, where $\mu(-,-,-)$ is the Maslov index of three maximal isotropic subspaces of $H_1(\Sigma'';\mathbb{Q})$, see [Tur94, IV.3.5] for a detailed definition. To a morphism $\Gamma : T \longrightarrow T'$ in Graphs, we may associate a functor $\mathsf{Surf}^{\mathsf{c}}(\Gamma) : \mathsf{Surf}^{\mathsf{c}}(T) \longrightarrow \mathsf{Surf}^{\mathsf{c}}(T')$ as follows: For $(\Sigma,\lambda) \in \mathsf{Surf}^{\mathsf{c}}(T)$, the graph Γ prescribes a gluing of Σ along those boundary components attached to legs arising from internal edges of Γ. We denote the glued surface by Σ^{Γ} and the quotient map by $q^{\Gamma} : \Sigma \longrightarrow \Sigma_{\Gamma}$. Having established this notation, we set $\mathsf{Surf}^{\mathsf{c}}(\Gamma)(\Sigma,\lambda) := (\Sigma^{\Gamma}, q^{\Gamma}_*\lambda)$. This way, we obtain the *(groupoid-valued) central extension of the surface operad* $\mathsf{Surf}^{\mathsf{c}} : \mathsf{Graphs} \longrightarrow \mathsf{Grpd}$, a modular operad that comes with a epimorphism $\mathsf{Surf}^{\mathsf{c}} \longrightarrow \mathsf{Surf}$. We denote by $\mathbf{Surf}^{\mathsf{c}}$ the associated topological modular operad. The well-known fact that the cocycle describing the framing anomaly vanishes on the handlebody subgroups of the mapping class groups can be phrased as follows in terms of modular operads:

Lemma 7.15. *The map* $\mathsf{Hbdy} \longrightarrow \mathsf{Surf}$ *of modular operads canonically lifts to a map* $\mathsf{Hbdy} \longrightarrow \mathsf{Surf}^{\mathsf{c}}$ *of modular operads.*

Proof. The desired map $\mathsf{Hbdy} \longrightarrow \mathsf{Surf}^{\mathsf{c}}$ sends a handlebody $H \in \mathsf{Hbdy}(T)$ to its boundary surface $\Sigma_H \in \mathsf{Surf}(T)$ plus the maximal isotropic subspace $\lambda(H) := \ker(H_1(\Sigma_H;\mathbb{Q}) \longrightarrow H_1(H;\mathbb{Q}))$. A morphism $\phi : H \longrightarrow H'$ in $\mathsf{Hbdy}(T)$ is sent to the induced map $\Sigma_\phi : \Sigma_H \longrightarrow \Sigma_{H'}$ and weight zero. The map Σ_ϕ then sends $\lambda(H)$ to $\lambda(H')$. For composable morphisms $H \xrightarrow{\phi_0} H' \xrightarrow{\phi_1} H''$ in $\mathsf{Hbdy}(T)$, we observe

$$(\Sigma_{\phi_1},0) \circ (\Sigma_{\phi_0},0) = (\Sigma_{\phi_1\phi_0}, \mu(\lambda(H''), \Sigma_{\phi_1*}\lambda(H'), \Sigma_{\phi_1\phi_0*}\lambda(H))) = (\Sigma_{\phi_1\phi_0}, \mu(\lambda(H''),\lambda(H''),\lambda(H'')))$$

But the Maslov index $\mu(\lambda(H''),\lambda(H''),\lambda(H''))$ is zero by equation (3.5.a) in [Tur94, IV.3.5]. This proves that the assignments actually yield a functor $\mathsf{Hbdy}(T) \longrightarrow \mathsf{Surf}^{\mathsf{c}}(T)$. It can be easily seen to also be a map of modular operads. $\square$

The modular operad $\mathsf{Surf}^{\mathsf{c}}$ allows us to give a concise definition of the notion of a modular functor:

Definition 7.16. A *($\mathsf{Lex}^{\mathsf{f}}$-valued) modular functor* is a modular $\mathsf{Surf}^{\mathsf{c}}$-algebra in $\mathsf{Lex}^{\mathsf{f}}$.

A modular functor, according to the above definition has an underlying category $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$, which we think of as assigned to the circle. The structure of a $\mathsf{Surf}^{\mathsf{c}}$-algebra on $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$ assigns to surfaces with p incoming and

q outgoing boundary components a map $\mathcal{C}^{\boxtimes p} \boxtimes \mathcal{C}^{\boxtimes q} \longrightarrow \mathsf{Vect}^{\mathsf{f}}$ in $\mathsf{Lex}^{\mathsf{f}}$ (which can be seen as a map $\mathcal{C}^{\boxtimes p} \longrightarrow \mathcal{C}^{\boxtimes q}$ by duality). Mapping classes of the surface translate to natural isomorphisms of this functor. The gluing of surfaces translates to left exact coends. This explains why our definition is in line with the ones given in [FS17, SW21a]. It is more general than the notion from [Tur94, BK01] that additionally builds in semisimplicity, simplicity of the unit and a normalization axiom (these additional assumptions are not really topological in the sense that they are not part of the modular operad $\mathsf{Surf}^{\mathsf{c}}$; we comment very briefly on how to include them in Corollary 7.18 below). On the topological side, i.e. as far as the definition of $\mathsf{Surf}^{\mathsf{c}}$ is concerned, Definition 7.16 is also essentially in line with [Til98]. Here, however, the key difference is that in [Til98] a different target category of linear categories is considered. This choice ultimately leads again to semisimplicity.

By definition a modular functor has an underlying category $\mathcal{C} \in \mathsf{Lex}^{\mathsf{f}}$. The structure of a modular functor endows this value on the circle with more structure about which we can make, using our previous results, the following statement:

Theorem 7.17. *The category obtained by evaluation of a $\mathsf{Lex}^{\mathsf{f}}$-valued modular functor on the circle naturally comes with a ribbon Grothendieck-Verdier structure.*

Proof. Any modular functor is by definition a $\mathsf{Surf}^{\mathsf{c}}$-algebra and hence can be restricted along the map $\mathsf{Hbdy}_0 \longrightarrow \mathsf{Surf}^{\mathsf{c}}_0$ obtained from Lemma 7.15 after restriction to genus zero. This proves that its evaluation on the circle comes with the structure of a cyclic Hbdy_0-algebra. By Lemma 7.6 this is a cyclic RBr-structure which amounts to a ribbon Grothendieck-Verdier structure by Theorem 5.12. $\qquad\square$

When imposing stronger assumptions on the value of a modular functor on the circle (either directly or indirectly by choice of a different target category), we recover the following result which — in a slightly different language — is part of [Til98, Section 3] and [BK01, Theorem 5.7.10]:

Corollary 7.18. *Consider a $\mathsf{Lex}^{\mathsf{f}}$-valued modular functor whose evaluation $\mathcal{C}$ on the circle is semisimple and has a simple monoidal unit. Under these assumptions, the ribbon Grothendieck-Verdier structure on $\mathcal{C}$ from Theorem 7.17 is a ribbon r-structure if and only if the modular functor is normalized in the sense that its value on the sphere is a one-dimensional vector space.*

Proof. By Theorem 7.8 we know that the vector space that the modular functor assigns to the sphere is the morphism space $\mathcal{C}(K, I)$. Hence, it remains to prove

$$\dim \mathcal{C}(K, I) = 1 \iff K \cong I . \tag{7.9}$$

To this end, recall that K is the image of I under the duality functor D, which is an anti-equivalence. Since I is simple, so is K. By semisimplicity of $\mathcal{C}$, we have

$$\mathcal{C}(K, I) \cong \left\{ \begin{array}{ll} k \, \mathrm{id}_I , & \text{if } K \cong I , \\ 0 , & \text{else .} \end{array} \right.$$

This implies (7.9). $\qquad\square$

We end the subsection by giving the proof of Proposition 7.14 that we still owe:

Proof of Proposition 7.14. By Lyubashenko's construction the modular category $\mathcal{C}$ gives rise to a modular functor, i.e. a modular $\mathsf{Lex}^{\mathsf{f}}$-valued algebra over $\mathsf{Surf}^{\mathsf{c}}$ that we denote by $\mathcal{C}^{\mathrm{Lyu}}$; this is essentially a reformulation of [Lyu95a, Lyu95b, Lyu96] using a different language.

Consider now the map $h : \mathsf{U}_{\int} \mathsf{Hbdy}^{\mathsf{a}}_0 \longrightarrow \mathsf{Hbdy}^{\mathsf{a}} \subset \mathsf{Hbdy} \longrightarrow \mathsf{Surf}^{\mathsf{c}}$ of modular operads. We will now compare the pullback $h^* \mathcal{C}^{\mathrm{Lyu}}$ with the modular $\mathsf{U}_{\int} \mathsf{Hbdy}^{\mathsf{a}}_0$-algebra $\widehat{\mathcal{C}^{\mathsf{a}}}$ featuring in Theorem 7.8. Once we prove

$$\widehat{\mathcal{C}^{\mathsf{a}}} \simeq h^* \mathcal{C}^{\mathrm{Lyu}} \tag{7.10}$$

we obtain by means of the Theorems 7.8 and 7.9 immediately the desired statement if $(g, n) \neq (0, 0), (0, 1), (1, 0)$. In the case $(g, n) = (0, 0), (0, 1)$, the statement can be easily verified directly. In the case, $(g, n) = (1, 0)$, it

follows from the fact that by $h^*\mathcal{C}^{\mathrm{Lyu}}$ by construction factors through Hbdy. Then by (7.10) the same is true for $\widehat{\mathcal{C}^{\mathrm{a}}}$.

Hence, it remains to prove (7.10): To this end, we use $h^*\mathcal{C}^{\mathrm{Lyu}} = \widehat{\left(h^*\mathcal{C}^{\mathrm{Lyu}}\right)_0}$ which follows from Remark 7.2. Here $\left(h^*\mathcal{C}^{\mathrm{Lyu}}\right)_0$ denotes the restriction of $h^*\mathcal{C}^{\mathrm{Lyu}}$ to a cyclic algebra. Note that Remark 7.2 uses that $h^*\mathcal{C}^{\mathrm{Lyu}}$ inverts all morphisms in the categories of operations (which is the case because $h^*\mathcal{C}^{\mathrm{Lyu}}$ comes from a modular algebra over $\mathsf{Hbdy}^{\mathrm{a}}$ which is groupoid-valued). Thanks to $(h^*\mathcal{C}^{\mathrm{Lyu}})_0 = (\mathcal{C}^{\mathrm{Lyu}})_0$ (this holds by definition), (7.10) now reads $\widehat{\mathcal{C}^{\mathrm{a}}} \simeq \widehat{\left(\mathcal{C}^{\mathrm{Lyu}}\right)_0}$ and hence can be reduced to the equivalence $\mathcal{C}^{\mathrm{a}} \simeq \left(\mathcal{C}^{\mathrm{Lyu}}\right)_0$ of cyclic $\mathsf{Hbdy}^{\mathrm{a}}_0$-algebras. This equivalence can be seen as follows: Clearly, both assign the same left exact functors to genus zero handlebodies. In fact, they also assign the same isomorphisms of such functors to isomorphisms of genus zero handlebodies (phrased equivalently, they agree on the mapping class groups of genus zero surfaces — which are ribbon braid groups). Explicitly, the braiding generators act through the braiding of $\mathcal{C}$, and the Dehn twist around a boundary component acts by the balancing. This description is tautologically true for $\mathcal{C}$ as a cyclic RBr-algebra (and therefore for $\mathcal{C}^{\mathrm{a}}$). In the Lyubashenko construction, i.e. for $\mathcal{C}^{\mathrm{Lyu}}$ and hence for $\left(\mathcal{C}^{\mathrm{Lyu}}\right)_0$, it holds by definition, see [Lyu95a, Lyu95b, Lyu96] or the Lego-Teichmüller version of the construction in [FS17, Section 2&3]. □

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