Reconstructing the orbit type stratification of a torus action from its equivariant cohomology

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Abstract

We investigate what information on the orbit type stratification of a torus action on a compact space is contained in its rational equivariant cohomology algebra. Regarding the (labelled) poset structure of the stratification we show that equivariant cohomology encodes the subposet of ramified elements. For equivariantly formal actions, we also examine what cohomological information of the stratification is encoded. In the smooth setting we show that under certain conditions – which in particular hold for a compact orientable manifold with discrete fixed point set – the equivariant cohomologies of the strata are encoded in the equivariant cohomology of the manifold.

1 Introduction

Motivated by Masuda's equivariant cohomological rigidity result for toric symplectic manifolds [12, Theorem 1.1], Franz–Yamanaka [7] recently showed that the isomorphism type of a GKM graph is encoded in its graph cohomology. Hence, for a GKM manifold, the equivariant cohomology contains complete information on the combinatorics of the one-skeleton which translates to a complete understanding of the combinatorial aspects of the entire orbit type stratification. Furthermore in [1] and [13], combinatorial aspects of actions of compact Lie groups are related to the spectrum of the equivariant cohomology ring, leading among other things to an algebraic criterion for uniformity of an action. These results naturally trigger the question in how far the combinatorics of a general torus action is determined by its equivariant cohomology. More specifically, we consider the connected orbit type stratification of an action of a compact torus T on a space X, i.e. the collection of all connected components of fixed point sets X^U where $U \subset T$ is some subtorus. It naturally carries the combinatorial structure of a poset as well as a function which remembers the kernel of the action on each element of the stratum. We ask whether this combinatorial data is encoded in the rational equivariant cohomology algebra. In full generality, such a statement cannot hold true, as the equivariant cohomology algebra of any torus action on a sphere with nonempty, connected fixed point set is the same as that of the trivial action, see Example 3.13. In this paper we argue that the reason for such behavior is to be found in the existence of unramified elements in the orbit type stratification. We define an element in the orbit type stratification to be *ramified* if it is either minimal, or, recursively, minimal with the property that it contains two distinct ramified elements – see Definition 3.11. Our first result, Theorem 3.14, which holds under mild topological assumptions on the space but without any further conditions on the action, states:

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Theorem. *The rational equivariant cohomology of a compact T-space X encodes the subposet $\overline{\chi}$ of ramified elements in the connected orbit type stratification, together with the function $\overline{\lambda}: \overline{\chi} \to \{$connected subgroups of $T\}$ that associates to an element $C \in \overline{\chi}$ the identity component of the kernel of the T-action on C.*

Our technique of choice to prove this theorem is the new notion of Thom system, which we introduce in Definition 3.2. It may be formulated in the general context of (graded) commutative rings and encodes algebraic properties of the equivariant Thom classes of the path-components of the fixed point set in case of a smooth action. While this connection to Thom classes plays a major role in the second half of the paper, the theorem above could also be proved using the machinery developed in [1] and [13], see Remark 3.1.

In more specialized settings, the subset of ramified elements may already contain further information on the stratification. We note that the ramification condition in the statement below, which is Corollary 3.17, is satisfied in particular if the fixed points of the action are isolated.

Corollary. *Let X be an equivariantly formal, compact T-space such that every isotropy codimension 1 element of the orbit type stratification poset χ is ramified. Then $H_T^*(X)$ encodes χ up to rational equivalence. If X is additionally a manifold and the T-action is smooth, then all of χ is encoded in $H_T^*(X)$.*

In Section 4 we investigate in how far the equivariant cohomology algebra of a T-space encodes all other (equivariant) cohomology algebras in the orbit type stratum. A starting point is the following result (cf. Proposition 4.3).

Proposition. *The sum of all Betti numbers of each indiviual path-component of X^T is encoded in $H_T^*(X)$. If X is equivariantly formal then the individual sums of all Betti numbers of the components of X^U are encoded in $H_T^*(X)$ for every subtorus $U \subset T$.*

In particular this implies the first half of the following corollary. The second part is a consequence of the theorem above and was proved first in [7].

Corollary. *For an equivariantly formal compact orientable T-manifold M the equivariant cohomology algebra $H_T^*(M)$ encodes whether the action is of GKM type or not. In case the action is GKM, $H_T^*(M)$ also encodes the GKM graph of the action.*

In general, one can not expect $H_T^*(X)$ to contain more specific information like individual Betti numbers of the strata or multiplicative structure, even for equivariantly formal actions on compact manifolds whose orbit type stratification consists only of ramified elements (see Example 4.5). However under stronger conditions, we show in Theorem 4.6:

Theorem. *Let M be an equivariantly formal, compact orientable T-manifold such that the map $H^*(M) \to H^*(X)$ is surjective for all components X of M^T. Then the equivariant cohomology $H_T^*(M)$ encodes the connected orbit type stratification χ of M as well as for any $C, D \in \chi$ with $C \subset D$ the respective equivariant cohomology algebras and the map $H_T^*(D) \to H_T^*(C)$ induced by the inclusion.*

The surjectivity condition is trivially satisfied in case the fixed point set of the action consists of isolated points.

In Section 5 we conclude the paper with some remarks on the question which additional information on the orbit type stratification can be obtained from the cohomology by considering integral instead of rational coefficients.

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2 Preliminaries

In this paper we consider continuous actions of a compact torus T. All spaces are assumed to be locally contractible, Hausdorff, and have finite-dimensional rational (singular) cohomology. In parts of Section 4 we will restrict to smooth manifolds. Cohomology is always singular with coefficients in $\mathbb{Q}$ if not specified otherwise and we usually suppress coefficients from the notation. When considering elements of a graded object, these will always be assumed to be homogeneous in absence of further specification.

Given a T-space X, as well as a subgroup $U \subset T$, we denote by X^U the set of points in X fixed by U. Note that if X is a smooth manifold and the action is smooth, then every component of X^U is a smooth submanifold. The components of all X^U form a poset:

Definition 2.1. Let X be a T-space. The connected orbit type stratification of X is the poset $\chi = \{\text{connected components of } X^U \mid U \subset T \text{ connected, closed subgroup}\}$, together with the function $\lambda\colon \chi \to \{\text{connected, closed subgroups of } T\}$ such that $\lambda(C)$ is the identity component of the kernel of the restricted T-action on C.

To any T-space X one associates its equivariant cohomology $H_T^*(X)$, which is the cohomology of its Borel construction $X_T := ET \times_T X$, equipped with the structure of $H^*(BT)$-algebra induced by the natural projection $ET \times_T X \to BT$. We will abbreviate $R := H^*(BT)$.

Lemma 2.2. *For a T-space X the following conditions are equivalent:*

1. *$H_T^*(X)$ is a free R-module.*
2. *$H_T^*(X) = R \otimes H^*(X)$ as an R-module.*
3. *The map $H_T^*(X) \to H^*(X)$ induced by a fiber inclusion $X \to X \times_T ET$ is surjective.*
4. *$\dim_{\mathbb{Q}} H^*(X^T) = \dim_{\mathbb{Q}} H^*(X)$.*

These equivalences are standard, see e.g. [2, Corollary 4.2.3] and [10, Corollary IV.2].

Definition 2.3. If a T-space satisfies the equivalent conditions of Lemma 2.2 then we say that it is equivariantly formal.

Let us recall the Borel localization theorem, see e.g. [2, Theorem 3.2.6]: for any multiplicatively closed subset $S \subset R$, we set

$$X^S = \{x \in X \mid S^{-1}H_T^*(Tx) \neq 0\},$$

where S^{-1} denotes localization at S.

Theorem 2.4. *Assume that either X is compact or that it is paracompact and that the set of identity components of isotropy subgroups is finite. Then the inclusion $X^S \to X$ induces an isomorphism of $S^{-1}R$-modules $S^{-1}H_T^*(X) \longrightarrow S^{-1}H_T^*(X^S)$.*

We will be particularly interested in the following situation: for a subtorus $U \subset T$, let $\mathfrak{p}_U = \ker(H^*(BT) \to H^*(BU))$, and $S = R \setminus \mathfrak{p}_U$. Then $X^S = X^U$, the fixed point set of the restricted U-action, and we obtain an isomorphism

$$S^{-1}H_T^*(X) \longrightarrow S^{-1}H_T^*(X^U). \tag{1}$$

For $U = T$ we obtain in particular that the kernel of the canonical map $H_T^*(X) \to H_T^*(X^T)$ is the torsion submodule of $H_T^*(X)$. Denoting by $X_1 = \{x \in X \mid \dim Xp \leq 1\}$ the one-skeleton of the action, we have a sequence

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X^T) \longrightarrow H_T^*(X_1, X), \tag{2}$$

which, for an equivariantly formal action, is exact at $H_T^*(X)$.

Lemma 2.5 (Chang-Skjelbred Lemma [6, Lemma 2.3])**.** *Assume that either X is compact or that it is paracompact and that the set of identity components of isotropy subgroups is finite. Then for an equivariantly formal action of a torus T on X the sequence (2) is also exact at $H_T^*(X^T)$.*

In the following proposition we collect some well-known properties of equivariantly formal actions.

Proposition 2.6. *Consider an equivariantly formal T-action on a space X. Then the following hold true:*

1. *For any subtorus $U \subset T$, the induced U-action on X is equivariantly formal.*
2. *For any subtorus $U \subset T$, the induced T-action on every component of X^U is equivariantly formal.*

Proof. We observe that for any subtorus $U \subset T$ the map $H_T^*(X) \to H^*(X)$ induced by fiber inclusion factors as $H_T^*(X) \to H_U^*(X) \to H^*(X)$. Thus, if $H_T^*(X) \to H^*(X)$ is surjective, so is $H_U^*(X) \to H^*(X)$. The first statement then follows from Lemma 2.2.

Lemma 2.2 then implies that $\dim H^*(X^T) = \dim H^*(X) = \dim H^*(X^U)$. As $(X^U)^T = X^T$, this implies, again via Lemma 2.2, that the T-action on X^U is equivariantly formal. In general, a T-action is equivariantly formal if and only if the action on every connected component is equivariantly formal. ☐

3 Orbit type stratification

The purpose of this section is to recover information on the combinatorics of the orbit type stratification of a T-action from the algebraic data of its equivariant cohomology algebra $H_T^*(X)$. To this end we introduce the notion of Thom system, see Definition 3.2. It is motivated by the fact that in certain smooth settings, a preferred choice of Thom system in the equivariant cohomology will be given by the equivariant Thom classes of the path-components of the fixed point set (cf. Lemma 4.11).

Remark 3.1. Somewhat dually to this approach, Quillen [13] and Allday [1], see also [2, Section 3.6], related the combinatorics of the action to the spectrum of the even degree part of the equivariant cohomology ring, by considering ideals of the form $\mathfrak{p}(K, c) = \ker(H_T^*(X) \to H^*(BK))$ where $K \subset T$ is a subtorus, c a component of X^K, and the map is induced by the inclusion of a point in c. The set of all such pairs (K, c) forms a poset $\mathcal{T}(X)$, where $(K, c) \leq (L, d)$ if and only if $K \subset L$ and $d \subset c$. The results in [1] can be used to reconstruct $\mathcal{T}(X)$ from $H_T^*(X)$, in some sense substituting for the roles of Theorems 3.8 and 3.10 in this section. The poset $\mathcal{T}(X)$ is not the same as the connected orbit type stratification (see Definition 2.1) as it does not detect whether, for $(K, c) \leq (L, d)$, the inclusion $d \subset c$ is strict. Rather, this poset corresponds, in the proof of Theorem 3.14 below, to the poset χ'. As we are interested in the more geometric orbit type stratification, we are led to the concept of ramification, see Definition 3.11 below.

Definition 3.2. For a (graded) commutative ring A, we call a (homogeneous) collection of elements $\tau_1, \ldots, \tau_k \in A$ a Thom system if

- $\tau_i \cdot \tau_j$ is nilpotent whenever $i \neq j$
- τ_i is not nilpotent
- for any system $\alpha_1, \ldots, \alpha_l \in A$ satisfying the two preceding properties we have $l \leq k$.

Lemma 3.3. *Let X be a path-connected space with trivial T-action and $\alpha \in H_T^*(X)$. The following are equivalent:*

(i) α is not nilpotent.

(ii) α restricts to a nontrivial element in $H_T^*(*)$ for any point in X.

(iii) multiplication with α is injective on $H_T^*(X)$

Proof. We have $H_T^*(X) = R \otimes H^*(X)$ as R-algebras. Thus an element is nilpotent if and only if it is contained in $R \otimes H^+(X)$. This proves the equivalence of (i) and (ii). Clearly also (iii) implies (i). Finally, assume that (ii) holds and write the nontrivial $R \otimes H^0(X)$-component of α as $f \otimes 1$. It follows that multiplying an element of $R \otimes H^{\geq k}(X)$ with α multiplies its $R \otimes H^k(X)$ component with f. Thus multiplication with α is injective. □

Proposition 3.4. *Let X be a space with trivial T-action. Then $H_T^*(X)$ admits a Thom system. A collection $\tau_1, \ldots, \tau_k \in H_T^*(X)$ is a Thom system if and only if X has k connected components $X_1, \ldots, X_k$ which can be numbered in a way such that for any choice of points $p_i \in X_i$*

- *τ_i restricts to 0 in $H_T^*(p_j)$ for $i \neq j$*
- *the restriction of τ_i to $H_T^*(p_i)$ is not 0.*

Proof. Let $\tau_1, \ldots, \tau_l \in H_T^*(X)$ be elements satisfying the first two conditions in the definition of a Thom system and let $X_1, \ldots, X_k$ be the components of X. Choose $p_i \in X_i$. As the τ_i are not nilpotent, Lemma 3.3 shows that they restrict nontrivially to at least one of the $H_T^*(p_i)$. Also, since the $H_T^*(p_i)$ are integral domains it follows that no two of the τ_i restrict nontrivially to the same point. Thus it follows that $l \leq k$ and that, if $l = k$, then the τ_i correspond bijectively to the X_i in the manner described in the proposition (after possibly adjusting the order). Thus it remains to argue that a Thom system has k elements. This follows from the fact that the first two conditions in the definition of a Thom system are satisfied by the elements $e_1, \ldots, e_k$ defined by the condition that e_i restricts to 1 in $H_T^*(X_i)$ and to 0 in $H_T^*(X_j)$ for $j \neq i$. □

For a subtorus $U \subset T$ of a torus T we denote

$$\mathfrak{p}_U := \ker(H^*(BT) \to H^*(BU)),$$

as well as $S_U := R \backslash \mathfrak{p}_U$. For a space X with T-action, we will also consider $H_U^*(X)$ as an algebra over R, via the map $H^*(BT) \to H^*(BU)$.

Lemma 3.5. *Let X be a compact T-space, and $U \subset T$ a subtorus which acts trivially on X. If $x \in \ker(H_T^*(X) \to H_U^*(X))$ then it is nilpotent in $H_T^*(X)/\mathfrak{p}_U H_T^*(X)$.*

Proof. Let S be a subtorus of T which is complementary to U, i.e. $T = U \times S$. Then $R = H^*(BU) \otimes H^*(BS)$ and $H_T^*(X) \cong H^*(BU) \otimes H_S^*(X)$ as R-algebras with the obvious R-algebra structure. Furthermore $H_U^*(X) \cong H^*(BU) \otimes H^*(X)$ and the restriction $r\colon H_T^*(X) \to H_U^*(X)$ corresponds to

$$\mathrm{id}_{H^*(BU)} \otimes r'\colon H^*(BU) \otimes H_S^*(X) \to H^*(BU) \otimes H^*(X)$$

where r' is the restriction $H_S^*(X) \to H^*(X)$. Both algebras above inherit a bigrading with respect to the tensor product. The bigrading is respected by r.

If x lies in the kernel of r then its $H^*(BU) \otimes H_S^0(X)$ component is zero. Hence $x \in H^*(BU) \otimes H_S^+(X)$. Note that for N large enough, any product of N elements of $H_S^+(X)$ lies in $H^+(BS) \cdot H_S^*(X)$. This follows from the fact that $H_S^*(X)$ is finitely generated as $H^*(BS)$-module [2, Prop. 3.10.1] where we put N larger than the highest degree in a $H^*(BS)$-generating set of $H_S^*(X)$. Consequently, x^N lies in $H^+(BS) \cdot H_T^*(X)$ and in particular in $\mathfrak{p}_U \cdot H_T^*(X)$. □

Lemma 3.6. *Let X be a compact T-space and $U \subset T$ a subtorus. Then for any $x \in H_T^*(X)$ the image of x in $H_U^*(X^U)$ is nilpotent if and only if the image of x in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ is nilpotent.*

Proof. As $\mathfrak{p}_U H_T^*(X)$ is contained in the kernel of $H_T^*(X) \to H_U^*(X)$, the restriction map $H_T^*(X) \to H_U^*(X^U)$ factors as

$$H_T^*(X) \to H_T^*(X)/\mathfrak{p}_U H_T^*(X) \to H_U^*(X^U).$$

Applying localization at S_U, it follows that if the image of x in $S_U^{-1}H_T^*(X)/S_U^{-1}\mathfrak{p}_U H_T^*(X)$ is nilpotent, then the same holds for the image in $S_U^{-1}H_U^*(X^U)$. An element of $H_U^*(X^U)$ is nilpotent if and only if it is nilpotent in $S_U^{-1}H_U^*(X^U)$ which proves one direction.

Assume conversely that the image of x in $H_U^*(X^U)$ is nilpotent. Then some power x^k maps to the kernel of $H_T^*(X^U) \to H_U^*(X^U)$. By Lemma 3.5, some higher power x^N satisfies $f(x^N) \in \mathfrak{p}_U H_T^*(X^U)$ where f is the map $H_T^*(X) \to H_T^*(X^U)$. By Borel localization the map

$$S_U^{-1}f\colon S_U^{-1}H_T^*(X) \to S_U^{-1}H_T^*(X^U)$$

is an isomorphism of $S_U^{-1}R$-modules, see (1). Since $S_U^{-1}f(x^N)$ is in $S_U^{-1}\mathfrak{p}_U \cdot S_U^{-1}H_T^*(X^U)$ it follows that the image of x^N in $S_U^{-1}H_T^*(X)$ lies in $S_U^{-1}\mathfrak{p}_U \cdot S_U^{-1}H_T^*(X)$. The Lemma follows from the fact that localization commutes with taking quotients, i.e. $S_U^{-1}H_T^*(X)/S_U^{-1}\mathfrak{p}_U \cdot H_T^*(X) \cong S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$. □

Remark 3.7. With regards to the above lemma and the theorem below, we remark that in general $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ and $S_U^{-1}H_U^*(X^U)$ are not isomorphic. Also, we only obtain criteria for elements to restrict to nilpotent elements in $H_U^*(X^U)$ without a precise description on the kernel. The reason for this is the fact that in general the kernel of the restriction $H_T^*(X) \to H_U^*(X)$ is larger than $\mathfrak{p}_U H_T^*(X)$ and may additionally contain certain Massey products involving elements of $\mathfrak{p}_U$. This phenomenon is discussed in [3] under the name of spherical actions.

Theorem 3.8. *Let X be a compact T-space and $U \subset T$ a subtorus. Then there are elements $\tau_1, \dots, \tau_k \in H_T^*(X)$ which restrict to a Thom system of $H_U^*(X^U)$. A set of elements $\tau_1, \dots, \tau_k \in H_T^*(X)$ has this property if and only if it restricts to a Thom system in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$.*

Proof. By Lemma 3.6, $\tau_1, \dots, \tau_r \in H_T^*(X)$ satisfy the first two conditions of a Thom system when restricted to $H_U^*(X^U)$ if and only if they do so in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$. Let $\eta_1, \dots, \eta_l \in S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ be a set satisfying the nilpotence conditions of a Thom system. Then multiplying the η_i with elements of S_U preserves these conditions. Consequently, we may assume that the η_i are restrictions from $H_T^*(X)$. Since $H_U^*(X^U)$ admits a Thom system by Proposition 3.4, it follows that $l \leq k$, where k is the number of elements in a Thom system of $H_U^*(X^U)$, i.e., the number of path components of X^U. In particular $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ admits a Thom system which lies in the image of $H_T^*(X) \to S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$.

The theorem follows if we show that the image of $H_T^*(X) \to H_U^*(X^U)$ contains a Thom system of $H_U^*(X^U)$. Let $X_1, \dots, X_k$ be the components of X^U and let $e_i \in H_T^*(X^U)$ be the element which restricts to $1 \in H_T^*(X_i)$ and to $0 \in H_T^*(X_j)$ for $i \neq j$. By Borel Localization there are polynomials $f_i \in S$ such that $f_i \cdot e_i$ is the restriction of some $\tau_i \in H_T^*(X)$. It remains to check that the τ_i restrict to a Thom system of $H_U^*(X^U)$. Choose points $p_i \in X_i$. The composition

$$H_T^*(X) \to H_T^*(X^U) \to H_U^*(X^U) \to H_U^*(p_i)$$

is R-linear. Thus τ_i maps to $f_i \cdot 1 \in H_U^*(p_i)$, which is nonzero because $f \in S_U$, and to $0 \in H_U^*(p_j)$ for $i \neq j$. Thus the τ_i restrict to a Thom system in $H_U^*(X^U)$ by Lemma 3.3. □

Definition 3.9. For a subtorus $U \subset T$, we call a set of elements $\tau_1, \dots, \tau_k \in H_T^*(X)$ with the property as in Theorem 3.8 a *U-local Thom system* (of $H_T^*(X)$). Given such a system, we denote by $F_U(\tau_i)$ the unique component of X^U such that τ_i restricts to a nonzero element in $H_U^*(p_i)$ for any point $p_i \in F_U(\tau_i)$.

Theorem 3.10. *Let X be a compact T-space and $H \subset U \subset T$ subtori. Let $\tau_1, \ldots, \tau_k \in H_T^*(X)$ be a U-local Thom system and $\eta_1, \ldots, \eta_l$ be an H-local Thom system. Then $F_U(\tau_i) \subset F_H(\eta_j)$ if and only if there is some $f \in S_H$ such that the image of $f\tau_i - \eta_j\tau_i$ in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ is nilpotent.*

Proof. For $i = 1, \ldots, k$, let $p_i \in F_U(\tau_i)$ and let $r_i \colon H_T^*(X) \to H_U^*(p_i)$ be the natural restriction map. We claim that for any $x \in H_T^*(X)$ and $f \in R$ we have $r_i(x) = f + \mathfrak{p}_U \in R/\mathfrak{p}_U \cong H_U^*(p_i)$ if and only if the image of $f\tau_i - x\tau_i$ in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ is nilpotent. To prove the claim, recall that by Lemma 3.6 the latter condition is equivalent to $f\tau_i - x\tau_i$ being nilpotent in $H_U^*(X^U)$. By Lemma 3.3 this is again equivalent to $r_j(f\tau_i - x\tau_i)$ being 0 for $j = 1, \ldots, k$. Since $r_j(\tau_i) = 0$ for $j \neq i$ this depends only on $r_i(f\tau_i - x\tau_i)$ being zero. But this is the case if and only if $r_i(f \cdot 1) = r_i(x)$, which proves the claim.

The inclusion $F_U(\tau_i) \subset F_H(\eta_j)$ holds if and only if $p_i \in F_H(\eta_j)$. This is the case if and only if the image of η_j in $H_H^*(p_i)$ is not 0. Since this restriction map factors through $H_U^*(p_i)$ the condition is equivalent to $r_i(\eta_j) \notin \ker(H^*(BU) \to H^*(BH)) = \overline{\mathfrak{p}_H}$, where the latter denotes the image of $\mathfrak{p}_H$ in $H^*(BU)$. This is equivalent to $r_i(\eta_j) = r_i(f \cdot 1)$ for some $f \in S_H$, proving the theorem. ∎

Let χ be the connected orbit type stratification, see Definition 2.1. Let us introduce the following recursive definition:

Definition 3.11. We call an element $C \in \chi$ *ramified* if it is either minimal in the poset χ, or there exist two ramified elements $D_1 \neq D_2 \in \chi$ with the property that $D_1, D_2 \subset C$ and C is minimal with respect to this property.

Definition 3.12. We define $\overline{\chi}$ to be the subposet of χ given by all ramified elements in χ.

Example 3.13. Consider a T-action on a sphere S^n with nonempty, connected fixed point set F. Then the action is automatically equivariantly formal – for even n any action on S^n is, for odd n this is because then the total Betti number of the fixed point set is necessarily 2. Note that all elements of the connected orbit type stratification except for the minimal one are unramified. We choose an R-module basis of $H_T^*(S^n)$ of the form $\{1, a\}$, with $a \in H_T^n(S^n)$. By replacing a by an element of the form $a + f$, with $f \in H^n(BT)$, we may assume that a restricts to an element in $R \otimes H^+(F)$. As the restriction map $H_T^*(S^n) \to H_T^*(F) = R \otimes H^*(F)$ is injective, this implies that $a^2 = 0$ (as $H^+(F)$ is concentrated in only one degree).

We have shown that the equivariant cohomology $H_T^*(S^n)$ is, as an R-algebra, isomorphic to that of the trivial T-action on S^n. In particular, equivariant cohomology can not distinguish these actions. However, among those indistinguishable actions, many different orbit type stratifications are possible.

Theorem 3.14. *The equivariant cohomology of a compact T-space X encodes the subposet $\overline{\chi}$ of ramified elements in the poset χ of orbit type strata, together with the restriction $\overline{\lambda} : \overline{\chi} \to$ {connected subgroups of T} of λ.*

Proof. We construct a poset $\overline{\chi}'$ and a map $\lambda' \colon \overline{\chi}' \to$ {connected subgroups of T} together with an isomorphism $\varphi \colon \overline{\chi}' \to \overline{\chi}$ of posets satisfying $\lambda' = \lambda \circ \varphi$. We fix a U-local Thom system $\tau_1^U, \ldots, \tau_{k_U}^U \in H_T^*(X)$ for every subtorus $U \subset T$ and define χ' to be be the set of tuples (τ_i^U, U). We write $(\tau_j^U, U) \leq (\tau_i^H, H)$ whenever $H \subset U$ and $f\tau_j^U - \tau_i^H\tau_j^U$ is nilpotent in $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$ for some $f \in S_H$. Then by Theorem 3.10 this turns χ' into a partially ordered set which corresponds bijectively to the poset of pairs (C, U), where C is a component of X^U. The map $(\tau_i^U, U) \mapsto F_U(\tau_i^U)$ corresponds to the forgetful map $(C, U) \mapsto C$ and gives a surjection $\varphi \colon \chi' \to \chi$ compatible with the poset structure.

In analogy with $\overline{\chi}$, we call an element $C \in \chi'$ ramified if it is either minimal in χ' or there exist two ramified elements $D_1 \neq D_2$ in χ' with the property that C is minimal among the elements containing D_1 and D_2. Furthermore we set $\lambda'(\tau_i^U, U) = U$. We claim that φ restricts to a bijection $\overline{\chi}' \to \overline{\chi}$ between ramified subsets and that $\lambda' = \lambda \circ \varphi$ on $\overline{\chi}'$. Note first that for any $C \in \chi'$ we have $\operatorname{codim} \lambda'(C) \geq \operatorname{codim} \lambda(\varphi(C))$ and that any element $C \in \overline{\chi}'$ has to satisfy $\lambda'(C) = \lambda(\varphi(C))$ due to the minimality condition. Also if $\varphi(C) = \varphi(D)$ for $C, D \in \overline{\chi}'$ then in particular $\lambda'(C) = \lambda'(D)$. But then the properties of Thom systems yield $C = D$ proving injectivity of $\varphi|_{\overline{\chi}'}$. It remains to prove that $\varphi(\overline{\chi}') = \overline{\chi}$.

To see this we use induction over the isotropy codimension, where the statement is obvious for the fixed points in isotropy codimension 0. Let $\overline{\chi}'^k$ (resp. $\overline{\chi}^k$) be the subset consisting of those elements x for which the codimension of $\lambda'(x)$ (resp. $\lambda(x)$) is k or less. Suppose we have shown that $\varphi(\overline{\chi}'^k) = \overline{\chi}^k$. Let $C \in \overline{\chi}^{k+1}$ with $\lambda(C)$ of codimension $k+1$. Then there is some $C' \in \chi'$ with $\varphi(C') = C$ and $\operatorname{codim} \lambda'(C') = k+1$. We show that C' is ramified and hence $\varphi(\overline{\chi}') \supset \overline{\chi}$. If C is minimal and $D' \leq C'$ then $\varphi(D') = C$. Thus $\operatorname{codim} \lambda'(D') \geq \operatorname{codim} \lambda(C) = \operatorname{codim} \lambda'(C')$ which implies $D' = C'$. If C is not minimal then there are $D_1, D_2 \in \overline{\chi}^k$ such that C is minimal among the elements containing D_1 and D_2. Let D_1', D_2' denote preimages in $\overline{\chi}'^k$. Then we have $D_1', D_2' \leq C'$. For any $D_1', D_2' \leq B' \leq C'$ we have that $\varphi(B') = C$ so $\operatorname{codim}\ \lambda'(B') \geq \operatorname{codim}\ \lambda(C) = \operatorname{codim}\ \lambda'(C')$ and thus $B' = C'$. This concludes the proof of $\varphi(\overline{\chi}') \supset \overline{\chi}$. In a similar fashion, the induction proves $\varphi(\overline{\chi}') \subset \overline{\chi}$. ☐

Definition 3.15. A map between two spaces is called a rational equivalence if it induces an isomorphism on rational cohomology.

Remark 3.16. The existence of a rational equivalence $X \to Y$ between two spaces is much stronger than the condition $H^*(X) \cong H^*(Y)$. Under appropriate conditions on the fundamental groups it implies that X and Y have the same rational homotopy type and in particular isomorphic homotopy groups up to torsion.

Corollary 3.17. *Let X be an equivariantly formal, compact T-space such that every isotropy codimension 1 element of χ is ramified. Then $H_T^*(X)$ encodes χ up to rational equivalence in the sense that for any $D \in \chi$ the inclusion $C \subset D$ of the unique maximal ramified element in D is a rational equivalence. If X is additionally a manifold and the T-action is smooth, then all of χ is encoded in $H_T^*(X)$.*

Proof. Every element $D \in \chi$ contains a minimal element. In particular it contains a ramified element $C \subset D$ which we assume to be maximal with this property. If there are two distinct ramified $C, C' \subset D$ which are maximal with respect to these properties then D would by definition be ramified itself, resulting in a contradiction. Thus C is unique. Then since every isotropy codimension 1 element is ramified it follows that C and D have the same 1-skeleton. By Proposition 2.6, C and D are both equivariantly formal so the Chang-Skjelbred Lemma 2.5 implies that the inclusion is a rational equivalence.

If X is a manifold, we observe that for some $N \in \chi$ we have $N^T \neq \emptyset$ and that for some $p \in N^T$ the isotropy T-representation of X at p decomposes into 2-dimensional subrepresentations of orbit dimension 1 and the tangent space of X^T. We deduce that the dimension of N is determined by the one-skeleton of N. In particular in the above setting C and D are submanifolds of the same dimension so $C = D$. ☐

4 Cohomology

In this section we discuss under which conditions equivariant cohomology contains cohomological information about the elements in the connected orbit type stratification. We continue to use the notation from the last section, i.e., for a subtorus $U \subset T$, $\mathfrak{p}_U = \ker(H^*(BT) \to H^*(BU))$ and $S_U = R \backslash \mathfrak{p}_U$.

Definition 4.1. We call a Thom system $\tau_1, \ldots, \tau_k$ *strict*, if $\tau_i\tau_j = 0$ for $i \neq j$. For some subtorus $U \subset T$, a collection $\tau_1, \ldots, \tau_k \in H_T^*(X)$ is called a *strict* U-local Thom system of $H_T^*(X)$ if the τ_i restrict to a strict Thom system of $H_U^*(X^U)$.

Clearly any graded commutative ring that admits a Thom system also admits a strict Thom system. However, not all statements on Thom systems transfer directly to their strict counterparts without imposing any additional conditions. The following lemma records some analogous properties.

Lemma 4.2. *Let X be a compact T-space, $U \subset T$ a subtorus, and $\tau_1, \ldots, \tau_k \in H_T^*(X)$.*

(i) The collection $\tau_1, \ldots, \tau_k$ forms a strict U-local Thom system if and only if every τ_i restricts to $0 \in H_U^*(X_j)$ for all components $X_j \subset X^U$ except for a single X_i where it is not nilpotent.

(ii) The τ_i form a strict T-local Thom system if and only if they induce a strict Thom system of $S_T^{-1}H_T^*(X)$. If the action is equivariantly formal then this is the case if and only if the τ_i are a strict Thom system of $H_T^*(X)$.

(iii) Assume that the action is equivariantly formal. Then the τ_i are a strict U-local Thom system if and only if they induce a strict Thom system of $S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X))$.

Proof. For the first statement, recall from Lemma 3.3 that multiplication with τ_j is injective on $H_U^*(X_j)$. Thus $\tau_i\tau_j = 0$ implies that τ_i restricts to 0 on $H_U^*(X_j)$.

The first half of (ii) follows from the localization theorem and the fact that $H_T^*(X^T) \to S_T^{-1}(X^T)$ is injective. The second half is due to the fact that $H_T^*(X) \to S_T^{-1}H_T^*(X)$ is injective in the equivariantly formal case.

Regarding statement (iii), if X is equivariantly formal, then we claim that $H_U^*(X) \cong H_T^*(X)/\mathfrak{p}_U H_T^*(X)$ as $H^*(BU) \cong R/\mathfrak{p}_U$-algebras. To see this, recall that any set of elements $x_i \in H_T^*(X)$ which restricts to a $\mathbb{Q}$-basis of $H^*(X)$ gives an R-basis of $H_T^*(X)$. Since the restricted U-action is again equivariantly formal by Proposition 2.6, the restriction of the x_i to $H_U^*(X)$ is an $H^*(BU)$-basis. Consequently the restriction $H_T^*(X) \to H_U^*(X)$ is surjective with kernel $\mathfrak{p}_U H_T^*(X)$, which proves the claim. The Borel localization theorem applied to the restricted U-action yields (note that localizing $H_U^*(X)$ at S_U is the same as localizing at the image of S_U in $H^*(BU)$)

$$S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X)) \cong S_U^{-1}H_U^*(X) \cong S_U^{-1}H_U^*(X^U).$$

The τ_i are a U-local Thom system if and only if they restrict to a Thom system of $S_U^{-1}H_U^*(X^U)$ thus (iii) follows. ☐

It follows from the above lemma that we can algebraically detect strict U-local Thom systems if the action is equivariantly formal or $U = T$. In this case the equivariant cohomology algebra encodes the total Betti numbers in the following way:

Proposition 4.3. *Let X be a compact T-space, $U \subset T$ a subtorus, and $\tau_1, \ldots, \tau_k \in H_T^*(X)$ a strict U-local Thom system corresponding to the components $F_U(\tau_1), \ldots, F_U(\tau_k)$ of X^U. Assume further that either X is additionally equivariantly formal or that $U = T$. Then*

$$\dim H^*(F_U(\tau_i)) = \mathrm{rk}_{S_U^{-1}R/\mathfrak{p}_U} I_i$$

where $I_i = \{x \in S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X)) \mid x\tau_j = 0,\ \text{for all } j \neq i\}$.

Proof. As argued in the proof of Lemma 4.2, we have

$$S_U^{-1}(H_T^*(X)/\mathfrak{p}_U H_T^*(X)) \cong S_U^{-1} H_U^*(X^U)$$

if $U = T$ or if the action is equivariantly formal. The ideal I_i corresponds to the kernel of the restriction $S_U^{-1}H_U^*(X^U) \to S_U^{-1}H_U^*(X^U - F_U(\tau_i))$. Since $S_U^{-1}H_U^*(X^U) = S_U^{-1}H_U^*(X^U - F_U(\tau_i)) \oplus S_U^{-1}H_U^*(F_U(\tau_i))$. It follows that I_i is isomorphic to $S_U^{-1}H_U^*(F_U(\tau_i))$ which, as an $S_U^{-1}R/\mathfrak{p}_U$-module, is isomorphic to $S_U^{-1}R/\mathfrak{p}_U \otimes H^*(F_U(\tau_i))$. □

The second statement of the following corollary was first proven in [7, Theorem 5.1]. Recall that an equivariantly formal compact orientable T-manifold is of GKM type [9] if it has only finitely many fixed points, and the one-skeleton M_1 of the action is a finite union of T-invariant 2-spheres.

Corollary 4.4. *For an equivariantly formal compact orientable T-manifold M the equivariant cohomology algebra $H_T^*(M)$ encodes if the action is of GKM type or not. In case the action is GKM, $H_T^*(M)$ also encodes the GKM graph of the action.*

Proof. By Proposition 4.3, in the situation at hand the equivariant cohomology algebra encodes if the fixed point set consists of isolated points. If all fixed point components are isolated points, then every isotropy codimension 1 element of the orbit type stratification χ is ramified, and Theorem 3.14 implies that the equivariant structure of the one-skeleton M_1 of the action is encoded in $H_T^*(M)$. □

Example 4.5. In general, we can not expect $H_T^*(X)$ to encode more specific information about the cohomology even if the action is equivariantly formal: consider the S^1-action on $S^4 \subset \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$ given by $s \cdot (v, w, h) = (sv, w, h)$. As argued in Example 3.13, the equivariant cohomology of this action agrees with the one of the trivial action on S^4. However the fixed point sets of the two actions are S^2 and S^4, which have different cohomologies.

With regards to our previous results it also seems reasonable to ask for an example where additionally the entirety of χ is ramified. Note that this is fulfilled for any S^1-action with at least 2 fixed point components. We obtain such examples from the previous ones by considering the diagonal action on $S^4 \times S^2$ with standard rotation on the right hand side. For these actions we obtain the cohomologically distinct fixed point sets $S^2 \coprod S^2$ and $S^4 \coprod S^4$.

The previous examples show that additional requirements are needed to reconstruct the cohomologies in the orbit type stratification from the global equivariant cohomology. Our main result with regards to cohomology is

Theorem 4.6. *Let M be an equivariantly formal, compact orientable T-manifold such that the map $H^*(M) \to H^*(X)$ is surjective for all components X of M^T. Then the equivariant cohomology $H_T^*(M)$ encodes the connected orbit type stratification χ of M as well as for any $C, D \in \chi$ with $C \subset D$ the respective equivariant cohomology algebras and the map $H_T^*(D) \to H_T^*(C)$ induced by the inclusion.*

Example 4.7. We remark that the surjectivity condition is automatically satisfied in case the fixed point set is finite. Let us describe two classes of examples with nondiscrete fixed point set for which it is fulfilled. Recall first that whenever a T-action on a compact manifold M admits a T-invariant Morse-Bott function $f : M \to \mathbb{R}$ with critical set M^T, then the action is equivariantly formal, and for the global minimum c of f, the restriction map $H_T^*(M) \to H_T^*(f^{-1}(c))$ is surjective – this follows from the arguments in [5, Section 1], see also [8, Theorem 7.1].

Now, consider a Hamiltonian T-action on a compact symplectic manifold for which every component of the fixed point set is mapped, via the momentum map, to the boundary of the momentum polytope. In this setting, for any such component X, one can choose a component of the momentum map which attains its global minimum exactly at X. Thus, such T-actions satisfy the assumptions of Theorem 4.6.

Similarly, given a toric symplectic manifold M, with acting torus T, the restriction of the T-action to any subtorus U fulfills the same assumptions. Indeed, every component of M^U corresponds to a face in the momentum polytope of the T-action, so that it occurs as the minimum of an appropriate component of the T-momentum map.

Before we come to the proof of Theorem 4.6, we need several lemmas. See e.g. [8] for the notion of Cohen-Macaulay module in the context of equivariant cohomology. Also recall that a T-equivariant vector bundle $V \to X$, i.e. a vector bundle with a T-action such that the transformations between fibers are linear, induces a vector bundle $V_T \to X_T$ over the Borel construction. The equivariant characteristic classes of V are defined as the regular characteristic classes of V_T in $H_T^*(X)$.

Lemma 4.8. *Let M be an equivariantly formal, compact T-manifold and $N \subset M$ a connected component of M^S for some subtorus $S \subset T$. Let SN be the sphere bundle of the normal bundle of $N \subset M$ and $e \in H_T^*(N)$ its equivariant Euler class. Then the bundle induces an isomorphism $H_T^*(SN) \cong H_T^*(N)/(e)$. Furthermore, $H_T^*(SN)$ is Cohen-Macaulay.*

Proof. There is a fiber bundle $S^n \to (SN)_T \to N_T$ of Borel constructions. In the associated Serre spectral sequence, the generator of $H^n(S^n)$ transgresses onto the equivariant Euler class $e \in H_T^*(N)$ of the normal bundle. Now multiplication with e is injective in $H_T^*(N)$: to see this it suffices to check that multiplication is injective on $H_T^*(N^T)$ and this is the case because e restricts to a nonzero element in $H_T^*(*)$ for any point in N^T (there it restricts to the monomial over all weights of the normal representation at $*$, see e.g. [11, Lemma 6.10]).

As a consequence, in the spectral sequence we are ultimately left with a single row and $H_T^*(SN) \cong H_T^*(N)/(e)$ as R-algebras. By [8, Lemma 5.2] we have

$$\operatorname{depth}(H_T^*(SN)) \geq \dim T - 1.$$

But also $\dim_R H_T^*(SN) \leq \dim T - 1$ as $(0) \subsetneq \operatorname{Ann}_R(H_T^*(SN))$ and (0) is prime. ∎

Lemma 4.9. *Let M be an equivariantly formal, compact T-manifold and $N \subset M$ a connected component of M^S for some subtorus $S \subset T$. Let e be the equivariant Euler class of the normal bundle of $N \subset M$. Then an element $x \in H_T^*(N)$ is divisible by e if and only if for any component $X \subset N^T$ the restriction $x|_{H_T^*(X)}$ is divisible by $e|_{H_T^*(X)}$.*

Proof. As argued previously x is divisible by e if and only if it restricts to 0 in $H_T^*(SN)$. As the latter is Cohen-Macaulay of dimension $\dim T - 1$, this is the case if and only if the restriction to $H_T^*((SN)_1)$ is 0, see [8, Theorem 6.1]. Now $(SN)_1$ is contained in the restriction $(SN)|_{N^T}$ of SN to N^T. If $x|_{H_T^*(X)}$ is divisible by $e|_{H_T^*(X)}$ for every component $X \subset M^T$ then it follows that $x|_{H_T^*((SN)|_{N^T})} = 0$ and thus $x|_{H_T^*((SN)_1)} = 0$. ∎

Lemma 4.10. *Let M be an equivariantly formal, compact orientable T-manifold such that the map $H^*(M) \to H^*(N)$ is surjective for some component N of M^T. Then the T-action on $M\backslash N$ is equivariantly formal.*

Proof. By Lemma 2.2, any T-space X with finite dimensional cohomology is equivariantly formal if and only if the sums over all Betti numbers satisfy $\dim_{\mathbb{Q}} H^*(X) = \dim_{\mathbb{Q}} H^*(X^T)$. Since $(M\backslash N)^T = M^T\backslash N$ it suffices to prove that $\dim H^*(M\backslash N) = \dim H^*(M) - \dim H^*(N)$. By assumption, the inclusion of N is injective on homology and thus the long exact homology sequence of the pair (M, N) splits into short exact sequences

$$0 \to H_*(N) \to H_*(M) \to H_*(M, N) \to 0.$$

But by Lefschetz duality, applied to M with a tubular neighborhood of N removed, we have $H_*(M,N) = H^{n-*}(M\backslash N)$, where n is the dimension of M. $\square$

Lemma 4.11. *Let M be an equivariantly formal, compact orientable T-manifold such that the map $H^*(M) \to H^*(X_i)$ is surjective for all components $X_1, \ldots, X_k$ of M^T. Then there is a strict Thom system $\tau_1, \ldots, \tau_k$ of $H_T^*(M)$ which is minimal in the sense that for any other strict Thom system $\tau_1' \ldots, \tau_k'$, after possibly changing the order, the cohomological degrees satisfy $|\tau_i| \leq |\tau_i'|$. It is, up to scalars from $\mathbb{Q}^\times$, uniquely given by the equivariant Thom classes of the components of M^T in M.*

Proof. There is a Mayer-Vietoris sequence

$$0 \to H_T^*(M) \to H_T^*(M\backslash X_i) \oplus H_T^*(X_i) \to H_T^*(SX_i) \to 0$$

where SX_i is the sphere bundle of the normal bundle of X_i in M. Let $e_i \in H_T^*(X_i)$ denote the equivariant Euler class of SX_i and let $\tau_i \in H_T^*(M)$ be the equivariant Thom class of $X_i \subset M$, i.e. the unique element that restricts to $(0, e_i)$ in the above sequence. Then by Lemma 4.2, $\tau_1, \ldots, \tau_k$ form a strict Thom system.

Assume τ_i' is a strict Thom system of $H_T^*(M)$. Then by Lemma 4.2 the element τ_i' restricts to 0 on all X_j except for X_i. The action on $M\backslash X_i$ is again equivariantly formal by Lemma 4.10. Thus τ_i' restricts to 0 in $H_T^*(M\backslash X_i)$ and therefore to the kernel of $H_T^*(X_i) \to H_T^*(SX_i)$. The latter is the ideal generated by the equivariant Euler class of SX, thus the claim follows. $\square$

Lemma 4.12. *Let M be an equivariantly formal compact, orientable T-manifold such that the map $H^*(M) \to H^*(X_i)$ is surjective for all components $X_1, \ldots, X_k$ of M^T. Then every isotropy codimension 1 element of the connected orbit type stratification χ of M is ramified.*

Proof. Suppose this is not the case. Then there is an isotropy codimension 1 element of χ which contains only a single X_i. Consequently the one-skeleton of $M\backslash X_i$ is disconnected. However the one-skeleton of a equivariantly formal action on a connected manifold is connected by Proposition 2.5 which contradicts Lemma 4.10. $\square$

For the following lemma, recall our convention that elements in a graded space are assumed to be homogeneous.

Lemma 4.13. *Let X be a path-connected, trivial T-space.*

(i) Fix $x \in H_T^*(X)$ as well as coprime elements $f_1, \ldots, f_k \in R$. If there are $x_1, \ldots, x_k \in H_T^*(X)$ with the properties that $x = \prod_{i=1}^k x_i$ and the $x_i - f_i$ are nilpotent, then $x_1, \ldots, x_k$ are unique.

(ii) Assume X is a smooth manifold and $E \to X$ is an effective T-equivariant vector bundle. Then V splits as $V = V_1 \oplus \ldots \oplus V_l$ for T-equivariant vector bundles $V_i \to X$ with the property that the identity component of every isotropy in V_i is the same codimension 1 subtorus $S_i \subset T$ and $S_i \neq S_j$ for $i \neq j$. Let e, resp. e_i, denote the equivariant Euler classes of V, resp. V_i and let $\alpha_i \in R^2$ be a nontrivial weight associated to S_i (i.e. which vanishes when restricted to $H^2(BS_i)$). Then $e = \prod_{i=1}^k e_i$ and $e_i - a_i\alpha_i^{k_i}$ is nilpotent for some $k_i \in \mathbb{N}$, $a_i \in \mathbb{Q}^\times$.

(iii) In the setting of (ii), suppose we have a decomposition $e = \prod_{i=1}^l x_i$ such that $x_i - b_i\beta_i^{l_i}$ is nilpotent for some $b_i \in \mathbb{Q}^\times$, $l_i \in \mathbb{N}$ and pairwise linearly independent $\beta_i \in R^2$. Then there are $c_i, d_i \in \mathbb{Q}^\times$, such that after possibly changing the order we have $\alpha_i = c_i\beta_i$ and $e_i = d_ix_i$, $i = 1, \ldots, l$.

Proof. Suppose first that we have already shown statement (i) for products of 2 elements. As the action is trivial, $H_T^*(X) \cong R \otimes H^*(X)$ inherits a multiplicative bigrading. By Lemma 3.3, $x_i - f_i$ being nilpotent is equivalent to the fact that x_i restricts to f_i in $H_T^*(*) = R$ for any point, i.e. the $R \otimes H^0(X)$ component of x_i is f_i. In particular $\prod_{i\geq l} x_i - \prod_{i\geq l} f_i$ is nilpotent for any l and (i) follows by inductively applying the result for two factors of the form x_l and $\prod_{i\geq l+1} x_i$.

For the proof with 2 factors, suppose we have coprime elements $f, g \in R$ and $a, b, a', b' \in H_T^*(X)$ such that $ab = a'b'$ and $a - f$, $b - g$, $a' - f$, $b' - g$ are nilpotent. We show inductively that $a \equiv a'$ and $b \equiv b' \mod R \otimes H^{\geq k}(X)$ for all k. Starting at $k = 1$ we note that an element is nilpotent if and only if it is contained in $R \otimes H^+(X)$. Thus the assumptions imply $a \equiv f \cdot 1 \equiv a'$ and $b \equiv g \cdot 1 \equiv b' \mod R \otimes H^+(X)$. Now suppose $a \equiv a'$, $b \equiv b' \mod R \otimes H^{k-1}(X)$ for some $k \geq 2$. For any element $y \in H_T^*(X)$ write y_i for its component in $R \otimes H^i(X)$ and $y_{\leq i} := y_0 + \ldots + y_i$. Then $(a_{\leq k} \cdot b_{\leq k})_k = (ab)_k = x_k = (a'b')_k = (a'_{\leq k} \cdot b'_{\leq k})_k$ and thus by induction

$$a_k \cdot g + f \cdot b_k = a'_k \cdot g + f \cdot b'_k.$$

Now choose a basis h_α of $H^k(X)$. We write $a_k = \sum u_\alpha h_\alpha$, $b_k = \sum v_\alpha h_\alpha$, $a'_k = \sum u'_\alpha h_\alpha$, and $b'_k = \sum v'_\alpha h_\alpha$ for unique $u_\alpha, v_\alpha, u'_\alpha, v'_\alpha \in R$. In particular we obtain the equations $u_\alpha g + f v_\alpha = u'_\alpha g + f v'_\alpha$ and thus $(u_\alpha - u'_\alpha)g = f(v_\alpha - v'_\alpha)$ in R. As f and g are coprime it follows that $f | (u_\alpha - u'_\alpha)$. However note that, as $f \otimes 1$ is the $R \otimes H^0(X)$-part of the homogeneous element a, the total degree satisfies $\deg(u_\alpha - u'_\alpha) = \deg a - k = \deg f - k < \deg f$. Thus $u_\alpha - u'_\alpha = 0$ and also $v_\alpha - v'_\alpha = 0$ which finishes the proof of (i).

For any $U \subset T$, the fixed point set V^U is a subbundle of V. This yields the decomposition $V = V_1 \oplus \ldots \oplus V_l$ as described in the lemma (see also [4, Lemma 1.6.7]). The product decomposition of e follows from the sum decomposition of V. Restricting V_i to a point gives a representation which decomposes into 2-dimensional representations associated to a weight which is some nonzero rational multiple of α_i. Hence the restriction of e_i to $H_T^*(*) \cong R$ is equal to $c_i \alpha^{k_i}$ for some $c_i \in \mathbb{Q}^\times$, $k_i \in \mathbb{N}$. Then the rest of (ii) follows by Lemma 3.3.

To prove (iii) note that for a product decomposition of e as in the lemma, the $R \otimes H^0(X)$ component of e is equal to $\prod b_i \beta_i^{l_i}$. Thus $\alpha_i^{k_i} = \beta_i^{l_i}$ up to scalars and order. Since the $\alpha_i^{k_i}$ are coprime, the claim now follows from (i). ☐

Proof of Theorem 4.6. By Corollary 3.17 and Lemma 4.12, χ is encoded in $H_T^*(M)$, so it remains to prove that equivariant cohomology is encoded as well. As in Lemma 4.11, we algebraically detect the actual equivariant Thom classes $\tau_1, \ldots, \tau_k$ (up to $\mathbb{Q}^\times$)) of the fixed point components $X_1, \ldots, X_k$ as a minimal strict Thom system. Since the restriction $H_T^*(M) \to H^*(M)$ is surjective by Lemma 2.2, the surjectivity assumption implies that $H_T^*(M) \to H_T^*(X_i)$ is surjective for all i. The kernel of this map is $K_i = \{x \in H_T^*(M) \mid x\tau_i = 0\}$. Thus the cohomology of all X_i is encoded as $H_T^*(X_i) \cong H_T^*(M)/K_i$. Now let $N \in \chi$ be a submanifold which without loss of generality contains exactly the components $X_1, \ldots, X_l$ and whose principal isotropy has a subtorus $S \subset T$ as identity component. We will compute $H_T^*(N)$ from this data.

Consider a small equivariant tube $DN \subset M$ around N with equivariant sphere bundle $SN \subset DN$. We have a Mayer-Vietoris sequence

$$0 \to H_T^*(X) \to H_T^*(M - N) \oplus H_T^*(N) \to H_T^*(SN) \to 0.$$

Let $J_N \subset H_T^*(N)$ denote the ideal generated by the equivariant Euler class e_N of the normal bundle of $N \subset M$. By Lemma 4.8, J_N is the kernel of $H_T^*(N) \to H_T^*(SN)$. Consequently, for $y \in J$, elements of the form $(0, y)$ in the middle term of the Mayer-Vietoris sequence lie in the image of the restriction map on $H_T^*(X)$. This proves that J_N is contained in the image of the restriction $r_N \colon H_T^*(M) \to H_T^*(N)$.

We now try to characterize $r_N^{-1}(J_N)$ algebraically. Let $r_i \colon H_T^*(M) \to H_T^*(M)/K_i \cong H_T^*(X_i)$, $i = 1, \ldots, l$, denote the restriction map. Recall that the class τ_i restricts in $H_T^*(X_i)$ to the equivariant euler class of the normal bundle of X_i in M, up to scalar. Then it follows from part (ii) of Lemma 4.13 that we find a product decomposition

$$r_i(\tau_i) = \prod_{j=1}^{n_j} x_{ij}$$

in which x_{ij} has the property that for some $c_{ij} \in \mathbb{Q}^\times$ and $l_{ij} \in \mathbb{N}$ the element $x_{ij} - c_i\alpha_{ij}^{l_{ij}}$ is nilpotent where $\alpha_{ij} \in R^2$ are the distinct weights associated to those codimension 1 isotropy groups which occur around X_i. Then by part (iii) of Lemma 4.13, the x_{ij} are up to scalars the equivariant Euler classes of certain subbundles V_{ij} of the normal bundle V_i of $X_i \subset M$ as in Lemma 4.13 part (ii). The restriction of the normal bundle of $N \subset M$ to X_i is equal to the sum over those V_i whose unique connected maximal isotropy group does not contain the principal isotropy S of N. Let x_i be the product over those x_{ij} for which α_{ij} does not vanish on S. Then $x_i \in H_T^*(X_i)$ is – up to scalar – the restriction of e_N. After possibly renormalizing the individual x_i we find an element $\tau_N \in H_T^*(M)$ for which $r_i(\tau_N) = x_i$, for all $i = 1, \ldots, l$. By Lemma 4.9, $r_N(\tau_N)$ agrees with e_N up to scalar and we have

$$r_N^{-1}(J_N) = \{x \in H_T^*(M) \mid r_i(\tau_N) | r_i(x) \text{ for } i = 1, \ldots, l\} =: I_N.$$

This description encodes $r_N^{-1}(J_N)$ algebraically, as it is independent of the particular choice of τ_N. As the ideal I_N restricts onto $J_N = e_N \cdot H_T^*(N)$, the map

$$I_N \to \bigoplus_{i=1}^{l} H_T^*(X_i) \cong H_T^*(N^T), \quad x \mapsto \left(\frac{r_1(x)}{r_1(\tau_N)}, \ldots, \frac{r_l(x)}{r_l(\tau_N)}\right)$$

is well defined and its image is that of the injective map $H_T^*(N) \to H_T^*(N^T)$. Thus we have constructed $H_T^*(N)$ out of $H_T^*(M)$. If furthermore $N' \subset N$ is another isotropy manifold containing without loss of generality the fixed point components $X_1, \ldots, X_{l'}$, $l' \leq l$, then there is a commutative diagram

$$\begin{array}{ccc} H_T^*(N) & \longrightarrow & \prod_{i=1}^{l} H_T^*(X_i) \\ \downarrow & & \downarrow \\ H_T^*(N') & \longrightarrow & \prod_{i=1}^{l'} H_T^*(X_i) \end{array}$$

where the horizontal maps are injective and the right hand map is projection onto the first l' factors. We have just argued that $H_T^*(M)$ encodes the image of the horizontal maps, hence it also encodes the left hand map. □

5 Remarks on the integral case

This section has the purpose of commenting on the question what additional information on the orbit type stratification can be deduced from the integral equivariant cohomology. Generalizing from our results in the rational case it seems natural to ask:

(i) What does $H_T^*(X; \mathbb{Z})$ know about the full orbit type stratification also considering disconnected isotropies?

(ii) Does $H_T^*(X; \mathbb{Z})$ encode the equivariant integral cohomologies of the strata in an equivariantly formal setting analogous to Theorem 4.6?

There are some subtleties regarding the right requirements and the notion of equivariant formality in the integral setting. E.g. unlike in the rational case, a module of the form $H^*(BT;\mathbb{Z})\otimes H^*(X;\mathbb{Z})$ is not necessarily free over $H^*(BT;\mathbb{Z})$ and in particular freeness is not necessarily implied by the degeneracy of the Serre spectral sequence of the Borel fibration. This problem however does not arise in case $H^*(X;\mathbb{Z})$ is free over $\mathbb{Z}$. Let us begin by pointing out the limitations of possible generalizations even under the assumption of torsion freeness.

Example 5.1. Consider T^2-actions on $S^4\subset\mathbb{C}^2\oplus\mathbb{R}$ given by $(s,t)\cdot(v,w,h)=(s^at^bv,s^ct^dw,h)$, for $a,b,c,d\in\mathbb{Z}$. Then $R=\mathbb{Z}[X,Y]$ and $H_T^*(S^4)$ is generated over R by $1\in H_T^0(S^4)$, $\alpha\in H_T^4(S^4)$ with a single relation $\alpha^2=(aX+bY)(cX+dY)\alpha$. To see this, note that the action above is a pullback of the standard T^2-action on S^4, i.e. with $(a,b,c,d)=(1,0,0,1)$, which has the relation $\alpha^2=XY\alpha$ (this follows e.g. from the integral GKM description). There is a map induced by the pullback between the two equivariant cohomologies, which maps $H(BT;\mathbb{Z})$ bases to one another and transforms $H(BT;\mathbb{Z})$ via $X\mapsto aX+bY$ and $Y\mapsto cX+dY$. Hence the relation for α^2 transforms accordingly as claimed above.

(i) Setting $(a,b,c,d)=(2,0,3,0)$ we obtain 5 different elements in the orbit type stratification: two fixed points, 2 two-spheres with isotropies $\mathbb{Z}_2\times S^1$ and $\mathbb{Z}_3\times S^1$ as well as the principal orbit type $\{1\}\times S^1$. For $(6,0,1,0)$ there are only 4 as only one two-sphere occurs with isotropy $\mathbb{Z}_6\times S^1$. Thus we see that the combinatorics of the orbit type stratification are not encoded even though all extensions are ramified. The connected orbit type stratification is of course encoded by the previous rational results.

(ii) Consider the actions with (a,b,c,d) equal to $(2,0,0,3)$ and $(6,0,0,1)$. Then, while the connected orbit type stratification is encoded in $H_T^*(S^4)$, the corresponding equivariant cohomology algebras are not: in both cases $(S^4)^{S^1\times\{1\}}$ is S^2 however the rotation speeds of the respective T-actions are different which yields nonisomorphic equivariant cohomology algebras. Thus without further restrictions on the combinatorics of the stratification no integral result analogous to Theorem 4.6 can be expected to hold.

The reason for the failure of these methods lies in the fact that for general subgroups of $S\subset T$ the Borel localization theorem does no longer establish a bijection between Thom systems of $H_S^*(X)$ and components of X^S. This is due to fact that the ring $H^*(BS;\mathbb{Z})$ might contain elements of positive degree which multiply to 0 and thus multiplicatively closed subsets available for localization are somewhat limited.

Example 5.2. Consider the above example for $(a,b,c,d)=(2,0,0,3)$ and the subgroup $S=S^1\times\mathbb{Z}_2$. One has $H^*(BS)=\mathbb{Z}[X,Y]/(2Y)$ and $H_S^*(S^4)=H^*(BS)\otimes_R H_T^*(S^4)\cong\mathbb{Z}[X,Y,\alpha]/(2Y,\alpha^2)$. The fixed point set $(S^4)^S$ consists of two discrete points. However there is no element in $H_S^*(S^4)$ which restricts to a nontrivial element on a single fixed point while vanishing on the other: α has to restrict to 0 on the fixed points due to being nilpotent while elements in the image of $H^*(BS)\to H_S^*(S^4)$ restrict to the same element on both fixed points. Thus the technique of using Thom systems to detect fixed point components does not apply for the subgroup S.

Despite these counterexamples, the integral cohomology $H_T^*(X;\mathbb{Z})$ does of course know more about the orbit type stratification than $H_T^*(X;\mathbb{Q})$. The correspondence between certain Thom systems and fixed point components of S does carry over in case $H^+(BS;\mathbb{Z})-\{0\}$ is multiplicatively closed, enabling Borel localization. The groups S for which this is the case are precisely tori and p-tori, i.e. subgroups of the form $(\mathbb{Z}_p)^r$, where p is prime. One way to go would therefore be to develop results analogous to those in this paper for p-tori (in fact, the references [13] and [2, Section 3.6] mentioned in the introduction and in Remark 3.1 also deal with p-torus actions), and deduce refined results on the orbit type stratification from $H_T^*(X;\mathbb{Z})$ via this route.

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