

*Pacific
Journal of
Mathematics*

**CONNECTED COMPONENTS OF MORSE BOUNDARIES
OF GRAPHS OF GROUPS**

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Volume 317 No. 2

April 2022

CONNECTED COMPONENTS OF MORSE BOUNDARIES OF GRAPHS OF GROUPS

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Let a finitely generated group G split as a graph of groups. If the edge groups are undistorted and do not contribute to the Morse boundary $\partial_M G$, we show that every connected component of $\partial_M G$ with at least two points originates from the Morse boundary of a vertex group.

Under stronger assumptions on the edge groups (such as wideness in the sense of Druţu–Sapir), we show that the Morse boundaries of the vertex groups are topologically embedded in $\partial_M G$.

1. Introduction

Morse boundaries $\partial_M G$ of finitely generated groups G were introduced by Charney and Sultan [2015] and Cordes [2017] in an attempt to extend to all groups some of the fundamental properties of Gromov boundaries of hyperbolic groups [Gromov 1987]. Importantly, every quasi-isometry between finitely generated groups extends to a homeomorphism of their Morse boundaries, which can provide a useful tool to distinguish quasi-isometry classes of groups.

When G is not Gromov-hyperbolic, the topology of $\partial_M G$ is rather unwieldy (for instance, it is not first countable or compact), which often makes an explicit computation difficult. Despite this, Charney, Cordes and Sisto [2019] recently showed that essentially all known examples of infinite, *totally disconnected* Morse boundaries fall into just two homeomorphism classes: the Cantor space and the ω -Cantor space.

While groups whose Morse boundary is a Cantor space are fully classified (they are hyperbolic, hence virtually free), it appears that the class of groups with ω -Cantor boundary is rather large. For instance, it includes all irreducible, nonfree right-angled Artin groups, as well as all nongeometric graph manifold groups [Charney et al. 2019].

Our first goal is to expand the class of finitely generated groups known to have totally disconnected Morse boundary. A natural source of examples is provided by graphs of groups. We restrict to the situation where no Morse ray in G stays

MSC2020: 20E08, 20F65, 20F67.

Keywords: Morse boundary, graph of groups, totally disconnected, relatively wide.

at bounded distance from an edge group, and show that all nontrivial connected components of $\partial_M G$ originate from vertex groups (Theorem A).

As motivation for our assumptions, recall that surface groups split as graphs of groups with cyclic edge groups and nonabelian free vertex groups. Thus, when edge groups are allowed to contain Morse rays in G , the Morse boundary $\partial_M G$ can be connected (here, a circle) even if all Morse boundaries of vertex groups are totally disconnected (here, Cantor sets).

More generally, we investigate necessary and sufficient conditions for Morse boundaries of vertex groups to be topologically embedded in $\partial_M G$, even when they are not totally disconnected. Our main result in this direction is Theorem C, which applies, for instance, to all graphs of groups with (undistorted, one-ended) solvable edge groups. This is new already for JSJ decompositions of irreducible 3-manifold groups.

As a set, $\partial_M G$ is defined as the collection of all Morse geodesic rays in a fixed Cayley graph of G , identifying rays at finite Hausdorff distance. Our topology of choice will always be the *direct limit* topology from [Charney and Sultan 2015; Cordes 2017].

We mention that an alternative topology on $\partial_M G$ was introduced by Cashen and Mackay [2019]. The latter has the advantage of being metrisable, while retaining quasi-isometric invariance. However, it appears to be more complicated to describe explicitly, and we are not aware of a single nonempty Morse boundary of a nonhyperbolic group for which the Cashen–Mackay topology can be given “intrinsic” characterisations in the spirit of [Charney et al. 2019].

The following relative version of Morse boundaries plays an important role in our results.

Definition (relative Morse boundary). Let $H \leq G$ be finitely generated groups, with H undistorted in G . The *relative Morse boundary* of H in G , denoted by $(\partial_M H, G)$, is the subset of $\partial_M H$ consisting of points represented by rays that remain Morse in the Cayley graphs of G .

We always endow $(\partial_M H, G)$ with the subspace topology coming from $\partial_M H$. *This is the opposite of the convention adopted in [Karrer 2021].*

Note that we always have a natural injection $(\partial_M H, G) \hookrightarrow \partial_M G$. Our first result greatly extends the main theorem of [Karrer 2021], while providing a significantly simpler proof.

Theorem A. *Let a finitely generated group G split as a graph of groups. Suppose that:*

- *All edge groups are finitely generated and undistorted in G .*
- *$(\partial_M E, G) = \emptyset$ for every edge group E .*

If a connected component $C \subseteq \partial_M G$ is not a singleton, then C is contained in the image of the natural injection $(\partial_M V, G) \hookrightarrow \partial_M G$ for a vertex group $V \leq G$. Furthermore, if V_1, V_2 are distinct vertex groups, then $(\partial_M V_1, G) \cap (\partial_M V_2, G) = \emptyset$.

The following consequence of Theorem A has some overlap with [Charney et al. 2019, Theorem 1.2]. The advantage is that here we do not require acylindricity of the splitting, nor that the vertex groups have trivial Morse boundary.

Corollary B. *Under the assumptions of Theorem A, suppose additionally that $(\partial_M V, G)$ is totally disconnected for every vertex group $V \leq G$. Then $\partial_M G$ is totally disconnected.*

We emphasise that the empty set is totally disconnected. Thus, in Corollary B, the relative Morse boundaries $(\partial_M V, G)$ are allowed to be empty and we make no claim that $\partial_M G$ will be nonempty.

Corollary B follows from Theorem A because, for every undistorted subgroup $H \leq G$, the natural inclusion $(\partial_M H, G) \hookrightarrow \partial_M G$ is an *open* map (see Lemma 2.3). However, even in the setting of Theorem A, the inclusions $(\partial_M V, G) \hookrightarrow \partial_M G$ need not be *continuous*, as demonstrated by the following example.

Example. Consider the group $G = \mathbb{Z}^2 * \mathbb{Z} = \langle x, y \rangle * \langle z \rangle$. It admits the splitting $G = \langle x, y \rangle *_{\langle y \rangle} \langle y, z \rangle$. Since $\langle y \rangle$ lies in the flat $\langle x, y \rangle$, it is undistorted and has trivial relative Morse boundary in G .

Consider the vertex group $V := \langle y, z \rangle \simeq F_2$, which is also undistorted. The inclusion

$$(\partial_M V, G) \hookrightarrow \partial_M G$$

is not continuous. In order to see this, consider the rays labelled by $z^n y^n z^\infty$ and z^∞ , which all lie in $(\partial_M V, G)$. Since V is hyperbolic, we have $z^n y^n z^\infty \rightarrow z^\infty$ in the topology of $\partial_M V$.

However, since the rays $z^n y^n z^\infty$ spend longer and longer in the flat $\langle x, y \rangle$, they are not uniformly Morse in G . It follows that they form a closed subset of $\partial_M G$ (every stratum of $\partial_M G$ contains only finitely many of them). Hence $z^n y^n z^\infty \not\rightarrow z^\infty$ in the topology of $\partial_M G$. This last argument is taken from [Murray 2019, Section 5].

Of course, it would be desirable to have conditions ensuring that the injections $(\partial_M V, G) \hookrightarrow \partial_M G$ in Theorem A are topological embeddings, as this would be an important step towards fully characterising $\partial_M G$ in terms of boundaries of vertex groups.

In relation to this, note that a key feature of the above example is that, although $(\partial_M \langle y \rangle, G) = \emptyset$, we have $(\partial_M \langle y \rangle, V) \neq \emptyset$. This leads us to suspect that this kind of issue should not present itself if all the edge groups have trivial relative Morse boundary in the incident *vertex* groups.

We prove this guess under the following, potentially stronger assumption.

Definition (relatively wide). Let $H \leq G$ be finitely generated groups, with H undistorted in G .

- The group G is *wide* if none of its asymptotic cones G_ω have cut points [Druţu and Sapir 2005].
- We say that H is *relatively wide* in G if, for every asymptotic cone G_ω , no two points of the limit $H_\omega \subseteq G_\omega$ are separated by a cut point of G_ω .

Note that H is relatively wide in G as soon as either H or G is wide.

Examples of wide groups include one-ended groups satisfying a law (e.g., solvable, uniformly amenable, Burnside, etc.) [Druţu and Sapir 2005], one-ended groups with infinite centre, and various higher-rank lattices [Druţu et al. 2010].

Wide groups have empty Morse boundary. It is a well-known open question whether the converse holds. We record here the relative version of this question, as it might be easier to find counterexamples in the relative case.

Question. Let $H \leq G$ be finitely generated groups, with H undistorted in G and $(\partial_M H, G) = \emptyset$. Is H relatively wide in G ?

Conversely, it is easy to see that $(\partial_M H, G) = \emptyset$ holds as soon as H is relatively wide in G . We can now state our second main result.

Theorem C. *Let a finitely generated group G split as a graph of groups. Consider a vertex group $V \leq G$. Suppose that all incident edge groups $E \leq V$ are finitely generated, undistorted in G , and relatively wide in V . Then:*

- (1) V is undistorted in G and $(\partial_M V, G) = \partial_M V$.
- (2) The inclusion $\partial_M V \hookrightarrow \partial_M G$ is a topological embedding.

Remark. We emphasise that *relative wideness* of E in V can be rephrased purely in terms of divergence: it is equivalent to the statement that geodesics in (a Cayley graph of) E , viewed as uniform quasigeodesics in V , have *uniform linear divergence* (see Lemma 2.11(3)).

Corollary D. *If every vertex group satisfies the assumptions of Theorem C, then every connected component of $\partial_M G$ is either a singleton or homeomorphic to a connected component of the Morse boundary of a vertex group.*

Corollary B gives many examples of groups with totally disconnected Morse boundary, but it does not further describe the topological spaces that may arise as boundaries. The remarkable [Charney et al. 2019, Theorem 1.4] shows instead that, for any finitely generated group G , the boundary $\partial_M G$ is an ω -Cantor space as soon as it is totally disconnected, noncompact, σ -compact, and contains a Cantor subspace. The last property is generally not hard to obtain: for instance, using acylindrical hyperbolicity of G (if given) to construct a stable free subgroup of G

[Sisto 2016; Dahmani et al. 2017], or by applying Theorem C to any Cantor subspace that the boundaries of the vertex groups may have.

This suggests studying the following problem.

Question. In the setting of Corollary D, suppose that all vertex groups have σ -compact Morse boundary. Is $\partial_M G$ then σ -compact?

2. Preliminaries

To economise on constants, we will speak of C -quasigeodesics when referring to (C, C) -quasigeodesics. In the whole section, X and Y are proper geodesic metric spaces.

We denote closed metric balls by $\mathcal{B}(x, r)$ and closed metric neighbourhoods of subsets by $\mathcal{N}(A, r)$. Where necessary, we may add a subscript $\mathcal{B}_X(x, r)$ or $\mathcal{N}_Y(A, r)$ to specify the relevant space.

2A. Morse boundaries. We refer the reader to [Cordes 2017] for further details.

A quasigeodesic $\gamma \subseteq X$ is N -Morse for a function $N : [1, +\infty) \rightarrow [0, +\infty)$ if every C -quasigeodesic with endpoints on γ is contained in the $N(C)$ -neighbourhood of γ . The function N is usually referred to as a *Morse gauge* for γ .

Fix a basepoint $p \in X$. We define $\partial_M^N X_p$ as the set of N -Morse geodesic rays based at p , identifying rays at finite Hausdorff distance. Endowed with the compact-open topology, this space is compact and metrisable. Let $\partial_M X_p$ be the union of all spaces $\partial_M^N X_p$, as N varies among all possible Morse gauges. We define a topology on $\partial_M X_p$ as follows: a subset $U \subseteq \partial_M X_p$ is open (resp. closed) if and only if all intersections $U \cap \partial_M^N X_p$ are open (resp. closed).

If $q \in X$ is a different basepoint, we have a natural homeomorphism $\partial_M X_p \rightarrow \partial_M X_q$ given by pairing rays at finite Hausdorff distance. Thus, the space $\partial_M X_p$ is independent of the choice of p and we simply denote it by $\partial_M X$. We refer to $\partial_M X$ as the *Morse boundary* of X .

We record here the following standard properties of Morse quasigeodesics for later use.

Lemma 2.1. (1) *Let $\alpha \subseteq X$ be an N -Morse geodesic. Let $\beta \subseteq X$ be a C -quasigeodesic with endpoints at distance $\leq C$ from those of α . Then $d_{\text{Haus}}(\alpha, \beta) \leq D$, where D only depends on C and N .*

(2) *Let $\alpha \subseteq X$ be an N -Morse geodesic ray. Let $\beta \subseteq X$ be C -quasigeodesic ray with the same starting point and $d_{\text{Haus}}(\alpha, \beta) < +\infty$. Then $d_{\text{Haus}}(\alpha, \beta) \leq D$ and β is N' -Morse, where D and N' only depend on C and N .*

(3) *Let $\alpha \subseteq X$ be an N -Morse C -quasigeodesic. Then there exists an N' -Morse geodesic $\beta \subseteq X$ with the same starting point and $d_{\text{Haus}}(\alpha, \beta) \leq D$, where D and N' only depend on C and N .*

(4) *The restriction of an N -Morse C -quasigeodesic to a subinterval of its domain is always N' -Morse, where N' only depends on N and C .*

Proof. Part (1) is easily deduced from [Cordes 2017, Lemma 2.1]. Part (2) is [Cordes 2017, Corollary 2.5]. Part (3) can be proved using [Charney and Sultan 2015, Lemma 2.5] as in the proof of [Cordes 2017, Lemma 2.9]. Part (4) is [Liu 2021, Lemma 3.1]. □

Let $f : Y \rightarrow X$ be a quasi-isometric embedding. If γ is a Morse geodesic in Y , the quasigeodesic $f \circ \gamma$ might still not be Morse in X . This motivates the following notion of *relative Morse boundary*, which is equivalent to the one from the Introduction.

Definition 2.2. Set $(\partial_M Y, f) := \{[\gamma] \in \partial_M Y \mid f \circ \gamma \text{ is Morse in } X\}$. We endow $(\partial_M Y, f)$ with the subspace topology coming from $\partial_M Y$.

If $[\gamma] \in (\partial_M Y, f)$, then the quasigeodesic ray $f \circ \gamma$ is Morse in X ; hence at finite Hausdorff distance from a Morse geodesic ray by Lemma 2.1(3). This defines an injection

$$f_* : (\partial_M Y, f) \hookrightarrow \partial_M X.$$

The example in the Introduction shows that f_* is not continuous in general. However, it is always a closed map (equivalently, an open map, since f_* is injective).

Lemma 2.3. *If $f : Y \rightarrow X$ is a quasi-isometric embedding, $f_* : (\partial_M Y, f) \hookrightarrow \partial_M X$ is a closed map.*

Proof. Fix a basepoint $q \in Y$ and set $p := f(q)$. Let $A \subseteq (\partial_M Y, f)$ be a closed subset. We need to show that the intersection $f_*(A) \cap \partial_M^N X_p$ is closed for every Morse gauge N . In fact, since $\partial_M^N X_p$ is metrisable, it suffices to show that $f_*(A) \cap \partial_M^N X_p$ is sequentially closed.

Let $\alpha_n \subseteq Y$ be geodesic rays based at q , so that $[\alpha_n] \in A$ and $f_*[\alpha_n] \in \partial_M^N X_p$ for some Morse gauge N . Suppose that $f_*[\alpha_n] \rightarrow \xi$ in $\partial_M X$. We need to show that $\xi \in f_*(A)$.

Let $\beta_n \subseteq X$ be N -Morse geodesic rays based at p representing $f_*[\alpha_n]$. By Lemma 2.1(2), the quasigeodesic rays $f \circ \alpha_n$ are at uniformly finite Hausdorff distance from β_n , say $\leq D$, and they are uniformly Morse in X . Since f is a quasi-isometric embedding, it follows that the α_n are uniformly Morse in Y , say N' -Morse.

Now by the Arzelà–Ascoli theorem, we can pass to a subsequence and assume that the β_n converge uniformly on compact sets to an N -Morse geodesic ray $\beta \subseteq X$ based at p . Similarly, the α_n converge to an N' -Morse ray $\alpha \subseteq Y$ based at q . In particular, $[\alpha_n] \rightarrow [\alpha]$ in the topology of $\partial_M Y$; hence $[\alpha] \in A$. Since $d_{\text{Haus}}(f \circ \alpha_n, \beta_n) \leq D$ for every n , and f is a quasi-isometric embedding, we have $d_{\text{Haus}}(f \circ \alpha, \beta) < +\infty$. Hence $\xi = [\beta] = f_*[\alpha] \in f_*(A)$, as required. □

2B. Divergence. It was shown in [Arzhantseva et al. 2017] that Morse quasi-geodesics can equivalently be characterised as quasigeodesics with completely superlinear divergence. The proof of Theorem C will require this equivalence to be *effective*, in the sense that the Morse gauge and the divergence function of the quasigeodesic should determine each other.

This kind of statement is proved in detail in Cashen’s habilitation thesis [2019, Survey: Corollary 3.5], but we also explain here how to deduce it from the proofs of various results in [Arzhantseva et al. 2017].

We begin with the definition of divergence.

Definition 2.4. Consider an L -quasigeodesic ray $\gamma \subseteq X$ and a parameter $0 < \epsilon < 1/(2L)$. The *divergence function* of γ is

$$\delta_\gamma(r, \epsilon) := \inf_{s \geq r} \inf\{\text{lengths of paths connecting } \gamma(s \pm r) \text{ in } X \setminus \mathcal{B}(\gamma(s), \epsilon r)\} \in [0, +\infty].$$

The proof of [Arzhantseva et al. 2017, Proposition 5.10] shows the following.

Lemma 2.5. *Let $\gamma \subseteq X$ be an L -quasigeodesic ray. Suppose that $\delta_\gamma(r, \epsilon) \leq Cr$ for some r, C and $\epsilon < 1/(4L)$. Then there exists $s \geq r$ such that $\gamma(s \pm r)$ are joined by an L' -quasigeodesic avoiding $\mathcal{B}(\gamma(s), \epsilon' r)$, where the constants L' and ϵ' depend only on L, C, ϵ (and not on r).*

Corollary 2.6. *Given constants $L, \epsilon < 1/(4L)$, and a Morse gauge N , there exists a weakly increasing, diverging function f such that the following holds. For every N -Morse L -quasigeodesic ray $\gamma \subseteq X$, we have $\delta_\gamma(r, \epsilon) \geq rf(r)$ for all $r \geq 0$.*

Proof. Fix $\bar{r} \geq 0$. Define $g(\bar{r})$ as the infimum of the ratio $\delta_\gamma(\bar{r}, \epsilon)/\bar{r}$ as γ varies among all N -Morse L -quasigeodesic rays in X . Let $\bar{\gamma}$ be one such quasigeodesic satisfying $\delta_{\bar{\gamma}}(\bar{r}, \epsilon)/\bar{r} \leq 2g(\bar{r})$.

Lemma 2.5 gives $s \geq \bar{r}$ and an L' -quasigeodesic joining $\bar{\gamma}(s \pm \bar{r})$ avoiding the ball $\mathcal{B}(\bar{\gamma}(s), \epsilon' \bar{r})$, where ϵ' and L' depend only on L, ϵ and the value $g(\bar{r})$. Since $\bar{\gamma}$ is N -Morse, we must have $\epsilon' \bar{r} \leq N(L')$. This implies that $g(r)$ diverges as $r \rightarrow +\infty$.

Now, let f be the largest weakly increasing function with $f \leq g$, namely

$$f(x) = \inf_{t \geq x} g(t).$$

Since g diverges, so does f . Finally, if $\gamma \subseteq X$ is an N -Morse L -quasigeodesic ray, it is clear that we have $\delta_\gamma(r, \epsilon) \geq rg(r) \geq rf(r)$ for all $r \geq 0$. □

The above corollary shows that the divergence function of a Morse quasigeodesic ray can be bounded uniformly (from below) in terms of the Morse gauge. In order to reverse this kind of result, we need to speak of *contracting* geodesics.

Definition 2.7. Let $\gamma \subseteq X$ be a quasigeodesic with closed image.

(1) The *nearest-point projection* $\pi_\gamma : X \rightarrow 2^\gamma$ is defined by

$$\pi_\gamma(x) = \{p \in \gamma \mid d(x, p) = d(x, \gamma)\}.$$

Since X is proper and γ is closed, the subset $\pi_\gamma(x) \subseteq \gamma$ is always nonempty.

(2) If ρ is a sublinear, weakly increasing, nonnegative function, we say that γ is ρ -*contracting* if, for all $x, y \in X$ with $d(x, y) \leq d(x, \gamma)$, we have

$$\text{diam}(\pi_\gamma(x) \cup \pi_\gamma(y)) \leq \rho(d(x, \gamma)).$$

From the proof of [Arzhantseva et al. 2017, Proposition 5.5], we obtain:

Lemma 2.8. *Let $\gamma \subseteq X$ be an L -quasigeodesic ray with closed image. Consider points $x, y \in X$ with $d(x, y) \leq d(x, \gamma)$ and projections $x' \in \pi_\gamma(x), y' \in \pi_\gamma(y)$. Then, if $d(x', y') \geq 5L^3$, we have*

$$4d(x, \gamma) \geq \delta_\gamma\left(r, \frac{1}{16L}\right),$$

for some $r \in [1/(4L) \cdot d(x', y'), L \cdot d(x', y')]$.

Finally, the following is [Arzhantseva et al. 2017, Proposition 4.1].

Proposition 2.9. *Let $\gamma \subseteq X$ be a quasigeodesic with closed image. If γ is ρ -contracting, then it is N -Morse with N depending only on ρ .*

We will use the combination of the previous results in the following form.

Lemma 2.10. *Let $\alpha \subseteq X$ and $\beta \subseteq Y$ be L -quasigeodesic rays. Suppose that there exist constants $K \geq 0$ and $0 < \epsilon_1, \epsilon_2 \leq 1/(16L)$ such that, for all $r > K$, we have*

$$\delta_\alpha(r, \epsilon_1) \leq K \cdot \delta_\beta(r, \epsilon_2).$$

If α is N -Morse in X , then β is N' -Morse in Y , with N' depending only on N, L, K, ϵ_1 .

Proof. We will show that β is ρ -contracting, with ρ depending only on N, L, K, ϵ_1 . The fact that β is N' -Morse then follows from Proposition 2.9.

By Corollary 2.6, there exists a weakly increasing, diverging function f such that $\delta_\alpha(r, \epsilon_1) \geq rf(r)$ for all $r \geq 0$. Recall that f depends only on L, ϵ_1, N . Setting $\eta(r) := rf(r)/K$, we have

$$\delta_\beta\left(r, \frac{1}{16L}\right) \geq \delta_\beta(r, \epsilon_2) \geq \frac{1}{K} \cdot \delta_\alpha(r, \epsilon_1) \geq \eta(r),$$

for all $r > K$. Note that η is strictly increasing and $\eta(r)/r \rightarrow +\infty$. Define

$$\rho'(x) := \sup\{r \mid \eta(r) \leq x\} \quad \text{and} \quad \rho(x) := \sup_{t \leq x} \rho'(t).$$

Observe that $\rho'(x)/x \rightarrow 0$ for $x \rightarrow +\infty$. In addition, ρ is weakly increasing and $\rho \geq \rho'$. Observing that ρ' is bounded on bounded sets, the fact that ρ' is sublinear implies that ρ is sublinear as well.

Now, consider points $x, y \in Y$ with $d(x, y) \leq d(x, \beta)$ and projections $x' \in \pi_\beta(x)$, $y' \in \pi_\beta(y)$. Suppose that $d(x', y') > \max\{5L^3, 4LK\}$. Recalling that η is monotone, Lemma 2.8 shows that

$$4d(x, \beta) \geq \eta\left(\frac{1}{4L} \cdot d(x', y')\right).$$

It follows that

$$d(x', y') \leq 4L \cdot \rho'(4d(x, \beta)) \leq 4L \cdot \rho(4d(x, \beta)).$$

This shows that β is ρ -contracting, where ρ depends only on η , and hence only on L, ϵ_1, N, K . □

2C. Relatively wide subgroups. Relatively wide subgroups were defined in the Introduction. This property will be required in the proof of Theorem C in the form of part (3) of the next lemma.

Lemma 2.11. *Let $H \leq G$ be finitely generated groups, with H undistorted in G . Let Λ and Γ be Cayley graphs for H and G , respectively. Let $i : \Lambda \rightarrow \Gamma$ be a quasi-isometric embedding corresponding to the inclusion $H \hookrightarrow G$. Then the following properties are equivalent:*

- (1) *The subgroup H is relatively wide in G .*
- (2) *For all $C \geq 1$, there exists $K = K(C) \geq 1$ such that the following holds. Let $\alpha : [-r, r] \rightarrow \Gamma$ be a C -quasigeodesic with $\alpha(\pm r) \in H \subseteq \Gamma^{(0)}$. If $r > K$, then $\alpha(\pm r)$ are joined by an edge path $\gamma \subseteq \Gamma$ that is disjoint from the ball $B_\Gamma(\alpha(0), r/K)$ and has length $|\gamma| \leq K \cdot r$.*
- (3) *There exists $K_0 \geq 1$ such that the following holds. Let $\beta : [-r, r] \rightarrow \Lambda$ be a geodesic. If $r > K_0$, then $i \circ \beta(\pm r) \in \Gamma^{(0)}$ are joined by an edge path $\gamma \subseteq \Gamma$ that is disjoint from the ball $B_\Gamma(i \circ \beta(0), r/K_0)$ and has length $|\gamma| \leq K_0 \cdot r$.*

Proof. Since H is undistorted, the implication (2) \Rightarrow (3) is clear. We show that (3) \Rightarrow (1) and (1) \Rightarrow (2).

(1) \Rightarrow (2) Suppose for the sake of contradiction that (2) fails for some constant C . Then there exists a sequence of C -quasigeodesics $\alpha_n : [-r_n, r_n] \rightarrow \Gamma$, with $r_n \rightarrow +\infty$, such that every path joining $\alpha_n(\pm r_n) \in H$ in $\Gamma \setminus B_\Gamma(\alpha_n(0), r_n/n)$ has length $> nr_n$.

Fix a nonprincipal ultrafilter ω . Let Γ_ω be the asymptotic cone of Γ given by basepoints $\alpha_n(0)$ and scaling factors $1/r_n$. Let $x_-, x_0, x_+ \in \Gamma_\omega$ be the points determined by the sequences $\alpha_n(-r_n), \alpha_n(0), \alpha_n(r_n)$, respectively. The α_n converge to a C -bi-Lipschitz curve $\alpha : [-1, 1] \rightarrow \Gamma_\omega$ with $\alpha(-1) = x_-, \alpha(0) = x_0$ and $\alpha(1) = x_+$.

Since H is relatively wide in G and x_{\pm} are limits of sequences in $H \subseteq \Gamma$, the points x_{\pm} lie in the same connected component of $\Gamma_{\omega} \setminus \{x_0\}$. Since Γ_{ω} is a geodesic space, it is locally path connected; hence there exists a continuous path in $\Gamma_{\omega} \setminus \{x_0\}$ joining x_{\pm} . Being compact, this path misses the 4ϵ -ball around x_0 for some $\epsilon > 0$. Thus, we can discretise it to a sequence $x_1, \dots, x_k \in \Gamma_{\omega} \setminus \mathcal{B}_{\Gamma_{\omega}}(x_0, 4\epsilon)$ with $d(x_i, x_{i+1}) \leq \epsilon$ and $x_1 = x_-, x_k = x_+$.

Choose approximations $x_i(n) \in \Gamma$ with

$$d(x_i(n), x_{i+1}(n)) \leq 2\epsilon r_n \quad \text{and} \quad d(x_i(n), \alpha_n(0)) \geq 3\epsilon r_n,$$

so that $x_1(n) = \alpha_n(-r_n)$ and $x_k(n) = \alpha_n(r_n)$. Joining consecutive points by geodesics, we obtain a path of length $\leq 2k\epsilon r_n$ avoiding $\mathcal{B}_{\Gamma}(\alpha_n(0), \epsilon r_n)$ and connecting $\alpha_n(\pm r_n)$. For large n , we have $1/n < \epsilon$ and $n > 2k\epsilon$, which contradicts our initial assumptions.

(3) \Rightarrow (1). Suppose towards a contradiction that H is not relatively wide in G . Thus, there exist an asymptotic cone Γ_{ω} and two points $x, y \in H_{\omega} \subseteq \Gamma_{\omega}$ that are separated by a third point $z \in \Gamma_{\omega}$.

Write $x = (x_n)$ and $y = (y_n)$ with $x_n, y_n \in H \subseteq \Gamma^{(0)}$ and choose geodesics $\beta_n : I_n \rightarrow \Lambda$ joining x_n and y_n for intervals I_n . The paths $i \circ \beta_n$ are uniform quasigeodesics in Γ , so they converge to a bi-Lipschitz path $\alpha : I \rightarrow H_{\omega} \subseteq \Gamma_{\omega}$ from x to y . Since z separates x and y , it must lie in $\alpha(I)$.

Thus, possibly replacing x and y with other points of $\alpha(I)$, reparametrising α and β_n , and modifying the scaling factors in the definition of Γ_{ω} (though not their growth rate), we can assume that $I = [-1, 1]$, $\alpha(0) = z$ and $I_n = [-r_n, r_n]$ for some sequence $r_n \rightarrow +\infty$. In particular, the points $i \circ \beta_n(0)$ converge to z .

Since (3) holds, there exist edge paths $\gamma_n \subseteq \Gamma$ joining x_n to y_n , avoiding $\mathcal{B}_{\Gamma}(i \circ \beta_n(0), r_n/K_0)$ and having length $|\gamma_n| \leq K_0 \cdot r_n$. These paths can be discretised to sequences of points $w_1(n), \dots, w_k(n) \in \Gamma$, with k independent of n , such that $d(w_i(n), i \circ \beta_n(0)) > r_n/K_0$ and $d(w_i(n), w_{i+1}(n)) \leq r_n/(2K_0)$ for all i . We choose these points so that $w_1(n) = x_n$ and $w_k(n) = y_n$.

Let $w_i \in \Gamma_{\omega}$ be the limit of the sequence $w_i(n)$. Note that $d(w_i, z) \geq 1/K_0$ and $d(w_i, w_{i+1}) \leq 1/(2K_0)$, so every geodesic joining w_i and w_{i+1} in Γ_{ω} avoids $\mathcal{B}_{\Gamma_{\omega}}(z, 1/(2K_0))$. Concatenating such geodesics, we obtain a path in Γ_{ω} from $w_1 = x$ to $w_k = y$ avoiding $\mathcal{B}_{\Gamma_{\omega}}(z, 1/(2K_0))$, which contradicts the assumption that x and y are separated by z . □

2D. Graphs of groups. Throughout the paper, we are interested in finitely generated groups G that split as a graph of groups. This is equivalent to the fact that G admits a nonelliptic action on a simplicial tree $G \curvearrowright T$ without inversions [Serre 1980]. In this case, there is a unique smallest G -invariant subtree [Culler and Morgan 1987]. Restricting to it, we can further assume that the action $G \curvearrowright T$ is *minimal*, i.e., that no proper subtree is G -invariant.

In this subsection, we consider the following setting.

Assumption 2.12. Let G be a group generated by a finite subset $S_G \subseteq G$ with $1 \in S_G$ and $S_G = S_G^{-1}$. Let Γ be the corresponding Cayley graph of G , endowed with its graph metric d_Γ . Suppose that we have a nonelliptic, minimal action without inversions on a simplicial tree $G \curvearrowright T$.

Vertex stabilisers for the action $G \curvearrowright T$ will be referred to as *vertex groups*, and usually denoted by V . Similarly, *edge groups* are G -stabilisers of edges of T , and will usually be denoted by E when the corresponding edge is not specified. To avoid confusion between edges of Γ and T , we will denote the latter by Fraktur letters ϵ .

Remark 2.13. The action $G \curvearrowright T$ is cocompact. Indeed, fixing a basepoint $p \in T$, let K be the convex hull of $S_G \cdot p$. This is a compact subtree of T . Since S_G generates G , the subset $G \cdot K$ is connected, and hence a G -invariant subtree. By minimality, we obtain $T = G \cdot K$.

The following results are certainly well known to experts, but we were not able to locate proofs in the literature at the required level of generality. Our arguments are inspired by the proof of [Bowditch 1998, Proposition 1.2] in the case of hyperbolic groups with quasiconvex edge groups.

Lemma 2.14. *Consider a vertex $p \in T$ and let V be its stabiliser. There exist a constant D_p and stabilisers E_1, \dots, E_k of edges incident to p such that the following holds. Every path $\gamma \subseteq \Gamma$ joining points of V can be decomposed as a concatenation $\gamma_1\gamma_2 \dots \gamma_m$ with the following property. The endpoints of each γ_i lie in the D_p -neighbourhood in Γ of a coset $v_i E_{j_i}$ with $v_i \in V$ and $1 \leq j_i \leq k$.*

Proof. By Remark 2.13, there are finitely many V -orbits of edges of T incident to p . Let $\epsilon_1, \dots, \epsilon_k$ be a finite list of representatives and let E_1, \dots, E_k be their stabilisers. Let $K \subseteq T$ be the convex hull of $S_G \cdot p$. If $\epsilon \subseteq T$ is an edge, define $\Omega(\epsilon) := \{g \in G \mid \epsilon \subseteq gK\}$.

Claim. *There exists a constant D_p such that, for every edge $\epsilon \subseteq T$ incident to p , the set $\Omega(\epsilon)$ is contained in the D_p -neighbourhood in Γ of some coset vE_i with $v \in V$ and $1 \leq i \leq k$.*

Proof of claim. Observe that $\Omega(\epsilon)$ is a union of at most N right cosets of E , where E is the stabiliser of ϵ and N is the number of edges in the compact tree K . Thus, there exists a constant D_p such that, for each $1 \leq i \leq k$, the set $\Omega(\epsilon_i)$ is contained in the D_p -neighbourhood of E_i in Γ .

For an arbitrary edge $\epsilon \ni p$, we can write $\epsilon = v\epsilon_i$ for some $v \in V$ and some i . In this case, $\Omega(\epsilon) = \Omega(v\epsilon_i) = v\Omega(\epsilon_i)$ is contained in the D_p -neighbourhood of the left coset vE_i . □

Now, we define a continuous G -equivariant map $f_p : \Gamma \rightarrow T$ as follows. For every $s \in S_G$, let $\pi_s \subseteq K$ be the geodesic from p to sp . If $g \in G$, we set $f_p(g) = gp$. Then, on each edge $[g, gs] \subseteq \Gamma$ with $g \in G$ and $s \in S_G$, we define f_p as the linear parametrisation of the path $g\pi_s$.

Up to decomposing γ , we can assume that γ meets V only at its endpoints. We can also suppose that γ is not a single edge, or the statement is clear. Thus, $f_p \circ \gamma$ is a nontrivial path in the tree T beginning and ending at the basepoint p .

Let $e_1, \dots, e_m \subseteq \Gamma$ be all edges of γ for which $f_p(e_i)$ contains p , in the order in which they appear along γ . Thus, e_1 and e_m are necessarily the initial and terminal edge of γ , respectively. Note that the geodesic $f_p(e_i) \subseteq T$ meets p exactly once and not at its endpoints, except for e_1 and e_m .

Define $\gamma_i \subseteq \gamma$ as the subsegment between the initial vertex of e_i and the initial vertex of e_{i+1} . The portion of the path $f_p(\gamma_i \cup e_{i+1})$ between the two occurrences of p is contained in a connected component of $T \setminus \{p\}$. Thus, it must begin and end by crossing the same edge of T incident to p .

We name this edge \mathfrak{f}_i . Say the endpoints of $e_i \subseteq \Gamma$ are $g_i \in G$ and $g_i s_i$ with $s_i \in S_G$. Since $\mathfrak{f}_i \subseteq f_p(e_i) \cap f_p(e_{i+1})$, we have $\mathfrak{f}_i \subseteq g_i \pi_{s_i} \cap g_{i+1} \pi_{s_{i+1}}$; hence $\{g_i, g_{i+1}\} \subseteq \Omega(\mathfrak{f}_i)$. Since g_i and g_{i+1} are the endpoints of γ_i , the claim concludes the proof. \square

If $\alpha \subseteq \Gamma$ is a path, we denote by $|\alpha|$ its *length*, i.e., the number of edges that it contains. Recall that a finitely generated subgroup of G is *undistorted* if the inclusion in G is a quasi-isometric embedding.

Lemma 2.15. *Let $p \in T$ be a vertex with stabiliser V . Suppose that the stabiliser of every edge $p \in \epsilon \subseteq T$ is finitely generated and undistorted in G . Then V is finitely generated and undistorted.*

Proof. We will prove the following.

Claim. *There exists a constant L such that, for every $v \in V$, there exists a path $\alpha \subseteq \Gamma$ from the identity to v such that α is contained in the L -neighbourhood of V and $|\alpha| \leq L \cdot d_\Gamma(1, v)$.*

Assuming the claim, we define Σ as the intersection of V and the $(2L + 1)$ -ball in Γ centred at the identity. Since Γ is locally finite, the set Σ is finite. Denoting by d_Σ the word metric on V induced by Σ , the claim shows that $d_\Sigma(1, v) \leq L \cdot d_\Gamma(1, v)$ for every $v \in V$. Thus, Σ generates V and V is undistorted in G .

Proof of claim. Now, let us prove the claim. Let the constant D_p and the edge groups $E_1, \dots, E_k \leq V$ be those provided by Lemma 2.14. Let $\gamma \subseteq \Gamma$ be a geodesic from the identity to some element $v \in V$. Take the decomposition $\gamma = \gamma_1 \gamma_2 \dots \gamma_m$ as in Lemma 2.14, with the endpoints of γ_i in the D_p -neighbourhood of a coset $v_i E_{j_i}$.

Since E_1, \dots, E_k are finitely generated and undistorted, there exists a constant L such that, for all i and all $x, y \in E_i$, the points x and y are joined by a path $\beta \subseteq \Gamma$ contained in the L -neighbourhood of E_i and satisfying $|\beta| \leq L \cdot d_\Gamma(x, y)$. The same holds if x and y lie in a left coset of some E_i .

It follows that we can replace each γ_i with a path γ'_i with the same endpoints so that γ'_i is contained in the L -neighbourhood of $v_i E_{j_i} \subseteq V$ and $|\gamma'_i| \leq L \cdot |\gamma_i|$. Define α as the concatenation of the γ'_i . Then, it is clear that α is contained in the L -neighbourhood of V and $|\alpha| \leq L \cdot |\gamma| = L \cdot d_\Gamma(1, v)$. This proves the claim, and hence the lemma. \square

Corollary 2.16. *If all edge groups are finitely generated and undistorted in G , then all vertex groups are finitely generated and undistorted in G .*

The fact that vertex groups are finitely generated as soon as edge groups are finitely generated is also proved in detail e.g., in [Cohen 1989, Lemma 8.32, p. 218] and [Dicks and Dunwoody 1989].

3. Proof of Theorem A

Assumption 3.1. Let G be a finitely generated group. Let $G \curvearrowright T$ be a minimal action on a simplicial tree without inversions. Suppose that all edge-stabilisers $E \leq G$ are finitely generated, undistorted, and satisfy $(\partial_M E, G) = \emptyset$.

Fix a finite generating set for G and let Γ be the corresponding Cayley graph. Choose a basepoint $p \in T^{(0)}$ and let $f_p : G \rightarrow T$ denote the orbit map $f_p(g) = gp$.

The complement in T of an open edge has exactly two connected components, which we refer to as *halfspaces*. Let $\mathcal{E}(T)$ denote the set of (unoriented) edges of T , and let $\mathcal{H}(T)$ be the set of halfspaces. Note that $\mathcal{H}(T)$ is naturally in bijection with the set of *oriented* edges of T . In particular, every edge $\epsilon \in \mathcal{E}(T)$ gives rise to two halfspaces $\mathfrak{h}, \mathfrak{h}^*$. The complement of \mathfrak{h} is always denoted by \mathfrak{h}^* .

We denote by $G_{\mathfrak{h}} \leq G$ the stabiliser of the halfspace \mathfrak{h} . Since G acts without inversions, $G_{\mathfrak{h}}$ coincides with the stabiliser G_ϵ of the edge ϵ associated to \mathfrak{h} .

Note, every $\mathfrak{h} \in \mathcal{H}(T)$ gives rise to a $G_{\mathfrak{h}}$ -invariant partition $G = f_p^{-1}(\mathfrak{h}) \sqcup f_p^{-1}(\mathfrak{h}^*)$.

Lemma 3.2. *We can choose a subgraph $\Gamma(\epsilon) \subseteq \Gamma$ for every $\epsilon \in \mathcal{E}(T)$ so that the following hold:*

- (1) $\Gamma(\epsilon)$ is connected, G_ϵ -invariant and G_ϵ -cocompact.
- (2) $\Gamma(g\epsilon) = g\Gamma(\epsilon)$ for all $g \in G$ and $\epsilon \in \mathcal{E}(T)$.
- (3) If $\mathfrak{h}, \mathfrak{h}^*$ are the two halfspaces determined by ϵ , then $\Gamma(\epsilon)$ contains every edge of Γ with an endpoint in $f_p^{-1}(\mathfrak{h})$ and the other in $f_p^{-1}(\mathfrak{h}^*)$.
- (4) For every $r \geq 0$, we have $\mathcal{N}_\Gamma(f_p^{-1}(\mathfrak{h}), r) \cap \mathcal{N}_\Gamma(f_p^{-1}(\mathfrak{h}^*), r) \subseteq \mathcal{N}_\Gamma(\Gamma(\epsilon), r)$.
- (5) Each edge $e \subseteq \Gamma$ lies in the subgraph $\Gamma(\epsilon)$ for only finitely many edges $\epsilon \in \mathcal{E}(T)$.

Proof. Recall that edge-stabilisers are finitely generated. If S_ϵ is a finite generating set of G_ϵ , we can construct a subgraph $\Gamma(\epsilon) \subseteq \Gamma$ satisfying (1) by choosing paths joining the identity $1 \in G$ to the elements of S_ϵ , and taking the union of all their G_ϵ -translates. Defining $\Gamma(\epsilon)$ in this fashion for one edge ϵ in every G -orbit of edges of T , and then setting $\Gamma(g\epsilon) := g\Gamma(\epsilon)$, guarantees that condition (2) is also satisfied.

Now, let us ensure that (3) holds. For this, it suffices to show that there are only finitely many G_ϵ -orbits of edges of Γ with an endpoint in $f_p^{-1}(\mathfrak{h})$ and the other in $f_p^{-1}(\mathfrak{h}^*)$, as these can then be added to $\Gamma(\epsilon)$. If S_G is the finite generating set of G giving rise to Γ , then every such edge is of the form $[g, gs]$ with $gp \in \mathfrak{h}$, $gsp \in \mathfrak{h}^*$ and $s \in S_G$. Note that $g^{-1}\epsilon$ is then one of the finitely many edges separating p and sp for some $s \in S_G$. It follows that g lies in a finite union of right cosets of G_ϵ , as required.

Note that (4) follows from (3): If a point $x \in \Gamma$ lies in the r -neighbourhood of both $f_p^{-1}(\mathfrak{h})$ and $f_p^{-1}(\mathfrak{h}^*)$, then it lies on a path of length $\leq r$ joining $f_p^{-1}(\mathfrak{h})$ to $f_p^{-1}(\mathfrak{h}^*)$. Since $G = f_p^{-1}(\mathfrak{h}) \sqcup f_p^{-1}(\mathfrak{h}^*)$, condition (3) implies that such a path must contain an edge of $\Gamma(\epsilon)$. Hence x lies in the r -neighbourhood of $\Gamma(\epsilon)$.

Finally, (5) follows from (1): Fix some $\epsilon \in \mathcal{E}(T)$. Suppose that $e \subseteq g\Gamma(\epsilon)$ for some $g \in G$. Since $g^{-1}e \subseteq \Gamma(\epsilon)$ and G_ϵ acts cocompactly on $\Gamma(\epsilon)$, we deduce that g must lie in a finite union of left cosets of G_ϵ . In other words, e is contained in $\Gamma(\epsilon')$ for only finitely many edges ϵ' in the G -orbit of ϵ . Since there are only finitely many G -orbits in $\mathcal{E}(T)$, by Remark 2.13, this proves (5). \square

Let us choose $\Gamma(\epsilon) \subseteq \Gamma$ for every $\epsilon \in \mathcal{E}(T)$ as in Lemma 3.2. If \mathfrak{h} is one of the two halfspaces determined by ϵ , it is convenient to define $\Gamma(\mathfrak{h}) \subseteq \Gamma$ as the set $f_p^{-1}(\mathfrak{h}) \setminus \Gamma(\epsilon) \subseteq G$ along with all (half-open) edges of $\Gamma \setminus \Gamma(\epsilon)$ that it intersects. Thus, we have a G_ϵ -invariant partition:

$$\Gamma = \Gamma(\mathfrak{h}) \sqcup \Gamma(\epsilon) \sqcup \Gamma(\mathfrak{h}^*).$$

Lemma 3.3. *For every Morse gauge N , there exists a constant $D(N)$ with the following property. If $\beta \subseteq \Gamma$ is an N -Morse geodesic with endpoints in $\Gamma(\epsilon)$ for some $\epsilon \in \mathcal{E}(T)$ (resp. in $\Gamma(\mathfrak{h})$ for some $\mathfrak{h} \in \mathcal{H}(T)$), then β is contained in the $D(N)$ -neighbourhood of $\Gamma(\epsilon)$ (resp. of $\Gamma(\mathfrak{h})$).*

Proof. By Lemma 2.1(4) and Lemma 3.2(3), the statement about $\Gamma(\mathfrak{h})$ follows from that on $\Gamma(\epsilon)$.

Let $\epsilon_1, \dots, \epsilon_k$ be representatives for the orbits of $G \curvearrowright \mathcal{E}(T)$ (recall Remark 2.13). Since edge-stabilisers are undistorted, there exists a constant L such that any two points of $\Gamma(\epsilon_i)$ can be connected by a path entirely contained in $\Gamma(\epsilon_i)$ that is an L -quasigeodesic in Γ .

Now, suppose then that β has endpoints $x, y \in \Gamma(\epsilon)$. There exist $g \in G$ and $1 \leq i \leq k$ such that $\Gamma(\epsilon) = g\Gamma(\epsilon_i)$. Thus, there exists an L -quasigeodesic $\alpha \subseteq \Gamma$ connecting x and y within $\Gamma(\epsilon)$. By Lemma 2.1(1), the Hausdorff distance between

β and α is at most $D(N)$, where $D(N)$ depends only on N and L . Since $\alpha \subseteq \Gamma(\epsilon)$, this proves the lemma. \square

Corollary 3.4. *Let $\gamma \subseteq \Gamma$ be a Morse geodesic ray. Then:*

- (1) *For every edge $\epsilon \in \mathcal{E}(T)$, the intersection $\gamma \cap \Gamma(\epsilon)$ is compact.*
- (2) *For every halfspace $\mathfrak{h} \in \mathcal{H}(T)$, a subray of γ is entirely contained in either $\Gamma(\mathfrak{h})$ or $\Gamma(\mathfrak{h}^*)$.*
- (3) *If a subray of γ is contained in $\Gamma(\mathfrak{h})$, then γ must get arbitrarily far from $\Gamma(\mathfrak{h}^*)$.*

Proof. If $\gamma \cap \Gamma(\epsilon)$ were noncompact, it would be unbounded and Lemma 3.3 would show that γ stays at bounded distance from $\Gamma(\epsilon)$. This would contradict the assumption that $(\partial_M G_\epsilon, G) = \emptyset$. This proves part (1). Parts (2) and (3) then follow, respectively, from properties (3) and (4) in Lemma 3.2. \square

By Corollary 3.4(3), which of the sets $\Gamma(\mathfrak{h})$ and $\Gamma(\mathfrak{h}^*)$ contains a subray of γ does not change if we replace γ with a ray at finite Hausdorff distance. This leads us to consider the following (well-defined) subset of $\partial_M G$ for each $\mathfrak{h} \in \mathcal{H}(T)$:

$$M(\mathfrak{h}) = \{[\gamma] \in \partial_M G \mid \gamma \text{ is eventually contained in } \Gamma(\mathfrak{h})\}.$$

Lemma 3.5. *For every $\mathfrak{h} \in \mathcal{H}(T)$, we have a partition $\partial_M G = M(\mathfrak{h}) \sqcup M(\mathfrak{h}^*)$ into closed subsets.*

Proof. From the above discussion, it is clear that $M(\mathfrak{h})$ and $M(\mathfrak{h}^*)$ are disjoint and cover $\partial_M G$. We only need to prove that they are closed. Choosing some $x \in \Gamma(\mathfrak{h})$ as basepoint, it suffices to show that $M(\mathfrak{h}) \cap \partial_M^N \Gamma_x$ is sequentially closed for every Morse gauge N .

Let $\gamma_n \subseteq \Gamma$ be a sequence of N -Morse geodesic rays based at x that converge uniformly on compact sets to an N -Morse geodesic ray $\gamma \subseteq \Gamma$. If $[\gamma_n] \in M(\mathfrak{h})$ for every n , then Lemma 3.3 guarantees that the γ_n are all contained in the $D(N)$ -neighbourhood of $\Gamma(\mathfrak{h})$. The same neighbourhood must then contain γ and, by Corollary 3.4(3), we conclude that $[\gamma] \in M(\mathfrak{h})$. \square

If $\xi \in \partial_M G$, we introduce the following subset of $\mathcal{H}(T)$:

$$\begin{aligned} \sigma(\xi) &:= \{\mathfrak{h} \in \mathcal{H}(T) \mid \xi \in M(\mathfrak{h})\} \\ &= \{\mathfrak{h} \in \mathcal{H}(T) \mid \text{if } \xi = [\gamma], \text{ then } \gamma \text{ is eventually contained in } \Gamma(\mathfrak{h})\}. \end{aligned}$$

Note that $\sigma(\xi)$ satisfies the following two properties:

- (1) If $\epsilon \in \mathcal{E}(T)$, then exactly one of the two halfspaces determined by ϵ lies in $\sigma(\xi)$.
- (2) If $\mathfrak{h}_1, \mathfrak{h}_2 \in \sigma(\xi)$, then $\mathfrak{h}_1 \cap \mathfrak{h}_2 \neq \emptyset$.

A subset of $\mathcal{H}(T)$ with these properties is known as an *ultrafilter* [Sageev 2014, Definition 2.1].

For every vertex $x \in T^{(0)}$, an important example of an ultrafilter is the set

$$\sigma_x := \{\mathfrak{h} \in \mathcal{H}(T) \mid x \in \mathfrak{h}\}.$$

Conversely, every ultrafilter $\sigma \subseteq \mathcal{H}(T)$ that does not contain infinite descending chains of halfspaces is of the form σ_x for a vertex $x \in T^{(0)}$; see for instance [Sageev 2014, Proposition 2.1].

Remark 3.6. If $\xi, \eta \in \partial_M G$ are in the same connected component, then $\sigma(\xi) = \sigma(\eta)$. Indeed, Lemma 3.5 implies that, for every $\mathfrak{h} \in \mathcal{H}(T)$, we have either $\{\xi, \eta\} \subseteq M(\mathfrak{h})$ or $\{\xi, \eta\} \subseteq M(\mathfrak{h}^*)$.

Lemma 3.7. *Let $\alpha \subseteq \Gamma$ be an N -Morse geodesic line whose endpoints at infinity $\alpha^\pm \in \partial_M G$ lie in the same connected component of $\partial_M G$. Set $\sigma := \sigma(\alpha^+)$, which coincides with $\sigma(\alpha^-)$ by Remark 3.6. Then the following hold:*

- (1) α is contained in the intersection of the $D(N)$ -neighbourhoods of the subsets $\Gamma(\mathfrak{h}) \subseteq \Gamma$ with $\mathfrak{h} \in \sigma$ (here $D(N)$ is the constant introduced in Lemma 3.3).
- (2) There exists a vertex $x \in T$ such that $\sigma = \sigma_x$.
- (3) α stays at bounded distance from the stabiliser $G_x \leq G$.

Proof. Part (1) is immediate from Lemma 3.3. We will prove part (2) by showing that σ does not contain any infinite descending chains of halfspaces.

Fix a vertex $g \in \alpha \subseteq \Gamma$. Consider the ultrafilter $\sigma_{gp} \subseteq \mathcal{H}(T)$ determined by the vertex $gp \in T$. If $\mathfrak{h} \in \sigma \setminus \sigma_{gp}$, then $g \in f_p^{-1}(\mathfrak{h}^*) \cap \alpha$, while α is contained in the $D(N)$ -neighbourhood of $\Gamma(\mathfrak{h})$ by part (1). If ϵ is the edge corresponding to \mathfrak{h} , Lemma 3.2(4) guarantees that $\Gamma(\epsilon)$ meets the $D(N)$ -ball around g . By Lemma 3.2(5), the latter can occur only for finitely many edges of T .

This shows that the set $\sigma \setminus \sigma_{gp}$ is finite. Since σ_{gp} does not contain infinite descending chains, neither does σ . This proves part (2).

Finally, let us prove part (3). Let $x \in T$ be the vertex provided by part (2). Let $\tau_x \subseteq \sigma_x$ be the subset of halfspaces whose corresponding edge of T is incident to x . Let $\Omega \subseteq \Gamma$ be the intersection of the $D(N)$ -neighbourhoods of the sets $\Gamma(\mathfrak{h})$ with $\mathfrak{h} \in \tau_x$. Part (1) guarantees that $\alpha \subseteq \Omega$. Since G_x fixes x , it leaves invariant τ_x , and thus it also leaves invariant $\Omega \subseteq \Gamma$. In order to prove part (3), it suffices to show that the action $G_x \curvearrowright \Omega$ is cocompact. This implies that the Hausdorff distance between Ω and G_x is finite, so α then stays at bounded distance from G_x .

Observe that $\{x\} = \bigcap_{\mathfrak{h} \in \tau_x} \mathfrak{h}$; hence $f_p^{-1}(x) = \bigcap_{\mathfrak{h} \in \tau_x} f_p^{-1}(\mathfrak{h})$. The latter is either empty (if x is not in the same G -orbit as our basepoint $p \in T$), or a single G_x -orbit. Thus, it suffices to show that G_x acts with finitely many orbits on the difference $\Omega^{(0)} \setminus f_p^{-1}(x)$.

For every vertex $y \in \Omega^{(0)} \setminus f_p^{-1}(x)$, there exists $\mathfrak{h} \in \tau_x$ such that y lies in the intersection of $f_p^{-1}(\mathfrak{h}^*)$ and the $D(N)$ -neighbourhood of $\Gamma(\mathfrak{h})$. If ϵ is the edge corresponding to \mathfrak{h} , Lemma 3.2(4) shows that y lies in the $D(N)$ -neighbourhood of $\Gamma(\epsilon)$. Recall that there are only finitely many G_x -orbits of edges ϵ corresponding to elements of τ_x , that these edges satisfy $G_\epsilon \leq G_x$, and that G_ϵ acts cocompactly on $\Gamma(\epsilon)$. This shows that G_x acts with finitely many orbits on $\Omega^{(0)} \setminus f_p^{-1}(x)$, as required. \square

Corollary 3.8. *If a connected component $\mathcal{C} \subseteq \partial_M G$ is not a singleton, then there exists a vertex $x \in T$ such that \mathcal{C} is contained in the image of the natural inclusion $(\partial_M G_x, G) \hookrightarrow \partial_M G$.*

Proof. By Remark 3.6, there exists an ultrafilter $\sigma \subseteq \mathcal{H}(T)$ such that $\sigma = \sigma(\xi)$ for all $\xi \in \mathcal{C}$. Any two points of \mathcal{C} are endpoints at infinity of a Morse geodesic line in Γ , by [Cordes 2017, Proposition 3.11]. Thus, Lemma 3.7(2) shows that $\sigma = \sigma_x$ for some $x \in T^{(0)}$, and Lemma 3.7(3) guarantees that every point of \mathcal{C} is represented by a ray at bounded distance from G_x . \square

With the following observation, Corollary 3.8 immediately implies Theorem A.

Lemma 3.9. *If $x, y \in T$ are distinct vertices, then $(\partial_M G_x, G) \cap (\partial_M G_y, G) = \emptyset$.*

Proof. Choose a halfspace $\mathfrak{h} \in \mathcal{H}(T)$ with $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^*$. A point in

$$(\partial_M G_x, G) \cap (\partial_M G_y, G)$$

would be represented by asymptotic Morse rays $r_x, r_y \subseteq \Gamma$ contained in neighbourhoods of $\Gamma(\mathfrak{h})$ and $\Gamma(\mathfrak{h}^*)$, respectively. This would contradict Corollary 3.4(3), so no such point can exist. \square

Corollary 3.8, Lemma 2.3 and Corollary 2.16 imply the following, which is Corollary B.

Corollary 3.10. *If $(\partial_M G_x, G)$ is totally disconnected for every $x \in T^{(0)}$, then $\partial_M G$ is totally disconnected.*

The following is not required in the proof of the main theorems, but it looks like a useful observation (for instance, when studying whether $\partial_M G$ is σ -compact).

Remark 3.11. Recall that we have defined an ultrafilter $\sigma(\xi) \subseteq \mathcal{H}(T)$ for every point $\xi \in \partial_M G$. This yields a map $f : \partial_M G \rightarrow \bar{T} := T^{(0)} \sqcup \partial T$, where ∂T is the set of ends of T .

Every halfspace $\mathfrak{h} \in \mathcal{H}(T)$ can be extended to a subset $\bar{\mathfrak{h}} \subseteq \bar{T}$ by adding the ends of T that it contains. This yields a partition $\bar{T} = \bar{\mathfrak{h}} \sqcup \bar{\mathfrak{h}}^*$. It is customary to endow \bar{T} with the topology having the collection of sets $\bar{\mathfrak{h}}$ with $\mathfrak{h} \in \mathcal{H}(T)$ as a subbasis.

Lemma 3.5 implies that $f : \partial_M G \rightarrow \bar{T}$ is continuous. Moreover, for every vertex $x \in T$, the proof of Lemma 3.7(3) shows that $f^{-1}(x)$ is exactly the set of points of

$\partial_M G$ that are represented by rays in Γ at bounded distance from the stabiliser G_x . If $\xi \in \partial T$, it is possible to show that $f^{-1}(\xi)$ is either empty or a singleton, but this requires a bit more work.

We emphasise that the sets $T^{(0)}$ and ∂T are neither open nor closed in \bar{T} , since T will normally be locally infinite in our setting.

Remark 3.12. The attentive reader might have noticed that the majority of the proof of Theorem A works more generally when G acts cocompactly on a CAT(0) cube complex and all hyperplane-stabilisers are finitely generated, undistorted and with trivial relative Morse boundary in G . We have chosen to restrict to actions on trees (the main case of interest) in order to avoid additional difficulties in the following two spots:

- (1) If G acts essentially and cocompactly on a CAT(0) cube complex with undistorted hyperplane-stabilisers, it is not clear if vertex-stabilisers will have to be undistorted as well (cf. Corollary 2.16). It is possible that this can be shown along the lines of [Groves and Manning 2018, Theorem A].
- (2) The last paragraph of the proof of Lemma 3.7(3) would be a bit more delicate in a CAT(0) cube complex. One would probably need to require stabilisers of all *intersections of pairwise-transverse hyperplanes* to be finitely generated.

4. Proof of Theorem C

In order to prove that the Morse boundary of the vertex group V topologically embeds in the Morse boundary of the ambient group G , we will show that uniformly Morse rays $\alpha \subseteq V$ remain uniformly Morse in G (see Corollary 4.3 below).

One possible approach to this would consider a quasigeodesic $\gamma \subseteq G$ with endpoints on α and use this to construct a quasigeodesic $\gamma' \subseteq V$ with the same endpoints and still getting roughly as far from α as γ . Lemma 2.14 allows us to decompose γ as a concatenation $\gamma_1 \gamma_2 \dots \gamma_m$, where the endpoints of each γ_i are near a coset of an edge group $v_i E_{j_i} \subseteq V$. We might thus hope to replace each γ_i with a quasigeodesic $\gamma'_i \subseteq v_i E_{j_i}$ with nearby endpoints, in order to form a quasigeodesic $\gamma' \subseteq V$. Since α is Morse in V , the quasigeodesic γ' would then stay close to α and, if edge groups have trivial relative Morse boundary in V , the γ'_i would all have to be quite short. Hence the γ_i would also be short and γ would stay close to γ' and α , showing that α is Morse in G .

The problem with this strategy is that, even if γ and the γ'_i are quasigeodesics, there is no guarantee that the concatenation $\gamma' = \gamma'_1 \gamma'_2 \dots \gamma'_m$ will be a quasigeodesic. Instead, we will follow a different approach based on divergence, which is the content of Proposition 4.2.

Assuming that α has a “slow” divergence function in G , we will show that it also has a slow divergence function in V (which contradicts the assumption that

α is Morse in V). The proof proceeds along similar lines to the above sketch. If $\gamma \subseteq G$ were a short path with endpoints on α avoiding a large ball, then we could decompose γ as $\gamma_1\gamma_2 \dots \gamma_m$, and replace each γ_i with a short path in V , using the assumption that edge groups are relatively wide in V (in the form of Lemma 2.11). The new path γ' still avoids a ball of roughly the same radius. Here, the advantage is that it does not matter whether or not γ' is a quasigeodesic. Of course, asking that edge groups be relatively wide in V might be stronger than simply asking that edge groups have trivial relative Morse boundary in V .

We now give precise proofs, considering the following setting.

Assumption 4.1. Let G be a finitely generated group with a nonelliptic, minimal action without inversions on a simplicial tree $G \curvearrowright T$. Fix a base vertex $p \in T$ with stabiliser V . We assume that the stabiliser of every edge $\epsilon \subseteq T$ incident to p is finitely generated and undistorted in G .

Let D_p and $E_1, \dots, E_k \leq V$ be as in Lemma 2.14. Choose finite generating sets $S_{E_i} \subseteq E_i, S_V \subseteq V$ and $S_G \subseteq G$, ensuring that:

$$S_{E_1} \cup \dots \cup S_{E_k} \subseteq S_V \subseteq S_G.$$

Let $\Theta_i \subseteq \Delta \subseteq \Gamma$ be the corresponding Cayley graphs of E_i, V and G , respectively. We denote their intrinsic path metrics by d_{Θ_i}, d_Δ and d_Γ .

The inclusions $(\Theta_i, d_{\Theta_i}) \hookrightarrow (\Delta, d_\Delta) \hookrightarrow (\Gamma, d_\Gamma)$ are all 1-Lipschitz. In view of Lemma 2.15, we can fix a constant $C \geq 1$ such that they are all C -bi-Lipschitz.

We assume that E_1, \dots, E_k are relatively wide in V . Let $K \geq 1$ be the maximum of the constants K_0 provided by Lemma 2.11(3) in relation to the pairs $E_i \leq V$ with $1 \leq i \leq k$ and the Cayley graphs Θ_i and Δ .

We write $\delta_\alpha^\Delta(\cdot, \cdot)$ and $\delta_\alpha^\Gamma(\cdot, \cdot)$ when we need to specify the graph used to compute divergence.

Proposition 4.2. *There exist $\eta \in (0, 1)$ and K' , both depending only on C, D_p, K , such that, for every geodesic ray $\alpha \subseteq (\Delta, d_\Delta)$ and every $r > K'$, we have*

$$\delta_\alpha^\Delta\left(r, \eta \cdot \frac{1}{16C}\right) \leq K' \cdot \delta_\alpha^\Gamma\left(r, \frac{1}{16C}\right).$$

Proof. Let $\alpha \subseteq \Delta$ be a geodesic ray based at a vertex. Recall that α is C -bi-Lipschitz as a path in (Γ, d_Γ) . Suppose that there exist integers s, r such that the vertices $\alpha(s \pm r)$ can be joined by a path $\gamma \subseteq \Gamma$ that avoids the ball $B_\Gamma(\alpha(s), r/(16C))$.

Our goal is to construct a path γ' joining $\alpha(s \pm r)$, so that γ' is entirely contained in Δ , avoids the ball $B_\Delta(\alpha(s), \eta \cdot r/(16C))$, and has length $|\gamma'| \leq K' \cdot |\gamma|$. The constants K' and η will be determined at the end of the proof. Note that we are only interested in the situation where $r > K'$.

By Lemma 2.14, we can take the decomposition $\gamma = \gamma_1\gamma_2 \dots \gamma_m$ with the endpoints γ_i^\pm in the neighbourhood $\mathcal{N}_\Gamma(v_i E_{j_i}, D_p)$ for some $v_i \in V$ and $1 \leq j_i \leq k$. Let

$\beta_i \subseteq v_i \Theta_{j_i}$ be a geodesic (for $d_{\Theta_{j_i}}$) whose endpoints β_i^\pm satisfy $d_\Gamma(\beta_i^\pm, \gamma_i^\pm) \leq D_p$. Note that β_i is contained in Δ and is C -bi-Lipschitz for d_Δ .

Suppose that $r > 32C(D_p + K)$. Choose $\epsilon > 0$ small enough that

$$\frac{1 - \epsilon}{16C} - \epsilon K \geq \frac{1}{32C}.$$

Claim. *There exists a path $\beta'_i \subseteq \Delta$ with the same endpoints as β_i , such that β'_i avoids the ball $B := \mathcal{B}_\Delta(\alpha(s), \epsilon \cdot r/(16C))$ and has length $|\beta'_i| \leq K|\beta_i|$.*

Proof of claim. If β_i is disjoint from B , we can simply take $\beta'_i = \beta_i$.

Otherwise, choose a vertex $y \in \beta_i \cap B$ and set $B' := \mathcal{B}_\Delta(y, 2\epsilon \cdot r/(16C))$. Observe that $B \subseteq B'$. Thus, it suffices to construct the path β'_i so that it avoids the ball B' .

Parametrise β_i by arc-length so that $\beta_i(0) = y$. Suppose first that the domain of β_i contains an interval $[-\rho, \rho]$ satisfying the three inequalities

$$\rho > K, \quad \rho > \frac{\epsilon Kr}{8C}, \quad \rho > \frac{\epsilon r}{8}.$$

Since $\rho > K$ and β_i is contained in a left coset of E_{j_i} in V , we can apply Lemma 2.11, which provides a path $\beta'_i \subseteq \Delta$ connecting the points $\beta_i(\pm\rho)$, avoiding $\mathcal{B}_\Delta(y, \rho/K)$ and with length $|\beta'_i| \leq K\rho$. By the second inequality, this path avoids the ball B' , which is contained in $\mathcal{B}_\Delta(y, \rho/K)$.

We can then prolong β'_i along β_i until it reaches β_i^\pm . Since β_i is C -bi-Lipschitz with respect to d_Δ , the third inequality ensures that β_i can only meet the ball B' at times in the interval $(-\rho, \rho)$. Thus, β'_i avoids B' even after prolonging. This is the path required by the claim.

We are left to consider the case when the domain of β_i does not contain a sufficiently long interval $[-\rho, \rho]$. Suppose without loss of generality that $d_\Gamma(y, \beta_i^-) \leq d_\Gamma(y, \beta_i^+)$. Then, recalling that β_i is 1-Lipschitz with respect to d_Γ , we must have

$$\begin{aligned} \max \left\{ K, \frac{\epsilon Kr}{8C}, \frac{\epsilon r}{8} \right\} &\geq d_\Gamma(y, \beta_i^-) \geq d_\Gamma(y, \gamma_i^-) - D_p \\ &\geq d_\Gamma(\gamma_i^-, \alpha(s)) - d_\Gamma(y, \alpha(s)) - D_p \\ &\geq (1 - \epsilon) \frac{r}{16C} - D_p. \end{aligned}$$

The last inequality is due to the fact that γ_i^- lies on γ , and hence outside the ball $\mathcal{B}_\Gamma(\alpha(s), r/(16C))$, whereas y lies within B , and hence in the larger ball $\mathcal{B}_\Gamma(\alpha(s), \epsilon \cdot r/(16C))$.

Given our choice of ϵ , and recalling that $r > 32C(D_p + K)$, we obtain a contradiction:

$$(1 - \epsilon) \frac{r}{16C} - D_p \geq \left[\frac{1}{32C} + \epsilon K \right] \cdot r - D_p > K + \epsilon Kr \geq \max \left\{ K, \frac{\epsilon Kr}{8C}, \frac{\epsilon r}{8} \right\}. \quad \square$$

Now, we complete the proof of the proposition. Recall that β'_i has the same endpoints as β_i , namely β_i^\pm . Let $\zeta_i \subseteq \Delta$ be a shortest path connecting the points β_i^+ and β_{i+1}^- . Joining the paths β'_i provided by the claim by the paths ζ_i , we form a path γ' entirely contained in Δ and with the same endpoints as γ .

Recalling that $\gamma_i^+ = \gamma_{i+1}^-$ and $d_\Gamma(\beta_i^\pm, \gamma_i^\pm) \leq D_p$, we have

$$d_\Delta(\beta_i^+, \beta_{i+1}^-) \leq C \cdot d_\Gamma(\beta_i^+, \beta_{i+1}^-) \leq 2CD_p.$$

It follows that $|\zeta_i| \leq 2CD_p$ for each i .

Now, suppose that $r > 32C^2D_p/\epsilon$. Then, since each β'_i avoids the ball $B = \mathcal{B}_\Delta(\alpha(s), \epsilon \cdot r/(16C))$, while each ζ_i has length at most $2CD_p$, the path γ' avoids the ball

$$\mathcal{B}_\Delta\left(\alpha(s), \epsilon \cdot \frac{r}{16C} - CD_p\right) \supseteq \mathcal{B}_\Delta\left(\alpha(s), \frac{\epsilon}{2} \cdot \frac{r}{16C}\right).$$

Thus, setting $\eta := \epsilon/2$, the path γ' avoids the ball $\mathcal{B}_\Delta(\alpha(s), \eta \cdot r/(16C))$.

We are left to bound the length of γ' :

$$\begin{aligned} |\gamma'| &\leq \sum |\zeta_i| + \sum |\beta'_i| \leq m \cdot 2CD_p + K \cdot \sum |\beta_i| \\ &= 2mCD_p + K \cdot \sum d_{\Theta_{j_i}}(v_i^{-1}\beta_i^-, v_i^{-1}\beta_i^+) \\ &\leq 2CD_p \cdot |\gamma| + KC^2 \cdot \sum d_\Gamma(\beta_i^-, \beta_i^+) \\ &\leq 2CD_p \cdot |\gamma| + KC^2 \cdot \sum (|\gamma_i| + 2D_p) \\ &\leq (2CD_p + KC^2 + 2KC^2D_p)|\gamma|. \end{aligned}$$

In conclusion, taking

$$K' := \max\left\{32C(D_p + K), \frac{32C^2D_p}{\epsilon}, 2CD_p + KC^2 + 2KC^2D_p\right\}$$

and assuming that $r > K'$, we have constructed a path γ' with the same endpoints as γ , entirely contained in Δ , avoiding the ball $\mathcal{B}_\Delta(\alpha(s), \eta \cdot r/(16C))$, and with length $|\gamma'| \leq K' \cdot |\gamma|$. This shows that

$$\delta_\alpha^\Delta\left(r, \eta \cdot \frac{1}{16C}\right) \leq K' \cdot \delta_\alpha^\Gamma\left(r, \frac{1}{16C}\right),$$

as required. □

Along with Lemma 2.10, Proposition 4.2 has the following consequence.

Corollary 4.3. *For every Morse gauge N , there exists a Morse gauge N' such that every N -Morse geodesic ray $\alpha \subseteq \Delta$ is an N' -Morse quasigeodesic in Γ .*

We are now ready to prove Theorem C.

Proof of Theorem C. By Corollary 2.16, V is undistorted in G . Recall that the relative Morse boundary $(\partial_M V, G)$ is by definition a subset of $\partial_M V$. Since all incident edge groups $E \leq V$ are relatively wide in V , we are in the setting of Assumption 4.1, so Corollary 4.3 implies that $(\partial_M V, G) = \partial_M V$.

In addition, Corollary 4.3 guarantees that the natural inclusion $\partial_M V \hookrightarrow \partial_M G$ is a Morse preserving map in the sense of [Cordes 2017, Definition 4.1]. Thus, the fact that $\partial_M V \hookrightarrow \partial_M G$ is a topological embedding follows from [Cordes 2017, Proposition 4.2]. \square

Acknowledgements

The authors are grateful to the referee for helpful suggestions. Fioravanti thanks the Max Planck Institute for Mathematics in Bonn, the University of Bonn and Ursula Hamenstädt for their hospitality and financial support while this work was being completed. Karrer was supported by the Israel Science Foundation (grant no. 1562/19).

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Received November 1, 2021. Revised January 20, 2022.

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
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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

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Volume 317 No. 2 April 2022

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