RELATING CUT AND PASTE INVARIANTS AND TQFTS

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Abstract

In this paper we shall be concerned with a relation between TQFTs and cut and paste invariants introduced in [KKNO73]. Cut and paste invariants, or SK invariants, are functions on the set of smooth manifolds that are invariant under the cutting and pasting operation. Central to the work in this paper are also SKK invariants, whose values on cut and paste equivalent manifolds differ by an error term depending only on the glueing diffeomorphism. Here we investigate a surprisingly natural group homomorphism between the group of invertible TQFTs and the group of SKK invariants and describe how these groups fit into an exact sequence. We conclude in particular that all positive real-valued SKK invariants can be realized as restrictions of invertible TQFTs.

All manifolds are smooth and oriented throughout unless stated otherwise.

1. Introduction

1.1. **Cut and Paste Invariants.** Our knowledge of cut and paste invariants is fruit of several decades of exciting developments in differential and algebraic topology. To understand these invariants, mathematicians needed a solid grasp of characteristic classes, cobordism theory, and the signature. The motivation to study these invariants came from the study of the index of elliptic operators.

The 1950s and 1960s saw incredible developments in all of these areas. In 1953, Vladimir Rokhlin uncovered much about the signature; he found a connection between Pontryagin classes and the signature and discovered the signature's importance in cobordism theory. Crucial here is also Thom's cobordism theory, which he developed during those same years. It was also at this time when Friedrich Hirzebruch proved his famous theorem relating the signature and the Pontryagin numbers, an immensely important development in the theory of characteristic classes. Work continued throughout the next decade, and Sergei Novikov first presented the additivity property

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of the signature in 1966. Once this result was published in 1970, our knowledge of the signature and cobordism was firm enough for work to be done on the SK and SKK invariants.

However, it was the study of the index of elliptic operators that actually brought attention to the study of these invariants. This topic had also been substantially developed in this time period, in particular by Michael Atiyah and Isadore Singer who proved the Atiyah-Singer Index Theorem in 1963. This was a groundbreaking development in both analysis and topology, and showed a deep connection between the two fields. In an attempt to investigate certain aspects this work, Klaus Jaenich noticed that the index of elliptic operators had the properties of invariants that were not influenced by cut and paste operations. In 1968 and 1969, he wrote two papers [Jan68], [Jan69] studying these invariants. These papers included very interesting results but lacked a systematic approach to these ideas.

This opened the door for four young authors, Ulrich Karras, Matthias Kreck, Walter Neumann, and Erich Ossa, to give these invariants a more thorough treatment. These four authors published a short book in 1973 titled *Cutting and Pasting of Manifolds; SK Groups*, in which they completely classified these invariants [KKNO73]. They were the first to use the terms "SK invariants" and "SKK invariants." The "SK" stands for "schneiden" and "kleben," which mean "cut" and "paste" in German. The second K in SKK stands for "kontrollierbar," the German word for "controllable."

1.2. **Topological Quantum Field Theories.** The idea of Topological Quantum Field Theories originated in the 1980s, an era of rapid development of our understanding of the relationship between geometry and physics. The most credit for the initial development of TQFTs is due to Edward Witten and Michael Atiyah. Both did work in mathematics and theoretical physics; Witten was more the physicist and Atiyah more the mathematician. Witten contributed by laying the mathematical foundation for super-symmetric quantum mechanics. This theory is rather complicated, and Atiyah contributed by looking at a simplified, purely mathematical version of Witten's theories. In [Ati88] Atiyah axiomatized TQFTs, which, unlike Witten's theory, did not involve any geometric ideas such as curvature or Riemannian metric.

1.3. **The origin of this project.** The relation between SKK invariants and invertible TQFTs was first investigated by Matthias Kreck, Stephan Stolz and Peter Teichner. The exact sequence presented in this paper was due to them, even if they had not published their work.

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2. Cut and Paste

In this section we will describe an equivalence relation on manifolds called the "cut and paste" relation. We will conclude by imposing further structure on these equivalence classes to form the SK and SKK groups.

We first briefly describe the cut and paste operation on oriented manifolds. To perform this operation on a manifold M, cut M along a codimension-1 submanifold Σ with trivial normal bundle and paste the resulting manifold back together via an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$.

Definition 2.1. Two closed oriented manifolds M and N are said to be *cut and paste equivalent* or *SK equivalent* if N can be obtained from M by a finite sequence of cut and paste operations. In this case we write $[M]_{SK} = [N]_{SK}$.

This is clearly an equivalence relation. A pictorial representation of a nontrivial cut and paste operation is shown in Figure 1. The figures on the top right and bottom are mapping tori of the map $f : \Sigma \to \Sigma$.

We now describe the cut and paste invariants. Let $\mathbb{M}_n$ denote the set of all closed oriented n-manifolds, and let $M, N \in \mathbb{M}_n$.

Definition 2.2. Let G be an abelian group. A function $\Theta : \mathbb{M}_n \to G$ is said to be an n-dimensional *cut and paste invariant* or *SK invariant* if the following hold.

- $\Theta(M) = \Theta(N)$ whenever M and N are cut and paste equivalent.
- $\Theta(M \coprod N) = \Theta(M) + \Theta(N)$

It follows from the additivity properties of the signature and Euler characteristic that both are SK invariants.

We can also define a weaker class of invariants, which will be especially interesting for later use.

Definition 2.3. Let G be an abelian group. A function $\xi : \mathbb{M}_n \to G$ is said to be an n-dimensional *SKK invariant* if the following hold.

- Suppose M and N are cut and paste equivalent, i.e. $M = X_1 \cup_f X_2$ and $N = X_1 \cup_g X_2$ so that f and g are the gluing diffeomorphisms of M and N, respectively. Then
$$\xi(M) - \xi(N) = \xi(f, g)$$
where $\xi(f, g) \in G$ depends only on f and g.
- $\xi(M \coprod N) = \xi(M) + \xi(N)$

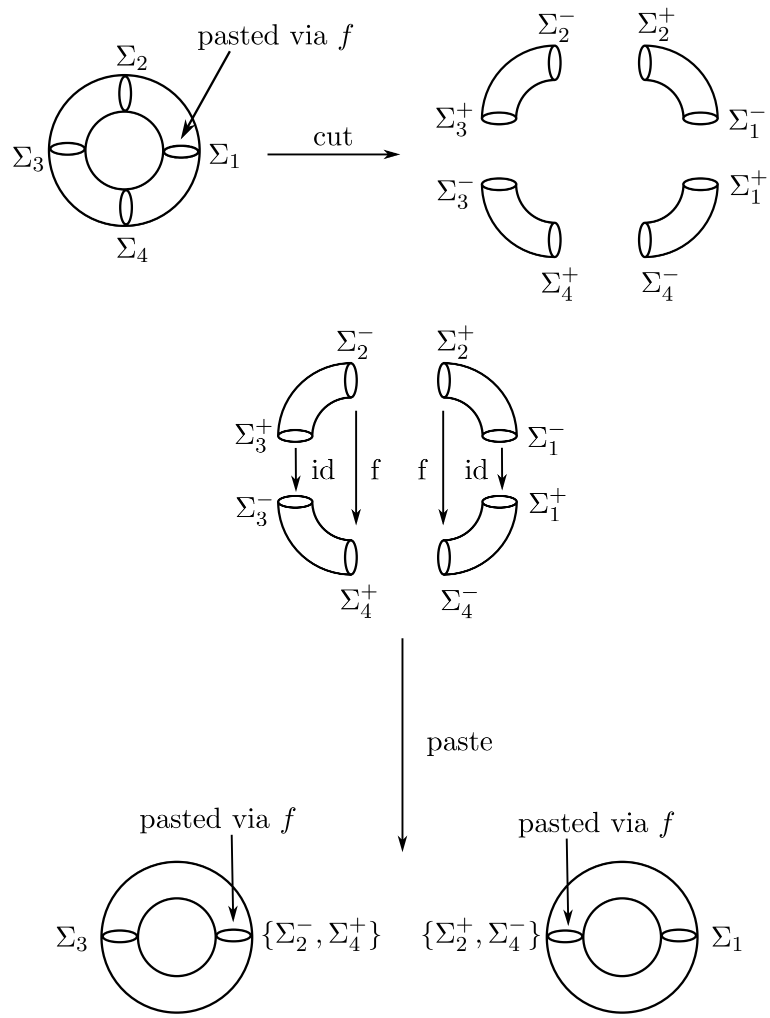


FIGURE 1. Cutting and pasting a mapping torus

It is clear from the definitions that every SK invariant is an SKK invariant. For an SK invariant, the "error" function $\xi(f, g)$ is always zero. Note that the converse is not true, since there exist SKK invariants which are not SK invariants. One such example is the Kervaire semicharacteristic.

Definition 2.4. Let M be a compact, oriented n-dimensional manifold. We define the *Kervaire semicharacteristic* $\chi_{1/2}$ by

$$\chi_{1/2}(M) = \begin{cases} \frac{1}{2}\chi(M) & n \text{ even} \\ \big(\sum_{i=0} \operatorname{rank} H_{2i}(M)\big) \bmod 2 & n \text{ odd}, \end{cases} \tag{2.1}$$

where χ is the Euler characteristic.

See [KKNO73, p. 44] for a proof that $\chi_{1/2}$ is an SKK invariant in dimensions $4n+1$. It is an SKK invariant in all dimensions other than dimensions $4n-1$.

In order to classify SK and SKK invariants, we must define the SK and SKK groups. $\mathbb{M}_n$ is a communtative monoid under the disjoint union operation, so we can form its Grothendieck group $\mathcal{G}(\mathbb{M}_n)$. We form quotient groups of this group, which will give us more concise and useful definitions of SK and SKK invariants.

Definition 2.5. Let R_n^{SK} denote the subgroup of $\mathcal{G}(\mathbb{M}_n)$ generated by all elements of the form $[M]\coprod -[N]$ where $[M]_{SK} = [N]_{SK}$. $\mathcal{G}(\mathbb{M}_n)/R_n^{SK} = SK_n$ is called the n^{th} *SK group*.

This definition gives us an alternative description of SK invariants.

Proposition 2.6. *Let G be an abelian group. A function $\Theta : \mathbb{M}_n \to G$ is an n-dimensional SK invariant if and only if it is an element of the group* $\mathrm{Hom}\,(SK_n, G)$.

We can now give a full description of SK invariants. In the following theorem, σ denotes the signature and χ denotes the Euler characteristic.

Theorem 2.7. [KKNO73, p. 7]

(a) *For n odd, we have*
$$SK_n = 0. \tag{2.2}$$
(b) *For $n \equiv 0 \mod 4$, we have*
$$\left(\frac{\chi - \sigma}{2}\right) \oplus \sigma : SK_n \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}, \tag{2.3}$$
so the signature σ and the Euler characteristic χ together form a complete set of invariants.

(c) *For $n \equiv 2 \mod 4$, we have*
$$\frac{\chi}{2} : SK_n \xrightarrow{\cong} \mathbb{Z}, \tag{2.4}$$
so χ is a complete invariant.

We would like to construct a quotient group of $\mathcal{G}(\mathbb{M}_n)$ that serves the same purpose as SK_n, but for SKK invariants. How can this be done? The stated condition
$$\xi([M]) - \xi([N]) = \xi(f, g) \tag{2.5}$$
when $[M]_{SK} = [N]_{SK}$ and f and g are the gluing diffeomorphisms of $M = X_1 \cup_f X_2$ and $N = X_1 \cup_g X_2$, respectively, is equivalent to the condition
$$\xi([M]) - \xi([N]) = \xi([M']) - \xi([N']) \tag{2.6}$$
when $[M']_{SK} = [N']_{SK}$ and f and g are also the gluing diffeomorphisms of $M' = Y_1 \cup_f Y_2$ and $N' = Y_1 \cup_g Y_2$, respectively. This leads to the following definition.

Definition 2.8. Let R_n^{SKK} denote the subgroup of $\mathcal{G}(\mathbb{M}_n)$ generated by all elements of the form

$$[M] \coprod -[N] \coprod -[M'] \coprod [N'], \tag{2.7}$$

where $[M]_{SK} = [N]_{SK}$, $[M']_{SK} = [N']_{SK}$ and the gluing diffeomorphisms are given by the above description. $\mathcal{G}(\mathbb{M}_n)/R_n^{SKK} = SKK_n$ is called the n^{th} *SKK group*.

Just as in the case of SK_n, we will let $[M]_{SKK}$ denote the SKK equivalence class of a closed manifold M. We now get our desired description of SKK invariants.

Proposition 2.9. *Let G be an abelian group. An n-dimensional SKK invariant is an element of the group* $\mathrm{Hom}\,(SKK_n, G)$.

3. Cobordisms and SKK Groups

In this section we briefly describe cobordisms, which is needed both for the classification of SKK invariants and the definition of TQFTs.

Definition 3.1. Let Σ_0 and Σ_1 be closed oriented $(n-1)$-manifolds. An *oriented n-cobordism* $M : \Sigma_0 \rightsquigarrow \Sigma_1$ is a manifold M along with orientation-preserving diffeomorphisms

$$\phi_{\text{in}} : \Sigma_0 \to \overline{\partial_{\text{in}} M} \quad \text{and} \quad \phi_{\text{out}} : \Sigma_1 \to \partial_{\text{out}} M, \tag{3.1}$$

where

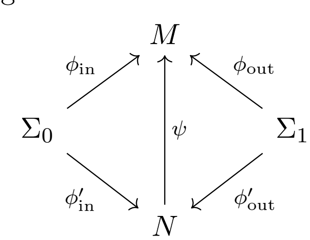
$$\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M. \tag{3.2}$$

Here $\partial_{\text{in}} M$ is called the *in-boundary* of M and $\partial_{\text{out}} M$ is called the *out-boundary* of M. Similarly, ϕ_{in} and ϕ_{out} are call the *in-boundary diffeomorphism* of M and the *out-boundary diffeomorphism* of M, respectively.

The cobordism relation is an equivalence relation and the set Ω_{n-1} consisting of all oriented cobordism classes of $(n-1)$-dimensional manifolds forms an abelian group with disjoint union as composition operation.

Note that there is a natural gluing operation on two cobordisms $M : \Sigma_0 \rightsquigarrow \Sigma_1$ and $N : \Sigma_1 \rightsquigarrow \Sigma_2$. The gluing diffeomorphism $\partial_{\text{out}} M \to \overline{\partial_{\text{in}} N}$ is the map $\phi'_{\text{in}} \circ \phi_{\text{out}}^{-1}$, where ϕ'_{in} is the in-boundary diffeomorphism $\Sigma_1 \to \overline{\partial_{\text{in}} N}$ of N. We will denote the resultant cobordism MN.

Definition 3.2. Two cobordisms $M : \Sigma_0 \rightsquigarrow \Sigma_1$ and $N : \Sigma_0 \rightsquigarrow \Sigma_1$ are said to be *equivalent* if there exists an orientation-preserving diffeomorphism ψ making the following diagram commute.



In this case we write $M \sim N$.

We now relate the cobordism groups to the SKK groups.

Theorem 3.3. [KKNO73, p. 44] *The homomorphism $SKK_n \to \Omega_n$ that assigns to each manifold its cobordism class is a surjective SKK invariant.*

Theorem 3.4. [KKNO73, p. 44] *The following sequence is exact:*

$$0 \to I_n \to SKK_n \to \Omega_n \to 0, \tag{3.3}$$

where

$$I_n = \begin{cases} \mathbb{Z} & n \equiv 0 \bmod 2 \\ \mathbb{Z}_2 & n \equiv 1 \bmod 4 \\ 0 & n \equiv 3 \bmod 4 \end{cases} \tag{3.4}$$

In addition, χ splits the sequence in dimensions divisible by 4, and $\chi_{1/2}$ splits the sequence in dimensions $n \equiv 1, 2 \bmod 4$.

Thus the SKK invariants are none other than linear combinations of the Euler characteristic, the Kervaire semicharacteristic, and bordism invariants.

4. Topological Quantum Field Theories and the Group of Invertible TQFTs

In this section we give the definition of a TQFT. Furthermore we will describe a multiplication operation on TQFTs, which will allow us to define the group of invertible TQFTs.

Definition 4.1. An *n-dimensional oriented TQFT* is a symmetric monoidal functor $\mathcal{T}$ from the n-cobordism category to the vector space category, assigning to each closed oriented $(n-1)$-manifold Σ a $\Bbbk$-vector space $\mathcal{T}(\Sigma)$ and to each oriented n-dimensional cobordism $M : \Sigma_0 \to \Sigma_1$ a linear map $\mathcal{T}(M) : \mathcal{T}(\Sigma_0) \to \mathcal{T}(\Sigma_1)$, satisfying the following properties:

(1) Two equivalent cobordisms have the same image.

$$M \sim N \implies \mathcal{T}(M) = \mathcal{T}(N) \tag{4.1}$$

(2) A glued cobordism goes to a composition of linear maps.

$$\mathcal{T}(MN) = \mathcal{T}(M) \circ \mathcal{T}(N) \tag{4.2}$$

(3) A cylinder cobordism gets sent to the identity map.

$$\mathcal{T}(\Sigma \times I) = Id_{\mathcal{T}(\Sigma)} \tag{4.3}$$

(4) Disjoint unions of $(n-1)$-manifolds and cobordisms get sent to a tensor product of vector spaces and linear maps, respectively.

$$\mathcal{T}(\Sigma \coprod \Sigma') = \mathcal{T}(\Sigma) \otimes \mathcal{T}(\Sigma') \tag{4.4}$$

$$\mathcal{T}(M \coprod N) = \mathcal{T}(M) \otimes \mathcal{T}(N) \tag{4.5}$$

(5) The empty manifold $\emptyset$ gets sent to the ground field $\Bbbk$.

$$\mathcal{T}(\emptyset) = \Bbbk \tag{4.6}$$

Similarly to the case of cobordisms, we will refer to all oriented TQFTs as "TQFTs" for brevity.

We can now form a *category* of TQFTs, which we will denote by $\mathbf{nTQFT}_{\Bbbk}$, where the objects are TQFTs and the arrows are natural transformations of TQFTs. This is a symmetric monoidal functor category, which has a natural symmetric monoidal structure.

Definition 4.2. The product of two TQFTs $\mathcal{T}_1$ and $\mathcal{T}_2$ is the TQFT

$$\mathcal{T}_1 \otimes \mathcal{T}_2 \tag{4.7}$$

that assigns to each $(n-1)$-manifold Σ the vector space

$$\mathcal{T}_1(\Sigma) \otimes \mathcal{T}_2(\Sigma) \tag{4.8}$$

and to each n-cobordism $M : \Sigma_0 \rightsquigarrow \Sigma_1$ the linear map

$$\mathcal{T}_1(M) \otimes \mathcal{T}_2(M) : \mathcal{T}_1(\Sigma_0) \otimes \mathcal{T}_2(\Sigma_0) \to \mathcal{T}_1(\Sigma_1) \otimes \mathcal{T}_2(\Sigma_1). \tag{4.9}$$

The trivial TQFT, which sends each $(n-1)$-manifold to $\Bbbk$ and each n-cobordism to the identity, is an identity element under the tensor product operation.

We wish to study the invertible objects in $\mathbf{nTQFT}_{\Bbbk}$, that is, the TQFTs $\mathcal{T}$ with an inverse $\mathcal{T}'$ such that $\mathcal{T} \otimes \mathcal{T}'$ is the trivial TQFT. It is a consequence of the axioms of TQFTs that TQFTs only assign finite-dimensional vector spaces to closed $(n-1)$-manifolds (See [Koc04, p. 31]). Since tensoring multiplies the dimension of finite-dimensional vector spaces, each vector space that an invertible TQFT $\mathcal{T}$ assigns to an $(n-1)$-manifold must be 1-dimensional, i.e. isomorphic to $\Bbbk$.

Since all linear maps $\Bbbk \to \Bbbk$ are simply scalar multiplication, each map that an invertible TQFT $\mathcal{T}$ assigns to an n-manifold can be canonically associated with a scalar. All linear maps assigned by invertible TQFTs must be invertible, that is, multiplication by a nonzero scalar.

It is clear that the set of invertible TQFTs forms a group under the composition operation in $\mathbf{nTQFT}_{\Bbbk}$. We will denote this group as $\mathbf{nTQFT}_{\Bbbk}^{\times}$.

What we will do next combines many of the ideas that we have presented so far. Our goal is to determine how invertible TQFTs evaluate two cut and paste equivalent closed manifolds, considered as cobordisms $\emptyset \rightsquigarrow \emptyset$.

Let $\mathcal{T}$ be an invertible TQFT and let M and N be cut and paste equivalent closed manifolds, with gluing diffeomorphisms f and g, respectively, as in Figure 2.

Also, let δ be the canonical isomorphism $\mathrm{Hom}_{\Bbbk}(\Bbbk, \Bbbk) \to \Bbbk$, where $\mathrm{Hom}_{\Bbbk}(\Bbbk, \Bbbk)$ is the vector space of $\Bbbk$-linear automorphisms of $\Bbbk$. Now using the fact that Σ has a collar neighborhood in both M and N, we can replace M and N with the equivalent cobordisms

$$M_1 C_f M_2 \quad \text{and} \quad M_1 C_g M_2, \tag{4.10}$$

where C_f and C_g denote the mapping cylinders of f and g respectively, as in Figure 4.

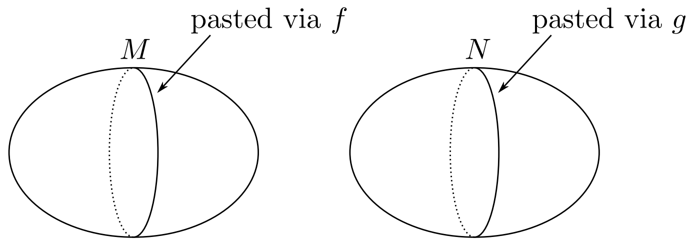


FIGURE 2. Cut and paste equivalent manifolds

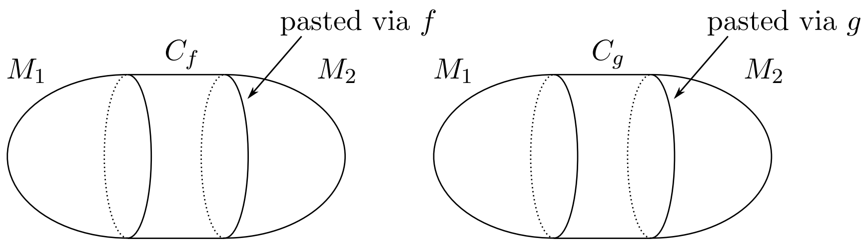


FIGURE 3. Equivalent cobordisms

Now evaluating $\mathcal{T}$ on both cobordisms and taking a quotient, we see an interesting relation.

$$\frac{\delta(\mathcal{T}(M_1C_fM_2))}{\delta(\mathcal{T}(M_1C_gM_2))} = \frac{\delta(\mathcal{T}(M_1)) \cdot \delta(\mathcal{T}(C_f)) \cdot \delta(\mathcal{T}(M_2))}{\delta(\mathcal{T}(M_1)) \cdot \delta(\mathcal{T}(C_g)) \cdot \delta(\mathcal{T}(M_2))} = \frac{\delta(\mathcal{T}(C_f))}{\delta(\mathcal{T}(C_g))} \tag{4.11}$$

Note that the above equation is valid because all scalars are required to be nonzero by the invertibility of $\mathcal{T}$.

5. RESULTS

We now relate the ideas of invertible TQFTs and SKK invariants. Let $\Bbbk^\times$ be the multiplicative group on $\Bbbk - \{0\}$. The relation will be expressed by means of a group homomorpism

$$\Psi_n : \mathbf{nTQFT}_{\Bbbk}^{\times} \to \mathrm{Hom}\,(SKK_n, \Bbbk^\times). \tag{5.1}$$

Let $\mathcal{T} \in \mathbf{nTQFT}_{\Bbbk}^{\times}$. If X is a smooth n-dimensional manifold, then we set $\Psi_n(\mathcal{T})([X]) = \delta(\mathcal{T}([X]))$. Now let M and N be cut and paste equivalent closed oriented n-manifolds, where f and g are the gluing diffeomorphisms of M and N, respectively. We have shown that

$$\frac{\Psi_n(\mathcal{T})(M)}{\Psi_n(\mathcal{T})(N)} = \frac{\delta(\mathcal{T}(M))}{\delta(\mathcal{T}(N))} = \frac{\delta(\mathcal{T}(C_f))}{\delta(\mathcal{T}(C_g))} = \Psi_n(\mathcal{T})(f, g). \tag{5.2}$$

The notation is a bit different since we are dealing with a multiplicative group, but this is precisely the first criterion in the description of SKK invariants in Definition 2.3.

Thus $\Psi_n(\mathcal{T})$ induces a homomorphism $SKK_n \to \Bbbk^\times$. Hence $\Psi_n(\mathcal{T})$ is an SKK invariant on n-manifolds, and Ψ_n is a homomorphism $\mathbf{nTQFT}_\Bbbk^\times \to \operatorname{Hom}(SKK_n, \Bbbk^\times)$.

This homomorphism Ψ_n gives a natural relationship between invertible TQFTs and SKK invariants. Note that this relationship only makes sense if G from Definition 2.3 is the multiplicative group of a field. This causes a problem for the SKK invariants we have given, since the target groups here include $\mathbb{Z}$, which is not isomorphic to the multiplication group of any field. However, the problem is solved by the exponential map $\exp : x \to e^x$. This exponential map exp sends $\mathbb{Z}$ monomorphically to a subgroup of $\mathbb{R}^\times$. This allows us to think of the our additive integer and rational-valued invariants as elements of $\operatorname{Hom}(SKK_n, \mathbb{R}^\times)$.

We will be interested in finding the kernel and image of Ψ_n. The kernel is easy to describe.

Theorem 5.1. *Let $\mathscr{T}_\Bbbk^n$ be the subgroup of $\mathbf{nTQFT}_\Bbbk^\times$ that consists of all $\mathcal{T} \in \mathbf{nTQFT}_\Bbbk^\times$ such that $\mathcal{T}(M) = id_\Bbbk$ for all n-cobordisms M with empty boundary. Then $\mathscr{T}_\Bbbk^n = \ker(\Psi_n)$.*

Proof. Let $\mathcal{T}$ be an invertible n-TQFT, and suppose $\Psi_n(\mathcal{T})$ is the trivial SKK_n-invariant, i.e. the invariant that sends all closed n-manifolds to $1 \in \Bbbk^\times$. Then obviously $\mathcal{T} \in \mathscr{T}_\Bbbk^n$. It is also clear that $\mathcal{T} \in \mathscr{T}_\Bbbk^n \implies \mathcal{T} \in \ker \Psi_n$. □

The proposition above expresses a degree of "forgetfulness" of Ψ_n. TQFTs assign values to all cobordisms, with or without boundary, and SKK invariants are only defined on closed manifolds. Thus by applying Ψ_n to $\mathcal{T}$, one loses some information about $\mathcal{T}$, as we will show in Theorem 5.3.

We start by proving the following Lemma.

Lemma 5.2. *If $\mathcal{T} \in \mathscr{T}_\Bbbk^n$, then for any cobordism M, $\mathcal{T}(M)$ only depends on the in-boundary and out-boundary of M.*

Proof. Let $M : \Sigma_0 \rightsquigarrow \Sigma_1$ and $N : \Sigma_0 \rightsquigarrow \Sigma_1$ be cobordisms. Consider the cobordism $\overline{M} : \overline{\Sigma}_0 \rightsquigarrow \overline{\Sigma}_1$. Also let

$$B_{\Sigma_0} : \emptyset \rightsquigarrow \Sigma_0 \coprod \overline{\Sigma}_0$$

and

$$B_{\Sigma_1} : \Sigma_1 \coprod \overline{\Sigma}_1 \rightsquigarrow \emptyset$$

be the cobordisms with cylinder n-manifolds and identity boundary diffeomorphisms. We have

$$\delta^{-1}(1) = \mathcal{T}(B_{\Sigma_0}(M \coprod \overline{M})B_{\Sigma_1}) = \mathcal{T}(B_{\Sigma_0})\mathcal{T}(M)\mathcal{T}(\overline{M})\mathcal{T}(B_{\Sigma_1})$$

so $\mathcal{T}(M) = (\mathcal{T}(B_{\Sigma_0})\mathcal{T}(\overline{M})\mathcal{T}(B_{\Sigma_1}))^{-1}$. The same reasoning shows that $\mathcal{T}(N) = (\mathcal{T}(B_{\Sigma_0})\mathcal{T}(\overline{M})\mathcal{T}(B_{\Sigma_1}))^{-1}$. □

Theorem 5.3. *$\mathscr{T}^n_{\Bbbk}$ is trivial if and only if $\Bbbk^\times$ is trivial.*

Proof. First suppose $\Bbbk^\times$ is trivial. Then there is obviously only one possible invertible TQFT, which is the trivial TQFT. In this case $\mathscr{T}^n_{\Bbbk}$ is clearly trivial.

Now suppose $\Bbbk^\times$ is nontrivial. We must define $\mathcal{T} \in \mathbf{nTQFT}^\times_{\Bbbk}$ so that $\mathcal{T}$ evaluates closed manifolds trivially and manifolds with boundary nontrivially. We proceed as follows.

Let $\mathcal{T}(\Sigma) = \Bbbk$ for all $(n-1)$-manifolds Σ. Then assign to each closed connected $(n-1)$-manifold Σ a scalar λ_Σ. We require $\lambda_\Sigma \notin \{0,1\}$ if $\Sigma \neq \emptyset$, and $\lambda_\emptyset = 1$. Now for any cobordism

$$M : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{j=0}^{m} \Sigma_j \tag{5.3}$$

we define

$$\mathcal{T}(M) = \delta^{-1}\Big(\prod_{i=0}^{n} \lambda_{\Sigma_i} \cdot \prod_{j=0}^{m} \lambda_{\Sigma_j}^{-1}\Big). \tag{5.4}$$

Now we check the axioms of Definition 4.1 in order. The value $\mathcal{T}$ assigns to an n-cobordism depends only on its in-boundary and out-boundary. Equivalent cobordisms must have the same in-boundary and out-boundary, so (1) holds.

Now consider axiom (2), and let

$$N : \coprod_{j=0}^{m} \Sigma_j \rightsquigarrow \coprod_{k=0}^{l} \Sigma_k \tag{5.5}$$

be another cobordism. We have

$$\begin{aligned}
\mathcal{T}(MN) &= \delta^{-1}\Big(\prod_{i=0}^{n} \lambda_{\Sigma_i} \cdot \prod_{k=0}^{m} \lambda_{\Sigma_k}^{-1}\Big) \\
&= \delta^{-1}\Big(\prod_{i=0}^{n} \lambda_{\Sigma_i} \cdot \prod_{j=0}^{m} \lambda_{\Sigma_j}^{-1} \cdot \prod_{j=0}^{m} \lambda_{\Sigma_j} \cdot \prod_{k=0}^{l} \lambda_{\Sigma_k}^{-1}\Big) \\
&= \delta^{-1}\Big(\prod_{i=0}^{n} \lambda_{\Sigma_i} \cdot \prod_{j=0}^{m} \lambda_{\Sigma_j}^{-1}\Big) \circ \delta^{-1}\Big(\cdot \prod_{j=0}^{m} \lambda_{\Sigma_j} \cdot \prod_{k=0}^{l} \lambda_{\Sigma_k}^{-1}\Big) \\
&= \mathcal{T}(M) \circ \mathcal{T}(N)
\end{aligned} \tag{5.6}$$

and (2) is satisfied. A connected cylinder cobordism of Σ is a cobordism $\Sigma \rightsquigarrow \Sigma$, so we have

$$\begin{aligned}
\mathcal{T}(\Sigma \times I) &= \delta^{-1}(\lambda_\Sigma \cdot \lambda_\Sigma^{-1}) \\
&= \delta^{-1}(1) \\
&= id_{\Bbbk^\times}
\end{aligned} \tag{5.7}$$

so (3) is satisfied. The disconnected case follows from (4), which clearly follows from Equation (5.4). (5) is satisfied trivially.

Thus $\mathcal{T}$ is an invertible n-TQFT, and for all closed manifolds M,

$$\begin{aligned}\mathcal{T}(M) &= \delta^{-1}(\lambda_{\emptyset} \circ \lambda_{\emptyset}^{-1}) \\ &= \delta^{-1}(1) \\ &= id_{\Bbbk^\times}.\end{aligned} \tag{5.8}$$

We chose $\mathcal{T}$ to take on nontrivial values, so $\mathcal{T}$ is not the trivial TQFT. □

We now want to try to figure out the image of Ψ_n. One problem, of course, comes from the "forgetfullness" of Ψ_n. Given an SKK invariant ξ, we would like to choose an invertible n-TQFT $\mathcal{T}$ such that $\Psi_n(\mathcal{T}) = \xi$. But ξ gives us no explicit information as to how $\mathcal{T}$ should evaluate cobordisms with nonempty boundary.

Some of the SKK invariants that we've listed, however, do give us information about how to evaluate such cobordisms. We cannot with perfect accuracy say that any of these invariants define TQFTs, since these invariants (composed with exponential functions, if necesarry) assign nonzero scalars to manifolds rather than invertible linear maps $\Bbbk \to \Bbbk$. This difference is, however, superficial, and we therefore choose to ignore it. Specifically, if $\mathbb{M}_n^\partial$ denotes the set of diffeomorphism classes of compact oriented n-manifolds with boundary, then we will say that a function $\Theta : \mathbb{M}_n^\partial \to \Bbbk^\times$ defines an invertible n-TQFT if the formula

$$\mathcal{T}(M) = \delta^{-1}(\Theta(M)) \tag{5.9}$$

defines an invertible n-TQFT. The following proposition shows us exactly when Θ has this property.

Theorem 5.4. *The function $\Theta : \mathbb{M}_n^\partial \to \Bbbk^\times$ defines an invertible n-TQFT if and only if*

$$\Theta([M \underset{f}{\cup} N]) = \Theta([M]) \cdot \Theta([N]) \tag{5.10}$$

for all $[M], [N] \in \mathbb{M}_n^\partial$. Here f is any orientation-preserving diffeomorphism $\partial_{\text{out}} M \to \overline{\partial_{\text{in}} N}$, where $\partial_{\text{out}} M$ and $\partial_{\text{in}} N$ are unions of boundary components of M and N, respectively.

Proof. First suppose that Equation (5.10) holds. We check the TQFT axioms of Definition 4.1.

(1) is clearly satisfied, since for any equivalent n-cobordisms M and N, there is an orientation-preserving diffeomorphism $\psi : M \to N$. Θ must evaluate such cobordisms equally. (2) is satisfied by Equation (5.10). To show (3), note that

$$\Theta([\Sigma \times I]) = \Theta([\Sigma \times I]) \cdot \Theta([\Sigma \times I]) \tag{5.11}$$

for all closed $(n-1)$-manifolds Σ. This true because the two identical cylinder cobordisms can be glued to produce another cylinder cobordism.

Thus we have $\Theta([\Sigma \times I]) = 1$. (4) is also satisfied by Equation (5.10), where f is an empty map. (5) is satisfied trivially.

Now suppose Θ defines an invertible n-TQFT $\mathcal{T}$. Then $\mathcal{T}$ must evaluate n-cobordisms based only on their oriented diffeomorphism class. Now let $f : \partial_{\text{out}} M \to \overline{\partial_{\text{in}} N}$ be an orientation-preserving diffeomorphism. We can easily form cobordisms $M : \overline{\partial_{\text{in}} M} \rightsquigarrow \partial_{\text{out}} M$ and $N : \partial_{\text{out}} M \rightsquigarrow \partial_{\text{out}} N$, where the in-boundary diffeomorphism of N is f and all other diffeomorphisms are the identity. The resulting glued cobordism has

$$M \underset{f}{\cup} N \tag{5.12}$$

as its n-manifold. Thus by axiom (2) of Definition 4.1, Equation (5.10) holds. □

Corollary 5.5. *If an oriented n-diffeomorphism invariant Θ on manifolds with boundary defines an invertible n-TQFT, then it restricts to a linear combination of the Euler characteristic and signature on closed manifolds.*

Proof. Because our choice of f was arbitrary in Theorem 5.4, Θ restricts to an SK invariant on closed manifolds. The result follows from Theorem 2.7. □

Corollary 5.6. *The Euler characteristic χ and semicharacteristic $\chi_{1/2}$ define invertible n-TQFTs if and only if n is even. The signature σ defines an n-TQFT.*

Proof. For n even, χ and $\chi_{1/2}$ satisfy Equation (5.10) by the union formula for the Euler characteristic and the fact that the Euler characteristic of any closed odd-dimensional manifold is zero. The result then follows from Theorem 5.4.

Now let n be odd. The n-disk has Euler characteristic 1. We can glue two n-disks via the identity of the boundary to form an n-sphere, which has Euler characteristic 0, contradicting Equation (5.10). Thus the Euler characteristic cannot does not define an invertible n-TQFT for n odd.

That $\chi_{1/2}$ cannot does not define an n-TQFT for n odd follows from Corollary 5.5.

The signature σ satisfies Equation (5.10) by Novikov additivity. □

Note that the above corollary is not a proof that Ψ_n is not surjective. We have only given conditions for an *oriented diffeomorphism invariant* on manifolds with boundary to define a TQFT. In general, an n-TQFT can pick up more information than just the oriented diffeomorphism class of the n-manifold; in particular, it notices the choice of boundary manifolds and diffeomorphisms. These choices completely determine how cobordisms are glued. This is why Θ does not notice how manifolds are glued.

We would like to describe

$$\Psi_n : \mathbf{nTQFT}_{\Bbbk}^{\times} \to \operatorname{Hom}\left(SKK_n, \Bbbk^{\times}\right) \tag{5.13}$$

by giving an exact sequence that includes these terms and the kernel and cokernel of Ψ_n. This is difficult to do in general, but under a few assumptions it is feasible. In particular, we shall see that if the target group of the SKK invariants is the multiplicative group of positive reals, then we can describe the sequence. Before we go into the description of this exact sequence we give a preliminary definition and lemma.

Definition 5.7. Let M be a compact oriented manifold with boundary. We define the *double* of M to be the closed manifold

$$D(M) = M \underset{id_{\partial M}}{\cup} \overline{M}. \tag{5.14}$$

The following lemma gives a relation in SKK_n that will be necessary for the description of our exact sequence.

Lemma 5.8. *Let X_1, X_2, and X_3 be oriented n-manifolds with boundaries Σ_1, Σ_2 and Σ_3, respectively. An example is pictured in Figure 4.*

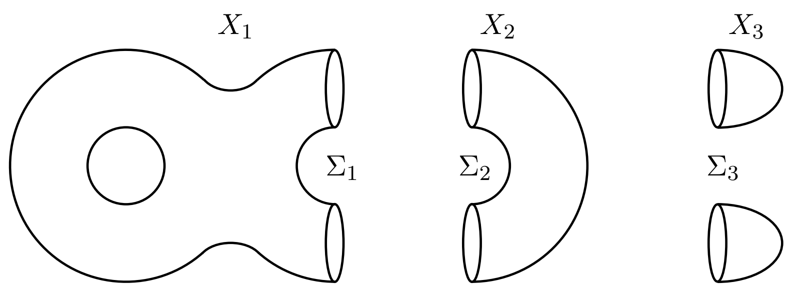


FIGURE 4. X_1, X_2, and X_3

Let

$$f : \Sigma_1 \to \Sigma_2 \text{ and } g : \Sigma_2 \to \overline{\Sigma}_3 \tag{5.15}$$

be orientation-preserving diffeomorphisms. Then in SKK_n, we have

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [X_2 \underset{g}{\cup} X_3] = [X_1 \underset{g\circ f}{\cup} X_3] + [D(X_2)]. \tag{5.16}$$

The desired relation is pictured in Figure 5.

Proof. Recall that in SKK_n, whenever M and N are pasted, respectively, via the same diffeomorphisms as M' and N',

$$[M] - [N] = [M'] - [N']. \tag{5.17}$$

We will make use of this equation to prove the lemma. To do this, we must start with two "cut" manifolds with boundary, glue each in two different ways, and apply (5.17). Our starting manifolds will be

$$X_1^* = X_1 \coprod X_2 \coprod X_2' \coprod \overline{X}_3, \tag{5.18}$$

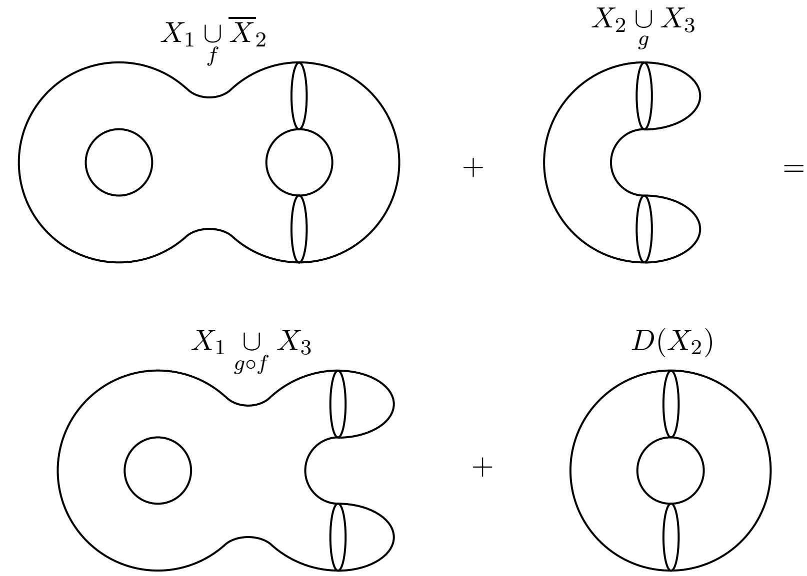


FIGURE 5. Our desired relation in SKK_n

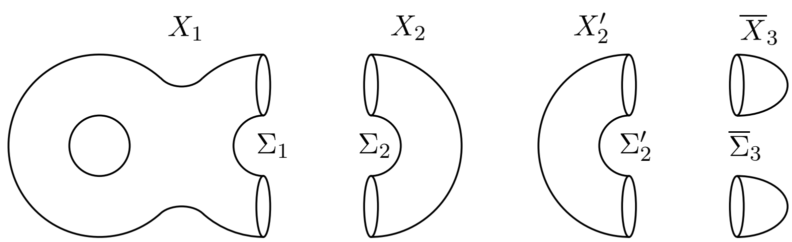


FIGURE 6. X_1^*

which is pictured in Figure 6,
and

$$X_2^* = X_1 \coprod X_2 \coprod X_2' \coprod C_{g^{-1}} \underset{id_{\Sigma_2}}{\cup} \overline{X}_2, \tag{5.19}$$

where $C_{g^{-1}}$ is the mapping cylinder of g^{-1}. This manifold is pictured in Figure 7.

Both have

$$\Sigma_1 \coprod \Sigma_2 \coprod \Sigma_2' \coprod \overline{\Sigma}_3 \tag{5.20}$$

as their boundaries. The two gluing diffeomorphisms are

$$F_1 = \begin{cases} f: & \Sigma_1 \to \Sigma_2 \\ g: & \Sigma_2' \to \overline{\Sigma}_3 \end{cases} \quad \text{and} \quad F_2 = \begin{cases} g \circ f: & \Sigma_1 \to \overline{\Sigma}_3 \\ id: & \Sigma_2 \to \Sigma_2' \end{cases} \tag{5.21}$$

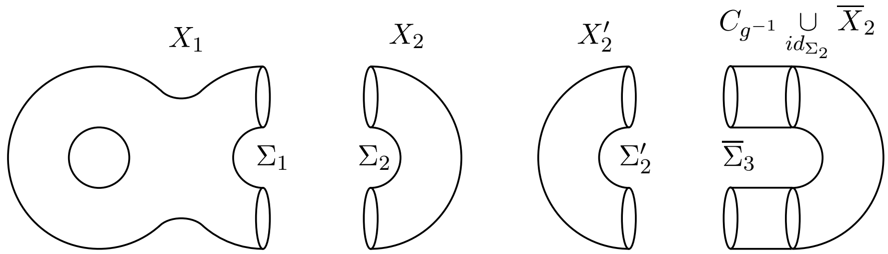


FIGURE 7. X_2^*

Gluing X_1^* via F_1, we obtain

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [X_2 \underset{g}{\cup} X_3], \tag{5.22}$$

as in Figure 8.

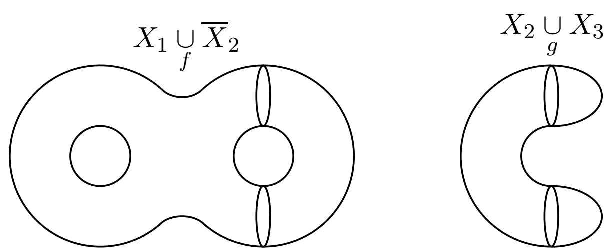


FIGURE 8. X_1^* glued via F_1

Gluing X_1^* via F_2, we obtain

$$[X_1 \underset{g\circ f}{\cup} X_3] + [D(X_2)], \tag{5.23}$$

as in Figure 9.

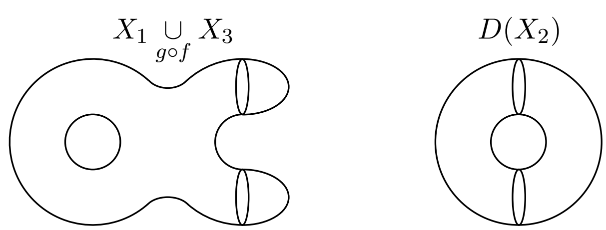


FIGURE 9. X_1^* glued via F_2

Gluing X_2^* via F_1, we obtain

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [D(X_2)], \tag{5.24}$$

as in Figure 10.

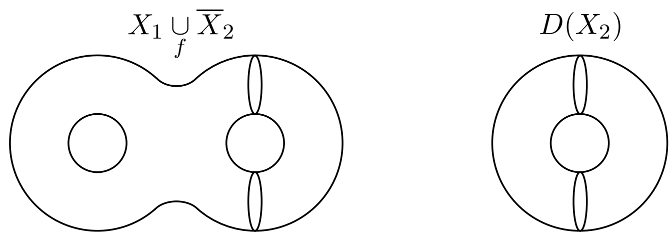


FIGURE 10. X_2^* glued via F_1

Gluing X_2^* via F_2, we obtain

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [D(X_2)], \tag{5.25}$$

as in Figure 11.

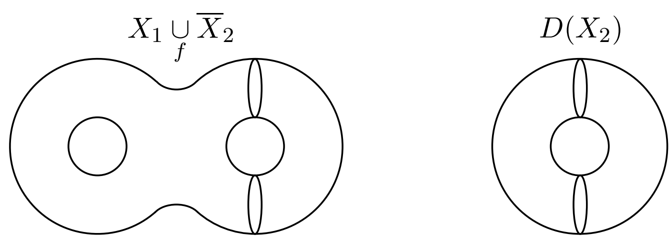


FIGURE 11. X_2^* glued via F_2

Now we apply Equation (5.17). We have

$$\begin{aligned}[X_1 \underset{f}{\cup} \overline{X}_2] + [X_2 \underset{g}{\cup} X_3] - [X_1 \underset{g\circ f}{\cup} X_3] - [D(X_2)] = [X_1 \underset{f}{\cup} \overline{X}_2] + [D(X_2)] \\ - [X_1 \underset{f}{\cup} \overline{X}_2] - [D(X_2)].\end{aligned}$$

Therefore

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [X_2 \underset{g}{\cup} X_3] - [X_1 \underset{g\circ f}{\cup} X_3] - [D(X_2)] = 0,$$

and

$$[X_1 \underset{f}{\cup} \overline{X}_2] + [X_2 \underset{g}{\cup} X_3] = [X_1 \underset{g\circ f}{\cup} X_3] + [D(X_2)],$$

as desired. □

We now have everything we need to describe our sequence.

Theorem 5.9. *Let $\mathscr{T}^n_{\mathbb{R}}(1,-1)$ denote the subgroup of $\mathbf{nTQFT}^\times_{\mathbb{R}}$ that consists of all invertible n-TQFTs that take on the values of $\delta^{-1}(1)$ and $\delta^{-1}(-1)$ on closed manifolds. Then the following sequence is split exact:*

$$0 \to \mathscr{T}^n_{\mathbb{R}}(1,-1) \xrightarrow{i_n} \mathbf{nTQFT}^\times_{\mathbb{R}} \xrightarrow{|\Psi_n|} \mathrm{Hom}\,(SKK_n, \mathbb{R}^\times_+) \to 0 \tag{5.26}$$

Here i_n is the inclusion map, $\mathbb{R}^\times_+$ is the multiplicative group of positive reals, and $|\Psi_n|$ is the homomorphism

$$|\Psi_n|(\mathcal{T}) := [M]_{SKK} \to |\delta(\mathcal{T}([M]_{SKK}))|. \tag{5.27}$$

Proof. Checking exactness at the term $\mathbf{nTQFT}^\times_{\mathbb{R}}$ is easy. Because of the absolute value bars in the definition of $|\Psi_n|$, it is clear that $|\Psi_n| \circ i_n$ is trivial. Also, if $|\Psi_n|(\mathcal{T})$ is the trivial SKK_n invariant, then

$$|\delta(\mathcal{T}([M]_{SKK}))| = 1 \tag{5.28}$$

on all closed manifolds X. This shows that

$$\mathcal{T}([M]_{SKK}) = \delta^{-1}(\pm 1) \tag{5.29}$$

and $\mathcal{T} \in \mathscr{T}^n_{\mathbb{R}}(1,-1)$.

The difficult part is defining the splitting homomorphism

$$\mathcal{S} : \mathrm{Hom}\,(SKK_n, \mathbb{R}^\times_+) \to \mathbf{nTQFT}^\times_{\mathbb{R}}. \tag{5.30}$$

That is, given an n-dimensional SKK invariant ξ with values in the positive reals, we must find an invertible TQFT $\mathcal{S}(\xi)$ such that $|\Psi_n|(\mathcal{S}(\xi)) = \xi$. This cannot be a completely arbitrary assignment, because $\mathcal{S}$ must be a homomorphism.

We first make use of Theorem 3.4 to describe $\mathrm{Hom}\,(SKK_n, \mathbb{R}^\times_+)$. We let χ^* denote the subgroup of $\mathrm{Hom}\,(SKK_n, \mathbb{R}^\times_+)$ given by multiples of the exponential map composed with the Euler characteristic. Since $\mathbb{R}^\times_+$ is torsion-free, we have

$$\mathrm{Hom}\,(SKK_n, \mathbb{R}^\times_+) = \begin{cases} 0 & n \text{ odd} \\ \chi^* & n \equiv 2 \bmod 4 \\ \chi^* \oplus \mathrm{Hom}\,(\Omega_n, \mathbb{R}^\times_+) & n \equiv 0 \bmod 4 \end{cases} \tag{5.31}$$

Here we are using the fact that Ω_n is finite for $n \not\equiv 0 \mod 4$.

We know exactly how to define $\mathcal{S}$ on the direct summands given by χ^*; $\exp \circ \chi$ is a generator of χ^*, so we simply take $S(\chi)(M) = \delta^{-1}(\exp(\chi(M)))$ for a cobordism with n-manifold M. By Theorem 5.4, this defines an invertible TQFT, and we obviously have

$$|\Psi_n| \circ \mathcal{S} = id \tag{5.32}$$

on these direct summands. So it only remains to define $\mathcal{S}$ on $\mathrm{Hom}\,(\Omega_n, \mathbb{R}^\times_+)$ where $4|n$. That is, we can assume $4|n$ and $\xi \in \mathrm{Hom}\,(\Omega_n, \mathbb{R}^\times_+)$. Now we take the necessary steps to define $\mathcal{T}$.

Under these assumptions, Ω_{n-1} is finite. Let l be the order of Ω_{n-1}. Then for every closed connected $(n-1)$-manifold Σ, we can choose an n-manifold B_Σ with boundary given by

$$\partial B_\Sigma = \coprod_l \Sigma. \tag{5.33}$$

For simplicity we require $B_\emptyset = \emptyset$. Now let

$$M : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{j=0}^{m} \Sigma_j \tag{5.34}$$

be a cobordism, where each Σ_i and Σ_j is connected. Let $\phi_{i,\text{in}}$ and $\phi_{j,\text{out}}$ be the in-boundary and out-boundary diffeomorphisms of the components of ∂M. Before we give the definition, we form a closed manifold $C(M)$ as follows: Let ϕ^*_{in} be the orientation-preserving diffeomorphism

$$\coprod_{i=0}^{n} \partial B_{\Sigma_i} \to \coprod_l \partial_{\text{in}} \overline{M} \tag{5.35}$$

given by the appropriate $\phi_{i,\text{in}}$ on each component. Also let ϕ^*_{out} be the orientation-preserving diffeomorphism

$$\coprod_{j=0}^{m} \partial B_{\Sigma_j} \to \coprod_l \partial_{\text{out}} M \tag{5.36}$$

given by the appropriate $\phi_{j,\text{out}}$ on each component. Now define

$$C(M) = \Big(\coprod_{i=0}^{n} B_{\Sigma_i}\Big) \bigcup_{\phi^*_{\text{in}}} \Big(\coprod_l M\Big) \bigcup_{(\phi^*_{\text{out}})^{-1}} \Big(\coprod_{j=0}^{m} \overline{B}_{\Sigma_j}\Big). \tag{5.37}$$

Now we can define $\mathcal{S}(\xi)(M)$ by the following equation

$$\mathcal{S}(\xi)(M) = \delta^{-1}(\xi(C(M))^{1/l}) \tag{5.38}$$

$\mathcal{S}$ is clearly a homomorphism. We need to check the axioms of Definition 4.1 to show that $\mathcal{S}(\xi)$ is an n-TQFT. Checking (1) first, suppose that

$$M : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{j=0}^{m} \Sigma_j \text{ and } N : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{j=0}^{m} \Sigma_j \tag{5.39}$$

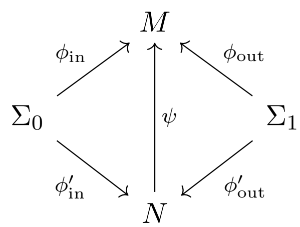
are equivalent coborisms. We will set

$$\Sigma_0 = \coprod_{i=0}^{n} \Sigma_i \text{ and } \Sigma_1 = \coprod_{j=0}^{m} \Sigma_j \tag{5.40}$$

for notational simplicity. In addition, we will have ϕ_{in} and ϕ_{out} be the boundary diffeomorphisms of M and ϕ'_{in} and ϕ'_{out} be the boundary diffeomorphisms of N. Since ξ evaluates diffeomorphic manifolds equally, we need only show that $C(M)$ and $C(N)$ are diffeomorphic. Since the in- and out-boundaries of M and N are the same, (1) will hold by Equation (5.38). We have diffeomorphisms for each of the gluing components of $C(M)$, which include

$$\begin{aligned} id : \coprod_{i=0}^{n} B_{\Sigma_i} &\to \coprod_{i=0}^{n} B_{\Sigma_i} \\ \psi : N &\to M \\ id : \coprod_{j=0}^{m} B_{\Sigma_j} &\to \coprod_{j=0}^{m} B_{\Sigma_j} \end{aligned} \tag{5.41}$$

for each of the gluing components of $C(M)$. Recall that the diagram



commutes. This amounts to the assertion that these diffeomorphisms of 5.41 agree on the glued areas of $C(M)$ and $C(N)$. Thus $C(M)$ and $C(N)$ are diffeomorphic, and (1) holds.

Now we check (2). Let

$$M : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{j=0}^{m} \Sigma_j \text{ and } N : \coprod_{j=0}^{m} \Sigma_j \rightsquigarrow \coprod_{k=0}^{p} \Sigma_k \tag{5.42}$$

be two cobordisms. To prove (2), we apply Lemma 5.8. Keeping the notation from the construction of $C(M)$, set

$$\begin{aligned} X_1 &= \Big(\coprod_{i=0}^{n} B_{\Sigma_i}\Big) \bigcup_{\phi^*_{\text{in}}} \Big(\coprod_{l} M\Big) \\ X_2 &= \coprod_{j=0}^{m} B_{\Sigma_j} \\ X_3 &= \Big(\coprod_{l} N\Big) \bigcup_{(\phi^{*\prime}_{\text{out}})^{-1}} \Big(\coprod_{i=0}^{n} \overline{B}_{\Sigma_i}\Big) \\ f &= (\phi^*_{\text{out}})^{-1} \\ g &= \phi^{*\prime}_{\text{in}}. \end{aligned} \tag{5.43}$$

Lemma 5.8 gives

$$[X_1 \cup_f \overline{X}_2] + [X_2 \cup_g X_3] = [X_1 \cup_{g \circ f} X_3] + [D(X_2)] \tag{5.44}$$

in SKK_n, which gives in our case

$$[C(M)] + [C(N)] = [C(MN)] + [D(X_2)]. \tag{5.45}$$

Now note that because $[D(X_2)]$ bounds and $\xi \in \operatorname{Hom}(\Omega_n, \mathbb{R}_+^\times)$, we have $\xi([D(X_2)]) = 1$. Thus

$$\begin{aligned}\mathcal{S}(\xi)(MN) &= \delta^{-1}(\xi(C(MN))^{1/l}) \\ &= \delta^{-1}(\xi(C(M))^{1/l} \cdot \xi(C(N))^{1/l}) \\ &= \delta^{-1}(\xi(C(M))^{1/l}) \circ \delta^{-1}(\xi(C(N))^{1/l}) \\ &= \mathcal{S}(\xi)(M) \circ \mathcal{S}(\xi)(N),\end{aligned} \tag{5.46}$$

as desired. Thus (2) holds.

It should be clear that

$$C(M \coprod N) = C(M) \coprod C(N). \tag{5.47}$$

From this (3) clearly follows. For (4), note that for a cylinder cobordism

$$\Big(\coprod_{i=0}^{n} \Sigma_i\Big) \times I : \coprod_{i=0}^{n} \Sigma_i \rightsquigarrow \coprod_{i=0}^{n} \Sigma_i, \tag{5.48}$$

we have

$$C\Big(\Big(\coprod_{i=0}^{n} \Sigma_i\Big) \times I\Big) = D\Big(\coprod_{i=0}^{n} B_{\Sigma_i}\Big). \tag{5.49}$$

Thus

$$\begin{aligned}\mathcal{S}(\xi)\Big(\Big(\coprod_{i=0}^{n} \Sigma_i\Big) \times I\Big) &= \delta^{-1}\Big(\xi\Big(C\Big(\Big(\coprod_{i=0}^{n} \Sigma_i\Big) \times I\Big)\Big)^{1/l}\Big) \\ &= \delta^{-1}\Big(\xi\Big(D\Big(\coprod_{i=0}^{n} B_{\Sigma_i}\Big)\Big)^{1/l}\Big) \\ &= \delta^{-1}(1) \\ &= id_{\mathbb{R}},\end{aligned} \tag{5.50}$$

and (4) holds. (5) holds trivially. Thus $\mathcal{S}(\xi)$ is an invertible n-TQFT.

Lastly, we need to check that

$$|\Psi_n| \circ \mathcal{S}(\xi) = \xi \tag{5.51}$$

for all $\xi \in \operatorname{Hom}(\Omega_n, \mathbb{R}_+^\times)$. By our requirement that $B_\emptyset = \emptyset$, we have for each closed manifold M

$$C(M) = \coprod_l M. \tag{5.52}$$

Thus

$$\begin{aligned}|\Psi_n| \circ \mathcal{S}(\xi)(M) &= |\Psi_n|(\delta^{-1}(\xi(C(M))^{1/l})) \\ &= |\Psi_n|(\delta^{-1}(\xi(M))) \\ &= \xi(M).\end{aligned} \tag{5.53}$$

□

The construction of the splitting homomorphism $\mathcal{S}$ is *not* independent of the choices of the B_Σ's. To illustrate this, let $\xi \in \operatorname{Hom}(SKK_8, \mathbb{R}_+^\times)$ be the bordism invariant given by $\xi(M) = \exp(p_2(M))$, where p_2 denotes the Pontryagin number given by the trivial partition of a two-element set. Also let D^8 and $\mathring{D}^8$ denote the closed and open 8-disks respectively. Now consider the cobordism

$$D^8 : \emptyset \rightsquigarrow S^7. \tag{5.54}$$

Since Ω_7 has order 1, D^8 and $\mathbb{C}P^4 - \mathring{D}^8$ are both sufficient choices for B_{S^7}. Now defining $\mathcal{S}$ with the choice D^8, we have

$$\begin{aligned} \mathcal{S}(\xi)(D^8) &= \delta^{-1}(\xi(S^8)) \\ &= \delta^{-1}(\exp(p_2(S^8))) \\ &= \delta^{-1}(\exp(0)) \\ &= id_{\mathbb{R}}. \end{aligned} \tag{5.55}$$

Defining $\mathcal{S}$ with the choice $\mathbb{C}P^4 - \mathring{D}^8$, we have

$$\begin{aligned} \mathcal{S}(\xi)(D^8) &= \delta^{-1}(\xi(\mathbb{C}P^4)) \\ &= \delta^{-1}(\exp(p_2(\mathbb{C}P^4))) \\ &= \delta^{-1}(\exp(10)) \\ &\neq id_{\mathbb{R}}. \end{aligned} \tag{5.56}$$

Given a closed manifold Σ that bounds, there is no canonical choice of manifold M with $\partial M = \Sigma$. This is why we do not explicitly define $\mathcal{S}$. An explicit definition is not necessary, however, for showing that the sequence splits.

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