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When do improved covariance matrix estimators enhance portfolio optimization? An empirical comparative study of nine estimators

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The use of improved covariance matrix estimators as an alternative to the sample estimator is considered an important approach for enhancing portfolio optimization. Here we empirically compare the performance of nine improved covariance estimation procedures using daily returns of 90 highly capitalized US stocks for the period 1997–2007. We find that the usefulness of covariance matrix estimators strongly depends on the ratio between the estimation period T and the number of stocks N, on the presence or absence of short selling, and on the performance metric considered. When short selling is allowed, several estimation methods achieve a realized risk that is significantly smaller than that obtained with the sample covariance method. This is particularly true when T/N is close to one. Moreover, many estimators reduce the fraction of negative portfolio weights, while little improvement is achieved in the degree of diversification. On the contrary, when short selling is not allowed and $T > N$, the considered methods are unable to outperform the sample covariance in terms of realized risk, but can give much more diversified portfolios than that obtained with the sample covariance. When $T < N$, the use of the sample covariance matrix and of the pseudo-inverse gives portfolios with very poor performance.

Keywords: Portfolio optimization; Correlation structures; Statistical methods; Econophysics

1. Introduction

Portfolio optimization (Markowitz 1952, 1959, Elton and Gruber 1995) is one of the main topics in quantitative finance. Markowitz's solution to the portfolio optimization problem, the mean–variance efficient portfolio, relies upon a series of assumptions and is constructed using first and second sample moments of financial asset returns. Although analytical and elegant, Markowitz solution to the portfolio optimization problem turns out to be highly sensitive to estimation errors of sample moments. For this reason, many moment estimators have been proposed to improve the performance of the portfolio optimization. Furthermore, the typical outcome of the Markowitz optimization procedure, especially for large portfolios, is characterized by large negative weights for a certain number of assets of the portfolio (Best and Grauer 1992, Green and Hollifield 1992, Jagannathan and Ma 2003). Negative portfolio weights require taking a short position (selling an asset without owning it), which is sometimes difficult to implement in practice, or forbidden to some classes of investors. For this reason it is quite widespread to constrain portfolio weights in the optimization procedure.

In the present study, we focus on the role played in the portfolio selection by estimation errors of the second moments of asset returns, both when taking short selling positions is allowed and when it is forbidden. We can ignore estimation errors of asset expected returns by restricting our attention to the global minimum variance

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portfolio, where asset expected returns are not involved (Ingersoll 1987). It should be noted that this choice is not a limiting one. In fact, the global minimum variance portfolio is typically characterized by an out-of-sample Sharpe ratio (the ratio between the portfolio return and its standard deviation, a key portfolio performance measure) that is as good as that of other efficient portfolios (Jorion 1985, Jagannathan and Ma 2003). Indeed, there is a consensus on the view that benefits of diversification can be achieved from risk reduction rather than from return maximization (Jorion 1985). Furthermore, the determination of expected returns is the role of the economist and of the portfolio manager, who are asked to generate or select valuable private information, while estimation of the covariance matrix is the task of the quantitative analyst (Ledoit and Wolf 2003).

The simplest estimator of the covariance matrix of N asset returns is the sample covariance estimator, which has $N\times(N+1)/2(\sim N^2/2$ when N is large) distinct elements. For an estimation time horizon of length T, the number of available data is $N\times T$. A very common circumstance in portfolio selection is that the number of assets N is of the same order of magnitude as the estimation time horizon T, for example because non-stationarity problems arise for large T, or because the portfolio is very large. In this case, the total number of parameters to be estimated is of the same order of magnitude as the total size of the available data. This unavoidable lack of data records generates large estimation errors in the sample covariance matrix, and thus covariance filtering methods are especially useful in order to reduce the estimation error. Here we discuss and compare the performance of portfolios obtained using several estimators of the covariance matrix. We perform the comparison of portfolio selection methods at different time horizons T, and we consider the portfolio optimization problem both with and without the inclusion of short selling constraints. Specifically, we apply portfolio optimization methods to 90 highly capitalized stocks traded on the New York Stock Exchange (NYSE) during the time period from January 1997 to December 2005. We find the global minimum variance portfolio both with and without short selling constraints at different time horizons. The investment and estimation horizons are chosen to be identical, and range from one month (approximately $T=20$ trading days) to two years (approximately $T=480$ trading days). We compare the performance of 10 covariance matrix estimators, namely the sample covariance estimator used in the Markowitz optimization, three estimators based on the spectral properties of the covariance matrix (Metha 1990, Laloux *et al.* 1999, Plerou *et al.* 1999, Rosenow *et al.* 2002, Potters *et al.* 2005), three estimators based on hierarchical clustering (Anderberg 1973, Mantegna 1999, Tumminello *et al.* 2007b, 2010, Basalto *et al.* 2008, Tola *et al.* 2008), and three estimators based on shrinking procedures (Jagannathan and Ma 2003, Ledoit and Wolf 2003, 2004a, b).

We find that the effectiveness of the last nine covariance estimators with respect to the sample estimator in portfolio optimization depends on the presence or absence of short selling, on the performance metric considered, and on the ratio T/N. Specifically, when short selling is allowed, several covariance estimators are able to give portfolios significantly less risky than the sample covariance portfolio. This is particularly true when T/N is close to one, in agreement with previous observations that sample covariance portfolio optimization can be quite problematic and ineffective in the $T/N\approx 1$ regime (Pafka and Kondor 2002, 2003, Papp *et al.* 2005, Kondor *et al.* 2007). Moreover, for a wide range of T/N, we verify that portfolios obtained using the proposed estimation procedures have a lower proportion of negative over positive weights (amount of short selling) (Jagannathan and Ma 2003) than the sample covariance portfolio, especially when $T/N\approx 1$. However, the degree of effective diversification of the portfolio is similar for different methods (including sample covariance).

The situation is significantly different when short selling is forbidden. When $T>N$, the realized risk of the sample covariance portfolio becomes comparable to that of the other portfolios. In this respect, the tested estimators are not able to give portfolios significantly less risky than the sample covariance portfolio and all the tested estimators have very similar risk. However, the portfolios obtained with these estimators are significantly more diversified than the sample covariance portfolio.

When $T<N$, the inverse of the sample covariance matrix does not exist because it has zero eigenvalues. It has been proposed to use the pseudo-inverse to extend the Markowitz optimization to the case $T<N$. We find that portfolios obtained with the pseudo-inverse are more risky and less diversified than the other portfolios.

By comparing portfolios with and without short selling we also verify and generalize the observation that including constraints (such as the no short selling constraint) in the portfolio optimization procedure is similar to performing an unconstrained optimization with a filtered covariance matrix (see Jagannathan and Ma (2003) and Schäfer *et al.* (2010) for shrinkage estimators).

The paper is organized as follows. In section 2 we discuss basic aspects of the Markowitz portfolio optimization procedure and set the notation. In section 3 we describe the investigated covariance matrix estimators. Section 4 presents the data set, the methodologies used to compare the different portfolios, and the empirical results. Section 5 concludes.

2. Markowitz portfolio optimization

In this section we briefly discuss some basic aspects of portfolio optimization in the Markowitz framework. This is also useful for setting the notation and stating the assumptions made and the methods used.

Given N stocks, at time t_0 an investor selects his/her portfolio of stocks by choosing a fraction of wealth w_i to invest in stock i, with $i=1,\ldots,N$, in order to have maximum profit and minimum risk from his/her investment at a fixed time t_0+T in the future. The N-dimensional column vector of the weights $\mathbf{w}$ is normalized as $\mathbf{w}^\top \mathbf{1}_N=1$, where $\mathbf{1}_N$ is the N-dimensional column vector of ones. The average return and the variance of the portfolio are

$$r_\mathrm{p} = \mathbf{w}^\top \mathbf{m}, \quad \sigma_\mathrm{p}^2 = \mathbf{w}^\top \Sigma \mathbf{w}, \tag{1}$$

respectively, where $\mathbf{m}$ and Σ are the N-dimensional column vector of the mean returns and the $N\times N$ covariance matrix of the stocks, respectively. The Markowitz optimization problem consists of finding the vector $\mathbf{w}$ that minimizes σ_p for a given value of r_p. The choice of using the standard deviation as a measure of risk is based on the assumption that returns follow a Gaussian distribution. If one does not set any constraint on the value of the weights, allowing them to be either positive or negative, the Markowitz solution to the optimization problem (Markowitz 1959) is

$$\mathbf{w}^* = \lambda \Sigma^{-1}\mathbf{1}_N + \gamma \Sigma^{-1}\mathbf{m}, \tag{2}$$

where

$$\lambda = \frac{C - r_\mathrm{p} B}{\Delta}, \quad \gamma = \frac{r_\mathrm{p} A - B}{\Delta},$$

$$A = \mathbf{1}_N^T \Sigma^{-1}\mathbf{1}_N, \quad B = \mathbf{1}_N^T \Sigma^{-1}\mathbf{m},$$

$$C = \mathbf{m}^T \Sigma^{-1}\mathbf{m}, \quad \Delta = AC - B^2.$$

The inverse of the parameter γ is usually referred to as the risk aversion.

When $\gamma=0$ (infinite risk aversion), the optimal portfolio is the global minimum variance portfolio and it does not depend on the expected returns. Since in this paper we aim to investigate the role of estimation risk of the covariance matrix, we focus on the global minimum variance portfolio, as done by Jorion (1985), Jagannathan and Ma (2003) and Ledoit and Wolf (2003), which obviously does not depend on the estimation error of the mean returns. Markowitz optimization typically gives both positive and negative portfolio weights and, especially for large portfolios, it usually gives large negative weights for a certain number of assets (Best and Grauer 1992, Green and Hollifield 1992, Jagannathan and Ma 2003). A negative weight corresponds to a short selling position (selling an asset without owning it) and it is sometimes difficult to implement in practice, or forbidden. For this reason, it is common practice to impose constraints on the portfolio weights in the optimization procedure. When one adds constraints on the range of variation of w_i, the optimization problem cannot be solved analytically, and quadratic programming must be used. Quadratic programming algorithms are implemented in most numerical programs, such as Matlab and R. In the following we will consider the portfolio optimization problem both with and without the no short selling constraint $w_i \geq 0$, $\forall i=1,\ldots,N$.

3. Covariance matrix estimators

One of the main problems of portfolio optimization is the estimation of the mean returns vector $\mathbf{m}$ and covariance matrix Σ. For the global minimum variance portfolio the investor needs only to estimate Σ. In what follows we estimate the covariance matrix using past return data. Specifically, at time t_0 we estimate the sample covariance matrix of daily returns in the T trading days preceding t_0. We then apply the different estimators and calculate the optimal portfolio. This portfolio is held until time t_0+T when we evaluate its performance. Note that our estimation and investment time horizons are chosen to be the same. In section 4 we will also comment on how the results change when we use a fixed evaluation period length and we vary the length of the estimation period. We consider three classes of estimators: (i) spectral estimators, (ii) hierarchical clustering estimators, and (iii) shrinkage estimators.

3.1. Sample covariance estimator

Let us first point out some aspects associated with the sample covariance direct optimization. In this case, the estimator of the covariance matrix at time t_0 is the sample covariance matrix estimated on the preceding T days. The input to the global minimum variance optimization problem is the inverse of the sample covariance matrix. When $T<N$, the inverse of the sample covariance matrix does not exist because of the presence of null eigenvalues. As suggested in the literature (for example, Ledoit and Wolf (2003)) in the optimization problem we use the pseudo-inverse, also called the generalized inverse (Mardia *et al.* 1979), of the covariance matrix. Replacing the inverse of the covariance matrix with the pseudo-inverse in the optimization problem allows one to obtain a unique combination of portfolio weights. It should be noted that, when $T<N$, the optimization problem remains undetermined and the pseudo-inverse solution is just a natural choice among the infinite undetermined solutions to the portfolio optimization problem.

In the same regime $T<N$, this problem does not arise for the other covariance estimators, because they typically give positive definite covariance matrices for any value of T/N, including $T/N<1$.

3.2. Spectral estimators

The first class of methods includes three different estimators of the covariance matrix, which make use of the spectral properties of the correlation matrix. The fundamental idea behind these methods is that the eigenvalues of the sample covariance matrix carry different economic information depending on their value.

The first method we consider is the single index model (see, for instance, Campbell *et al.* (1997) and Ledoit and Wolf (2003, 2004b)). In this model, stock returns $r_i(t)$ are described by the set of linear equations $r_i(t)=\alpha_i+\beta_i f(t)+\varepsilon_i(t)$, $i=1,\ldots,N$, where returns are therefore given by the linear combination of a single random variable, the index $f(t)$, and of an idiosyncratic stochastic term $\varepsilon_i(t)$. The parameters β_i can be estimated by linear regression of stock return time series on the index return. The covariance matrix associated with the model is $\mathbf{S}^{(\mathrm{SI})} = \sigma_{00}\boldsymbol{\beta}\boldsymbol{\beta}^\top + \mathbf{D}$, where σ_{00} is the variance of the index, $\boldsymbol{\beta}$ is the vector of parameters β_i, and $\mathbf{D}$ is the diagonal matrix of variances of ε_i. We denote this method hereafter as SI. It can be shown that this method gives an estimated covariance matrix very similar to that obtained with the method RMT-0 (see below) when only the largest eigenvalue of the sample covariance is assumed to carry reliable economic information.

The other two spectral methods make use of the Random Matrix Theory (RMT) (Metha 1990, Laloux *et al.* 1999, Plerou *et al.* 1999). Specifically, if the N variables of the system are i.i.d. with finite variance σ^2, then in the limit $T, N \to \infty$, with a fixed ratio T/N, the eigenvalues of the sample covariance matrix are bounded from above by the value

$$\lambda_{\max} = \sigma^2(1 + N/T + 2\sqrt{N/T}), \tag{3}$$

where $\sigma^2 = 1$ for correlation matrices. In most practical cases, one finds that the largest eigenvalue λ_1 of the sample correlation matrix of stocks is definitely inconsistent with RMT, i.e. $\lambda_1 \gg \lambda_{\max}$. In fact, the largest eigenvector is typically identified with the market mode. To cope with this evidence, Laloux *et al.* (1999) proposes modifying the null hypothesis of RMT so that system correlations can be described in terms of a one-factor model instead of a pure random model. Under such a less-restrictive null hypothesis, the value of $\lambda_{\max}$ is still given by equation (3), but now $\sigma^2 = 1 - \lambda_1/N$. Here we consider two different procedures that apply RMT to the covariance estimation problem.

The first procedure was proposed by Rosenow *et al.* (2002) and works as follows. One diagonalizes the sample correlation matrix and replaces all the eigenvalues smaller than $\lambda_{\max}$ by 0. One then transforms back the modified diagonal matrix in the standard basis, obtaining the matrix $\mathbf{H}^{(\mathrm{RMT-0})}$. The filtered correlation matrix $\mathbf{C}^{(\mathrm{RMT-0})}$ is obtained by simply forcing to 1 the diagonal elements of $\mathbf{H}^{(\mathrm{RMT-0})}$. Finally, the filtered covariance matrix $\mathbf{S}^{(\mathrm{RMT-0})}$ is the matrix of elements $\sigma_{ij}^{(\mathrm{RMT-0})} = c_{ij}^{(\mathrm{RMT-0})}\sqrt{\sigma_{ii}\sigma_{jj}}$, where $c_{ij}^{(\mathrm{RMT-0})}$ are the entries of $\mathbf{C}^{(\mathrm{RMT-0})}$ and σ_{ii} and σ_{jj} are the sample variances of variables i and j, respectively. In the following we will refer to this method as the RMT-0 method.

The second way to reduce the impact of eigenvalues smaller than $\lambda_{\max}$ on the estimate of portfolio weights was proposed by Potters *et al.* (2005). In this case, one diagonalizes the sample correlation matrix and replaces all the eigenvalues smaller than $\lambda_{\max}$ by their average value. Then one transforms back the modified diagonal matrix in the original basis, obtaining the matrix $\mathbf{H}^{(\mathrm{RMT-M})}$ of elements $h_{ij}^{(\mathrm{RMT-M})}$. It should be noted that replacing the eigenvalues smaller than $\lambda_{\max}$ by their average value preserves the trace of the matrix. Finally, the filtered correlation matrix $\mathbf{C}^{(\mathrm{RMT-}M)}$ is the matrix of elements $c_{ij}^{(\mathrm{RMT-M})} = h_{ij}^{(\mathrm{RMT-M})}/\sqrt{h_{ii}^{(\mathrm{RMT-M})}h_{jj}^{(\mathrm{RMT-M})}}$. The covariance matrix $\mathbf{S}^{(\mathrm{RMT-M})}$ to be used in the portfolio optimization is the matrix of elements $\sigma_{ij}^{(\mathrm{RMT-M})} = c_{ij}^{(\mathrm{RMT-M})}\sqrt{\sigma_{ii}\sigma_{jj}}$, where σ_{ii} and σ_{jj} are again the sample variances of variables i and j, respectively. We will refer to this method as the RMT-M method.

3.3. *Agglomerative hierarchical clustering estimators*

The second class of methods comprises three different estimators of the covariance matrix based on agglomerative hierarchical clustering (Anderberg 1973). Agglomerative hierarchical clustering methods are clustering procedures based on pair grouping where elements are iteratively merged together in clusters of increasing size according to their degree of similarity. Hierarchical clustering procedures therefore depend on the chosen similarity measure between elements of the system. In the present study we consider the correlation as a measure of similarity between two elements in the system. Hierarchical clustering algorithms work as follows. Given a data set of N time series, at the beginning each element defines a cluster. The similarity between two clusters is defined as the correlation coefficient between the corresponding two time series. Then the two clusters with the largest correlation are merged together in a single cluster. At the second iteration, one has to tackle the subtler problem of defining the similarity between clusters. Different similarities between clusters can be defined, each one characterizing a specific hierarchical clustering procedure. Once the similarity between two clusters is consistently defined, then the two clusters with the largest similarity are merged together, and the procedure is iterated until, after $N-1$ iterations, all the elements are grouped together in one cluster, corresponding to the whole data set.

We consider here three hierarchical clustering procedures that differ in the definition of similarity between clusters. In the unweighted pair group method with arithmetic mean (UPGMA), if a new cluster L is formed from clusters A and B, then the similarity between cluster L and any other cluster F is given by

$$\rho_{\mathrm{L,F}} = \frac{N_{\mathrm{A}}\rho_{\mathrm{A,F}} + N_{\mathrm{B}}\rho_{\mathrm{B,F}}}{N_{\mathrm{A}} + N_{\mathrm{B}}}, \tag{4}$$

where N_{A} and N_{B} are the number of elements in cluster A and B, respectively. Within this rule, the similarity between cluster L and cluster F is given by the arithmetic mean of the set $\{\rho_{ij}, \forall i \in \mathrm{L}, \text{ and } \forall j \in \mathrm{F}\}$. In the weighted pair group method with arithmetic mean (WPGMA), the average is weighted in such a way to get rid of the possibly different sizes of A and B:

$$\rho_{\mathrm{L,F}} = \frac{\rho_{\mathrm{A,F}} + \rho_{\mathrm{B,F}}}{2}. \tag{5}$$

Finally, in the Hausdorff linkage cluster analysis (Basalto *et al.* 2008), the similarity between cluster L and cluster F is obtained in terms of the Hausdorff distance between the two clusters:

$$\rho_{\mathrm{L,F}} = \min\left\{\min_{i\in \mathrm{L}}\max_{j\in \mathrm{F}}\ \rho_{ij},\ \max_{i\in \mathrm{L}}\min_{j\in \mathrm{F}}\ \rho_{ij}\right\}. \tag{6}$$

The output of any hierarchical clustering procedure is a dendrogram where each node α_k is associated with the similarity ρ_{α_k} between the two clusters of elements merging together in the node α_k. One can therefore construct a filtered similarity matrix $\mathbf{C}^<$ associated with a specific dendrogram as follows. Each entry $\rho_{ij}^<$ of $\mathbf{C}^<$ is set to ρ_{α_k}, where α_k is the node of the dendrogram corresponding to the smallest cluster in which the elements i and j merge together. The matrix $\mathbf{C}^<$ is positive definite provided that its entries are non-negative numbers (Tumminello *et al.* 2007b) and that the dendrogram does not show reversals (Anderberg 1973). The first condition is typically observed in the financial case, while the latter condition is always satisfied by the UPGMA and the WPGMA, while it could be violated in the Hausdorff method. When reversals are present in the dendrogram associated with the Hausdorff method, we remove such reversals using the minimum spanning tree associated with the hierarchical clustering procedure (Tumminello *et al.* 2007a). Since our procedure generates positive definite matrices, they can be interpreted as correlation matrices. Once $\mathbf{C}^<$ is constructed, we obtain an estimate of the covariance matrix by multiplying the entries of $\mathbf{C}^<$ by the sample standard deviations. Hierarchical clustering procedures have been shown to be effective in extracting financial information from the correlation matrix of stock returns (Mantegna 1999). Finally, it should be noted that hierarchical clustering methods have already been considered in portfolio optimization (Tola *et al.* 2008).

3.4. Shrinkage estimators

The last class of estimators comprises linear shrinkage methods. Linear shrinkage is a well-established technique in high-dimensional inference problems, when the size of the data is small compared with the number of unknown parameters in the model. In such cases, the sample covariance matrix is the best estimator in terms of actual fit to the data, but it is suboptimal because the number of parameters to be fitted is larger than the amount of data available (Stein 1956). The idea is to construct a more robust estimate $\mathbf{Q}$ of the covariance matrix by shrinking the sample covariance matrix $\boldsymbol{S}^{(\mathrm{S})}$ to a target matrix $\mathbf{T}$, which is typically positive definite and has a lower variance. The shrinking is obtained by computing

$$\mathbf{Q} = \alpha\mathbf{T} + (1-\alpha)\boldsymbol{S}^{(\mathrm{S})}, \tag{7}$$

where α is a parameter called the shrinkage intensity. We consider three different shrinkage estimates of the covariance matrix, each characterized by a specific target matrix.

The shrinkage to a single index uses the target matrix $\mathbf{T} = \mathbf{S}^{(\mathrm{SI})} = \sigma_{00}\boldsymbol{\beta}\boldsymbol{\beta}^\top + \mathbf{D}$, i.e. the single index covariance matrix previously discussed. This target was first proposed in the context of portfolio optimization by Ledoit and Wolf (2003). The second method is called shrinkage to common covariance. The target $\mathbf{T}$ is a matrix where the diagonal elements are all equal to the average of the sample variances, while non-diagonal elements are equal to the average of the sample covariances. In the shrinkage to common covariance, the heterogeneity of stock variances and of stock covariances is therefore minimized. The method has been proposed for the analysis of bioinformatic data (Schäfer and Stimmer 2005) and, to the best of our knowledge, it has never been used in the context of financial data analysis. The third method, termed shrinkage to constant correlation, has a more structured target and was used by Ledoit and Wolf (2004b). The estimator is obtained by first shrinking the correlation matrix to a target called the constant correlation, and then multiplying the shrunk correlation matrix by the sample standard deviations. The constant correlation target is a matrix with diagonal elements equal to one, and off-diagonal elements equal to the average sample correlation between the elements of the system. As α (the shrinkage intensity) we use the unbiased estimate calculated analytically by Schäfer and Stimmer (2005).

In conclusion, we consider 10 covariance matrix estimators that we label: sample covariance, SI, RMT-0, RMT-M, UPGMA, WPGMA, Hausdorff, shrinkage to SI, shrinkage to common covariance, and shrinkage to constant correlation.

4. Optimization process: Empirical results

In this section we present repeated portfolio optimizations performed using the covariance estimators discussed in the previous section. A set of highly liquid stocks traded on the NYSE is used.

4.1. Data

Our dataset consists of the daily returns of $N = 90$ highly capitalized stocks traded on the NYSE and included in the NYSE US 100 Index. For these stocks the closure prices are available in the 11-year period from 1 January 1997 to 31 December 2007.† The ticker symbols of the investigated stocks are AA, ABT, AIG, ALL, APA, AXP, BA, BAC, BAX, BEN, BK, BMY, BNI, BRK-B, BUD, C, CAT, CCL, CL, COP, CVS, CVX, D, DD, DE, DIS, DNA, DOW, DVN, EMC, EMR, EXC, FCX, FDX, FNM, GD, GE, GLW, HAL, HD, HIG, HON, HPQ, IBM, ITW, JNJ, JPM, KMB, KO, LEH, LLY, LMT, LOW, MCD, MDT, MER, MMM, MO, MOT, MRK, MRO, MS, NWS-A, OXY, PCU, PEP, PFE, PG, RIG, S, SGP, SLB, SO, T, TGT, TRV, TWX, TXN, UNH, UNP, USB, UTX, VLO, VZ, WAG, WB, WFC, WMT, WYE, XOM. As reference index in the SI model and in the shrinkage to a single index we use the Standard & Poor's 500 index, which is a widely used broadly based market index.

†The data, already preprocessed, were downloaded from `Yahoo Finance`.

At time t_0 the portfolio is selected by choosing the optimal weights that solve the global minimum variance problem with or without short selling constraints. The input to the optimization problem is the covariance matrix estimator $\boldsymbol{S}^{(f)}$ calculated using the T days preceding t_0 and obtained with one of the methods (i.e. $f \in$ {sample covariance (S), SI, RMT-0, RMT-M, UPGMA, WPGMA, Hausdorff, shrinkage to SI, shrinkage to common covariance, shrinkage to constant correlation}. We call $\boldsymbol{S}^{(f)}$ the estimated covariance matrix. The output of the optimization problem is

$$\mathbf{w}^{(f)} = \arg\min_{\mathbf{w}} \mathbf{w}^{\top}\mathbf{S}^{(f)}\mathbf{w}, \tag{8}$$

with the appropriate constraints. The *ex post* covariance matrix $\hat{\boldsymbol{S}}$ is defined as the sample covariance matrix calculated using the T days following t_0. The predicted portfolio risk is

$$s_{\mathrm{p}}^{(f)} = \sqrt{\mathbf{w}^{(f)\top}\mathbf{S}^{(f)}\mathbf{w}^{(f)}}, \tag{9}$$

and the realized portfolio risk is

$$\hat{s}_{\mathrm{p}}^{(f)} = \sqrt{\mathbf{w}^{(f)\top}\hat{\mathbf{S}}\mathbf{w}^{(f)}}. \tag{10}$$

Thus both $s_{\mathrm{p}}^{(f)}$ and $\hat{s}_{\mathrm{p}}^{(f)}$ are estimated using a time window of length T. The time window T is varied over a wide range. In our empirical study, we use seven different time windows T of 1, 2, 3, 6, 9, 12, and 24 months. In other words, we select the portfolio monthly ($T \simeq 20$), bimonthly ($T \simeq 40$), quarterly ($T \simeq 60$), six-monthly ($T \simeq 125$), nine-monthly ($T \simeq 187$), yearly ($T \simeq 250$), and biannually ($T \simeq 500$). Since the total number of trading days is 2761, we consider 131, 65, 43, 13, 21, 10, and 8 portfolio optimizations for the time horizon T equal to 1, 2, 3, 6, 9, 12, and 24 months, respectively (for the 24 month case, in order to improve the statistics, we repeated the optimization process starting from 1 January 1998). In order to compare risk levels at different time horizons, we report annualized risks in all figures and tables.

4.2. *Performance estimators*

To evaluate the performance of different covariance estimators we compare portfolio realized risk, portfolio reliability (i.e. the agreement between realized and predicted risk), and effective portfolio diversification of the portfolios $\mathbf{w}^{(f)}$. From now on we will drop the superscripts (f). Clearly a portfolio is less risky than another when its realized risk is smaller. Therefore, our first performance metric is the realized risk. Moreover, it is important that the portfolio is reliable, i.e. the *ex-ante* prediction is close to the *ex-post* observation of the portfolio risk. We consider both an absolute measure, $|\hat{s}_{\mathrm{p}} - s_{\mathrm{p}}|$ and a relative measure, $|\hat{s}_{\mathrm{p}} - s_{\mathrm{p}}|/\hat{s}_{\mathrm{p}}$, of reliability. Note that, in the relative measure, we normalize with respect to the realized risk instead of the predicted risk because the predicted risk can be very small or even zero when $T < N$. A third aspect for evaluating the performance of a portfolio is a high level of diversification across stocks of the portfolio. Thus we measure the effective portfolio diversification of the different covariance estimator methods. Following Bouchaud and Potters (2003), the effective number N_{eff} of stocks with a significant amount of money invested is defined as

$$N_{\mathrm{eff}} = \frac{1}{\sum_{i=1}^{N} w_i^2}. \tag{11}$$

This quantity is 1 when all the wealth is invested in one stock, whereas it is N when the wealth is equally divided among the N stocks, i.e. $w_i = 1/N$. When all weights are positive, i.e. when short selling is not allowed, the quantity N_{eff} has a clear meaning. On the other hand, when short selling is allowed there might be some ambiguity in the interpretation of N_{eff}.† For this reason, we introduce another measure of portfolio diversification. Specifically, we consider the absolute value of the weights and we compute the smallest number of stocks for which the sum of absolute weights is larger than a given percentage q of the sum of the absolute value of all the weights. In other words, we define

$$N_q = \arg\min_{l} \sum_{i=1}^{l} |w_i| \geq q \sum_{i=1}^{N} |w_i|. \tag{12}$$

In the following we consider $q = 0.9$ and we denote this indicator N_{90}. N_{90} is the minimum number of stocks in the portfolio such that their absolute weight accumulates to 90% of the total of asset absolute weights.

4.3. *Realized risk and reliability of different covariance estimators*

In this section we present the results obtained for repeated portfolio optimizations performed using the covariance estimators described in section 3. Let us first discuss the general qualitative behavior of the realized risk for different estimators, different time horizons T (and thus different ratios T/N) and different short selling conditions. Later we perform more rigorous statistical tests.

Figure 1 shows the mean value of the realized risk (averaged over different portfolio selection times t_0) as a function of the time horizon T in the case of short selling (top panel) and no short selling (bottom panel). When short selling is allowed (top panel), the performance of the sample covariance portfolio is very poor and clearly different from that of the portfolios obtained with the other investigated covariance estimators. The sample covariance direct optimization procedure gives the highest realized risk at each time window T, with the exception of $T = 2$ years. Furthermore, while the realized risk curves of

†For instance, consider a portfolio of $N = 2M + 1$ stocks where M weights are equal to $-x$, M weights are equal to x and the remaining one is equal to 1 with $x > 1$. The weights are normalized to one. In this limit example, the quantity in equation (11) is equal to $N_{\mathrm{eff}} = 1/(2Mx^2 + 1)$, which can be much smaller than 1, even if the portfolio is concentrated in $2M$ stocks. This example shows that N_{eff} is a meaningful measure of portfolio diversification only when short selling is not allowed.

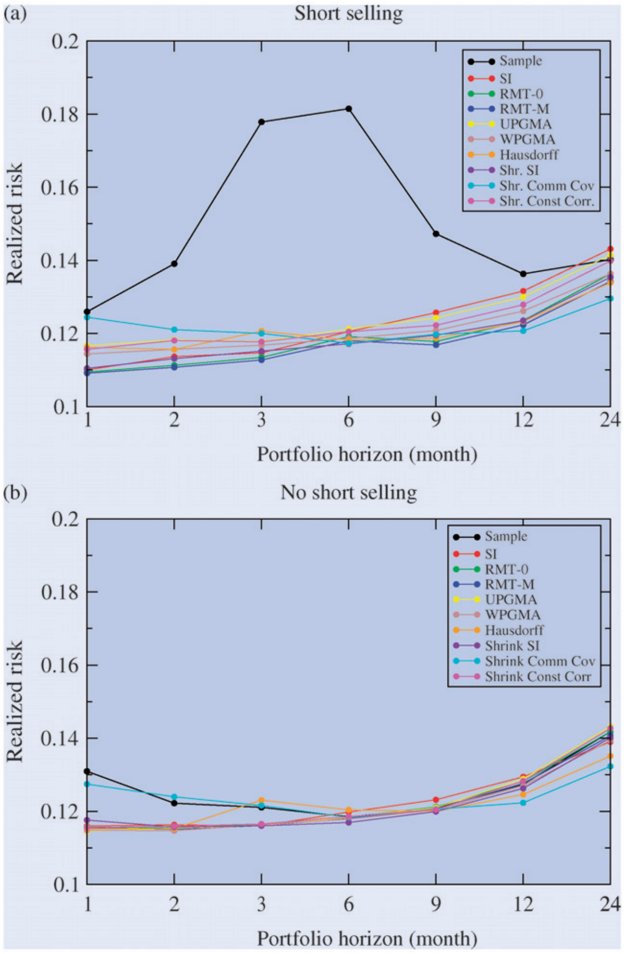

Figure 1. Mean realized (annualized) risk $\hat{s}_p$ for portfolios obtained with the 10 different methods as a function of the horizon T. $T = 1$, 2, 3, 6, 9, 12, 24 months corresponding to $T/N \approx 0.2, 0.4, 0.7, 2.1, 2.8, 5.6$, respectively. The top panel considers portfolios where short selling is allowed and the bottom panel considers portfolios where short selling is forbidden.

the other optimization procedures are approximately increasing functions of T (except shrinkage to common covariance), the realized risk of the sample covariance portfolio is non-monotonic: the realized risk is very high at $T = 3$ and 6 and decreases around those values. The non-monotonic behavior of the sample covariance direct optimization method can be explained as follows. When short selling is allowed, a high realized risk at $T \approx 4.5$ months is expected because $T \approx N$ (i.e. $T \approx 90$ days=4.5 months in our case) is the crossing point from non-singular to singular covariance matrices. In fact, from the work of Pafka and Kondor (2002, 2003), Papp *et al.* (2005) and Kondor *et al.* (2007), a divergence of the realized risk is shown to occur in the limit $T \to \infty$, $N \to \infty$ and $T/N \to 1$ from the right. Here we verify this behavior and we observe the divergence also when $T/N \to 1$ from the left. Note that the pseudo-inverse method sets to zero those eigenvalues of the covariance matrix inverse corresponding to the null eigenvalues of the sample covariance matrix. Therefore, the impact of the null eigenvalues of the sample covariance matrix on the portfolio weights is strongly reduced by using the pseudo-inverse. When the length T of the data series is sizably smaller than N, the pseudo-inverse mainly retains information about the few largest eigenvalues, which are actually the only non-null eigenvalues. The large majority of these non-null large eigenvalues are also retained by spectral methods. This observation explains why spectral methods and the pseudo-inverse method give similar results when $T \ll N$. From the top panel of figure 1 we can also see how spectral and hierarchical clustering methods show a similar performance in terms of realized risk. Shrinkage methods have a performance similar to that of the other algorithms, but the shrinkage to common covariance method shows a relatively poorer performance for low values of T, while it shows one of the best performances for high values of T.

The bottom panel of figure 1 shows the mean realized risk as a function of the time horizon T when the no short selling condition is imposed. Also in this case, the realized risk of all portfolios approximately increases with T except again for the sample covariance portfolio and the shrinkage to common covariance method. It is worth mentioning that the observed increase of the realized risk for large time horizons T is related to our choice of setting the same length for the estimation and the evaluation period. With this choice, a longer T implies a larger uncertainty of the future risk due to the non-stationarity of the correlations. When we repeat the same analysis by keeping the evaluation period length fixed and equal to one month while still having different estimation period lengths T, we find that realized risk is approximately constant or decreasing when T increases. Moreover, for T larger than N, all the methods are roughly equivalent in terms of realized risk. For $T < N$, sample covariance and shrinkage to common covariance portfolios clearly have a high realized risk, while the other methods are again essentially equivalent (with the possible exception of the Hausdorff estimator for $T = 3$ months). Finally, overall, except for the sample covariance portfolio, a comparison of the top and bottom panels of figure 1 shows that the realized risk of all portfolios turns out to be approximately the same both when constraints on short selling are applied and when they are not.

In the previous analysis we considered the average realized risk over repeated optimizations for different time horizons T. Now, we fix T and consider the realized risk time series to explore the role and nature of its fluctuations in different market conditions. We compare these time series for different values of the time horizon T.

In figure 2 we show the time series of the realized risk as a function of the optimization time t_0 for the sample covariance portfolio and for two representative covariance estimation methods (the shrinkage to common covariance and the RMT-M) when $T = 1$, 3, 6, and 12 months and short selling is allowed. From the figure it is evident that, for a given method, the temporal fluctuations in the time series of the realized risk are typically larger than the typical differences between the realized risk of the different methods. The same is true if we compare other estimators and also when short selling is

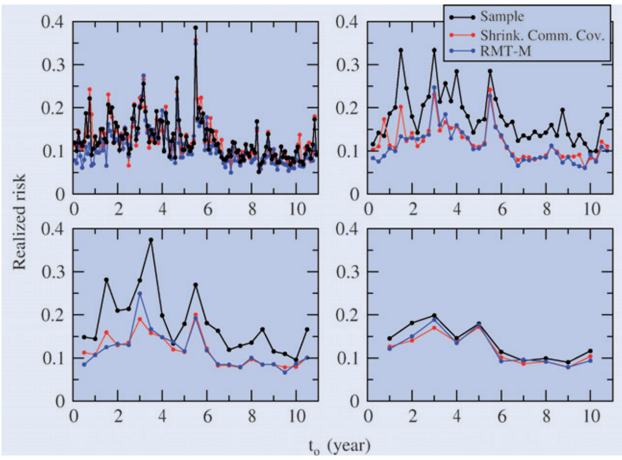

Figure 2. Time series of the realized risk $\hat{s}_p$ over the 11 years of the sample covariance, the RMT-M, and the shrinkage to common covariance portfolios for a portfolio horizon T equal to 1 (top left panel), 3 (top right panel), 6 (bottom left panel), and 12 (bottom right panel) months. In these optimizations, short selling is allowed.

not allowed. The observed large fluctuations in the realized risk indicate that, for a detailed comparison of different portfolio performances, a comparison of the relative differences between portfolio realized risks is more appropriate than a comparison of the average realized risk (averaged over different portfolio selection times). For example, let us consider the yearly case (bottom right panel of figure 2). The realized risks of the sample covariance (black circles) and shrinkage to common covariance (red circles) portfolios averaged over the 11-year time period are $13.6 \pm 1.3\%$ and $12.1 \pm 1.1\%$, respectively, where errors are standard errors. From these numbers, one would conclude that the two methods are equivalent in terms of realized risk. On the contrary, from the time series in the bottom right panel of figure 2, one concludes that the realized risk of the shrinkage to common covariance portfolio is systematically smaller than that of the sample covariance portfolio. In fact, our results show that, for a yearly investment horizon when short selling is allowed, the shrinkage to common covariance method outperforms all of the other methods.

For these reasons we measure portfolio performances relative to the sample covariance portfolio by means of quantity $1 - \hat{s}_p/\hat{s}_p^{(S)}$, where $\hat{s}_p$ is the realized risk of the investigated portfolio and $\hat{s}_p^{(S)}$ is the realized risk for the sample covariance portfolio in the same period and conditions. This quantity measures how the investigated portfolio outperforms the sample covariance portfolio (in percentage) in terms of realized risk. To assess the statistical robustness of the difference observed between a result obtained with a given covariance estimator and the sample covariance estimator, we perform a t-test to evaluate whether the difference $\hat{s}_p^{(S)} - \hat{s}_p$ has mean value equal to zero. Similarly, in order to test whether a given portfolio is more reliable than the sample covariance portfolio we perform a t-test to evaluate whether the difference $|\hat{s}_p^{(S)} - s_p^{(S)}| - |\hat{s}_p - s_p|$ is different from zero. Here s_p and $s_p^{(S)}$ are the predicted risk for the investigated and the sample covariance portfolio, respectively.

A quantitative comparison of all the covariance estimator methods is provided in tables 1, 2, and 3 for the cases $T=1$ year, 6 months, and 1 month, respectively, for both the case when short selling is allowed and when it is not. Since $N=90$, in the first two cases we have $T>N$, while in the third case it is $T<N$.

Let us discuss first the case in which short selling is allowed. Comparing the mean values of $1 - \hat{s}_p/\hat{s}_p^{(S)}$ (third column in the tables) and the results of the t-tests, we conclude that relative portfolio performances depend on the investment horizon T. For a yearly horizon, all methods except SI and UPGMA outperform the sample covariance portfolio and the best method is shrinkage to common covariance (as already noted above), which has a realized risk 11% smaller, on average, than the sample covariance portfolio. Note that when T is equal to one year, RMT-M also performs similarly well. In fact, the average realized risk for this method is 10.4% smaller than the sample covariance method. However, for shorter time horizons a different pattern emerges. When $T=6$ months (table 2), all portfolios perform equally well compared with the sample covariance portfolio, being roughly 33% less risky than the sample covariance portfolio. When $T=1$ month (see table 3), all methods except shrinkage to common covariance outperform sample covariance. The spectral methods SI, RMT-0, and RMT-M perform the best and equally well. Among the shrinkage methods, shrinkage to SI and shrinkage to constant correlation perform almost as well as the spectral methods, while the shrinkage to common covariance portfolio is the worst, having a realized risk that is statistically indistinguishable from the sample covariance portfolio. By considering the reliability, which is given in the last column of the tables, we conclude that all the methods outperform the sample covariance estimator with a single exception observed for the SI covariance estimator when $T=1$ year. Again, the degree of improvement is enhanced when $T=6$ months.

We now consider the no short selling case. As anticipated in the previous discussion, for $T>N$ all portfolios have similar realized risks and the observed values are quite close to those observed in the absence of the no short selling constraint. This is confirmed by the results shown in the bottom part of tables 1 and 2. For $T=1$ year, the quantity $1 - \hat{s}_p/\hat{s}_p^{(S)}$ is statistically consistent with zero for all portfolios. When $T=6$ months, only the shrinkage to single index estimator performs slightly better than the sample covariance estimator at the 5% confidence level. For $T=1$ month (table 3), a different result emerges. In fact, all portfolios have a significantly smaller realized risk than the sample covariance portfolio. The only notable exception is the shrinkage to common covariance portfolio, which presents the same (bad) performance as the sample covariance portfolio. The best results for the realized risk are observed for hierarchical clustering methods and for the shrinkage to constant correlation method. Moreover, the spectral

Table 1. Different portfolio performance measures that combine (annualized) predicted s_p and realized $\hat{s}_p$ risks. Ten different methods are compared for an horizon of $T = 1$ year. The numbers are averages over the different portfolios and the errors are standard errors. For $\hat{s}_p$ and $|\hat{s}_p - s_p|$ we report the result of a t-test evaluating whether the difference of each quantity and the corresponding quantity for the sample covariance portfolio has mean value equal to zero. **The p-value of the null hypothesis is below the 1% threshold. *The p-value of the null hypothesis is below the 5% threshold.

	s_p	$\hat{s}_p$	$1 - \frac{\hat{s}_p}{\hat{s}_p^{(S)}}$	$\lvert\hat{s}_p - s_p\rvert$
Year—s.s.				
Sample	6.97 ± 0.63	13.6 ± 1.3	0 ± 0	6.7 ± 1.1
SI	5.94 ± 0.41	13.2 ± 1.3	2.7 ± 5.0	7.2 ± 1.2
RMT-0	7.18 ± 0.67	$12.4 \pm 1.2^{**}$	9.5 ± 2.5	$5.2 \pm 1.1^{**}$
RMT-M	7.24 ± 0.68	$12.2 \pm 1.2^{**}$	10.4 ± 2.4	$5.1 \pm 1.0^{**}$
UPGMA	8.23 ± 0.88	13.0 ± 1.3	5.0 ± 2.3	$4.8 \pm 1.1^{**}$
WPGMA	7.88 ± 0.82	$12.6 \pm 1.3^{*}$	7.6 ± 2.6	$4.8 \pm 1.1^{**}$
Hausdorff	7.57 ± 0.80	$12.3 \pm 1.2^{*}$	9.3 ± 3.0	$4.75 \pm 0.99^{**}$
Shr. to SI	7.59 ± 0.70	$12.3 \pm 1.1^{**}$	9.09 ± 0.90	$4.76 \pm 0.98^{**}$
Shr. c. cov.	10.54 ± 0.91	$12.1 \pm 1.1^{**}$	11.0 ± 1.7	$2.57 \pm 0.69^{**}$
Shr. c. corr.	8.33 ± 0.81	$12.8 \pm 1.2^{**}$	6.3 ± 1.0	$4.5 \pm 1.0^{**}$
Year—no s.s.				
Sample	9.46 ± 0.88	12.7 ± 1.2	0 ± 0	4.06 ± 0.93
SI	7.90 ± 0.64	12.9 ± 1.2	-2.2 ± 3.0	5.5 ± 1.2
RMT-0	9.18 ± 0.84	12.8 ± 1.2	-0.34 ± 0.97	4.33 ± 0.98
RMT-M	9.08 ± 0.83	12.8 ± 1.2	0.07 ± 0.95	4.33 ± 0.98
UPGMA	9.9 ± 1.0	12.9 ± 1.3	-0.70 ± 0.98	3.93 ± 0.97
WPGMA	9.01 ± 0.89	12.7 ± 1.2	0.2 ± 1.5	4.11 ± 0.98
Hausdorff	8.68 ± 0.91	12.5 ± 1.1	1.7 ± 2.1	4.14 ± 0.95
Shr. to SI	9.35 ± 0.85	12.6 ± 1.1	0.75 ± 0.42	4.01 ± 0.93
Shr. c. cov.	11.7 ± 1.0	12.2 ± 1.1	3.4 ± 1.9	2.40 ± 0.72
Shr. c. corr.	10.05 ± 0.98	12.8 ± 1.2	-0.43 ± 0.90	3.92 ± 0.92

Table 2. Different portfolio performance measures that combine (annualized) predicted s_p and realized $\hat{s}_p$ risks. Ten different methods are compared for an horizon of $T = 6$ months. The numbers are averages over the different portfolios and the errors are standard errors. For $\hat{s}_p$ and $|\hat{s}_p - s_p|$ we report the result of a t-test evaluating whether the difference of each quantity and the corresponding quantity for the sample covariance portfolio has mean value equal to zero. **The p-value of the null hypothesis is below the 1% threshold. *The p-value of the null hypothesis is below the 5% threshold.

	s_p	$\hat{s}_p$	$1 - \frac{\hat{s}_p}{\hat{s}_p^{(S)}}$	$\lvert\hat{s}_p - s_p\rvert$
6 *months—s.s.*				
Sample	4.23 ± 0.30	18.1 ± 1.5	0 ± 0	13.9 ± 1.4
SI	5.52 ± 0.33	$12.05 \pm 0.92^{**}$	31.3 ± 3.0	$6.53 \pm 0.83^{**}$
RMT-0	6.10 ± 0.42	$11.91 \pm 0.96^{**}$	32.4 ± 3.3	$5.81 \pm 0.82^{**}$
RMT-M	6.17 ± 0.43	$11.80 \pm 0.95^{**}$	33.0 ± 3.2	$5.63 \pm 0.82^{**}$
UPGMA	7.46 ± 0.57	$12.12 \pm 0.91^{**}$	31.1 ± 3.1	$4.66 \pm 0.76^{**}$
WPGMA	7.22 ± 0.56	$11.86 \pm 0.86^{**}$	32.3 ± 3.1	$4.65 \pm 0.74^{**}$
Hausdorff	6.48 ± 0.55	$11.82 \pm 0.82^{**}$	32.4 ± 3.0	$5.34 \pm 0.77^{**}$
Shr. to SI	6.41 ± 0.43	$11.72 \pm 0.82^{**}$	33.4 ± 2.4	$5.30 \pm 0.65^{**}$
Shr. c. cov.	10.77 ± 0.76	$11.73 \pm 0.80^{**}$	33.2 ± 2.4	$2.82 \pm 0.55^{**}$
Shr. c. corr.	7.51 ± 0.53	$12.05 \pm 0.88^{**}$	31.7 ± 2.7	$4.54 \pm 0.67^{**}$
6 *months—no s.s.*				
Sample	8.57 ± 0.63	11.85 ± 0.87	0 ± 0	3.94 ± 0.69
SI	7.40 ± 0.52	11.98 ± 0.86	-1.7 ± 1.5	$4.92 \pm 0.78^{*}$
RMT-0	8.27 ± 0.62	11.83 ± 0.86	-0.1 ± 1.0	4.17 ± 0.72
RMT-M	8.20 ± 0.61	11.81 ± 0.86	0.1 ± 1.0	4.21 ± 0.72
UPGMA	9.19 ± 0.72	11.83 ± 0.89	0.26 ± 0.96	3.57 ± 0.72
WPGMA	8.42 ± 0.67	11.79 ± 0.87	0.4 ± 1.0	3.75 ± 0.78
Hausdorff	7.45 ± 0.67	12.04 ± 0.82	-2.5 ± 1.5	$4.88 \pm 0.83^{*}$
Shr. to SI	8.48 ± 0.61	$11.69 \pm 0.87^{*}$	1.31 ± 0.51	3.87 ± 0.71
Shr. c. cov.	11.79 ± 0.84	11.84 ± 0.85	-0.6 ± 2.2	3.30 ± 0.63
Shr. c corr	9.48 ± 0.71	11.86 ± 0.93	0.5 ± 1.1	$3.42 \pm 0.73^{*}$

Table 3. Different portfolio performance measures that combine predicted s_p and the realized $\hat{s}_p$ annualized risk. Ten different methods are compared for an horizon of $T = 1$ month. The numbers are averages over the different portfolios and the errors are standard errors. For $\hat{s}_p$ and $|\hat{s}_p - s_p|$ we report the result of a t-test evaluating whether the difference of each quantity and the corresponding quantity for the sample covariance portfolio has mean value equal to zero. **The p-value of the null hypothesis is below the 1% threshold. *The p-value of the null hypothesis is below the 5% threshold.

	s_p	$\hat{s}_p$	$1 - \frac{\hat{s}_p}{\hat{s}_p^{(S)}}$	$\lvert\hat{s}_p - s_p\rvert$
Month—s.s.				
Sample	0 ± 0	12.59 ± 0.41	0 ± 0	12.59 ± 0.41
SI	4.15 ± 0.12	$11.00 \pm 0.42^{**}$	12.1 ± 1.5	$6.85 \pm 0.37^{**}$
RMT-0	3.84 ± 0.11	$10.94 \pm 0.39^{**}$	12.5 ± 1.4	$7.10 \pm 0.34^{**}$
RMT-M	3.90 ± 0.12	$10.91 \pm 0.39^{**}$	12.8 ± 1.4	$7.01 \pm 0.34^{**}$
UPGMA	5.01 ± 0.17	$11.66 \pm 0.45^{**}$	6.6 ± 2.1	$6.65 \pm 0.38^{**}$
WPGMA	4.74 ± 0.17	$11.44 \pm 0.44^{**}$	8.3 ± 1.9	$6.70 \pm 0.37^{**}$
Hausdorff	4.98 ± 0.17	$11.62 \pm 0.45^{**}$	7.0 ± 2.1	$6.64 \pm 0.37^{**}$
Shr. to SI	3.48 ± 0.15	$11.04 \pm 0.39^{**}$	11.8 ± 1.2	$7.57 \pm 0.35^{**}$
Shr. c. cov.	13.1 ± 0.47	12.44 ± 0.42	0.5 ± 1.5	$3.64 \pm 0.30^{**}$
Shr. c. corr.	5.87 ± 0.20	$11.56 \pm 0.45^{**}$	7.4 ± 1.9	$5.70 \pm 0.37^{**}$
Month—no s.s.				
Sample	4.38 ± 0.24	13.09 ± 0.52	0 ± 0	8.73 ± 0.53
SI	5.60 ± 0.20	$11.60 \pm 0.44^{**}$	9.3 ± 1.4	$6.04 \pm 0.39^{**}$
RMT-0	5.48 ± 0.21	$11.57 \pm 0.42^{**}$	9.5 ± 1.2	$6.11 \pm 0.38^{**}$
RMT-M	5.49 ± 0.21	$11.54 \pm 0.42^{**}$	9.7 ± 1.2	$6.07 \pm 0.38^{**}$
UPGMA	7.11 ± 0.25	$11.45 \pm 0.44^{**}$	10.8 ± 1.3	$4.54 \pm 0.37^{**}$
WPGMA	6.15 ± 0.22	$11.48 \pm 0.44^{**}$	10.6 ± 1.2	$5.39 \pm 0.38^{**}$
Hausdorff	6.73 ± 0.23	$11.53 \pm 0.43^{**}$	10.3 ± 1.2	$4.87 \pm 0.34^{**}$
Shr. to SI	5.72 ± 0.21	$11.76 \pm 0.43^{**}$	8.64 ± 0.91	$6.06 \pm 0.38^{**}$
Shr. c. cov.	13.39 ± 0.48	12.74 ± 0.44	-2.6 ± 2.6	$3.76 \pm 0.30^{**}$
Shr. c. corr.	8.20 ± 0.29	$11.56 \pm 0.47^{**}$	10.3 ± 1.4	$3.93 \pm 0.35^{**}$

methods perform slightly worse than the others with respect to risk forecasting.

Note that when $T/N \approx 1$, the bad performance of the sample covariance portfolio, observed when short selling constraints are not imposed, is no longer present. The no short selling constraint makes the Markowitz optimization procedure essentially equivalent to an optimization procedure that has been performed with more robust covariance estimators. Again, this observation is in agreement with the conclusion that imposing the no short selling constraint on the portfolio optimization procedure is somehow equivalent to minimizing estimation errors in the input to the optimization problem (Jagannathan and Ma 2003).

4.4. Portfolio diversification

One further aspect to investigate concerns the degree of diversification of the portfolios. As for the realized risk, for the sample covariance estimator and for any given covariance estimator, we observe large fluctuations of the participation ratio as the portfolio estimation time t_0 varies. We therefore consider both the mean and the standard error of $N_{\rm eff}$ for each method across time and the mean value of $N_{\rm eff}/N_{\rm eff}^{(S)} - 1$ in percentage, where $N_{\rm eff}^{(S)}$ is the participation ratio for the sample covariance portfolio. This variable is a relative measure that quantifies the portfolio diversification with respect to the diversification of the benchmark sample covariance portfolio. Also in this case, we perform a t-test in order to evaluate whether the observed difference $N_{\rm eff}^{(S)} - N_{\rm eff}$ is compatible with a null hypothesis assuming that its mean value is zero.

In table 4 we report the average and standard error for $N_{\rm eff}$ and $N_{\rm eff}/N_{\rm eff}^{(S)} - 1$ for the 10 optimization methods and for $T = 1$ month, 6 months, and 1 year, together with the related results for the t-test. The table shows different behavior at different values of the investment time window T. Specifically, at $T = 1$ month, all methods present a participation ratio that is higher than that observed for the sample covariance portfolio. When $T = 6$ months, all methods still outperform the sample covariance with the exception of the shrinkage to constant correlation. When $T = 1$ year, there are still several methods that outperform the sample covariance, namely SI, WPGMA, Hausdorff, shrinkage to single index and shrinkage to common covariance. The method with the highest participation ratio at any time horizon is the shrinkage to common covariance. For example, when $T = 1$ month it has a participation ratio that is 530% higher than the sample covariance portfolio, on average. This high diversification is not shared with the other two shrinkage methods. This is probably due to the fact that the target matrix of the shrinkage to common covariance assumes that all the stocks are equivalent. SI among the spectral methods and WPGMA among the hierarchical clustering methods have the highest participation ratio of the other classes of covariance estimators.

In the above discussion, we have used $N_{\rm eff}$ to quantify the portfolio diversification under no short selling constraint. In fact, we have already stated that this

Table 4. Absolute and relative participation ratio measure N_{eff} of the portfolios obtained with the 10 covariance estimators for different horizons of $T=1$, 6 and 12 months. Short selling is not allowed. The numbers are averages over the different portfolios and the errors are standard errors. For N_{eff} we report the result of a t-test evaluating whether the difference with the corresponding quantity for the sample covariance portfolio has mean value equal to zero. **The p-value of the null hypothesis is below the 1% threshold. *The p-value of the null hypothesis is below the 5% threshold.

	One month		Six months		One year	
	N_{eff}	$\frac{N_{eff}}{N_{eff}^{(S)}}-1$	N_{eff}	$\frac{N_{eff}}{N_{eff}^{(S)}}-1$	N_{eff}	$\frac{N_{eff}}{N_{eff}^{(S)}}-1$
Sample	6.80 ± 0.22	0.0 ± 0.0	9.8 ± 1.0	0.0 ± 0.0	9.9 ± 1.5	0.0 ± 0.0
SI	$14.91\pm0.98^{**}$	104.0 ± 8.4	$14.0\pm2.1^{**}$	36.8 ± 7.5	13.8 ± 2.7	$33.4\pm9.2^{*}$
RMT-0	$13.45\pm0.80^{**}$	85.4 ± 6.2	$11.2\pm1.3^{**}$	13.4 ± 2.7	10.6 ± 1.7	6.8 ± 4.0
RMT-M	$13.63\pm0.81^{**}$	87.9 ± 6.2	$11.6\pm1.3^{**}$	16.9 ± 2.9	10.9 ± 1.7	10.1 ± 4.0
UPGMA	$8.90\pm0.44^{**}$	26.5 ± 3.5	$10.2\pm1.1^{**}$	5.1 ± 3.7	10.7 ± 1.8	6.7 ± 4.6
WPGMA	$11.62\pm0.53^{**}$	67.6 ± 4.3	$12.1\pm1.1^{**}$	26.3 ± 5.2	13.0 ± 1.9	$30.5\pm3.6^{**}$
Hausdorff	$9.55\pm0.34^{**}$	42.4 ± 3.3	$13.1\pm1.4^{**}$	36.0 ± 5.5	13.0 ± 1.8	$34.9\pm4.6^{**}$
Shr. to SI	$11.7\pm0.67^{**}$	60.9 ± 5.1	$11.3\pm1.4^{**}$	11.8 ± 2.2	10.7 ± 1.8	$7.3\pm1.8^{**}$
Shr. c. cov.	$37.3\pm1.4^{**}$	530 ± 45	$18.9\pm1.5^{**}$	159 ± 64	15.5 ± 1.8	$100\pm51^{**}$
Shr. c. corr.	$7.64\pm0.43^{**}$	7.5 ± 3.8	10.1 ± 1.2	-0.1 ± 2.6	10.0 ± 1.7	-1.3 ± 2.8

Table 5. Absolute and relative participation ratio measure N_{90} of the portfolios obtained with the 10 covariance estimators for different horizons of $T=1$, 6 and 12 months. Short selling is not allowed. The numbers are averages over the different portfolios and the errors are standard errors. For N_{90} we report the result of a t-test evaluating whether the difference with the corresponding quantity for the sample covariance portfolio has mean value equal to zero. **The p-value of the null hypothesis is below the 1% threshold. *The p-value of the null hypothesis is below the 5% threshold.

	One month		Six months		One year	
	N_{90}	$\frac{N_{90}}{N_{90}^{(S)}}-1$	N_{90}	$\frac{N_{90}}{N_{90}^{(S)}}-1$	N_{90}	$\frac{N_{90}}{N_{90}^{(S)}}-1$
Short selling						
Sample	59.41 ± 0.18	0.0 ± 0.0	56.81 ± 0.52	0.0 ± 0.0	55.3 ± 0.99	0.0 ± 0.0
SI	$52.85\pm0.31^{**}$	-10.95 ± 0.59	55.48 ± 0.71	-2.2 ± 1.4	55.1 ± 1.2	-0.3 ± 1.6
RMT-0	$53.87\pm0.29^{**}$	-9.23 ± 0.54	55.57 ± 0.67	-2.1 ± 1.2	55.1 ± 0.95	-0.2 ± 1.6
RMT-M	$53.85\pm0.29^{**}$	-9.26 ± 0.54	$55.38\pm0.68^{*}$	-2.4 ± 1.2	55.1 ± 0.97	-0.2 ± 1.6
UPGMA	$52.27\pm0.29^{**}$	-11.91 ± 0.55	$54.57\pm0.49^{**}$	-3.8 ± 1.1	55.6 ± 0.97	0.7 ± 1.9
WPGMA	$51.64\pm0.28^{**}$	-12.96 ± 0.56	$54.14\pm0.67^{**}$	-4.6 ± 1.3	54.9 ± 1.0	-0.6 ± 2.0
Hausdorff	$52.03\pm0.26^{**}$	-12.31 ± 0.52	$52.48\pm0.70^{**}$	-7.6 ± 1.2	53.7 ± 1.1	-2.7 ± 2.1
Shr. to SI	$53.45\pm0.29^{**}$	-9.97 ± 0.50	$54.38\pm0.63^{**}$	-4.2 ± 1.0	55.0 ± 1.1	-0.5 ± 1.5
Shr. c. cov.	$60.89\pm0.35^{**}$	2.57 ± 0.61	57.81 ± 0.49	1.9 ± 1.3	57.2 ± 1.0	3.6 ± 2.1
Shr. c. corr	$52.97\pm0.31^{**}$	-10.71 ± 0.62	$53.95\pm0.64^{**}$	-5.0 ± 1.1	54.6 ± 1.0	-1.24 ± 0.94
No short selling						
Sample	8.40 ± 0.19	0.0 ± 0.0	12.81 ± 1.00	0.0 ± 0.0	13.4 ± 1.5	0.0 ± 0.0
SI	$18.9\pm1.1^{**}$	113.3 ± 8.2	$17.0\pm2.0^{**}$	31.1 ± 7.6	16.4 ± 2.6	18.7 ± 8.0
RMT-0	$17.21\pm0.85^{**}$	95.9 ± 6.1	$13.8\pm1.2^{*}$	8.9 ± 4.0	13.4 ± 1.7	-0.7 ± 3.7
RMT-M	$17.40\pm0.85^{**}$	98.3 ± 6.0	$14.3\pm1.2^{**}$	12.5 ± 3.9	13.9 ± 1.7	3.1 ± 3.3
UPGMA	$11.55\pm0.48^{**}$	33.3 ± 3.4	12.9 ± 1.1	-0.8 ± 3.6	13.2 ± 1.9	-4.5 ± 5.0
WPGMA	$15.39\pm0.59^{**}$	79.8 ± 4.4	$15.6\pm1.2^{**}$	23.5 ± 5.7	$16.1\pm1.7^{**}$	20.5 ± 3.4
Hausdorff	$12.61\pm0.34^{**}$	51.5 ± 2.9	$17.4\pm1.4^{**}$	37.4 ± 4.9	$16.4\pm1.4^{**}$	25.7 ± 4.9
Shr. to SI	$15.24\pm0.74^{**}$	72.4 ± 5.2	$14.6\pm1.4^{**}$	12.5 ± 3.0	14.4 ± 1.9	5.7 ± 2.7
Shr. c. cov.	$37.4\pm1.2^{**}$	363 ± 20	$21.3\pm1.3^{**}$	85 ± 22	$18.8\pm1.7^{**}$	46 ± 10
Shr. c. corr	$10.00\pm0.51^{**}$	14.3 ± 3.9	12.7 ± 1.3	-4.2 ± 4.0	13.5 ± 1.9	-1.8 ± 4.8

indicator is not meaningful when short selling is allowed. For this reason, we now consider the second participation ratio indicator, N_{90}, introduced above. Table 5 reports the mean and the standard error of N_{90} for each method averaged across investment time and, as before, a relative measure both when short selling is allowed and when it is forbidden. We also perform a t-test to evaluate whether the difference $N_{90}^{(S)}-N_{90}$ has a mean value significantly different from zero.

When short selling is not allowed, N_{90} gives results very close to those observed for N_{eff}. In fact, when $T=1$ month, all the methods give a portfolio more diversified than that obtained with the sample covariance. When $T=6$ months, all the methods outperform sample covariance with the exception of shrinkage to constant correlation and UPGMA, whereas when $T=1$ year, only WPGMA, Hausdorff and shrinkage to common covariance still outperform sample covariance. When short

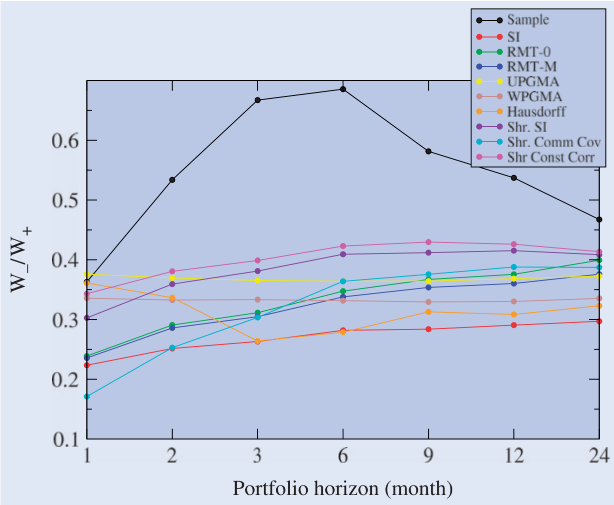

Figure 3. Mean value of the ratio w_-/w_+ between the sum of the absolute value of negative weights and the sum of positive weights for the portfolios obtained with the 10 different methods as a function of the horizon T.

selling is allowed, the sample covariance estimator provides portfolios characterized by a N_{90} value slightly higher or statistically compatible with the value observed for the other methods. The only exception is shrinkage to common covariance when $T=1$ month but, also in this case, the difference observed, although statistically validated, is very small.

In summary, when short selling is allowed the weights have a similar structure independently of the method, and the wealth (positive or negative) is roughly concentrated in 55 stocks. When short selling is not allowed, a large variety of behavior is observed depending on the method and on the investment time horizon. In general, the shrinkage to common covariance method has the largest participation ratio.

When short selling is allowed, it is also worth analysing the amount of short selling required by the optimization procedures of the global minimum variance portfolio. To quantify this aspect, in figure 3 we show, for each method, the average value of the ratio w_-/w_+, where w_- is the sum of the absolute value of all negative weights present in the portfolio and w_+ is the sum of all positive weights. The ratio w_-/w_+ ranges from 0 (absence of short selling) to about 1 (negative weights of the same size as positive weights).

Figure 3 shows that the sample covariance estimator gives the highest fraction of short selling positions. This property is maximal when $T/N \approx 1$. All the other methods present a significantly lower mean value of w_-/w_+. The specific values depend on the specific covariance estimation method and are slightly affected by the value of the investment horizon T. In fact, a slight increase of w_-/w_+ is observed when T is increasing. The lowest value $w_-/w_+ \approx 0.28$ is observed for the SI model, whereas the highest value $w_-/w_+ \approx 0.40$ is observed for the shrinkage to constant correlation method. The region of worst performance of the sample covariance portfolio is therefore associated with the maximal amount of portfolio wealth allocated in stocks that need to be sold short.

These results provide empirical support for the conclusion that the sample covariance portfolio in the presence of short selling suffers an overexposure to short selling. This overexposure is maximal when $T/N \approx 1$ and is progressively mitigated both when $T > N$ and when $T < N$. On the contrary, reducing the estimation errors on the covariance matrix estimation implicitly limits the amount of short selling positions requested in the optimal portfolio. According to the results obtained by Jagannathan and Ma (2003) and the empirical results obtained in this study, we observe that the reverse is also true. In fact, imposing no short selling conditions on the sample covariance portfolio reduces the estimation errors in the covariance matrix for any value of T, and especially when $T/N \approx 1$. Finally, it has been suggested (Schäfer *et al*. 2010) that the no short selling constraint acts in a similar way as shrinkage estimators only when correlations are positive (as in our investigated set). We have performed a portfolio optimization on synthetic multivariate time series of returns with negative correlations. For sample covariance portfolios, we have found that the no short selling constraint reduces the realized risk also in the presence of negative correlations. Moreover, in some cases, we observe that shrinkage portfolios where short selling is allowed have a detectably smaller realized risk than the corresponding sample covariance portfolio without the no short selling constraint. However, these results depend on the specific model used to generate the multivariate time series and on the time horizon. Further work is needed to fully clarify this important aspect.

5. Conclusions

The portfolio optimization problem is significantly affected by estimation errors of the covariance matrix. For this reason, many estimators alternative to the sample covariance matrix have been proposed in the literature. In this respect, two important and related questions are: (i) which aspects of the portfolio optimization can be improved with improved covariance matrix estimators?; and (ii) when, i.e. under which conditions, are improved covariance estimators really useful in enhancing the performance of the corresponding optimal portfolios? We have investigated these questions by considering nine different methods for estimating the covariance matrix and we have quantitatively compared the relative efficiency of the corresponding portfolios with respect to the benchmark sample covariance portfolio on a series of repeated investment exercises over 11 years. The portfolio optimization has been performed under different conditions: different estimation-investment horizons T, i.e. different values of T/N ($N=90$), and the presence/absence of short selling constraints. Despite the realized risk and the degree of portfolio diversification of the resulting portfolios constructed with the different covariance estimators show large fluctuations, the relative performances of the different methods turn out to be quite persistent over time. Under different market conditions, certain persistent behavior can be observed. For a specific choice of both the length of the estimation-investment horizon and the presence/absence of constraints on short selling, an estimator might be useful in improving a specific aspect of the optimization, but under a different choice the same method might not lead to a significant improvement of the same aspect.

Specifically, when $T/N > 1$, various covariance estimators lead to optimal portfolios with similar realized risk and portfolio diversification. In this regime, the sample covariance portfolio has an overall good performance both with and without short selling constraints. When short selling is allowed, a portfolio less risky than the sample covariance portfolio can be obtained using improved covariance estimators, and, when short selling is forbidden, the investigated estimators are not able to reduce the risk of the portfolio with respect to the sample covariance portfolio. In this last case, some covariance estimators lead to higher portfolio diversification.

On the other hand, when T/N is close to 1, portfolio performances are greatly influenced by the addition of no short selling constraints. Specifically, when short selling is allowed, we observe how the sample covariance portfolio has the worst performance. This result is consistent with the theoretical observations given by Jagannathan and Ma (2003) and with the observation of the divergence of estimation errors of the covariance matrix associated with this regime (Pafka and Kondor 2002, 2003, Papp *et al.* 2005, Kondor *et al.* 2007). Under this condition, all the investigated covariance estimators provide portfolios with lower realized risk, greater reliability and smaller exposure to short selling. Their performances are quite similar with respect to realized risk, reliability and portfolio diversification, but differences are observed with respect to the degree of exposure to short selling. When no short selling constraints are applied, we observe a different scenario. All covariance estimators lead to portfolios with realized risks and reliabilities that are statistically consistent with those obtained for the sample covariance portfolio. However, portfolios constructed with the investigated methods have a higher degree of diversification than those observed for the sample covariance one. This result is consistent with the theoretical and empirical conclusions reached by Jagannathan and Ma (2003), where it was shown that adding short selling constraints to the sample covariance portfolios can have the same effect as using a better estimate of the covariance matrix (using the shrinkage estimator in their case). Our results suggest that, indeed, this conclusion successfully applies also to other covariance estimators such as the methods investigated in this paper.

When T/N is less than one, the worst performance with respect to realized risk is obtained for the sample covariance and shrinkage to common covariance portfolios. This result indicates that one should not use the sample covariance matrix in this regime (with or without short selling). Also, the use of the pseudo-inverse gives portfolios with very poor performance. All the other methods lead to portfolios with better performances with respect to realized risk and reliability in realized risk forecasts both in the presence and in the absence of short selling. When the no short selling constraint is imposed, portfolio diversification is better achieved when filtered covariance estimators are used. This last observation is also true for the shrinkage to common covariance estimator both when short selling is allowed and when it is forbidden. Indeed, this method presents the highest degree of portfolio diversification. It is therefore worth noting that the observation that the sample covariance and shrinkage to common covariance portfolios are characterized by similar values of the realized risk does not imply that they have a similar composition. In fact, the portfolio obtained with the shrinkage to common covariance method is systematically more diversified. The conclusion reached by Jagannathan and Ma (2003) and empirically observed by the present authors when $T/N \approx 1$ does not seem to hold when T/N is less than one. In fact, portfolios obtained with the sample covariance estimator are characterized by realized risks, reliability of risk forecasts and portfolio diversification that are worse than most of other methods based on covariance estimators also when short selling is forbidden.

In summary, the use of efficient covariance estimators improves different aspects of the portfolio optimization process. The degree of improvement depends on the selected method, the value of the parameter T/N, and the presence or absence of the no short selling constraint. The improvements achieved refer to one or more of the following key portfolio indicators: (i) realized risk, (ii) reliability of realized risk predictions, (iii) degree of portfolio diversification and (iv) fraction of short selling when short selling is allowed.

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