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On the Grothendieck-Serre conjecture about principal bundles and its generalizations

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Let U be a regular connected affine semilocal scheme over a field k. Let $\boldsymbol{G}$ be a reductive group scheme over U. Assuming that $\boldsymbol{G}$ has an appropriate parabolic subgroup scheme, we prove the following statement. Given an affine k-scheme W, a principal $\boldsymbol{G}$-bundle over $W \times_k U$ is trivial if it is trivial over the generic fiber of the projection $W \times_k U \to U$.

We also simplify the proof of the Grothendieck–Serre conjecture: let U be a regular connected affine semilocal scheme over a field k. Let $\boldsymbol{G}$ be a reductive group scheme over U. A principal $\boldsymbol{G}$-bundle over U is trivial if it is trivial over the generic point of U.

We generalize some other related results from the simple simply connected case to the case of arbitrary reductive group schemes.

1. Introduction and main results

The conjecture of Grothendieck and Serre on principal bundles asserts that if $\boldsymbol{G}$ is a reductive group scheme over a regular affine semilocal scheme U and $\mathcal{E}$ is a rationally trivial principal $\boldsymbol{G}$-bundle over U, then $\mathcal{E}$ is trivial. We refer the reader to Section 1F for the precise definitions. The conjecture has been proved in the case, when U is a scheme over a field k (see [Fedorov and Panin 2015; Panin 2020a]).

One of the main goals of this paper is to generalize this result to families as we now explain. Let U and $\boldsymbol{G}$ be as before and denote the generic point of U by Ω. Let W be an affine k-scheme. Then a principal $\boldsymbol{G}$-bundle $\mathcal{E}$ over $W \times_k U$ is trivial, provided its restriction to $W \times_k \Omega$ is trivial, $\boldsymbol{G}$ satisfies some isotropy condition, and U is geometrically regular over k.

We note that our result is [Panin et al. 2015, Theorem 1.1] and [Panin 2019, Theorem 7.1], provided that our group scheme is isotropic, simple, and simply connected, and U is the spectrum of a semilocal ring of finitely many closed points on an irreducible smooth affine k-variety.

We will also give a streamlined and simplified proof of the conjecture of Grothendieck and Serre.

1A. ***Strongly locally isotropic semisimple group schemes.*** We start with formulating precisely the isotropy condition mentioned above. Let $\boldsymbol{G}$ be a semisimple group scheme over a connected scheme U. Let $\boldsymbol{Z}$ be the center of $\boldsymbol{G}$ and $\boldsymbol{G}^{\mathrm{ad}} := \boldsymbol{G}/\boldsymbol{Z}$ be the adjoint group scheme of $\boldsymbol{G}$ (see [SGA 3_{III} 1970,

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Exposé XXII, §4.3]). By [SGA 3_{III} 1970, Exposé XXIV, Proposition 5.10] there is a sequence $U_1, \dots, U_r$ of finite étale connected U-schemes such that

$$\boldsymbol{G}^{\text{ad}} \simeq \prod_{i=1}^{r} \boldsymbol{G}^i,$$

where $\boldsymbol{G}^i$ is the Weil restriction of a simple U_i-group scheme. Note that the group schemes $\boldsymbol{G}^i$ are uniquely defined by $\boldsymbol{G}$ up to isomorphism.

Definition 1.1. We say that a semisimple U-group scheme $\boldsymbol{G}$ is *strongly locally isotropic* if each factor $\boldsymbol{G}^i$ of $\boldsymbol{G}^{\text{ad}}$ is isotropic Zariski locally over U.

Remarks 1.2. (i) If $\boldsymbol{G}$ is a simple group scheme over U (or more generally, is the Weil restriction of a simple group scheme via a finite étale morphism $U' \to U$ with connected U' and U), then it is strongly locally isotropic if and only if Zariski locally over U it contains a proper parabolic subgroup scheme; see [SGA 3_{III} 1970, Exposé XXVI, Corollaire 6.14].

(ii) It follows from the previous remark that if a semisimple groups scheme is strongly locally isotropic, then it is locally isotropic.

(iii) Equivalently, one can show that a semisimple group scheme $\boldsymbol{G}$ is strongly locally isotropic if and only if Zariski locally over U it contains a proper parabolic subgroup scheme whose image in any nontrivial quotient of $\boldsymbol{G}$ is a proper subgroup scheme of the quotient.

1B. ***The Grothendieck–Serre conjecture for families.*** Here is our first main result.

Theorem 1. *Let U be a connected affine semilocal scheme geometrically regular over a field k. Denote by Ω the generic point of U. Let $\mathbf{G}$ be a reductive group scheme over U such that $\mathbf{G}^{\text{ad}}$ is strongly locally isotropic. Let W be an affine k-scheme. Let $\mathcal{E}$ be a principal $\mathbf{G}$-bundle over $W \times_k U$. If the restriction of $\mathcal{E}$ to $W \times_k \Omega$ is trivial, then $\mathcal{E}$ is trivial.*

This theorem will be proved in Section 3B. It is known that the requirement that $\boldsymbol{G}^{\text{ad}}$ be strongly locally isotropic is necessary (see counterexamples in [Fedorov 2016, §2.3]). However, we conjecture that a similar statement is true even when U is not a scheme over a field (that is, in the mixed characteristic case).

1C. ***A simplified proof of the Grothendieck–Serre conjecture.*** We will also present a simplified proof of the Grothendieck–Serre conjecture in Section 3A. Precisely, we will reprove the following theorem.

Theorem 2 [Fedorov and Panin 2015; Panin 2020a]. *Let U be a regular connected affine semilocal scheme over a field. Let Ω be the generic point of U. Let $\mathbf{G}$ be a reductive group scheme over U. Let $\mathcal{E}$ be a principal $\mathbf{G}$-bundle over U. If the restriction of $\mathcal{E}$ to Ω is trivial, then $\mathcal{E}$ is trivial.*

Theorem 2 is derived from Theorem 4 (the "section theorem") below using the results of [Panin 2019]. Theorem 4 was only known before for simple simply connected group schemes. Thus, to prove Theorem 2, one had first to reduce to the simple simply connected case, using the so-called purity theorems [Panin 2010; 2020b]. We will show that Theorem 4 holds for all reductive group schemes, thus eliminating the difficult reduction to the simple simply connected case. We will outline the strategy of the proof of Theorem 4 after its formulation in Section 1E.

In the case when $\boldsymbol{G}$ is a torus, the Grothendieck–Serre conjecture was settled in [Colliot-Thélène and Sansuc 1987]. It seems that our proof is new even in this case.

1D. *An application: principal bundles over affine spaces.* The following theorem is a generalization of [Panin et al. 2015, Corollary 1.7].

Theorem 3. *Let U be a regular connected affine scheme over $\mathbb{Q}$ and let $\boldsymbol{G}$ be a reductive group scheme over U such that $\boldsymbol{G}^{\mathrm{ad}}$ is strongly locally isotropic. Let n be a nonnegative integer, and let $\mathcal{E}$ be a principal $\boldsymbol{G}$-bundle over the affine space $\mathbb{A}^n_U$ whose restriction to the origin $U \times 0 \subset \mathbb{A}^n_U$ is trivial. Then $\mathcal{E}$ is trivial.*

Proof. The proof is by induction on n. The case $n = 0$ is obvious. Assume that the theorem is proved for $n-1$. Let $\mathcal{E}$ be a principal $\boldsymbol{G}$-bundle over $\mathbb{A}^n_U$. Write $\mathbb{A}^n_U = \mathbb{A}^{n-1}_U \times_U \mathbb{A}^1_U$. Let H be the zero section $\mathbb{A}^{n-1}_U \times 0$ so that we identify $\mathbb{A}^n_U = \mathbb{A}^1_H$. Note that H is integral. Let Ω be the generic point of H. We have a commutative diagram

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathbb{A}^1_\Omega \\ \downarrow & & \downarrow \\ H & \longrightarrow & \mathbb{A}^1_H = \mathbb{A}^n_U \end{array}$$

where the horizontal arrows are embeddings of the zero sections. By induction hypothesis the restriction of $\mathcal{E}$ to H is trivial, so its restriction to Ω is trivial as well. Since the restriction of $\mathcal{E}$ to Ω is trivial, its restriction to $\mathbb{A}^1_\Omega$ is also trivial by Raghunathan–Ramanathan theorem (see [Raghunathan and Ramanathan 1984; Gille 2002], we are using that U has characteristic zero).

Next, let ξ be any point of H and let W be the spectrum of $\mathcal{O}_{H,\xi}$. The restriction of $\mathcal{E}$ to $\mathbb{A}^1_W$ via the obvious morphism is trivial by our Theorem 1, since it is trivial over $\mathbb{A}^1_\Omega$. Further, U is normal so, according to [Thomason 1987, Corollary 3.2], we can embed $\boldsymbol{G}$ into $\mathrm{GL}_{n,U}$ for some n. Thus we can apply [Moser 2008, Korollar 3.5.2] to see that the principal $\boldsymbol{G}$-bundle $\mathcal{E}$ is trivial over $\mathbb{A}^1_H = \mathbb{A}^n_U$. □

1E. *Section theorems.* The following section theorem will be used in the proof of Theorem 2 in Section 3A.

Theorem 4. *Let U be an affine semilocal scheme. Assume that either U is a scheme over an infinite field, or U is a scheme over a finite field and the residue fields of all the closed points of U are finite. Let $\boldsymbol{G}$ be a reductive group scheme over U. Assume that Z is a closed subscheme of $\mathbb{A}^1_U$ finite over U. Let $\mathcal{E}$ be a principal $\boldsymbol{G}$-bundle over $\mathbb{A}^1_U$ trivial over $\mathbb{A}^1_U - Z$. Then for every section $\Delta : U \to \mathbb{A}^1_U$ of the projection $\mathbb{A}^1_U \to U$ the principal $\boldsymbol{G}$-bundle $\Delta^*\mathcal{E}$ is trivial.*

This is a generalization of [Fedorov and Panin 2015, Theorem 2] and of [Panin 2020a, Theorem 1.6] from simple simply connected to reductive group schemes. This theorem will be proved in Section 2E.

For not necessarily semilocal U we have a weaker statement, which will be used in Section 3B to prove Theorem 1.

Theorem 5. *Let U be an affine Noetherian connected scheme over a field. Let $\boldsymbol{G}$ be a reductive group scheme over U such that $\boldsymbol{G}$ can be embedded into $\mathrm{GL}_{n,U}$ for some n. Assume that $\boldsymbol{G}^{\mathrm{ad}}$ is strongly locally isotropic. Assume that $Z \subset \mathbb{A}^1_U$ is a closed subscheme finite over U. Let $\mathcal{E}$ be a principal $\boldsymbol{G}$-bundle over $\mathbb{A}^1_U$ trivial over $\mathbb{A}^1_U - Z$. Then for every section $\Delta : U \to \mathbb{A}^1_U$ of the projection $\mathbb{A}^1_U \to U$ the principal $\boldsymbol{G}$-bundle $\Delta^*\mathcal{E}$ is trivial.*

This section theorem will be proved in Section 2F.

Remark 1.3. The condition that $\boldsymbol{G}$ can be embedded into $\mathrm{GL}_{n,U}$ for some n is satisfied in many cases: e.g., if $\boldsymbol{G}$ is semisimple or if U is normal, see [Thomason 1987, Corollary 3.2].

The idea of the proofs of the section theorems above is the following: first, we extend the principal $\boldsymbol{G}$-bundle $\mathcal{E}$ to a principal $\boldsymbol{G}$-bundle $\hat{\mathcal{E}}$ over $\mathbb{P}^1_U$. If $\boldsymbol{G}$ is not simply connected, then the usual proof goes through with some modifications, provided that the restrictions of $\hat{\mathcal{E}}$ to the closed fibers of $\mathbb{P}^1_U \to U$ are in the neutral connected component of the stack of principal bundles. This can always be achieved by pulling back $\hat{\mathcal{E}}$ via a cover $\mathbb{P}^1_U \to \mathbb{P}^1_U$ of a sufficiently divisible degree.

1F. ***Definitions, conventions, and notation.*** All rings in this paper are commutative and unital. A semilocal ring is a Noetherian ring having only finitely many maximal ideals. An *affine semilocal scheme* is a scheme isomorphic to the spectrum of a semilocal ring.

A group scheme $\boldsymbol{G}$ over a scheme U is called *reductive* if $\boldsymbol{G}$ is affine and smooth as a U-scheme and, moreover, the geometric fibers of $\boldsymbol{G}$ are connected reductive algebraic groups (see [SGA 3_{III} 1970, Exposé XIX, Définition 2.7]). A smooth group scheme over a field k is called *a k-group*.

A U-scheme $\mathcal{E}$ with a left action $\mathrm{act} : \boldsymbol{G} \times \mathcal{E} \to \mathcal{E}$ is called *a principal $\boldsymbol{G}$-bundle over U* if $\mathcal{E}$ is faithfully flat and quasicompact over U and the action is simply transitive, that is, the morphism $(\mathrm{act}, p_2) : \boldsymbol{G} \times_U \mathcal{E} \to \mathcal{E} \times_U \mathcal{E}$ is an isomorphism (see [Grothendieck 1966, §6]). A principal $\boldsymbol{G}$-bundle $\mathcal{E}$ over U is *trivial* if $\mathcal{E}$ is isomorphic to $\boldsymbol{G}$ as a U-scheme with an action of $\boldsymbol{G}$. This is well-known to be equivalent to the projection $\mathcal{E} \to U$ having a section. We will use the term "principal $\boldsymbol{G}$-bundle over T" to mean a principal $\boldsymbol{G}_T$-bundle over T. We usually drop the adjective "principal".

A subgroup scheme $\boldsymbol{P} \subset \boldsymbol{G}$ is *parabolic* if $\boldsymbol{P}$ is smooth over U and for all geometric points $\operatorname{Spec} k \to U$ the quotient $\boldsymbol{G}_k/\boldsymbol{P}_k$ is proper over k (here k is an algebraically closed field). This coincides with [SGA 3_{III} 1970, Exposé XXVI, Définition 1.1].

2. Proofs of Theorems 4 and 5

We need some preliminaries.

2A. ***Topologically trivial principal bundles over $\mathbb{P}^1$.*** Let G be a semisimple group scheme over a field k. Let $\varphi : G^{\mathrm{sc}} \to G$ be the simply connected central cover. In other words, G^{sc} is simply connected and φ is a central isogeny (in particular, φ is finite and flat).

Definition 2.1. A Zariski locally trivial G-bundle E over $\mathbb{P}^1_k$ is called *topologically trivial* if it can be lifted to a Zariski locally trivial G^{sc}-bundle. More precisely, this means that there is a Zariski locally trivial G^{sc}-bundle E^{sc} over $\mathbb{P}^1_k$ such that $\varphi_* E^{\mathrm{sc}} \simeq E$.

Remark 2.2. If k is the field of complex numbers, then a principal bundle over $\mathbb{P}^1_k$ is topologically trivial in the sense of Definition 2.1 if and only if it is topologically trivial in the usual sense, that is, it has a continuous section (see [Sorger 2000, Corollary 4.1.2]), which justifies the name.

We need the following proposition.

Proposition 2.3. *For every Zariski locally trivial G-bundle E over $\mathbb{P}^1_k$ and for every finite morphism $\psi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ whose degree is divisible by the degree of φ, the G-bundle $\psi^* E$ is topologically trivial.*

Before giving the proof of the proposition we recall the description of Zariski locally trivial G-bundles over $\mathbb{P}^1_k$. Let $T \subset G$ be a maximal split torus of G. Let E be a Zariski locally trivial G-bundle over $\mathbb{P}^1_k$. Then by [Gille 2002, Théorème 3.8(b)], there is a cocharacter $\lambda : \mathbb{G}_{m,k} \to T$ such that $E \simeq \lambda_* \mathcal{O}(1)^\times$. Here $\mathcal{O}(1)$ is the hyperplane line bundle over $\mathbb{P}^1_k$; the $\mathbb{G}_{m,k}$-bundle $\mathcal{O}(1)^\times$ is the complement of the zero section in $\mathcal{O}(1)$. We are slightly abusing the notation, denoting the composition $\mathbb{G}_{m,k} \xrightarrow{\lambda} T \hookrightarrow G$ by λ as well.

Proof of Proposition 2.3. Put $d := \deg \varphi$. Let T^{sc} be a maximal split torus of G^{sc}. By [Borel and Tits 1972, Théorème 2.20(ii)], $T := \varphi(T^{\mathrm{sc}})$ is a maximal split torus of G. The k-group scheme $T \times_G G^{\mathrm{sc}}$ is of multiplicative type by [SGA 3_{II} 1970, Exposé XVII, Proposition 7.1.1(b)] and the isogeny $T \times_G G^{\mathrm{sc}} \to T$ also has degree d. It is clear that T^{sc} is the toral part of $T \times_G G^{\mathrm{sc}}$. It is also clear that $\varphi|_{T^{\mathrm{sc}}} : T^{\mathrm{sc}} \to T$ is an isogeny whose degree divides d (indeed, we can check it over an algebraic closure of k in which case we may assume that $T \times_G G^{\mathrm{sc}}$ is diagonalizable).

Denote the degree of the isogeny $\varphi|_{T^{\mathrm{sc}}} : T^{\mathrm{sc}} \to T$ by d'. It is also the index of the cocharacter lattice $X_*(T^{\mathrm{sc}})$ in $X_*(T)$. Let E be a Zariski locally trivial G-bundle over $\mathbb{P}^1_k$. As we have already mentioned, by [Gille 2002, Théorème 3.8(b)] there is a cocharacter $\lambda : \mathbb{G}_{m,k} \to T$ such that $E \simeq \lambda_* \mathcal{O}(1)^\times$. Let $\psi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a finite morphism of degree n. Then

$$\psi^* E \simeq \lambda_* \mathcal{O}(n)^\times \simeq (n\lambda)_* \mathcal{O}(1)^\times,$$

where $\mathcal{O}(n)$ is the n-th tensor power of $\mathcal{O}(1)$. If d divides n, then d' divides n as well, so $n\lambda$ is a cocharacter of $X_*(T^{\mathrm{sc}})$ and it is clear that $\psi^* E$ can be lifted to a G^{sc}-bundle. Proposition 2.3 is proved. □

It is clear from the proof that it is enough to require that the degree of ψ is divisible by the exponent of the kernel of φ.

2B. *Recollection on affine Grassmannians.* We will use affine Grassmannians of group schemes defined in [Fedorov 2016] in the proof of Theorem 6. We only consider the affine Grassmannians for semisimple group schemes. The results below should hold in bigger generality, for example, if the group scheme is reductive and can be embedded into the general linear group scheme. Since we are not aware of a reference, we will restrict ourselves to the semisimple case.

For an affine scheme $T = \operatorname{Spec} S$, put $D_T := \operatorname{Spec} S[[t]]$ and $\dot{D}_T := \operatorname{Spec} S((t))$, where $S((t)) := S[[t]](t^{-1})$.

Recall the definition of affine Grassmannians from [Fedorov 2016, §5.1]. Consider a connected affine scheme $U = \operatorname{Spec} R$; let Aff/U be the (big) étale site of affine schemes over U and $\text{ét}/U$ be the (big) étale site of schemes over U. Recall that a *U-space* is a sheaf of sets on $\text{ét}/U$. We can equivalently view it as a sheaf on Aff/U (see [SGA 4_2 1972, Exposé VII, Proposition 3.1]). Let $\boldsymbol{G}$ be a smooth affine U-group scheme. The *affine Grassmannian* $\mathrm{Gr}_{\boldsymbol{G}}$ is defined as the sheafification of the presheaf, sending an affine U-scheme T to the set $\boldsymbol{G}(\dot{D}_T)/\boldsymbol{G}(D_T)$. (The morphism $\dot{D}_T \to D_T$ induces a morphism $\boldsymbol{G}(D_T) \to \boldsymbol{G}(\dot{D}_T)$. It is obvious that this morphism is injective and we identify $\boldsymbol{G}(D_T)$ with its image.) If $\boldsymbol{G}$ is semisimple, then $\mathrm{Gr}_{\boldsymbol{G}}$ is an inductive limit of schemes over U (see [Fedorov 2016, Proposition 5.11]). These schemes may be chosen projective over U, though we will not use it.

Let Y be a finite and étale over U subscheme of $\mathbb{A}^1_U$ (automatically closed). Assume also that $Y \neq \varnothing$, then the projection $Y \to U$ is surjective (being both open and closed). Let $\mathcal{E}$ be a $\boldsymbol{G}$-bundle over $\mathbb{P}^1_U$. A *modification* of $\mathcal{E}$ at Y is a pair $(\mathcal{F}, \tau)$, where $\mathcal{F}$ is a $\boldsymbol{G}$-bundle over $\mathbb{P}^1_U$ and τ is an isomorphism

$$\mathcal{F}|_{\mathbb{P}^1_U - Y} \xrightarrow{\tau} \mathcal{E}|_{\mathbb{P}^1_U - Y}$$

(see [Fedorov 2016, §7.3]). We have an obvious notion of an isomorphism of modifications of $\mathcal{E}$ at Y.

Fix a $\boldsymbol{G}$-bundle $\mathcal{E}$ over $\mathbb{P}^1_U$ and assume that it is trivial in a Zariski neighborhood of $Y \subset \mathbb{A}^1_U$. Fix such a trivialization σ. Let Ψ_σ be the functor, sending a U-scheme T to the set of isomorphism classes of modifications of $\mathcal{E}|_{\mathbb{P}^1_T}$ at $Y \times_U T$. Recall [Fedorov 2016, Proposition 7.5]:

Proposition 2.4. *The functor Ψ_σ is canonically isomorphic to the functor sending a U-scheme T to* $\mathrm{Gr}_{\boldsymbol{G}}(Y \times_U T)$.

Note that this isomorphism depends on the trivialization σ of $\mathcal{E}$ in a neighborhood of Y. Let σ' be another trivialization on a (possibly different) Zariski neighborhood of Y. The restrictions of σ and σ' to the formal neighborhood of Y differ by a jet $\alpha \in L^+\boldsymbol{G}(Y)$, where the jet group scheme $L^+\boldsymbol{G}$ represents the functor $T \mapsto \boldsymbol{G}(D_T)$. Note that $L^+\boldsymbol{G}$ acts on $\mathrm{Gr}_{\boldsymbol{G}}$. The proof of the following lemma is clear from the proof of [Fedorov 2016, Proposition 5.1].

Lemma 2.5. *The functors $\Psi_{\sigma'}$ and $\Psi_\sigma \circ \tilde{\alpha}$ are canonically isomorphic, where $\tilde{\alpha}$ stands for the automorphism of* $\mathrm{Gr}_{\boldsymbol{G}}$ *given by the action of α.*

Remarks 2.6. To identify the modifications with sections of affine Grassmannian, it is enough to trivialize $\mathcal{E}$ on a formal neighborhood of Y. Such a trivialization exists if and only if $\mathcal{E}|_Y$ is trivial (because $\mathcal{E}$ is smooth over $\mathbb{P}^1_U$). If $\mathcal{E}$ is not trivial on Y, then the modifications are parametrized by a twist of the affine Grassmannian.

The unit section of $\boldsymbol{G}$ gives rise to a unit section $\mathrm{Id}_{\mathrm{Gr}} \in \mathrm{Gr}_{\boldsymbol{G}}(Y)$. This section corresponds to the trivial modification $(\mathcal{E}, \mathrm{Id}_{\mathcal{E}}|_{\mathbb{P}^1_U - Y})$ under the above isomorphism.

It is clear that we have a natural isomorphism $\mathrm{Gr}_{\boldsymbol{G}_1 \times_U \boldsymbol{G}_2} = \mathrm{Gr}_{\boldsymbol{G}_1} \times_U \mathrm{Gr}_{\boldsymbol{G}_2}$.

Note that there is a canonical automorphism of $\mathbb{P}^1_U$ switching $\mathbb{P}^1_U - (U \times 0)$ and $\mathbb{A}^1_U$. We use this automorphism to identify points of $\mathrm{Gr}_{\boldsymbol{G}}(U)$ with modifications of the trivial $\boldsymbol{G}$-bundle at $U \times \infty$, that is, with pairs $(\mathcal{E}, \tau)$, where $\mathcal{E}$ is a $\boldsymbol{G}$-bundle over $\mathbb{P}^1_U$, and τ is a trivialization of $\mathcal{E}$ over $\mathbb{A}^1_U$.

The following is a slight generalization of [Fedorov 2016, Proposition 7.1].

Lemma 2.7. *Let Y be an affine scheme; let $y_1, \dots, y_n \in Y$ be closed points. Let $\boldsymbol{H}$ be a simple simply connected Y-group scheme and assume that $\boldsymbol{H}$ contains a parabolic subgroup scheme that is proper on every connected component of Y. Then the restriction morphism*

$$\mathrm{Gr}_{\boldsymbol{H}}(Y) \to \prod_{i=1}^{n} \mathrm{Gr}_{\boldsymbol{H}}(y_i)$$

is surjective.

Proof. The proof is very similar to the proof of [Fedorov 2016, Proposition 7.1] but we give it for the sake of completeness. Let $\boldsymbol{P}^+$ be a parabolic subgroup scheme of $\boldsymbol{H}$ proper on every connected component of Y. Since Y is an affine scheme, by [SGA 3_{III} 1970, Exposé XXVI, Corollaire 2.3, Théorème 4.3.2(a)], there is an opposite to $\boldsymbol{P}^+$ parabolic subgroup scheme $\boldsymbol{P}^- \subset \boldsymbol{H}$. Let $\boldsymbol{U}^+$ be the unipotent radical of $\boldsymbol{P}^+$, and let $\boldsymbol{U}^-$ be the unipotent radical of $\boldsymbol{P}^-$. We will write $\boldsymbol{E}$ for the functor, sending a Y-scheme T to the subgroup $\boldsymbol{E}(T)$ of the group $\boldsymbol{H}(T)$ generated by the subgroups $\boldsymbol{U}^+(T)$ and $\boldsymbol{U}^-(T)$ of the group $\boldsymbol{H}(T)$ (see [Fedorov and Panin 2015, Definition 5.23; Fedorov 2016, Definition 7.2]). As in the proof of [Fedorov 2016, Proposition 7.1], we have a diagram

$$\begin{array}{ccc} \boldsymbol{E}(\dot{D}_Y) & \longrightarrow & \prod_{i=1}^{n} \boldsymbol{E}(\dot{D}_{y_i}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{\boldsymbol{H}}(Y) & \longrightarrow & \prod_{i=1}^{n} \mathrm{Gr}_{\boldsymbol{H}}(y_i) \end{array}$$

By [Fedorov 2016, Lemma 7.3] (whose easy proof is valid for any reductive group scheme) the top horizontal map is surjective. Thus it is enough to show that the map

$$\boldsymbol{E}(\dot{D}_{y_i}) \to \mathrm{Gr}_{\boldsymbol{H}}(y_i)$$

is surjective for each i. Set $k := k(y_i)$ and $H := \boldsymbol{H}_{y_i}$. Consider an element of $\mathrm{Gr}_{\boldsymbol{H}}(y_i) = \mathrm{Gr}_H(k)$, represented by a pair $(\mathcal{E}, \tau)$, where $\mathcal{E}$ is an H-bundle over $\mathbb{P}^1_k$, and τ is a trivialization of $\mathcal{E}$ over $\mathbb{A}^1_k$. By [Gille 2002, Théorème 3.8(a)], $\mathcal{E}$ is Zariski locally trivial. Let us trivialize $\mathcal{E}$ in a formal neighborhood of ∞, this trivialization and τ differ by an element $\beta \in H\big(k((t))\big)$. By construction, the image of β under the projection $H\big(k((t))\big) \to \mathrm{Gr}_{\boldsymbol{H}}(y_i)$ is $(\mathcal{E}, \tau)$.

Next, H is simple and simply connected and the field $k((t))$ is infinite. Thus we may use [Gille 2009, Lemme 4.5(1) and Fait 4.3(2)] to conclude that we can write $\beta = \beta'\beta''$ with $\beta' \in \boldsymbol{E}\big(k((t))\big) = \boldsymbol{E}(\dot{D}_{y_i})$, $\beta'' \in H\big(k[[t]]\big)$. Clearly, β' lifts $(\mathcal{E}, \tau)$ and we are done. □

Note that, instead of using [Gille 2002, Théorème 3.8(a)] in the proof above, one can use the Grothendieck–Serre conjecture for discrete valuation rings [Nisnevich 1984]. The same applies to the reference in the proof of Theorem 4.

2C. *Lifting modifications to the simply connected central cover.* Let, as before, $\varphi : G^{\mathrm{sc}} \to G$ be the simply connected central cover of a semisimple k-group scheme G, where k is a field. This gives a morphism of ind-schemes $\mathrm{Gr}_{G^{sc}} \to \mathrm{Gr}_G$. The goal of this section is to prove the following proposition (cf. Lemma 2.5).

Proposition 2.8. *Let K be any field containing k. The image of the set $\mathrm{Gr}_{G^{sc}}(K)$ in $\mathrm{Gr}_G(K)$ is $L^+G(K)$-invariant.*

Proof. Since $L^+G(K) = L^+G_K(K)$ and $\mathrm{Gr}_G(K) = \mathrm{Gr}_{G_K}(K)$, performing a base change we may assume that $K = k$.

For split group schemes there is a well-known stratification of Grassmannians by L^+G-orbits; the orbits are parametrized by the Weyl group orbits in the cocharacter lattice. If the group scheme is not an inner form, we have a coarser stratification constructed in [Fedorov 2016]. We will recall this stratification now.

Let G^{spl} be the split semisimple k-group scheme of the same type as G. Let $T^{\mathrm{spl}} \subset G^{\mathrm{spl}}$ be a maximal (split) torus. Following [Fedorov 2016, §5.4.2] put

$$X_* := \mathrm{Hom}(\mathbb{G}_{m,k}, T^{\mathrm{spl}}) \subset T^{\mathrm{spl}}\big(k((t))\big).$$

For $\lambda \in X_*$ denote by t^λ the corresponding element of $T^{\mathrm{spl}}\big(k((t))\big)$. Abusing notation, we also denote by t^λ the projection to $\mathrm{Gr}_{G^{\mathrm{spl}}}(k)$ of

$$t^\lambda \in T^{\mathrm{spl}}\big(k((t))\big) \subset G^{\mathrm{spl}}\big(k((t))\big).$$

Denote by $\mathrm{Gr}^\lambda_{G^{\mathrm{spl}}}$ the L^+G^{spl}-orbit of t^λ; this is a locally closed subscheme of $\mathrm{Gr}_{G^{\mathrm{spl}}}$. We have $\mathrm{Gr}^\lambda_{G^{\mathrm{spl}}} = \mathrm{Gr}^\mu_{G^{\mathrm{spl}}}$ if and only if λ and μ are in the same W-orbit (here W is the Weyl group of G^{spl}). By [Fedorov 2016, Proposition 5.7], we get a stratification (in the sense of [Fedorov 2016, §5.3])

$$\mathrm{Gr}_{G^{\mathrm{spl}}} = \bigcup_{\lambda \in X_*/W} \mathrm{Gr}^\lambda_{G^{\mathrm{spl}}}. \tag{1}$$

Next, G is a twist of G^{spl} by an $\mathbf{Aut}(G^{\mathrm{spl}})$-bundle $\mathcal{T}$ over $\operatorname{Spec} k$, so by [Fedorov 2016, Proposition 5.4] we get $\mathrm{Gr}_G = \mathcal{T} \times^{\mathbf{Aut}(G^{\mathrm{spl}})} \mathrm{Gr}_{G^{\mathrm{spl}}}$. Unfortunately, the orbits $\mathrm{Gr}^\lambda_{G^{\mathrm{spl}}}$ are not $\mathbf{Aut}(G^{\mathrm{spl}})$-invariant, so we need a coarser stratification. Note that $\mathbf{Out} := \mathbf{Aut}(G^{\mathrm{spl}})/G^{\mathrm{spl,ad}}$ acts on W so we get a semidirect product $W \rtimes \mathbf{Out}$. For $\hat\lambda \in X_*/(W \rtimes \mathbf{Out})$, write $\mathrm{Orb}(\hat\lambda)$ for the corresponding $\mathbf{Out}$-orbit on X_*/W and put

$$\mathrm{Gr}^{\hat\lambda}_{G^{\mathrm{spl}}} := \bigsqcup_{\lambda \in \mathrm{Orb}(\hat\lambda)} \mathrm{Gr}^\lambda_{G^{\mathrm{spl}}}.$$

Note that, if $\lambda_1, \lambda_2 \in \mathrm{Orb}(\hat\lambda)$, then $\mathrm{Gr}^{\lambda_1}_G$ is isomorphic to $\mathrm{Gr}^{\lambda_2}_G$, so these orbits have the same dimension. It follows that $\mathrm{Gr}^{\lambda_1}_G$ cannot lie in the closure of $\mathrm{Gr}^{\lambda_2}_G$. Thus, the above is, in fact, a disjoint union of schemes.

The locally closed subsets $\mathrm{Gr}^{\hat\lambda}_{G^{\mathrm{spl}}}$ are $\mathbf{Aut}(G^{\mathrm{spl}})$-invariant so we put

$$\mathrm{Gr}^{\hat\lambda}_G := \mathcal{T} \times^{\mathbf{Aut}(G^{\mathrm{spl}})} \mathrm{Gr}^{\hat\lambda}_{G^{\mathrm{spl}}}.$$

Now the stratification (1) gives rise to a stratification [Fedorov 2016, Proposition 5.12]

$$\mathrm{Gr}_G = \bigcup_{\hat\lambda \in X_*/(W \rtimes \mathbf{Out})} \mathrm{Gr}^{\hat\lambda}_G. \tag{2}$$

Let $G^{\mathrm{sc,spl}}$ be the simply connected central cover of G^{spl}, $T^{\mathrm{sc,spl}}$ be the preimage of T^{spl} in $G^{\mathrm{sc,spl}}$ (this is a maximal split torus in $G^{\mathrm{sc,spl}}$), and X_*^{sc} be the cocharacter lattice of $T^{\mathrm{sc,spl}}$. Then, similarly to the above,

$$\mathrm{Gr}_{G^{\mathrm{sc}}} = \bigcup_{\hat\lambda \in X_*^{\mathrm{sc}}/(W \rtimes \mathbf{Out})} \mathrm{Gr}_{G^{\mathrm{sc}}}^{\hat\lambda};$$

this decomposition is compatible with (2) and the projection $\pi : \mathrm{Gr}_{G^{\mathrm{sc}}} \to \mathrm{Gr}_G$.

Now we return to the proof of Proposition 2.8. Consider a point $\alpha \in \mathrm{Gr}_G(k)$. By (2) it belongs to $\mathrm{Gr}_G^{\hat\lambda}(k)$ for some $\hat\lambda \in X_*/(W \rtimes \mathbf{Out})$. We claim that α lifts to a point of $\mathrm{Gr}_{G^{\mathrm{sc}}}(k)$ if and only if $\hat\lambda \in X_*^{\mathrm{sc}}/(W \rtimes \mathbf{Out})$ (we identify X_*^{sc} with a sublattice of X_*). The proposition follows from this statement because $\mathrm{Gr}_G^{\hat\lambda}$ is manifestly L^+G-invariant.

Recall that the projection $\pi : \mathrm{Gr}_{G^{\mathrm{sc}}} \to \mathrm{Gr}_G$ takes $\mathrm{Gr}_{G^{\mathrm{sc}}}^{\hat\lambda}$ to $\mathrm{Gr}_G^{\hat\lambda}$. This proves the "only if" part of our claim. For the converse, it suffices to prove the following lemma.

Lemma 2.9. *Assume that $\hat\lambda \in X_*^{\mathrm{sc}}/(W \rtimes \mathbf{Out})$. Then π induces an isomorphism of schemes $\mathrm{Gr}_{G^{\mathrm{sc}}}^{\hat\lambda} \to \mathrm{Gr}_G^{\hat\lambda}$.*

Proof. First of all, it is enough to prove the statement after passing to an algebraic closure of k, in which case G is split and we have a finer stratification (1). Thus we assume that k is algebraically closed and show that for $\lambda \in X_*^{\mathrm{sc}}/W$ the canonical morphism $\pi' : \mathrm{Gr}_{G^{\mathrm{sc}}}^{\lambda} \to \mathrm{Gr}_G^{\lambda}$ is an isomorphism.

We say that a parabolic subgroup scheme $P \subset G$ is of type λ if the Weyl group of a Levi factor of P is the stabilizer of λ in W. Let F_G^λ be the scheme of parabolic subgroups of type λ. In [Fedorov 2016, §5.4.3] we constructed a morphism $\mathrm{Gr}_G^\lambda \to F_G^\lambda$. We have a similar morphism for G^{sc} and a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{G^{\mathrm{sc}}}^{\lambda} & \xrightarrow{\pi'} & \mathrm{Gr}_G^{\lambda} \\ \downarrow & & \downarrow \\ F_{G^{\mathrm{sc}}}^{\lambda} & \longrightarrow & F_G^{\lambda} \end{array} \tag{3}$$

Note that the lower horizontal morphism is an isomorphism (the proof is analogous to [Conrad 2014, Exercise 5.5.8]). Since the left projection in the diagram is G^{sc}-equivariant and G^{sc} acts transitively on $F_{G^{\mathrm{sc}}}^{\lambda}$, the generic flatness implies that this projection if flat. Similarly, the right projection is flat. Thus it is enough to check that π' induces isomorphism of fibers.

Fix a lift of λ to X_*^{sc} so that we have a point $t^\lambda \in \mathrm{Gr}_{G^{\mathrm{sc}}}^{\lambda}(k)$ and a point $t^\lambda \in \mathrm{Gr}_G^\lambda(k)$. Let C^{sc} be the fiber of the morphism $\mathrm{Gr}_{G^{\mathrm{sc}}}^{\lambda} \to F_{G^{\mathrm{sc}}}^{\lambda}$ containing t^λ; let C^{sc} be the fiber of the morphism $\mathrm{Gr}_G^\lambda \to F_G^\lambda$ containing t^λ. It is enough to show that π' induces an isomorphism $C^{\mathrm{sc}} \to C$ because diagram (3) is G^{sc}-equivariant.

For a k-group scheme H, we denote by $H^{(1)}$ the kernel of the evaluation map $L^+H \to H$. We note that this is just the group scheme of jets into H based at the identity. We claim that C is the $G^{(1)}$-orbit of t^λ. Indeed, we have a semidirect product decomposition $L^+G = G^{(1)} \rtimes G$. As explained in [Fedorov 2016, §5.4.3], the morphism $\mathrm{Gr}_G^\lambda \to F_G^\lambda$ is induced by the evaluation map

$$L^+G = G^{(1)} \rtimes G \to G \to G \cdot t^\lambda = F_G^\lambda.$$

Let P^λ be the stabilizer of t^λ in G. We see that $C = G^{(1)} P^\lambda \cdot t^\lambda = G^{(1)} \cdot t^\lambda$.

Next, let U be a unipotent subgroup scheme of G opposite to P^λ. We claim that $C = U^{(1)} \cdot t^\lambda$. To this end, it is enough to check that $G^{(1)} = U^{(1)} \cdot P^{(1)}$ and that $P^{(1)}$ stabilizes t^λ. For the first statement we note that the multiplication map $U \times P \to G$ induces an isomorphism on the level of Lie algebras, so it induces an isomorphism of jets based at the identity. Similarly, the second statement reduced to a statement about the Lie algebras.

Let $U^{\mathrm{sc}} \subset G^{\mathrm{sc}}$ be the preimage of U under the projection $G^{\mathrm{sc}} \to G$. Then U^{sc} is a unipotent subgroup scheme opposite to the stabilizer of t^λ in G^{sc}. Similarly to the above we check that $C^{\mathrm{sc}} = (U^{\mathrm{sc}})^{(1)} \cdot t^\lambda$.

Note that the central isogeny $\varphi : G^{\mathrm{sc}} \to G$ induces an isomorphism $U^{\mathrm{sc}} \to U$. Thus we have an isomorphism $(U^{\mathrm{sc}})^{(1)} \to U^{(1)}$. The stabilizer of t^λ in $U^{(1)}$ is

$$U^{(1)} \cap (t^\lambda \cdot L^+G \cdot t^{-\lambda}) = U^{(1)} \cap (t^\lambda \cdot L^+U \cdot t^{-\lambda})$$

and we have a similar formula for the stabilizer in $(U^{\mathrm{sc}})^{(1)}$. Thus the above isomorphism identifies stabilizers, so it induces an isomorphism $C^{\mathrm{sc}} \to C$. The lemma follows. □

The lemma completes the proof of the claim. Proposition 2.8 is proved. □

Remark 2.10. If the characteristic of k does not divide the order of $\pi_1(G)$, it is known that $\pi : \mathrm{Gr}_{G^{\mathrm{sc}}} \to \mathrm{Gr}_G$ induces an isomorphism between $\mathrm{Gr}_{G^{\mathrm{sc}}}$ and the neutral connected component of Gr_G. On the other hand, it is not difficult to derive from the above proof that in general π is a morphism from $\mathrm{Gr}_{G^{\mathrm{sc}}}$ to the neutral connected component of Gr_G inducing an isomorphism on K-points for every field K. The above proposition follows from this fact because the neutral component is preserved under the action of the connected group scheme L^+G. One expects that this morphism is a universal homeomorphism. We refer the reader to [Haines and Richarz 2019, Proposition 3.5] for a similar statement.

2D. ***Principal bundles with topologically trivial fibers over families of affine lines.*** In this section we prove an analogue of [Fedorov and Panin 2015, Theorem 3] and of [Panin 2020a, Theorem 1.8] where the group scheme is allowed to be arbitrary reductive but the $\boldsymbol{G}$-bundle is required to be topologically trivial on closed fibers. Recall that a semisimple group scheme over a scheme U is called *isotropic* if it contains a one-dimensional torus $\mathbb{G}_{m,U}$. If U is connected affine, and semilocal, then by [SGA 3_{III} 1970, Exposé XXVI, Corollaire 6.14] this is equivalent to the group scheme containing a proper parabolic subgroup scheme. For any scheme S we denote by $\mathrm{Pic}(S)$ the group of isomorphism classes of line bundles over S. Recall that $\boldsymbol{Z}$ is the center of $\boldsymbol{G}$ and $\boldsymbol{G}^{\mathrm{ad}} = \boldsymbol{G}/\boldsymbol{Z}$.

Theorem 6. *Let U be a connected affine semilocal scheme over a field. Let $\mathbf{G}$ be a reductive group scheme over U; write*

$$\boldsymbol{G}^{\mathrm{ad}} \simeq \prod_{i=1}^{r} \boldsymbol{G}^i,$$

where $\boldsymbol{G}^i$ is the Weil restriction of a simple U_i-group scheme $\bar{\boldsymbol{G}}^i$ via a finite étale morphism $U_i \to U$. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial and such that for all closed points $u \in U$ the $\mathbf{G}^{\mathrm{ad}}_u$-bundle $(\mathcal{G}|_{\mathbb{P}^1_u})/\boldsymbol{Z}_u$ is topologically trivial.

Let $Y \subset \mathbb{A}^1_U$ be a closed subscheme finite and étale over U. Assume that $Y \cap Z = \varnothing$. Assume further that for each $i = 1, \dots, r$ there is an open and closed subscheme $Y^i \subset Y \times_U U_i$ satisfying two properties:

(i) *the pullback of $\bar{\boldsymbol{G}}^i$ to each connected component of Y^i is isotropic, and*

(ii) *for every closed point $v \in U_i$ such that $\bar{\boldsymbol{G}}^i_v$ is isotropic we have* $\operatorname{Pic}(\mathbb{P}^1_v - Y^i_v) = 0$.

Finally, assume that the relative line bundle $\mathcal{O}_{\mathbb{P}^1_U}(1)$ trivializes on $\mathbb{P}^1_U - Y$. Then the restriction of $\mathcal{G}$ to $\mathbb{P}^1_U - Y$ is also trivial.

Remarks 2.11. (i) The condition that $\mathcal{O}_{\mathbb{P}^1_U}(1)$ trivializes on $\mathbb{P}^1_U - Y$ is necessary. Indeed, if we take $\boldsymbol{G} = \mathbb{G}_{m,U}$, $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1_U}(1)^\times$, $Y = \varnothing$, then $\mathcal{G}$ is not trivial over $\mathbb{P}^1_U - Y$. (Note that Y satisfies the other conditions of the theorem because $r = 0$.)

(ii) Note that the $\boldsymbol{G}^{\mathrm{ad}}_u$-bundle $(\mathcal{G}|_{\mathbb{P}^1_u})/\boldsymbol{Z}_u$ is Zariski locally trivial, because $\mathcal{G}|_{\mathbb{P}^1_u}$ is trivial over $\mathbb{P}^1_u - Z_u$.

(iii) Assume that the residue fields of the closed points of U are infinite. Then we may start with $Y, Z \subset \mathbb{P}^1_U$. Indeed, applying a projective transformation of $\mathbb{P}^1_U$ we can always achieve $Y, Z \subset \mathbb{A}^1_U$. The condition $Y \cap Z = \varnothing$ is also not necessary in this case; see Remark 2 after [Fedorov and Panin 2015, Theorem 3].

(iv) The proof of this theorem is much simpler in many cases: for example, if U is local or normal. When U is not normal, the problem is that a line bundle on $\mathbb{P}^1_U - Y$ need not be trivial, unless it can be extended to $\mathbb{P}^1_U$.

We need a proposition, which is a slight generalization of [Panin et al. 2015, Proposition 9.6].

Proposition 2.12. *Let, as above, U be a connected affine semilocal scheme over a field. Let $\boldsymbol{H}$ be a semisimple U-group scheme. Let $\mathcal{H}$ be an $\boldsymbol{H}$-bundle over $\mathbb{P}^1_U$ such that for every closed point $u \in U$ the restriction of $\mathcal{H}$ to $\mathbb{P}^1_u$ is a trivial $\boldsymbol{H}_u$-bundle. Then $\mathcal{H}$ is isomorphic to the pullback of an $\boldsymbol{H}$-bundle over U.*

Proof. Since $\boldsymbol{H}$ is semisimple, there is an embedding $\boldsymbol{H} \hookrightarrow \mathrm{GL}_{n,U}$ for some n by [Thomason 1987, Corollary 3.2]. The rest of the proof is completely analogous to that of [Panin et al. 2015, Proposition 9.6]. □

Proof of Theorem 6. Step 1. Let $\widetilde{\boldsymbol{G}}^i$ be the simply connected central cover of the group scheme $\boldsymbol{G}^i$ (see [Conrad 2014, Exercise 6.5.2]). Then $\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i$ is the simply connected central cover of $\boldsymbol{G}^{\mathrm{ad}}$. *We claim that the covering homomorphism $\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i \to \boldsymbol{G}^{\mathrm{ad}}$ lifts to a homomorphism $\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i \to \boldsymbol{G}$.* Indeed, let $[\boldsymbol{G}, \boldsymbol{G}]$ be the derived subgroup scheme of $\boldsymbol{G}$, then the morphism $[\boldsymbol{G}, \boldsymbol{G}] \to \boldsymbol{G}^{\mathrm{ad}}$ is a central isogeny, so the simply connected central cover of $[\boldsymbol{G}, \boldsymbol{G}]$ is also the simply connected central cover of $\boldsymbol{G}^{\mathrm{ad}}$. Hence, $\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i \to \boldsymbol{G}^{\mathrm{ad}}$ factors through $[\boldsymbol{G}, \boldsymbol{G}]$ and the statement follows. Thus we have a sequence of homomorphisms

$$\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i \to \boldsymbol{G} \to \boldsymbol{G}^{\mathrm{ad}} = \prod_{i=1}^r \boldsymbol{G}^i.$$

Let $\widehat{\boldsymbol{G}}^i$ be the simply connected central cover of $\bar{\boldsymbol{G}}^i$. It is easy to see that $\widetilde{\boldsymbol{G}}^i$ is the Weil restriction of $\widehat{\boldsymbol{G}}^i$ via $U_i \to U$.

Step 2. Let $u \in U$ be a closed point and put $\mathcal{G}_u := \mathcal{G}|_{\mathbb{P}^1_u}$. By assumption (see Definition 2.1), the $\boldsymbol{G}^{\mathrm{ad}}_u$-bundle $\mathcal{G}_u/\boldsymbol{Z}_u$ lifts to a Zariski locally trivial $\prod_{i=1}^r \widetilde{\boldsymbol{G}}^i_u$-bundle $\widetilde{\mathcal{G}}_u$ over $\mathbb{P}^1_u$. This corresponds to a sequence $(\widetilde{\mathcal{G}}^1_u, \dots, \widetilde{\mathcal{G}}^r_u)$, where $\widetilde{\mathcal{G}}^i_u$ is a $\widetilde{\boldsymbol{G}}^i_u$-bundle. Let $\mathcal{G}^i_u$ be the pushforward of $\widetilde{\mathcal{G}}^i_u$ to $\boldsymbol{G}^i_u$. According to [SGA 3_{III} 1970, Exposé XXIV, Proposition 8.4], $\widetilde{\boldsymbol{G}}^i$-bundles over any scheme T correspond to $\widehat{\boldsymbol{G}}^i$-bundles over $T \times_U U_i$. Fix i and consider the finite scheme $\boldsymbol{u} := u \times_U U_i$. Let $\widehat{\mathcal{G}}^i_{\boldsymbol{u}}$ be the $\widehat{\boldsymbol{G}}^i$-bundle corresponding to $\widetilde{\mathcal{G}}^i_u$ and let $\bar{\mathcal{G}}^i_{\boldsymbol{u}}$ be the $\bar{\boldsymbol{G}}^i$-bundle corresponding to $\mathcal{G}^i_u$.

We claim that $\widehat{\mathcal{G}}^i_{\boldsymbol{u}}$ is trivial over $\mathbb{P}^1_{\boldsymbol{u}} - Y^i_{\boldsymbol{u}}$ for all i. Indeed, let $v \in \boldsymbol{u}$, it is enough to show that every Zariski locally trivial $\widehat{\boldsymbol{G}}^i$-bundle over $\mathbb{P}^1_v - Y^i_v$ is trivial. If $\widehat{\boldsymbol{G}}^i_v$ is anisotropic, this follows immediately from [Gille 2002, Théorème 3.10(a)]. If $\widehat{\boldsymbol{G}}^i_v$ is isotropic, then $\bar{\boldsymbol{G}}^i_v$ is also isotropic (see [Borel and Tits 1972, Théorème 2.20]) so $\operatorname{Pic}(\mathbb{P}^1_v - Y^i_v) = 0$, and the statement again follows from [Gille 2002, Théorème 3.10(a)].

For $i = 1, \dots, r$ choose a trivialization $\hat{\tau}^i_{\boldsymbol{u}}$ of $\widehat{\mathcal{G}}^i_{\boldsymbol{u}}$ over $\mathbb{P}^1_{\boldsymbol{u}} - Y^i_{\boldsymbol{u}}$. These trivializations induce trivializations of $\bar{\mathcal{G}}^i_{\boldsymbol{u}}$ on $\mathbb{P}^1_{\boldsymbol{u}} - Y^i_{\boldsymbol{u}}$. Denote these trivializations by $\bar{\tau}^i_{\boldsymbol{u}}$.

Step 3. Let $\bar{\mathcal{F}}^i_{\boldsymbol{u}}$ be the trivial $\bar{\boldsymbol{G}}^i_{\boldsymbol{u}}$-bundle over $\mathbb{P}^1_{\boldsymbol{u}}$. Then $(\bar{\mathcal{F}}^i_{\boldsymbol{u}}, \bar{\tau}^i_{\boldsymbol{u}})$ is a modification of $\bar{\mathcal{G}}^i_{\boldsymbol{u}}$ at $Y^i_{\boldsymbol{u}}$. Choose a trivialization of $\mathcal{G}$ over $\mathbb{P}^1_U - Z$. Since $Y \cap Z = \varnothing$, this gives a trivialization of $\mathcal{G}_u$ (and, in turn, of $\mathcal{G}^i_u$) in a neighborhood of $Y_u \subset \mathbb{P}^1_u$. Finally, we get a trivialization of $\bar{\mathcal{G}}^i_{\boldsymbol{u}}$ in a neighborhood of $Y^i_{\boldsymbol{u}} \subset \mathbb{P}^1_{\boldsymbol{u}}$. The latter trivialization allows us to identify modifications with sections of the affine Grassmannian, so that $(\bar{\mathcal{F}}^i_{\boldsymbol{u}}, \bar{\tau}^i_{\boldsymbol{u}})$ corresponds to $\bar{\alpha}^i_{\boldsymbol{u}} \in \mathrm{Gr}_{\bar{\boldsymbol{G}}^i}(Y^i_{\boldsymbol{u}})$.

Lemma 2.13. *$\bar{\alpha}^i_{\boldsymbol{u}}$ can be lifted to $\hat{\alpha}^i_{\boldsymbol{u}} \in \mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y^i_{\boldsymbol{u}})$.*

Proof. Consider any trivialization $\hat{\sigma}^i_{\boldsymbol{u}}$ of $\widehat{\mathcal{G}}^i_{\boldsymbol{u}}$ in a Zariski neighborhood of $Y^i_{\boldsymbol{u}}$. This induces a trivialization $\bar{\sigma}^i_{\boldsymbol{u}}$ of $\bar{\mathcal{G}}^i_{\boldsymbol{u}}$ in the same neighborhood. These trivializations allow us to identify modifications with sections of affine Grassmannians. In particular, denoting by $\widehat{\mathcal{F}}^i_{\boldsymbol{u}}$ the trivial $\widehat{\boldsymbol{G}}^i_{\boldsymbol{u}}$-bundle over $\mathbb{P}^1_{\boldsymbol{u}}$, we get a modification $(\widehat{\mathcal{F}}^i_{\boldsymbol{u}}, \hat{\tau}^i_{\boldsymbol{u}})$ of $\widehat{\mathcal{G}}^i_{\boldsymbol{u}}$ and thus a section $\hat{\beta}^i_{\boldsymbol{u}} \in \mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y^i_{\boldsymbol{u}})$.

Let $\bar{\beta}^i_{\boldsymbol{u}}$ be the image of $\hat{\beta}^i_{\boldsymbol{u}}$ under the projection $\mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y^i_{\boldsymbol{u}}) \to \mathrm{Gr}_{\bar{\boldsymbol{G}}^i}(Y^i_{\boldsymbol{u}})$. It follows from the construction that $\bar{\alpha}^i_{\boldsymbol{u}}$ and $\bar{\beta}^i_{\boldsymbol{u}}$ correspond to the same modification of the same $\bar{\boldsymbol{G}}^i_{\boldsymbol{u}}$-bundle but with respect to different trivializations of this bundle near $Y^i_{\boldsymbol{u}}$. According to Lemma 2.5, $\bar{\alpha}^i_{\boldsymbol{u}}$ differs from $\bar{\beta}^i_{\boldsymbol{u}}$ by an action of an element of $L^+\bar{\boldsymbol{G}}^i(Y^i_{\boldsymbol{u}})$. The lemma follows from Proposition 2.8, applied to each point of $Y^i_{\boldsymbol{u}}$, and the fact that $\bar{\beta}^i_{\boldsymbol{u}}$ lifts to $\hat{\beta}^i_{\boldsymbol{u}}$. □

Step 4. Let $\hat{\alpha}^i_{\boldsymbol{u}}$ be as in the above lemma. The group scheme $(\widehat{\boldsymbol{G}}^i)_{Y^i}$ contains a proper parabolic subgroup scheme because $(\bar{\boldsymbol{G}}^i)_{Y^i}$ does (see [Conrad 2014, Exercise 5.5.8]). Thus, by Lemma 2.7, the collection $(\hat{\alpha}^i_{\boldsymbol{u}})$ lifts to a point $\hat{\alpha}^i \in \mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y^i)$. We extend this to a point of $\mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y \times_U U_i)$ by setting $\hat{\alpha}^i|_{Y \times_U U_i - Y^i} = \mathrm{Id}_{\mathrm{Gr}}$. It is easy to see that $\mathrm{Gr}_{\widehat{\boldsymbol{G}}^i}(Y \times_U U_i) = \mathrm{Gr}_{\widetilde{\boldsymbol{G}}^i}(Y)$, so $\hat{\alpha}^i$ corresponds to $\tilde{\alpha}^i \in \mathrm{Gr}_{\widetilde{\boldsymbol{G}}^i}(Y)$. Now the collection $(\tilde{\alpha}^i | i = 1, \dots, r)$ gives rise to a section $\alpha \in \mathrm{Gr}_{\boldsymbol{G}}(Y)$. Since we have trivialized $\mathcal{G}$ in a neighborhood of Y, this gives a modification $(\mathcal{F}, \tau)$ of $\mathcal{G}$ at Y. By construction the $\boldsymbol{G}^{\mathrm{ad}}_u$-bundle $(\mathcal{F}|_{\mathbb{P}^1_u})/\boldsymbol{Z}_u$ is trivial for every closed point u of U. Now, by Proposition 2.12, the $\boldsymbol{G}^{\mathrm{ad}}$-bundle $\mathcal{F}/\boldsymbol{Z}$ is isomorphic to the pullback of a $\boldsymbol{G}^{\mathrm{ad}}$-bundle under the projection $\mathbb{P}^1_U \to U$. On the other hand, since $\mathcal{F}$ is a modification of $\mathcal{G}$ at Y, the $\boldsymbol{G}^{\mathrm{ad}}$-bundle $(\mathcal{F}/\boldsymbol{Z})|_{U \times \infty} \simeq (\mathcal{G}/\boldsymbol{Z})|_{U \times \infty}$ is trivial. It follows that $\mathcal{F}/\boldsymbol{Z}$ is trivial. Now, it follows from the exact sequence for nonabelian cohomology groups, that there is a $\boldsymbol{Z}$-bundle $\mathcal{Z}$ over $\mathbb{P}^1_U$ such that $\mathcal{F}$ is isomorphic to the pushforward of $\mathcal{Z}$.

Step 5. Note that the center of a reductive group scheme is a group scheme of multiplicative type. Recall that the relative line bundle $\mathcal{O}_{\mathbb{P}^1_U}(1)$ trivializes on $\mathbb{P}^1_U - Y$. The following lemma is somewhat similar to [Colliot-Thélène and Sansuc 1987, Lemma 2.4].

Lemma 2.14. *Let U and Y be as before; let $\boldsymbol{Z}$ be a group scheme of multiplicative type over U. Let $\mathcal{Z}$ be a $\boldsymbol{Z}$-bundle over $\mathbb{P}^1_U$. Then $\mathcal{Z}|_{\mathbb{P}^1_U - Y}$ is isomorphic to the pullback of a $\boldsymbol{Z}$-bundle over U.*

Proof. Since $\boldsymbol{Z}$ is not smooth in general, we will work in the fppf topology over U. We claim that there is a unique cocharacter $\lambda : \mathbb{G}_{m,U} \to \boldsymbol{Z}$ such that $\mathcal{Z}' := \lambda_* \mathcal{O}_{\mathbb{P}^1_U}(1)^\times$ and $\mathcal{Z}$ are isomorphic locally in the fppf topology over U. Indeed, the statement is local over U, so we may assume that $\boldsymbol{Z}$ is split. Then the question reduces to the cases $\boldsymbol{Z} = \mathbb{G}_{m,U}$ and $\boldsymbol{Z} = \mu_{n,U}$, where $\mu_{n,U}$ is the group scheme of n-th roots of unity. The first case is a statement about line bundles; we leave it to the reader. The second case reduces to the statement that a $\mu_{n,U}$-bundle over $\mathbb{P}^1_U$ is trivial fppf locally over the base, which follows easily from the exact sequence $1 \to \mu_{n,U} \to \mathbb{G}_{m,U} \to \mathbb{G}_{m,U} \to 1$; the claim is proved.

We see that $\mathcal{Z} \simeq \mathcal{Z}' \otimes p^*\mathcal{Z}''$, where $p : \mathbb{P}^1_U \to U$ is the projection, $\mathcal{Z}''$ is a $\boldsymbol{Z}$-bundle over U (note that $\boldsymbol{Z}$ is a commutative group scheme so the tensor product of $\boldsymbol{Z}$-bundles makes sense). It remains to notice that $\mathcal{Z}' = \lambda_* \mathcal{O}_{\mathbb{P}^1_U}(1)^\times$ is trivial on $\mathbb{P}^1_U - Y$ because $\mathcal{O}_{\mathbb{P}^1_U - Y}(1)$ is trivial. Lemma 2.14 is proved. □

We see that $\mathcal{F}|_{\mathbb{P}^1_U - Y}$ is isomorphic to the pullback of a $\boldsymbol{G}$-bundle over U. Since $\mathcal{F}$ and $\mathcal{G}$ are isomorphic over $U \times \infty$ and $\mathcal{G}$ is trivial over $U \times \infty$, we see that $\mathcal{F}|_{\mathbb{P}^1_U - Y}$ is trivial. Finally, $\mathcal{G}$ and $\mathcal{F}$ are isomorphic over $\mathbb{P}^1_U - Y$, and Theorem 6 is proved. □

2E. *Proof of Theorem 4.* We use the notation from the formulation of the theorem. We may assume that U is connected. Applying an affine transformation to $\mathbb{A}^1_U$, we may assume that Δ is the horizontal section $\Delta(U) = U \times 1$. We can extend the $\boldsymbol{G}$-bundle $\mathcal{E}$ to a $\boldsymbol{G}$-bundle $\tilde{\mathcal{E}}$ over $\mathbb{P}^1_U$ by gluing it with the trivial $\boldsymbol{G}$-bundle over $\mathbb{P}^1_U - Z$. Let $\varphi : \boldsymbol{G}^{\text{sc}} \to \boldsymbol{G}^{\text{ad}}$ be the simply connected central cover (see [Conrad 2014, Exercise 6.5.2]); let d be the degree of φ. Consider the morphism $\mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}} : z \mapsto z^d$; let $\psi : \mathbb{P}^1_U \to \mathbb{P}^1_U$ be the base change of this morphism. Consider the $\boldsymbol{G}$-bundle $\psi^*\tilde{\mathcal{E}}$ over $\mathbb{P}^1_U$. For a closed point $u \in U$ write $\tilde{\mathcal{E}}_u := \tilde{\mathcal{E}}|_{\mathbb{P}^1_u}$. Then by [Gille 2002, Théorème 3.8(a)] the $\boldsymbol{G}^{\text{ad}}_u$-bundle $\tilde{\mathcal{E}}_u/\boldsymbol{Z}_u$ is Zariski locally trivial. By Proposition 2.3 the $\boldsymbol{G}^{\text{ad}}_u$-bundle $\psi^*\tilde{\mathcal{E}}_u/\boldsymbol{Z}_u$ is topologically trivial. Since the morphism ψ has a section over $U \times 1$, it is enough to show that $\psi^*\tilde{\mathcal{E}}|_{U\times 1}$ is trivial.

Case 1. U is a scheme over an infinite field k. We use notations from the formulation of Theorem 6. By [Fedorov and Panin 2015, Proposition 4.1] for $i = 1, \dots, r$, we can find a scheme Y^i finite and étale over U_i such that $(\bar{\boldsymbol{G}}^i)_{Y^i}$ is isotropic and for every closed point $v \in U_i$ such that $\bar{\boldsymbol{G}}^i_v$ is isotropic we have a $k(v)$-rational point on the fiber Y^i_v.

View Y^i as a U-scheme via $Y^i \to U_i \to U$ and consider a closed U-embedding $Y^i \to \mathbb{P}^1_U$. Since k is infinite and U is semilocal, we can shift the subschemes Y^i so that they do not intersect each other, $\psi^{-1}(Z)$, and $U \times 1$. Again, since k is infinite, we have $a \in k$ such that $U \times a$ does not intersect $\psi^{-1}(Z)$.

Take $Y = \bigsqcup_{i=1}^{r} Y^i \sqcup (U \times a)$. Note that Y^i is an open and closed subscheme of $Y \times_U U_i$. If v is a closed point of U_i such that $\bar{\boldsymbol{G}}^i_v$ is isotropic, then Y^i_v contains a rational point, so $\operatorname{Pic}(\mathbb{P}^1_v - Y^i_v) = 0$. Thus we can apply Theorem 6 to $\psi^*\tilde{\mathcal{E}}$.

Case 2. The residue fields of points of U are finite over k. Note that for all closed points $v \in \mathbb{P}^1_{U_i}$ the group scheme $\bar{\boldsymbol{G}}^i_v$ is quasisplit, since $k(v)$ is a finite field. A Borel subgroup of $\bar{\boldsymbol{G}}^i_v$ gives a k-rational point on the v-fiber of the U_i-scheme of Borel subgroup schemes of $\bar{\boldsymbol{G}}^i$. Thus, using [Panin 2020a, Lemma 3.1], we find a finite and étale over U_i scheme $\widetilde{Y}^i$ such that $(\bar{\boldsymbol{G}}^i)_{\widetilde{Y}^i}$ is quasisplit and for all closed points $v \in U_i$ the fiber $\widetilde{Y}^i_v$ has a $k(v)$-rational point.

Now we construct inductively for $i = 1, \dots, r$ finite field extensions k'_i and k''_i of k of coprime degrees and a closed embedding

$$Y^i = (\widetilde{Y}^i \times_k \operatorname{Spec} k'_i) \sqcup (\widetilde{Y}^i \times_k \operatorname{Spec} k''_i) \hookrightarrow \mathbb{A}^1_U.$$

such that Y^i does not intersect $\bigcup_{j=1}^{i-1} Y^j \cup (U \times 1)$ and for all closed points $v \in U_i$ the algebras $k(v) \otimes_k k'_i$ and $k(v) \otimes_k k''_i$ are fields. (We identify schemes Y^i with their images in $\mathbb{A}^1_U$.)

This is accomplished by applying the proof of [Panin 2020a, Lemma 2.1] (note that this lemma requires Y^i to have a rational point on every closed fiber but this is only needed to conclude that $\operatorname{Pic}(\mathbb{A}^1_U - Y^i) = 0$, which we do not claim).

By [Panin 2020a, Lemma 2.1] applied to the identity morphism $U \to U$, we can find field extensions $k' \supset k$ and $k'' \supset k$ of coprime degrees and a closed U-embedding

$$(U \times_k \operatorname{Spec} k') \sqcup (U \times_k \operatorname{Spec} k'') \hookrightarrow \mathbb{A}^1_U$$

such that the image Y^0 of this embedding does not intersect $\psi^{-1}(Z)$, $U \times 1$, and any of Y^i. Note that the relative line bundle $\mathcal{O}(1)$ trivializes on $\mathbb{P}^1_U - Y^0$.

Take $Y = \bigcup_{i=0}^{r} Y^i$. Note that Y^i is an open and closed subscheme of $Y \times_U U_i$ and by construction $\bar{\boldsymbol{G}}^i$ is quasisplit over Y^i. Thus, $\bar{\boldsymbol{G}}^i$ is isotropic over each connected component of Y^i by [SGA 3_{III} 1970, Exposé XXVI, Corollaire 6.14]. Also, for each closed point $v \in U_i$, the fiber Y^i_v has two points of coprime degree over $k(v)$ (namely, $\operatorname{Spec}(k(v) \otimes_k k'_i)$ and $\operatorname{Spec}(k(v) \otimes_k k''_i)$). Thus, $\operatorname{Pic}(\mathbb{A}^1_v - Y^i_v) = 0$. It remains to apply Theorem 6 to Y and $\psi^*\tilde{\mathcal{E}}$. □

2F. *Proof of Theorem 5.* We use the notation from the formulation of the theorem. As in the proof of Theorem 4, we extend the $\boldsymbol{G}$-bundle $\mathcal{E}$ to a $\boldsymbol{G}$-bundle $\tilde{\mathcal{E}}$ over $\mathbb{P}^1_U$ and assume that $\Delta(U) = U \times 1$. Let ψ and $\tilde{\mathcal{E}}_u$ be as in the proof of Theorem 4, then $\psi^*\tilde{\mathcal{E}}_u/\boldsymbol{Z}_u$ is topologically trivial for every closed point $u \in U$. It is enough to show that $\psi^*\tilde{\mathcal{E}}|_{\mathbb{P}^1_U - (U \times 0)}$ is trivial. By assumption, we can embed $\boldsymbol{G}$ into $\mathrm{GL}_{n,U}$. By [Moser 2008, Korollar 3.5.2] we may assume that U is local (note that $\mathbb{P}^1_U - (U \times 0) \simeq \mathbb{A}^1_U$).

In the same way as in the proof of Theorem 4 we find a closed subscheme $Y \subset \mathbb{P}^1_U$ finite and étale over U such that $\psi^*\tilde{\mathcal{E}}$ is trivial over $\mathbb{P}^1_U - Y$. Note that such Y may be chosen so that it does not intersect any given closed subscheme of $\mathbb{A}^1_U$ as long as this subscheme is finite over U. In particular, we may assume that $Y \cap (U \times 0) = \varnothing$. Since $\boldsymbol{G}^{\mathrm{ad}}$ is strongly locally isotropic and U is local, each $\boldsymbol{G}^i$ is locally isotropic. Thus, we can apply Theorem 6 taking Y for Z and $U \times 0$ for Y. We see that $\psi^* \tilde{\mathcal{E}}$ is trivial over $\mathbb{P}^1_U - (U \times 0)$, which completes the proof of the theorem. □

3. Proofs of Theorems 1 and 2

In this section we derive Theorems 2 and 1 from Theorems 4 and 5 respectively. The proofs are based on [Panin 2019, Theorem 1.5]. Note that these derivations are similar to those given in [Fedorov and Panin 2015; Panin 2020a; Panin et al. 2015]; we present them here for the sake of completeness.

3A. *Proof of Theorem 2.* *Step 1.* We may assume that U is the semilocal scheme of finitely many closed points $x_1, \dots, x_n$ on a smooth irreducible k-variety X, where k is a field. Indeed, let $U = \operatorname{Spec} R$ and let k be the prime field of R (or any other perfect field contained in R). Then, by Popescu's theorem [Popescu 1986; Swan 1998; Spivakovsky 1999], we can write $U = \varprojlim U_\alpha$, where U_α are affine schemes smooth and of finite type over k. Modifying the system (U_α), we may assume that U_α are integral schemes. A standard argument shows that there is an index α, a reductive group scheme $\boldsymbol{G}_\alpha$ over U_α such that $\boldsymbol{G}_\alpha|_U = \boldsymbol{G}$, and a $\boldsymbol{G}_\alpha$-bundle $\mathcal{E}_\alpha$ over U_α trivial over the generic point of U_α and such that the pullback of $\mathcal{E}_\alpha$ to U is isomorphic to $\mathcal{E}$. Let $y_1, \dots, y_n \in U_\alpha$ be the images of all closed points of U. For $i = 1, \dots, n$ choose a closed point $x_i \in U_\alpha$ in the Zariski closure of y_i. Let R' be the semilocal ring of $x_1, \dots, x_n$ on $X := U_\alpha$. Let $\boldsymbol{G}'$ be the restriction of $\boldsymbol{G}_\alpha$ to $U' := \operatorname{Spec} R'$. The morphism $U \to U_\alpha$ factors through U'. Thus it is enough to prove the theorem for U', $\boldsymbol{G}'$, and $\mathcal{E}' := \mathcal{E}_\alpha \times \times_{U_\alpha} U'$.

Step 2. Replacing X by a Zariski neighborhood of $\{x_1, \dots, x_n\}$, we may assume that there are a group scheme $\boldsymbol{G}_X$ over X such that $\boldsymbol{G}_X|_U = \boldsymbol{G}$, a $\boldsymbol{G}_X$-bundle $\mathcal{E}'$ over X such that $\mathcal{E}'|_U = \mathcal{E}$, and a nonzero function $f \in H^0(X, \mathcal{O}_X)$ such that the restriction of $\mathcal{E}'$ to X_f is a trivial bundle.

Step 3. We keep the notation from Step 2. Multiplying f by an appropriate function, we may assume that f vanishes at each x_i. Our goal is to construct a $\boldsymbol{G}$-bundle $\mathcal{G}$ over $\mathbb{A}^1_U$ by étale descent such that $\mathcal{G}|_{U \times 0} \simeq \mathcal{E}$. Then we can apply Theorem 4 to conclude that $\mathcal{E}$ is trivial. The construction of this $\mathcal{E}$ is standard and is achieved by using a certain diagram. Precisely, by [Panin 2019, Theorem 1.5] there is a monic polynomial $h \in H^0(U, \mathcal{O}_U)[t]$, a commutative diagram with an irreducible affine U-smooth Y:

$$\begin{array}{ccccc}
(\mathbb{A}^1_U)_h & \longleftarrow & Y_{\tau^*(h)} & \xrightarrow{p_X|_{Y^*_\tau(h)}} & X_f \\
\Big\downarrow{\scriptstyle \text{inc}} & & \Big\downarrow{\scriptstyle \text{inc}} & & \Big\downarrow{\scriptstyle \text{inc}} \\
\mathbb{A}^1_U & \xleftarrow{\ \tau\ } & Y & \xrightarrow{\ p_X\ } & X
\end{array} \tag{4}$$

and a morphism $\delta : U \to Y$ satisfying the following conditions:

(i) The left square is an elementary distinguished square in the category of affine U-smooth schemes in the sense of [Morel and Voevodsky 1999, §3.1, Definition 1.3]; this means that the vertical maps are open embeddings, the horizontal maps are étale, and τ induces an isomorphism

$$\tau^{-1}(\{h = 0\})_{\text{red}} \to \{h = 0\}_{\text{red}}.$$

(ii) $p_X \circ \delta = \mathrm{can} : U \to X$, where can is the canonical morphism.

(iii) $\tau \circ \delta = i_0 : U \to \mathbb{A}^1_U$ is the zero section of the projection $\mathrm{pr}_U : \mathbb{A}^1_U \to U$.

(iv) For $p_U := \mathrm{pr}_U \circ \tau$ there is a Y-group scheme isomorphism $\Phi : p_U^*(\boldsymbol{G}) \to p_X^*(\boldsymbol{G}_X)$ with $\delta^*(\Phi) = \mathrm{id}_{\boldsymbol{G}}$.

Step 4. We use part (iv) of Step 3 to view $p_X^*\mathcal{E}'$ as a $\boldsymbol{G}$-bundle. We use the left square from part (i) of Step 3 to glue the trivial $\boldsymbol{G}$-bundle over $(\mathbb{A}^1_U)_h$ with $p_X^*\mathcal{E}'$ to get a $\boldsymbol{G}$-bundle $\mathcal{G}$ over $\mathbb{A}^1_U$. We have

$$\mathcal{E} = \mathrm{can}^* \mathcal{E}' = \delta^* p_X^* \mathcal{E}' = \delta^* \tau^* \mathcal{G} = i_0^* \mathcal{G} \tag{5}$$

so it remains to show that $i_0^*\mathcal{G}$ is trivial. But $\{h = 0\}$ is a closed subscheme of $\mathbb{A}^1_U$ and it is finite over U because h is monic. The residues of all closed points of U are finite extensions of k, so they are finite if k is finite. Thus we can apply Theorem 4 and conclude that $i_0^*\mathcal{G}$ is trivial. □

Remark 3.1. A priori, (5) is an isomorphism of U-schemes. This is enough for our purposes because a principal bundle is trivial if and only if it has a section, so that triviality does not depend on the group scheme action. On the other hand, using the equation $\delta^*(\Phi) = \mathrm{id}_{\boldsymbol{G}}$, one can show that (5) is compatible with the action of the group scheme, see [Panin 2019, §6].

3B. *Proof of Theorem 1.* *Step 1.* We may assume that W is of finite type over k. Indeed, write $W = \varprojlim W_\alpha$, where W_α are k-schemes of finite type. Since $\mathcal{E}$ is affine and finitely presented over $W \times_k U$, there is an index α and a $\boldsymbol{G}$-bundle $\mathcal{E}_\alpha$ over $W_\alpha \times_k U$ such that $\mathcal{E}$ is isomorphic to the pullback of $\mathcal{E}_\alpha$ to $W \times_k U$. Next, there is an index $\beta > \alpha$ such that the pullback of $\mathcal{E}_\alpha$ to $W_\beta \times_k U$ (call it $\mathcal{E}_\beta$) is trivial over $W_\beta \times_k \Omega$. We see that it is enough to prove the theorem with W and $\mathcal{E}$ replaced by W_β and $\mathcal{E}_\beta$.

Step 2. Similarly to Step 1 of the proof of Theorem 2, we may assume that U is the semilocal scheme of finitely many closed points $x_1, \dots, x_n$ on a smooth irreducible k-variety X. In more detail, by Popescu's theorem we can write $U = \varprojlim U_\alpha$, where U_α are affine schemes smooth and of finite type over k. We may assume that U_α are integral schemes. Then we find an index α, a reductive group scheme $\boldsymbol{G}_\alpha$ over U_α such that $\boldsymbol{G}^{\mathrm{ad}}$ is strongly locally isotropic and such that $\boldsymbol{G}_\alpha|_U = \boldsymbol{G}$, and a $\boldsymbol{G}_\alpha$-bundle $\mathcal{E}_\alpha$ over $W \times_k U_\alpha$ trivial over $W \times_k \Omega_\alpha$, where Ω_α is the generic point of U_α and such that the pullback of $\mathcal{E}_\alpha$ to $W \times_k U$ is isomorphic to $\mathcal{E}$. Then it is enough to prove the theorem with U replaced by an appropriate semilocal ring of finitely many closed points of U_α.

Step 3. Set $U' := W \times_k U$, $X' := W \times_k X$. Similarly to Step 2 of the proof of Theorem 2, we may assume that there is a group scheme $\boldsymbol{G}_X$ over X such that $\boldsymbol{G}_X|_U = \boldsymbol{G}$, a $\boldsymbol{G}_X$-bundle $\mathcal{E}'$ over X' such that $\mathcal{E}'|_{U'} = \mathcal{E}$, and a nonzero function $f \in H^0(X, \mathcal{O}_X)$ such that the restriction of $\mathcal{E}'$ to X'_f is a trivial bundle.

Step 4. Similarly to Step 3 of the proof of Theorem 2, we find a monic polynomial $h \in H^0(U, \mathcal{O}_U)[t]$, a commutative diagram (4) with an irreducible affine U-smooth Y, and a morphism $\delta : U \to Y$ satisfying the same conditions.

Step 5. Set $Y' := W \times_k Y$. The diagram (4) is a diagram over k. Thus we can multiply this diagram by W, getting a monic polynomial $h' \in H^0(U', \mathcal{O}_{U'})[t]$ and a commutative diagram

$$\begin{array}{ccccc} (\mathbb{A}^1_{U'})_{h'} & \longleftarrow & Y'_{(\tau')^*(h')} & \xrightarrow{p_{X'}|_{Y'_{(\tau')^*(h')}}} & X'_f \\ \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathbb{A}^1_{U'} & \xleftarrow{\tau'} & Y' & \xrightarrow{p_{X'}} & X' \end{array}$$

We also get a morphism $\delta' : U' \to Y'$. These data satisfy the following conditions:

(1) The left-hand side square is an elementary distinguished square in the category of affine U'-smooth schemes in the sense of [Morel and Voevodsky 1999, §3.1, Definition 1.3].

(2) $p_{X'} \circ \delta' = \mathrm{can} : U' \to X'$, where can is the canonical morphism.

(3) $\tau' \circ \delta' = i'_0 : U' \to \mathbb{A}^1_{U'}$ is the zero section of the projection $\mathrm{pr}_{U'} : \mathbb{A}^1_{U'} \to U'$.

Step 6. We use part (iv) of Step 4 of the proof of Theorem 2 to view $p^*_{X'}\mathcal{E}'$ as a $\boldsymbol{G}$-bundle. We use the left square from part (i) of Step 5 to glue the trivial $\boldsymbol{G}$-bundle over $(\mathbb{A}^1_{U'})_{h'}$ with $p^*_{X'}\mathcal{E}'$ to get a $\boldsymbol{G}$-bundle $\mathcal{G}$ over $\mathbb{A}^1_{U'}$. We have

$$\mathcal{E} = \mathrm{can}^*\,\mathcal{E}' = (\delta')^* p^*_{X'}\mathcal{E}' = (\delta')^*(\tau')^*\mathcal{G} = (i'_0)^*\mathcal{G},$$

so it remains to show that $(i'_0)^*\mathcal{G}$ is trivial. But $\boldsymbol{G}$ can be embedded into $\mathrm{GL}_{n,U}$ for some n because U is regular and, in particular, normal (see [Thomason 1987, Corollary 3.2]). Thus $\boldsymbol{G}_{U'}$ can be embedded into $\mathrm{GL}_{n,U'}$. Next, $\{h' = 0\}$ is a closed subscheme of $\mathbb{A}^1_{U'}$ and it is finite over U' because h' is monic. Thus we can apply Theorem 5 and conclude that $(i'_0)^*\mathcal{G}$ is trivial. □

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