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DERIVATIVES OF EISENSTEIN SERIES OF WEIGHT 2 AND INTERSECTIONS OF MODULAR CORRESPONDENCES

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ABSTRACT. We give a formula for certain values and derivatives of Siegel series and use them to compute Fourier coefficients of derivatives of the Siegel Eisenstein series of weight $\frac{g}{2}$ and genus g. When $g = 4$, the Fourier coefficient is approximated by a certain Fourier coefficient of the central derivative of the Siegel Eisenstein series of weight 2 and genus 3, which is related to the intersection of 3 arithmetic modular correspondences. Applications include a relation between weighted averages of representation numbers of symmetric matrices.

1. INTRODUCTION

1.1. **Motivation : On the modular correspondences.** Let $j = j' = j(\tau)$ be the elliptic modular function on the upper half plane. For $m \geq 1$ let $\varphi_m \in \mathbb{Z}[j, j']$ be the classical modular polynomial defined by

$$\varphi_m(j(\tau), j(\tau')) = \prod_{A \in \mathrm{M}_2(\mathbb{Z}) \pmod{\mathrm{SL}_2(\mathbb{Z})},\ \det A = m} (j(\tau) - j(A\tau')).$$

Put $S = \operatorname{Spec} \mathbb{Z}[j, j']$ and $S_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[j, j']$. Let T_m and $T_{m,\mathbb{C}}$ be the arithmetic and geometric divisors defined by $\varphi_m = 0$. We can view S as an arithmetic threefold $\mathcal{S} = \mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{M}$, where $\mathcal{M}$ is the moduli stack of elliptic curves over $\mathbb{Z}$, and T_m as the moduli stack $\mathcal{T}_m$ of isogenies of elliptic curves of degree m. In the 19th century Hurwitz has computed the intersection

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) := \dim_{\mathbb{C}} \mathbb{C}[j, j']/(\varphi_{m_1}, \varphi_{m_2})$$

of complex curves. Gross and Keating [3] discovered that $(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})$ is related to the Fourier coefficients of the Siegel Eisenstein series of weight 2 for $Sp_2(\mathbb{Z})$. Moreover, they gave an explicit expression for the intersection

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) := \log \sharp \mathbb{Z}[j, j']/(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3})$$

of 3 arithmetic modular correspondences. It is already mentioned in the introduction of [3] that computations of Kudla or Zagier strongly suggest that $\deg \mathscr{Z}(B)$ equals the B-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 for $Sp_3(\mathbb{Z})$, up to multiplication by a

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constant which is independent of B. A complete proof of this identity has been given in [17] (cf. [11]).

The purpose of this paper is to compute the Fourier coefficients of the derivative of the Siegel Eisenstein series of weight 2 for $Sp_4(\mathbb{Z})$. One may expect that these coefficients are related to the intersection of 4 modular correspondences. However, the number

$$\log \sharp\mathbb{Z}[j,j']/(\varphi_{m_1},\varphi_{m_2},\varphi_{m_3},\varphi_{m_4}),$$

does not seem to be naturally expanded to a sum over positive semi-definite symmetric half-integral matrices of size 4 and does not seem to be a right object. The fiber product $\mathcal{T}_{m_1}\times_{\mathcal{S}}\mathcal{T}_{m_2}\times_{\mathcal{S}}\mathcal{T}_{m_3}\times_{\mathcal{S}}\mathcal{T}_{m_4}$ has a disjoint sum decomposition according to the values of the fundamental matrices:

$$\mathcal{T}_{m_1}\times_{\mathcal{S}}\mathcal{T}_{m_2}\times_{\mathcal{S}}\mathcal{T}_{m_3}\times_{\mathcal{S}}\mathcal{T}_{m_4}=\bigsqcup_T \mathscr{Z}(T),$$

where T extends over the set of positive semi-definite symmetric half-integral matrices of size 4 with diagonal entries m_1,m_2,m_3,m_4. If T is positive definite, then $\mathscr{Z}(T)$ is empty unless $\det T$ is a square and T is split except over a single prime. If T is positive definite and $\det T$ is a square, then the T-th Fourier coefficient is zero unless T is anisotropic only at a prime p, in which case the T-th Fourier coefficient is approximately equal to $\deg \mathscr{Z}(T')$, where T' is some positive semi-definite symmetric half-integral matrix of size 3 (see Theorem 1.3). Our result may imply that for each point of the intersection, where 4 surfaces intersect properly, in a small neighborhood of the point, the intersection multiplicity behaves like the intersection multiplicity of 3 surfaces of them.

In the intervening years Kudla and others have gone a long way towards proving such relations in much greater generality. In [8], he introduced a certain family of Eisenstein series of genus g and weight $\frac{g+1}{2}$. They have an odd functional equation and hence have a natural zero at their center of symmetry. The central derivatives of such series, which he refers to as incoherent Eisenstein series, have a connection with arithmetic algebraic geometry of cycles on integral models of Shimura varieties attached to orthogonal groups of signature $(2,g-1)$, at least when $g\le 4$. We refer the reader to [14] for $g=1$, to [8, 12, 15] for $g=2$, to [11, 24, 17] for $g=3$, and to [13] for $g=4$. However, there are serious problems with the construction of arithmetic models of these Shimura varieties as soon as $g\ge 5$.

1.2. **The Fourier coefficients of derivative of Eisenstein series.** In this paper we compute the Fourier coefficients of derivatives of incoherent Eisenstein series of genus g and weight $\frac{g}{2}$. In this introductory section we will consider classical Eisenstein series of level 1. Let g be a positive integer that is divisible by 4. Let

$$E_g(Z,s)=\sum_{\{C,D\}}\det(CZ+D)^{-g/2}|\det(CZ+D)|^{-s}(\det Y)^{s/2}$$

be the Siegel Eisenstein series of genus g, where $\{C, D\}$ runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of degree g, and Z is a complex symmetric matrix of degree g with positive definite imaginary part Y. This series converges absolutely for $\Re s > \frac{g}{2} + 1$ and admits a meromorphic continuation to the whole s-plane by the general theory of Langlands.

If $\frac{g}{4}$ is even, then $E_g(Z, s)$ is holomorphic at $s = 0$ and the T-th Fourier coefficient of $E_g(Z, 0)$ is equal to

$$2\Big(\sum_i \frac{1}{N(L_i, L_i)}\Big)^{-1} \sum_i \frac{N(L_i, T)}{N(L_i, L_i)} \tag{1.1}$$

by the Siegel formula (see [23, 10, 27]), where $\{L_i\}$ is the set of isometry classes of positive definite even unimodular lattices of rank g. Here $N(L, L')$ denotes the number of isometries $L' \to L$ for two quadratic spaces L, L' over $\mathbb{Z}$. In particular, the nondegenerate Fourier coefficients are supported on a single rational equivalence class.

On the other hand, if $\frac{g}{4}$ is odd, then $E_g(Z, s)$ has a zero at $s = 0$. Our main object of study in this paper is the derivative

$$\frac{\partial}{\partial s} E_g(Z, s)|_{s=0} = \sum_{T>0} C_g(T) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)} + \sum_{\text{other } T} C_g(T, Y) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)}.$$

Fix a positive definite symmetric half-integral $n \times n$ matrix T and a rational prime p. Let $\mathbb{Q}^{(p)}$ be a subring of $\mathbb{Q}$, consisting of the numbers of the form $\frac{a}{p^n}$ with $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. We define the additive character $\mathbf{e}_p$ of $\mathbb{Q}_p$ by setting $\mathbf{e}_p(x) = e^{-2\pi\sqrt{-1}y}$ with $y \in \mathbb{Q}^{(p)}$ such that $x - y \in \mathbb{Z}_p$. The Siegel series attached to T and p is defined by

$$b_p(T, s) = \sum_{z \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(-\mathrm{tr}(Tz))\nu[z]^{-s},$$

where $\nu[z]$ is the product of denominators of elementary divisors of z. Put $D_T = (-4)^{[n/2]} \det T$. We denote the primitive Dirichlet character corresponding to $\mathbb{Q}(\sqrt{D_T})$ by χ_T and its conductor by $\mathfrak{d}^T$. Put $\xi_p^T = \chi_T(p)$. Let $e_p^T = \mathrm{ord}_p D_T$ or $e_p^T = \mathrm{ord}_p D_T - \mathrm{ord}_p \mathfrak{d}^T$ according as n is odd or even. There exists a polynomial $F_p^T(X) \in \mathbb{Z}[X]$ such that

$$b_p(T, s) = \gamma_p^T(p^{-s}) F_p^T(p^{-s}),$$

where

$$\gamma_p^T(X) = (1 - X) \prod_{j=1}^{[n/2]} (1 - p^{2j} X^2) \times \begin{cases} 1 & \text{if } n \text{ is odd}, \\ \frac{1}{1 - \xi_p^T p^{n/2} X} & \text{if } n \text{ is even}. \end{cases}$$

The symbol η_p^T stands for the normalized Hasse invariant of T over $\mathbb{Q}_p$ (see Definition 2.1). We write $\mathrm{Diff}(T)$ for the finite set of prime numbers p such that $\eta_p^T = -1$. A direct calculation gives the following formula:

Proposition 5.1. *Assume that $\frac{g}{4}$ is odd. Let T be a positive definite symmetric half-integral matrix of size g.*

(1) *If $\chi_T = 1$, then $C_g(T) = 0$ unless $\mathrm{Diff}(T)$ is a singleton.*
(2) *If $\chi_T = 1$ and $\mathrm{Diff}(T) = \{p\}$, then*

$$C_g(T) = -\frac{2^{(g+2)/2} p^{-(g+e_p^T)/2} \log p}{\zeta(1-\frac{g}{2}) \prod_{i=1}^{(g-2)/2} \zeta(1-2i)} \frac{\partial F_p^T}{\partial X}(p^{-g/2}) \prod_{p \neq \ell | D_T} \ell^{-e_\ell^T/2} F_\ell^T(\ell^{-g/2}).$$

(3) *If $\chi_T \neq 1$, then*

$$C_g(T) = -\frac{2^{(g+2)/2} L(1, \chi_T)}{\zeta(1-\frac{g}{2}) \prod_{i=1}^{(g-2)/2} \zeta(1-2i)} \prod_{\ell | D_T} p^{-e_\ell^T/2} F_\ell^T(\ell^{-g/2}).$$

Remark 1.1. If $\chi_T \neq 1$, then $L(1, \chi_T) = \frac{\sqrt{\mathfrak{d}^T}}{\log \epsilon} h$ by Dirichlet's class number formula, where h is the class number of the real quadratic field $\mathbb{Q}(\sqrt{\det T})$ and $\epsilon = \frac{t+u\sqrt{\mathfrak{d}^T}}{2}$ ($t > 0$, $u > 0$) is the solution to the Pell equation $t^2 - \mathfrak{d}^T u^2 = 4$ for which u is smallest.

The following theorem is a special case of Theorem 4.3 and allows us to compute $\frac{\partial F_p^T}{\partial X}(\xi_p^T p^{-g/2})$. For simplicity we here assume p to be odd.

Theorem 1.2. *Let p be an odd rational prime and $T = \mathrm{diag}[t_1, \dots, t_g]$ with $0 \leq \mathrm{ord}_p t_1 \leq \cdots \leq \mathrm{ord}_p t_g$. Put $T' = \mathrm{diag}[t_1, \dots, t_{g-1}]$. Suppose that g is even and $p \nmid \mathfrak{d}^T$. Then*

$$F_p^T(\xi_p^T p^{-g/2}) = p^{e_p^T/2} F_p^{T'}(\xi_p^T p^{-g/2}).$$

If $\eta_p^T = -1$, then

$$\frac{\xi_p^T}{p^{g/2}} \frac{\partial F_p^T}{\partial X}\left(\frac{\xi_p^T}{p^{g/2}}\right) = \frac{F_p^{T'}(\xi_p^T p^{(2-g)/2})}{p-1} - p^{e_p^T/2} \frac{\xi_p^T}{p^{g/2}} \frac{\partial F_p^{T'}}{\partial X}\left(\frac{\xi_p^T}{p^{g/2}}\right).$$

Our key ingredient is the explicit formula for $F_p^T(X)$, given by Ikeda and Katsurada in [5], which expresses the polynomial F_p^T in terms of the (naive) extended Gross–Keating datum H of T over $\mathbb{Z}_p$. The polynomial $F_p^{T'} = F_p^{H'}$ is defined in terms of a subset $H' \subsetneq H$ for any p in a uniform way. Actually, if $g = 4$, then the values $\frac{\partial F_p^{H'}}{\partial X}(p^{-2})$ and $F_p^{H'}(p^{-1})$ depend only on (a_1, a_2, a_3) if we write (a_1, a_2, a_3, a_4) for the Gross–Keating invariant of T over $\mathbb{Z}_p$.

1.3. **Applications.**

1.3.1. *On the average of the representation numbers.* Theorem 1.2 combined with the Siegel formula will identify (1.1) with four times the average of the representation numbers of a symmetric matrix of size $g-1$ (see Conjecture 5.4 and Proposition 5.5). The following result is a special case of Proposition 5.5.

Corollary 5.6. If T is a positive definite symmetric half-integral matrix of size 4 which satisfies $\chi^T = 1$ and $\eta_\ell^T = 1$ for $\ell \neq p$, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that

$$\sum_{(E',E)} \frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')} = 2\sum_{(E',E)} \frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

1.3.2. *On the Fourier coefficients and the modular correspondences.* The factor $\frac{\partial F_p^{H'}}{\partial X}(\xi_p^T p^{-g/2})$ appears in Fourier coefficients of central derivatives of incoherent Eisenstein series of genus $g-1$ and weight $\frac{g}{2}$, which have close connection with arithmetical geometry on Shimura varieties at least for $g \leq 5$ as mentioned above. We will be mostly interested in the case $g = 4$. When T_{m_1}, T_{m_2} and T_{m_3} intersect properly, the formula of Gross and Keating in [3] can be stated as follows:

$$(T_{m_1}\cdot T_{m_2}\cdot T_{m_3}) = \sum_B \deg \mathscr{Z}(B),$$

where B extends over all positive definite symmetric half-integral matrices with diagonal entries m_1, m_2, m_3. Here $\deg \mathscr{Z}(B) = 0$ unless $\mathrm{Diff}(B)$ consists of a single rational prime p, in which case

$$\deg \mathscr{Z}(B) = -\frac{(\log p)}{2p^2}\frac{\partial F_p^B}{\partial X}\left(\frac{1}{p^2}\right)\sum_{(E,E')}\frac{N(\mathrm{Hom}(E',E),B)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}. \tag{1.2}$$

The degree $\deg \mathscr{Z}(B)$ equals the B-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 and genus 3 up to a negative constant (cf. Theorem 2.2 of [17]). We combine (1.2), Theorem 5.3 and Corollary 5.6 to obtain the following formula:

Theorem 1.3. *If T is a positive definite symmetric half-integral matrix of size 4, $\chi_T = 1$ and $\mathrm{Diff}(T)$ consists of a single prime number p, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that*

$$\frac{C_4(T)}{-2^8\cdot 3^2} = \deg \mathscr{Z}(T') + \frac{F_p^{T'}(p^{-1})}{2\sqrt{p}^{e_p^T}(p-1)}\log p \sum_{(E,E')}\frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Since $\mathrm{Hom}(E',E)$ is a quaternary quadratic space, if S has rank greater than 4, then $N(\mathrm{Hom}(E,E'),S) = 0$. Therefore when $g \geq 5$, the nature of Fourier coefficients of the derivative of Eisenstein series of weight 2 and genus g should be much different. The case $g = 4$ should be a boundary case. We will explicitly compute $F_p^{T'}(p^{-1})$ in Lemma 5.7 and show that

$$\left|\frac{C_4(T)}{-2^8\cdot 3^2\cdot \deg \mathscr{Z}(T')}-1\right|<\frac{20}{p\sqrt{p}}.$$

Moreover, Corollary 5.8 says that for a fixed prime number p

$$\lim_{\mathrm{ord}_p(\det T)\to\infty}\frac{C_4(T)}{-2^8\cdot 3^2\cdot \deg \mathscr{Z}(T')}=1.$$

1.4. **Organizations.** We now explain the lay-out of this paper. Section 2 extends the notion of incoherent Eisenstein series to the case where the point at which the Eisenstein series is evaluated lies within the left half-plane. We calculate the Fourier coefficients of those Eisenstein series and their derivatives. In Section 3 we derive a general formula for Fourier coefficients of derivatives of incoherent Eisenstein series. Section 4 is devoted to a local study of the Siegel series. We give the inductive expression for the special value of the derivative of the Siegel series. Section 5 is devoted to proving Theorem 5.3.

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Notations

For a finite set A, we denote by $\sharp A$ the number of elements in A. For a ring R we denote by $\mathrm{M}_{i,j}(R)$ the set of $i\times j$-matrices with entries in R and write $\mathrm{M}_m(R)$ in place of $\mathrm{M}_{m,m}(R)$. The group of all invertible elements of $\mathrm{M}_m(R)$ and the set of symmetric matrices of size m with entries in R are denoted by $\mathrm{GL}_m(R)$ and $\mathrm{Sym}_m(R)$, respectively. Let $\mathcal{E}_m(R)$ be the set of elements $(a_{ij})\in\mathrm{Sym}_m(R)$ such that $a_{ii}\in 2R$ for every i. For matrices $B\in\mathrm{Sym}_m(R)$ and $G\in\mathrm{M}_{m,n}(R)$ we use the abbreviation $B[G]={}^tGBG$, where ${}^tG$ is the transpose of G. If $A_1,\dots,A_r$ are square matrices, then $\mathrm{diag}[A_1,\dots,A_r]$ denotes the matrix with $A_1,\dots,A_r$ in the diagonal blocks and 0 in all other blocks. Let $\mathbf{1}_m$ be the identity matrix of degree m. Put

$$Sp_g(R)=\left\{G\in\mathrm{GL}_{2g}(R)\;\middle|\;G\begin{pmatrix}0&\mathbf{1}_g\\-\mathbf{1}_g&0\end{pmatrix}{}^tG=\begin{pmatrix}0&\mathbf{1}_g\\-\mathbf{1}_g&0\end{pmatrix}\right\},$$
$$M_g(R)=\left\{\mathbf{m}(A)=\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}\;\middle|\;A\in\mathrm{GL}_g(R)\right\},$$
$$N_g(R)=\left\{\mathbf{n}(B)=\begin{pmatrix}\mathbf{1}_g&B\\0&\mathbf{1}_g\end{pmatrix}\;\middle|\;B\in\mathrm{Sym}_g(R)\right\}.$$

Let $\mathbb{Z}$ be the set of integers and μ_n the group of n-th roots of unity. If x is a real number, then we put $[x]=\max\{m\in\mathbb{Z}\mid m\le x\}$.

2. Eisenstein series

Let k be a totally real number field with integer ring $\mathfrak{o}$. The set of real places of k is denoted by $\mathfrak{S}_\infty$. The completion of k at a place v is denoted by k_v. Let $(\ ,\ )_{k_v} : k_v^\times \times k_v^\times \to \mu_2$ denote the Hilbert symbol. We let $\mathfrak{p}$ denote a finite prime of k and do not use the letter $\mathfrak{p}$ for a real place. Let $q_\mathfrak{p} = \sharp\mathfrak{o}/\mathfrak{p}$ be the order of the residue field. We define the character $\mathbf{e}_\mathfrak{p}$ of $k_\mathfrak{p}$ by $\mathbf{e}_\mathfrak{p}(x) = \mathbf{e}(-y)$ with $y \in \mathbb{Q}^{(p)}$ such that $\mathrm{Tr}_{k_\mathfrak{p}/\mathbb{Q}_p}(x) - y \in \mathbb{Z}_p$ if p is the rational prime divisible by $\mathfrak{p}$. Put $\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbb{C}$ and $\mathbf{e}_\infty(z) = \prod_{v\in\mathfrak{S}_\infty} \mathbf{e}(z_v)$ for $z \in \prod_{v\in\mathfrak{S}_\infty} \mathbb{C}$.

Once and for all we fix a positive integer $g \geq 2$. Let $(V, (\ ,\ ))$ be a quadratic space of dimension m over k_v. Whenever we speak of a quadratic space, we always assume that $(\ ,\ )$ is nondegenerate, i.e., $(u, V) = 0$ implies that $u = 0$. Put $s_0 = \frac{1}{2}(m - g - 1)$. Given $u = (u_1, \dots, u_g) \in V^g$, we write (u, u) for the $g \times g$ symmetric matrix with (i, j) entry equal to (u_i, u_j). We write $\det V$ for the element in $k_v^\times/k_v^{\times 2}$ represented by the determinant of the matrix representation of the bilinear form $(\ ,\ )$ with respect to any basis for V over k_v. We define the character $\chi^V : k_v^\times \to \mu_2$ by

$$\chi^V(t) = (t, (-1)^{m(m-1)/2} \det V)_{k_v}. \tag{2.1}$$

We normalize our Hasse invariant η^V so that it depends only on the isomorphism class of an anisotropic kernel of V (cf. [2, 22]).

Definition 2.1. We associate to the quadratic space V over $k_\mathfrak{p}$ of dimension m an invariant $\eta^V \in \mu_2$ according to the type of V as follows:

- If m is odd, then an anisotropic kernel of V has dimension $2 - \eta^V$.
- If m is even and $\chi^V \neq 1$ and if we choose an element $c \in k_\mathfrak{p}^\times$ such that $\chi^V(c) = \eta^V$, then V is the orthogonal sum of a split form of dimension $m - 2$ with the norm form scaled by the factor c on the quadratic extension of $k_\mathfrak{p}$ corresponding to χ^V.
- If m is even and $\chi^V = 1$, then V is split or the orthogonal sum of the norm form on the quaternion algebra over $k_\mathfrak{p}$ with a split form of dimension $m - 4$ according as $\eta^V = 1$ or -1.

We denote the set of positive definite symmetric matrices over $\mathbb{R}$ of rank g by $\mathrm{Sym}_g(\mathbb{R})^+$. Let

$$\mathfrak{H}_g = \{X + \sqrt{-1}Y \in \mathrm{Sym}_g(\mathbb{C}) \mid Y \in \mathrm{Sym}_g(\mathbb{R})^+\}$$

be the Siegel upper half-space of genus g. The real symplectic group $Sp_g(\mathbb{R})$ acts transitively on $\mathfrak{H}_g$ by $GZ = (AZ + B)(CZ + D)^{-1}$ for $Z \in \mathfrak{H}_g$ and $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{R})$. We define the maximal compact subgroups by

$$K_\mathfrak{p} = Sp_g(\mathfrak{o}_\mathfrak{p}), \qquad K_v = \{G \in Sp_g(k_v) \mid G(\sqrt{-1}\mathbf{1}_g) = \sqrt{-1}\mathbf{1}_g\}$$

for $v \in \mathfrak{S}_\infty$. We have the Iwasawa decomposition

$$Sp_g(k_v) = M_g(k_v)N_g(k_v)K_v.$$

Denote the two-fold metaplectic cover of $Sp_g(k_v)$ by Mp_v. There is a canonical splitting $N_g(k_v) \to \mathrm{Mp}_v$. When $\mathfrak{p}$ does not divide 2, we have a canonical splitting $K_\mathfrak{p} \to \mathrm{Mp}_\mathfrak{p}$. We still use $N_g(k_v)$ and $K_\mathfrak{p}$ to denote the images of these splittings. Let $\tilde{K}_v$ denote the pull-back of K_v in Mp_v. Define the map $\mathrm{Mp}_v \to \mathbb{R}_+^\times$ by writing $\tilde{G} = \mathbf{n}(b)\tilde{m}\tilde{k} \in \mathrm{Mp}_v$ with $b \in \mathrm{Sym}_g(k_v)$, $a \in \mathrm{GL}_g(k_v)$, $\tilde{m} = (\mathbf{m}(a), \zeta)$ and $\tilde{k} \in \tilde{K}_v$ and setting $|a(\tilde{G})| = |\det a|_v$. We refer to Section 1.1 of [27] for additional explanation.

Let V be a quadratic space over k_v and ω_v the Weil representation of Mp_v with respect to $\mathbf{e}_v$ on the space $\mathcal{S}(V^g)$ of the Schwartz functions on V^g. We associate to $\varphi \in \mathcal{S}(V^g)$ the function on $\mathrm{Mp}_v \times \mathbb{C}$ by

$$f_\varphi^{(s)}(\tilde{G}) = (\omega_v(\tilde{G})\varphi)(0)|a(\tilde{G})|^{s-s_0}.$$

The real metaplectic group acts on the half-space $\mathfrak{H}_g$ through $Sp_g(\mathbb{R})$. There is a unique factor of automorphy $\jmath_v : \mathrm{Mp}_v \times \mathfrak{H}_g \to \mathbb{C}^\times$ whose square descends to the automorphy factor on $Sp(k_v) \times \mathfrak{H}_g$ given by $\jmath_v(G_v, Z_v)^2 = \det(C_v Z_v + D_v)$ for $G_v = \begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix} \in Sp(k_v)$. We define an automorphy factor $\jmath : \prod_{v \in \mathfrak{S}_\infty} (\mathrm{Mp}_v \times \mathfrak{H}_g) \to \mathbb{C}^\times$ by $\jmath(\tilde{G}, Z) = \prod_v \jmath_v(\tilde{G}_v, Z_v)$.

Let $\mathbb{A}$ be the adele ring of k and $\mathbb{A}_\mathbf{f}$ the finite part of the adele ring. We arbitrarily fix a quadratic character χ of $\mathbb{A}^\times/k^\times$ such that $\chi_v = \mathrm{sgn}^{m(m-1)/2}$.

Definition 2.2. Let $\mathcal{C} = \{\mathcal{C}_v\}$ be a collection of local quadratic spaces of dimension m such that $\chi^{\mathcal{C}_v} = \chi_v$ for all v, such that $\mathcal{C}_v$ is positive definite for $v \in \mathfrak{S}_\infty$ and such that $\eta^{\mathcal{C}_\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. We say that $\mathcal{C}$ is coherent if it is the set of localizations of a global quadratic space. Otherwise we call $\mathcal{C}$ incoherent.

One can derive the following criterion from the theorem of Minkowski-Hasse (see Theorem 4.4 of [21]).

Lemma 2.3. *Put $d = [k : \mathbb{Q}]$. When m is odd, $\mathcal{C}$ is coherent if and only if $(-1)^{d(m^2-1)/8} \prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = 1$. When m is even, $\mathcal{C}$ is coherent if and only if $(-1)^{dm(m-2)/8} \prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = 1$.*

There is a unique splitting $Sp_g(k) \hookrightarrow \mathrm{Mp}_g$ by which we regard $Sp_g(k)$ as the subgroup of the two-fold metaplectic cover Mp_g of $Sp_g(\mathbb{A})$. Let $P_g = M_g N_g$ be the Siegel parabolic subgroup of Sp_g. Given any pure tensor $\varphi = \otimes_\mathfrak{p} \varphi_\mathfrak{p} \in \otimes'_\mathfrak{p} \mathcal{S}(\mathcal{C}_\mathfrak{p}^g)$, we consider the function

$$f_\varphi^{(s)}(\tilde{G}) = \prod_\mathfrak{p} f_{\varphi_\mathfrak{p}}^{(s)}(\tilde{G}_\mathfrak{p}), \qquad f_{\varphi_\mathfrak{p}}^{(s)}(\tilde{G}_\mathfrak{p}) = (\omega_\mathfrak{p}(\tilde{G}_\mathfrak{p})\varphi_\mathfrak{p})(0)|a(\tilde{G}_\mathfrak{p})|^{s-s_0}$$

on $\mathrm{Mp}_g \times \mathbb{C}$ and the Eisenstein series on $\prod_{v \in \mathfrak{S}_\infty} \mathfrak{H}_g$

$$E(Z, f_\varphi^{(s)}) = (\det Y)^{(s-s_0)/2} \sum_{\gamma \in P_g(k) \backslash Sp_g(k)} |\jmath(\gamma, Z)|^{s_0 - s} \jmath(\gamma, Z)^{-g} f_\varphi^{(s)}(\gamma),$$

where Y is the imaginary part of Z. The series is absolutely convergent for $\Re s > \frac{g+1}{2}$. It admits a meromorphic continuation to the whole plane and its Laurent coefficients define automorphic forms. Moreover, it is holomorphic at $s = s_0$, and if $\mathcal{C}$ is coherent, then the Siegel–Weil formula holds by [10].

From now on we require that $m \leq g+1$. Let V be a totally positive definite quadratic space of dimension m over k. We normalize the invariant measure $\mathrm{d}h$ on $\mathrm{O}(V,k)\backslash\mathrm{O}(V,\mathbb{A})$ to have total volume 1 and define the integral

$$I(Z,\varphi) = \int_{\mathrm{O}(V,k)\backslash\mathrm{O}(V,\mathbb{A})} \Theta(Z,h;\varphi)\,\mathrm{d}h$$

of the theta function

$$\Theta(Z,h;\varphi) = \sum_{u \in V(k)^g} \varphi(h^{-1}u)\mathbf{e}_\infty(\mathrm{tr}((u,u)Z)).$$

Since we are under coherent situation, the Siegel–Weil formula can now be stated as follows:

$$E(Z, f^{(s)}_\varphi)|_{s=s_0} = 2I(Z,\varphi). \tag{2.2}$$

The reader who is interested in this identity can consult Theorem 2.2(i) of [27]. On the other hand, if $\mathcal{C}$ is incoherent, then the series $E(Z, f^{(s)}_\varphi)$ has a zero at $s = s_0$ by Corollary 5.5 of [27].

Consider the Fourier expansions

$$E(Z, f^{(s)}_\varphi) = \sum_{T \in \mathrm{Sym}_g(k)} A(T,Y,\varphi,s)\mathbf{e}_\infty(\mathrm{tr}(TZ)),$$

$$\frac{\partial}{\partial s} E(Z, f^{(s)}_\varphi)|_{s=s_0} = \sum_{T \in \mathrm{Sym}_g(k)} C(T,Y,\varphi)\mathbf{e}_\infty(\mathrm{tr}(TZ)),$$

where

$$Z = X + \sqrt{-1}Y, \qquad C(T,Y,\varphi) = \frac{\partial}{\partial s} A(T,Y,\varphi,s)|_{s=s_0}.$$

Put $\mathrm{Sym}^{\mathrm{nd}}_g = \mathrm{Sym}_g(k) \cap \mathrm{GL}_g(k)$. When $T \in \mathrm{Sym}^{\mathrm{nd}}_g$, the Fourier coefficient has an explicit expression as an infinite product

$$A(T,Y,\varphi,s) = a(T,Y,s) \prod_{\mathfrak{p}} W_T\left(f^{(s)}_{\varphi_\mathfrak{p}}\right)$$

for $\Re s \gg 0$, where

$$W_T\left(f^{(s)}_{\varphi_\mathfrak{p}}\right) = \int_{\mathrm{Sym}_g(k_\mathfrak{p})} f^{(s)}_{\varphi_\mathfrak{p}}\left(\begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix} \mathbf{n}(z_\mathfrak{p})\right) \overline{\mathbf{e}_\mathfrak{p}(\mathrm{tr}(Tz_\mathfrak{p}))}\,\mathrm{d}z_\mathfrak{p}$$

and $a(T,Y,s)\mathbf{e}_\infty(\sqrt{-1}\mathrm{tr}(TY))$ is a product of the confluent hypergeometric functions investigated in [18]. Given $T \in \mathrm{Sym}^{\mathrm{nd}}_g$, we define the quadratic form on $V^T = k^g$ by $u \mapsto T[u]$ and define the Hecke character $\chi^T = \prod_v \chi^T_v$ and the Hasse invariants $\eta^T_\mathfrak{p}$, where χ^T_v is defined in (2.1). Let $\mathrm{Diff}(T,\mathcal{C})$ denote the set of places v of k such that T is not represented by $\mathcal{C}_v$. Let

Sym_g^+ denote the set of totally positive definite symmetric $g \times g$ matrices over k.

Lemma 2.4. *Let* $\varphi_\mathfrak{p} \in \mathcal{S}(\mathcal{C}_\mathfrak{p}^g)$ *and* $T \in \mathrm{Sym}_g^{\mathrm{nd}}$.

(1) $a(T, Y, s)$ *and* $W_T\Big(f_{\varphi_\mathfrak{p}}^{(s)}\Big)$ *are entire functions in* s.

(2) $\lim_{s \to s_0} W_T\Big(f_{\varphi_\mathfrak{p}}^{(s)}\Big) = 0$ *unless* T *is represented by* $\mathcal{C}_\mathfrak{p}$.

(3) *If* $m = g$, $T \in \mathrm{Sym}_g^+$, $\chi^T = \chi$ *and* $\mathcal{C}$ *is incoherent, then* $\mathrm{Diff}(T, \mathcal{C})$ *is a finite set of odd cardinality.*

Proof. The first part is well-known (see [6, 18]). Lemma on p. 73 of [16] implies (2). By assumption $\mathrm{Diff}(T, \mathcal{C}) = \{\mathfrak{p} \mid \eta^{\mathcal{C}_\mathfrak{p}} = -\eta_\mathfrak{p}^T\}$. Since $\mathcal{C}$ is incoherent, Lemma 2.3 implies $\prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = -\prod_\mathfrak{p} \eta_\mathfrak{p}^T$, which proves (3). □

Let $T \in \mathrm{Sym}_g^+$. Then both $a(T, Y, s_0)$ and $C(T, Y, \varphi)$ are independent of Y. Put

$$c_m(T) = a(T, Y, s_0), \qquad C(T, \varphi) = C(T, Y, \varphi), \qquad D_T = \mathrm{N}_{k/\mathbb{Q}}(\det(2T)).$$

Let $\mathfrak{d}_k$ denote the absolute value of the discriminant of k. Note that

$$c_g(T) = c_g D_T^{-1/2}, \qquad c_g = \mathfrak{d}_k^{-g(g+1)/4}\left(\mathbf{e}\left(\frac{g^2}{8}\right)\frac{2^g \pi^{g^2/2}}{\Gamma_g\left(\frac{g}{2}\right)}\right)^d \tag{2.3}$$

by (4.34K) of [18], where $\Gamma_g(s) = \pi^{g(g-1)/4} \prod_{i=0}^{g-1} \Gamma\big(s - \frac{i}{2}\big)$.

Proposition 2.5. *Let* $m = g$ *and* $T \in \mathrm{Sym}_g^+$. *Suppose that* $\mathcal{C}$ *is incoherent. If* $\chi^T = \chi$, *then* $C(T, \varphi) = 0$ *unless* $\mathrm{Diff}(T, \mathcal{C})$ *is a singleton. Moreover, if* $\mathrm{Diff}(T, \mathcal{C}) = \{\mathfrak{p}\}$, *then*

$$C(T, \varphi) = c_g D_T^{-1/2} \lim_{s \to -1/2} \frac{\partial W_T\Big(f_{\varphi_\mathfrak{p}}^{(s)}\Big)}{\partial s} \prod_{\mathfrak{l} \neq \mathfrak{p}} W_T\Big(f_{\varphi_\mathfrak{l}}^{(s)}\Big).$$

Proof. For given φ and T, let $\mathfrak{S}$ be a finite set of rational primes of k such that if $\mathfrak{q} \notin \mathfrak{S}$, then $\mathfrak{q}$ does not divide 2, $\chi_\mathfrak{q}$ is unramified, $\mathbf{e}_\mathfrak{q}$ is of order 0, $T \in \mathrm{GL}_g(\mathfrak{o}_\mathfrak{q})$ and the restriction of $f_{\varphi_\mathfrak{q}}^{(s)}$ to $K_\mathfrak{q}$ is 1. Since T cannot be unimodular at $\mathfrak{p} \in \mathrm{Diff}(T, \mathcal{C})$, the set $\mathfrak{S}$ necessarily contains $\mathrm{Diff}(T, \mathcal{C})$. The T-th Fourier coefficient of $E(Z, f_\varphi^{(s)})$ is given by

$$A(T, Y, \varphi, s) = \beta^T(s) a(T, Y, s) \prod_{\mathfrak{q} \in \mathfrak{S}} \beta_\mathfrak{q}^T(s) W_T\Big(f_{\varphi_\mathfrak{q}}^{(s)}\Big), \tag{2.4}$$

where

$$\beta^T(s) = \frac{L\big(s + \frac{1}{2}, \chi^T \chi\big)}{\prod_{j=1}^{[(g+1)/2]} \zeta(2s + 2j - 1)} \times \begin{cases} 1 & \text{if } 2 \nmid g, \\ L\big(s + \frac{g+1}{2}, \chi\big)^{-1} & \text{if } 2 | g, \end{cases}$$

$$\beta_\mathfrak{q}^T(s) = \frac{\prod_{j=1}^{[(g+1)/2]} \zeta_\mathfrak{q}(2s + 2j - 1)}{L\big(s + \frac{1}{2}, \chi_\mathfrak{q}^T \chi_\mathfrak{q}\big)} \times \begin{cases} 1 & \text{if } 2 \nmid g, \\ L\big(s + \frac{g+1}{2}, \chi_\mathfrak{q}\big) & \text{if } 2 | g. \end{cases}$$

Notice that the product $\beta_{\mathfrak{q}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ is holomorphic at $s=-\frac{1}{2}$. Indeed, if $\chi_{\mathfrak{q}}^T=\chi_{\mathfrak{q}}$, then $\beta_{\mathfrak{q}}^T(s)$ is holomorphic at $s=-\frac{1}{2}$ while if $\chi_{\mathfrak{q}}^T\neq\chi_{\mathfrak{q}}$, then $\beta_{\mathfrak{q}}^T(s)$ has a simple pole at $s=-\frac{1}{2}$, but $W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ has a zero at $s=-\frac{1}{2}$ by Lemma 2.4(2).

Assume that $\chi^T=\chi$. Then $\beta^T(s)$ is holomorphic and has no zero at $s=-\frac{1}{2}$. If $\mathfrak{q}\in\mathrm{Diff}(T,\mathcal{C})$, then $\beta_{\mathfrak{q}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ has a zero at $s=-\frac{1}{2}$ by Lemma 2.4(2), which combined with (2.4) proves the first statement. We obtain the first formula by differentiating (2.4) at $s=-\frac{1}{2}$. □

Corollary 2.6. If $m=g$, $\mathcal{C}$ is incoherent and $T\in\mathrm{Sym}_g^+$ with $\chi^T\neq\chi$, then

$$C(T,\varphi)=c_gD_T^{-1/2}\lim_{s\to-1/2}\frac{\partial\beta^T}{\partial s}(s)\prod_{\mathfrak{p}}\beta_{\mathfrak{p}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{p}}}^{(s)}\Big).$$

Proof. Since $\beta^T(s)$ has a zero at $s=-\frac{1}{2}$ if $\chi\neq\chi^T$, we can deduce Corollary 2.6 from (2.4). □

3. Fourier coefficients of derivatives of Eisenstein series

Let $\gamma_v(t)$ be the Weil constant associated to the character of second degree $u\mapsto\mathbf{e}_v(tu^2)$, and $\varepsilon_v(\mathcal{C}_v)$ the unnormalized Hasse invariant of $\mathcal{C}_v$. Put

$$\gamma(\mathcal{C}_v)=\varepsilon_v(\mathcal{C}_v)\gamma_v\left(\frac{1}{2}\right)^{m-1}\gamma_v\left(\frac{1}{2}\det\mathcal{C}_v\right).$$

Let $L_{\mathfrak{p}}$ be an integral lattice of $\mathcal{C}_{\mathfrak{p}}$, i.e., a finitely generated $\mathfrak{o}_{\mathfrak{p}}$-submodule of $\mathcal{C}_{\mathfrak{p}}$ which spans $\mathcal{C}_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$ and such that $(u,u)\in\mathfrak{o}_{\mathfrak{p}}$ for every $u\in L_{\mathfrak{p}}$. Let

$$L_{\mathfrak{p}}^*=\{u\in\mathcal{C}_{\mathfrak{p}}\mid 2(u,w)\in\mathfrak{o}_{\mathfrak{p}}\text{ for every }w\in L_{\mathfrak{p}}\}$$

be its dual lattice. Let $\mathrm{ch}\langle L_{\mathfrak{p}}^g\rangle\in\mathcal{S}(\mathcal{C}_{\mathfrak{p}}^g)$ be the characteristic function of $L_{\mathfrak{p}}^g$. We write $S_{\mathfrak{p}}$ for the matrix for the quadratic form on $\mathcal{C}_{\mathfrak{p}}$ with respect to a fixed basis of $L_{\mathfrak{p}}$. For nondegenerate symmetric matrices $T\in\frac{1}{2}\mathcal{E}_g(\mathfrak{o}_{\mathfrak{p}})$ and $S\in\frac{1}{2}\mathcal{E}_m(\mathfrak{o}_{\mathfrak{p}})$ the local density of representing T by S is defined by

$$\alpha_{\mathfrak{p}}(S,T)=\lim_{i\to\infty}q_{\mathfrak{p}}^{ig((g+1)-2m)/2}A_i(S,T),$$

where

$$A_i(S,T)=\sharp\{X\in\mathrm{M}_{m,g}(\mathfrak{o}/\mathfrak{p}^i)\mid S[X]\equiv T\pmod{\mathfrak{p}^i}\}.$$

Proposition 3.1 (cf. [8]). *Put $\mathcal{V}_r=\mathcal{C}_{\mathfrak{p}}\oplus\mathcal{H}(k_{\mathfrak{p}})^r$, where $\mathcal{H}$ is the split binary quadratic space. We choose an integral lattice $L_{\mathfrak{p}}^g\oplus\mathrm{M}_{2r,g}(\mathfrak{o}_{\mathfrak{p}})$ of full rank in $\mathcal{V}_r^g$. Then*

$$\lim_{s\to r+s_0}W_T\Big(f^{(s)}_{\mathrm{ch}\langle L_{\mathfrak{p}}^g\oplus\mathrm{M}_{2r,g}(\mathfrak{o}_{\mathfrak{p}})\rangle}\Big)=\frac{\alpha_{\mathfrak{p}}\left(S_{\mathfrak{p}}\perp\frac{1}{2}\begin{pmatrix} & \mathbf{1}_r\\ \mathbf{1}_r & \end{pmatrix},T\right)}{\gamma(\mathcal{C}_{\mathfrak{p}})^g\mathfrak{d}_k^{-g/2}[L_{\mathfrak{p}}^*:L_{\mathfrak{p}}]^{g/2}}.$$

Here, s_0 is associated to $\mathcal{C}_{\mathfrak{p}}$.

Proof. This result can be deduced from the proof of [28, Lemma 8.3(2)]. □

Let $\mathcal{V}$ be a totally positive definite quadratic space of dimension g over k. Fix an integral lattice L in $\mathcal{V}$. Put

$$L_{\mathfrak{p}} = L \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}, \qquad \mathrm{ch}\langle L^g\rangle = \otimes_{\mathfrak{p}} \mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle.$$

For $h \in \mathrm{O}(\mathcal{V}, \mathbb{A})$ we write hL for the lattice defined by $(hL)_{\mathfrak{p}} = h_{\mathfrak{p}} L_{\mathfrak{p}}$. Put

$$K_L = \{h \in \mathrm{SO}(\mathcal{V}, \mathbb{A}) \mid hL = L\}, \quad \mathrm{SO}(L) = \{h \in \mathrm{SO}(\mathcal{V}, k) \mid hL = L\}.$$

Definition 3.2. We mean by the genus (resp. class) of L the set of all lattices of the form hL with $h \in \mathrm{O}(\mathcal{V}, \mathbb{A})$ (resp. $h \in \mathrm{O}(\mathcal{V}, k)$). The proper class of L consists of all lattices of the form hL with $h \in \mathrm{SO}(\mathcal{V}, k)$.

We write $\Xi'(L)$ and $\Xi(L)$ for the sets of classes and proper classes in the genus of L, respectively. Define the mass of the genus of L by

$$\mathfrak{m}'(L) = \sum_{\mathscr{L} \in \Xi'(L)} \frac{1}{\sharp \mathrm{O}(\mathscr{L})}, \qquad \mathfrak{m}(L) = \sum_{\mathscr{L} \in \Xi(L)} \frac{1}{\sharp \mathrm{SO}(\mathscr{L})}.$$

Remark 3.3. For each finite prime $\mathfrak{p}$ there is $h \in \mathrm{O}(\mathcal{V}, k_{\mathfrak{p}})$ with $\det h = -1$ such that $hL_{\mathfrak{p}} = L_{\mathfrak{p}}$. The genus of L therefore consists of lattices hL with $h \in \mathrm{SO}(\mathcal{V}, \mathbb{A})$. We identify $\Xi(L)$ with double cosets for $\mathrm{SO}(\mathcal{V}, k)\backslash \mathrm{SO}(\mathcal{V}, \mathbb{A})/K_L$ via the map $h \mapsto hL$.

Lemma 5.6(1) of [20] says that

$$\mathfrak{m}(L) = 2\mathfrak{m}'(L). \tag{3.1}$$

We consider the following sums of representation numbers of $T \in \mathrm{Sym}_g(k)$:

$$R'(L, T) = \sum_{\mathscr{L} \in \Xi'(L)} \frac{N(\mathscr{L}, T)}{\sharp \mathrm{O}(\mathscr{L})}, \qquad R(L, T) = \sum_{\mathscr{L} \in \Xi(L)} \frac{N(\mathscr{L}, T)}{\sharp \mathrm{SO}(\mathscr{L})},$$

where $N(L, T) = \sharp\{u \in L^g \mid (u, u) = T\}$.

Proposition 3.4. *Notation being as above, we have*

$$2\frac{R(L, T)}{\mathfrak{m}(L)} = c_g D_T^{-1/2} \lim_{s \to -1/2} \prod_{\mathfrak{p}} W_T\Big(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle}\Big).$$

Proof. This equality is nothing but the Siegel formula. Nevertheless we reproduce its proof here because of its importance for us. Since both sides are zero unless $V^T \simeq \mathcal{V}$ by Lemma 2.4(2), we may identify V^T with $\mathcal{V}$. As is well-known, there exists $h \in \mathrm{O}(V^T, k_{\mathfrak{p}})$ such that $hL_{\mathfrak{p}} = L_{\mathfrak{p}}$ and $\det h = -1$. Since $\mathrm{SO}(V^T, \mathbb{A})\backslash \mathrm{O}(V^T, \mathbb{A}) = \mu_2(\mathbb{A})$, we have

$$I(Z, \mathrm{ch}\langle L^g\rangle) = \frac{1}{2}\int_{\mathrm{SO}(V^T, k)\backslash \mathrm{SO}(V^T, \mathbb{A})} \Theta(Z, h; \mathrm{ch}\langle L^g\rangle)\,\mathrm{d}h.$$

Choose a finite set of double coset representatives $h_i \in \mathrm{SO}(V^T, \mathbb{A}_{\mathbf{f}})$ so that

$$\mathrm{SO}(V^T, \mathbb{A}) = \bigsqcup_i \mathrm{SO}(V^T, k) h_i K_L.$$

Then

$$I(Z,\mathrm{ch}\langle L^g\rangle)=\frac{1}{2}\mathrm{vol}(K_L)\sum_i\frac{\Theta(Z,h_i;\mathrm{ch}\langle L^g\rangle)}{\sharp\mathrm{SO}(h_iL)}.$$

Since $\mathfrak{m}(L)=2\mathrm{vol}(K_L)^{-1}$, the T-th Fourier coefficient of $I(Z,\mathrm{ch}\langle L^g\rangle)$ is equal to $\frac{R(L,T)}{\mathfrak{m}(L)}$. The Siegel–Weil formula (2.2) proves the declared identity. □

An examination of the proof of Proposition 3.4 confirms that

$$\frac{R(L,T)}{\mathfrak{m}(L)}=\frac{R'(L,T)}{\mathfrak{m}'(L)}. \tag{3.2}$$

We can prove the following result by combining Propositions 2.5 and 3.4.

Proposition 3.5. *We assume that* $\mathrm{Diff}(T,\mathcal{C})=\{\mathfrak{p}\}$*, notation and assumption being as in Proposition 2.5. Take an integral lattice* L *in* V^T *such that*

$$\lim_{s=-1/2}W_T\Big(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle}\Big)\neq 0.$$

If $\varphi_{\mathfrak{l}}=\mathrm{ch}\langle L^g_{\mathfrak{l}}\rangle$ *for every prime ideal* $\mathfrak{l}$ *distinct from* $\mathfrak{p}$*, then*

$$C(T,\varphi)=2\frac{R(L,T)}{\mathfrak{m}(L)}\lim_{s\to-1/2}W_T\Big(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle}\Big)^{-1}\frac{\partial W_T\Big(f^{(s)}_{\varphi_{\mathfrak{p}}}\Big)}{\partial s}.$$

4. Siegel series

In this section we drop the subscript $_{\mathfrak{p}}$. Thus k is a nonarchimedean local field of characteristic zero with integer ring $\mathfrak{o}$. We denote the maximal ideal of $\mathfrak{o}$ by $\mathfrak{p}$ and the order of the residue field $\mathfrak{o}/\mathfrak{p}$ by q. Fix a prime element ϖ of $\mathfrak{o}$. We define the additive order $\mathrm{ord}:k^\times\to\mathbb{Z}$ by $\mathrm{ord}(\varpi^i\mathfrak{o}^\times)=i$.

Let $T\in\frac{1}{2}\mathcal{E}_g(\mathfrak{o})$ with $\det T\neq 0$. Denote the conductor of χ^T by $\mathfrak{d}^T$. Put

$$D_T=(-4)^{[g/2]}\det T,$$

$$e^T=\begin{cases}\mathrm{ord}\,D_T & \text{if } g \text{ is odd,}\\ \mathrm{ord}\,D_T-\mathrm{ord}\,\mathfrak{d}^T & \text{if } g \text{ is even,}\end{cases}$$

$$\xi^T=\begin{cases}1 & \text{if } D_T\in k^{\times 2},\\ -1 & \text{if } D_T\notin k^{\times 2} \text{ and } \mathfrak{d}^T=\mathfrak{o},\\ 0 & \text{if } D_T\notin k^{\times 2} \text{ and } \mathfrak{d}^T\neq\mathfrak{o}.\end{cases}$$

The Siegel series associated to T is defined by

$$b(T,s)=\sum_{z\in\mathrm{Sym}_g(k)/\mathrm{Sym}_g(\mathfrak{o})}\psi(-\mathrm{tr}(Tz))\nu[z]^{-s},$$

where $\nu[z]=[z\mathfrak{o}^g+\mathfrak{o}^g:\mathfrak{o}^g]$ and ψ is an arbitrarily fixed additive character on k which is trivial on $\mathfrak{o}$ but nontrivial on $\mathfrak{p}^{-1}$. As is well-known, there exists a polynomial $\beta(T,X)\in\mathbb{Z}[X]$ such that $\beta(T,q^{-s})=b(T,s)$. Moreover, this polynomial $\beta(T,X)$ is divisible by the following polynomial

$$\gamma^T(X)=(1-X)\prod_{j=1}^{[g/2]}(1-q^{2j}X^2)\times\begin{cases}1 & \text{if } g \text{ is odd,}\\ \frac{1}{1-\xi^T q^{g/2}X} & \text{if } g \text{ is even.}\end{cases}$$

Put

$$\beta(T,X)=\gamma^T(X)F^T(X),\qquad \mathcal{F}^T(X)=X^{-e^T/2}F^T(q^{-(g+1)/2}X).$$

If g is even, then $\mathcal{F}^T\in\mathbb{Q}[\sqrt{q}][X+X^{-1}]$. If g is odd, then $\mathcal{F}^T\in\mathbb{Q}\big[\sqrt{X},\frac{1}{\sqrt{X}}\big]$.

Let $\mathcal{C}$ be a g-dimensional quadratic space over k. Recall that S is the matrix for the quadratic form on $\mathcal{C}$ with respect to a fixed basis of L, where L is an integral lattice of $\mathcal{C}$ as explained at the beginning of Section 3. If g is even, $\chi=\chi^{\mathcal{C}}$ is unramified and $\det(2S)\in\mathfrak{o}^\times$, then Lemma 14.8 combined with Proposition 14.3 of [19] gives

$$\alpha\left(S\perp\frac{1}{2}\begin{pmatrix} & \mathbf{1}_r\\ \mathbf{1}_r & \end{pmatrix},T\right)=\beta(T,\chi(\varpi)q^{-(g+2r)/2}). \tag{4.1}$$

For the rest of this paper we require g to be even.

Proposition 4.1. *If g is even, χ is unramified, $\chi^T=\chi$, $\eta^T=-1$, $\eta^{\mathcal{C}}=1$ and L is a self-dual lattice of $\mathcal{C}$, then*

$$\frac{\partial}{\partial s}W_T\Big(f^{(s)}_{\mathrm{ch}\langle L^g\rangle}\Big)\Big|_{s=-1/2}=-\frac{\sqrt{\mathfrak{d}_k}^{\,g}\log q}{\gamma(\mathcal{C})^g}\frac{\xi^T}{\sqrt{q}^g}\gamma^T\Big(\frac{\xi^T}{\sqrt{q}^g}\Big)\frac{\partial F^T}{\partial X}\Big(\frac{\xi^T}{\sqrt{q}^g}\Big).$$

Proof. By assumption $\lim_{s\to-1/2}W_T(f^{(s)}_\varphi)=0$ in view of Lemma 2.4(2). We combine Proposition 3.1 and (4.1) with Lemmas A.2-A.3 of [8] to see that

$$\begin{aligned}W_T\Big(f^{(s)}_\varphi\Big)&=\gamma(\mathcal{C})^{-g}\sqrt{\mathfrak{d}_k}^{\,g}\beta\Big(T,\xi^Tq^{-(g+1+2s)/2}\Big)\\&=\gamma(\mathcal{C})^{-g}\sqrt{\mathfrak{d}_k}^{\,g}\gamma^T\Big(\xi^Tq^{-(g+1+2s)/2}\Big)F^T\Big(\xi^Tq^{-(g+1+2s)/2}\Big).\end{aligned}$$

Since $\chi^T=\chi$, we see that $F^T(\xi^Tq^{-g/2})=0$. We can obtain the stated identity by differentiating this equality at $s=-\frac{1}{2}$. □

Definition 4.2. Let $T=(t_{ij})\in\frac{1}{2}\mathcal{E}_g(\mathfrak{o})\cap\mathrm{GL}_g(k)$. We denote by $S(T)$ the set of all nondecreasing sequences $(a_1,\dots,a_g)$ of nonnegative integers such that $\operatorname{ord}t_{ii}\geq a_i$ and $\operatorname{ord}(2t_{ij})\geq\frac{a_i+a_j}{2}$ for $1\leq i,j\leq g$. The Gross–Keating invariant $\mathrm{GK}(T)$ of T is the greatest element of $\bigcup_{U\in\mathrm{GL}_g(\mathfrak{o})}S(T[U])$ with respect to the lexicographic order.

Here, the lexicographic order is defined as follows: $(y_1,\dots,y_g)$ is greater than $(z_1,\dots,z_g)$ if there is an integer $1\leq j\leq g$ such that $y_i=z_i$ for $i<j$ and $y_j>z_j$. Ikeda and Katsurada [5] define a set $\mathrm{EGK}(T)$ of invariants of T attached to $\mathrm{GK}(T)$, which they call the extended Gross–Keating datum of

T. They associated to an extended Gross–Keating datum H a polynomial $\mathcal{F}^H(Y,X) \in \mathbb{Z}[Y^{1/2}, Y^{-1/2}, X, X^{-1}]$ and show that

$$\mathcal{F}^{\mathrm{EGK}(T)}(\sqrt{q}, X) = \mathcal{F}^T(X).$$

When g is even and $\mathfrak{d}^T = \mathfrak{o}$, one can associate to $\mathrm{EGK}(T)$ truncated extended Gross–Keating datum $\mathrm{EGK}(T)'$ of length $g-1$ by Proposition 4.4 of [5]. By Definitions 4.2-4.4 of [5]

$$\begin{aligned}\mathcal{F}^{\mathrm{EGK}(T)}(Y,X) =& Y^{\mathfrak{e}'/2} X^{-(\mathfrak{e}-\mathfrak{e}'+2)/2} \frac{1-\xi^T Y^{-1} X}{X^{-1}-X} \mathcal{F}^{\mathrm{EGK}(T)'}(Y, YX) \\ &+ Y^{\mathfrak{e}'/2} X^{(\mathfrak{e}-\mathfrak{e}'+2)/2} \frac{1-\xi^T Y^{-1} X^{-1}}{X-X^{-1}} \mathcal{F}^{\mathrm{EGK}(T)'}(Y, YX^{-1}),\end{aligned}$$

where $\mathrm{GK}(T) = (a_1, \cdots, a_g)$, $\mathfrak{e} = 2\left[\frac{a_1+\cdots+a_g}{2}\right]$ and $\mathfrak{e}' = a_1 + \cdots + a_{g-1}$. It is worth noting that since $\mathfrak{d}^T = \mathfrak{o}$, we have $\mathfrak{e} = a_1 + \cdots + a_g = e^T$. We put

$$F^H(X) = (q^{(g+1)/2}X)^{\mathfrak{e}/2} \mathcal{F}^H(\sqrt{q}, q^{(g+1)/2}X).$$

If q is odd, then T is equivalent to a diagonal matrix $\mathrm{diag}[t_1, \cdots, t_g]$ with $\mathrm{ord}\, t_1 \le \cdots \le \mathrm{ord}\, t_g$ and the (naive) extended Gross–Keating datum $\mathrm{EGK}(T) = (a_1, \cdots, a_g; \varepsilon_1, \ldots, \varepsilon_g)$ is given by

$$a_i = \mathrm{ord}\, t_i, \qquad T^{(i)} = \mathrm{diag}[t_1, \cdots, t_i], \qquad \varepsilon_i = \begin{cases} \eta^{T^{(i)}} & \text{if } i \text{ is odd,} \\ \xi^{T^{(i)}} & \text{if } i \text{ is even} \end{cases}$$

and $\mathrm{EGK}(T)' = (a_1, \cdots, a_{g-1}; \varepsilon_1, \ldots, \varepsilon_{g-1})$.

Theorem 4.3. *Assume that g is even and that $\mathfrak{d}^T = \mathfrak{o}$. Then*

$$F^H(\xi^T q^{-g/2}) = q^{e^T/2} F^{H'}(\xi^T q^{-g/2}),$$

where we put $H = \mathrm{EGK}(T)$ and $H' = \mathrm{EGK}(T)'$. If $\eta^T = -1$, then

$$\frac{\xi^T}{\sqrt{q}^g} \frac{\partial F^H}{\partial X}\left(\frac{\xi^T}{\sqrt{q}^g}\right) = \frac{F^{H'}(\xi^T q^{(2-g)/2})}{q-1} - \sqrt{q}^{e^T} \frac{\xi^T}{\sqrt{q}^g} \frac{\partial F^{H'}}{\partial X}\left(\frac{\xi^T}{\sqrt{q}^g}\right).$$

Proof. Substituting $Y = \sqrt{q}$ into $\mathcal{F}^H(Y,X)$, we get

$$\begin{aligned}\mathcal{F}^H(\sqrt{q}, X) =& X^{-(\mathfrak{e}+2)/2} \frac{1-\xi^T q^{-1/2} X}{X^{-1}-X} (\sqrt{q}X)^{\mathfrak{e}'/2} \mathcal{F}^{H'}(\sqrt{q}, \sqrt{q}X) \\ &+ X^{(\mathfrak{e}+2)/2} \frac{1-\xi^T q^{-1/2} X^{-1}}{X-X^{-1}} (\sqrt{q}X^{-1})^{\mathfrak{e}'/2} \mathcal{F}^{H'}(\sqrt{q}, \sqrt{q}X^{-1}) \\ =& X^{-(e^T+2)/2} \frac{1-\xi^T q^{-1/2} X}{X^{-1}-X} F^{H'}(q^{(1-g)/2}X) \\ &+ X^{(e^T+2)/2} \frac{1-\xi^T q^{-1/2} X^{-1}}{X-X^{-1}} F^{H'}(q^{(1-g)/2}X^{-1}).\end{aligned}$$

By letting $X = \xi^T \sqrt{q}$, we get

$$(\xi^T \sqrt{q})^{-e^T/2} F^H(\xi^T q^{-g/2}) = \mathcal{F}^H(\sqrt{q}, \xi^T \sqrt{q}) = (\xi^T \sqrt{q})^{e^T/2} F^{H'}(\xi^T q^{-g/2}).$$

In the proof of Proposition 4.1 we have seen that if $\eta^T = -1$, then

$$\mathcal{F}^H(\sqrt{q}, \xi^T\sqrt{q}) = \mathcal{F}^T(\xi^T\sqrt{q}) = (\xi^T\sqrt{q})^{-e^T/2}F^T(\xi^T q^{-g/2}) = 0,$$

and hence $F^{H'}(\xi^T q^{-g/2}) = 0$. We can prove the stated identity by differentiating the equality above at $X = \xi^T\sqrt{q}$. □

We will use the following result in the next section.

Lemma 4.4. *If T is a split symmetric half-integral matrix of size 4 over $\mathbb{Z}_p$, then there exists a nondegenerate isotropic symmetric half-integral matrix B of size 3 over $\mathbb{Z}_p$ such that $F_p^B = F_p^{\mathrm{EGK}_p(T)'}$.*

Proof. If $p = 2$, then the existence of such B follows from Proposition 6.4 of [4] and Theorem 1.1 of [5]. If p is odd, then T is equivalent to a diagonal matrix $\mathrm{diag}[t_1, \cdots, t_4]$ with $\mathrm{ord}\, t_1 \leq \cdots \leq \mathrm{ord}\, t_4$. Then we may choose B as $\mathrm{diag}[t_1, \cdots, t_3]$ by using the argument explained in the paragraph just before Theorem 4.3. □

5. The case $g = 4$

We discuss the classical Eisenstein series of Siegel. For this it is simplest to work over $k = \mathbb{Q}$. Provided that g is a multiple of 4, we consider the series

$$E_g(Z, s) = \sum_{\{C,D\}} \det(CZ + D)^{-g/2}|\det(CZ + D)|^{-s}(\det Y)^{s/2}.$$

Here the sum extends over all symmetric coprime pairs modulo $\mathrm{GL}_g(\mathbb{Z})$. Let $\mathcal{C}_p = \mathcal{H}(\mathbb{Q}_p)^{g/2}$ be the split quadratic space of dimension g over $\mathbb{Q}_p$. Define $\varphi = \otimes_p\varphi_p$ by taking $\varphi_p = \mathrm{ch}\langle \mathrm{M}_{g,g}(\mathbb{Z}_p)\rangle \in \mathcal{S}(\mathcal{C}_p^g)$. It is known that $E_g\big(Z, s + \frac{1}{2}\big) = E(Z, f_\varphi^{(s)})$ (see §IV.2 of [9]). The series is incoherent if and only if $\frac{g}{4}$ is odd due to Lemma 2.3.

Fix a positive definite symmetric half-integral matrix T of size g. Recall that χ_T stands for the primitive Dirichlet character corresponding to χ^T. The T-th Fourier coefficient of $E_g(Z, s)$ is given by

$$A(T, Y, s) = \frac{a\big(T, Y, s - \frac{1}{2}\big)L(s, \chi_T)}{\zeta\big(s + \frac{g}{2}\big)\prod_{i=1}^{g/2}\zeta(2s + 2i - 2)}\prod_{p|D_T} F_p^T(p^{-(2s+g)/2}).$$

The T-th Fourier coefficient of $\frac{\partial}{\partial s}E_g(Z, s)|_{s=0}$ is given by

$$C_g(T) = \frac{\partial}{\partial s}A(T, Y, s)|_{s=0}.$$

Recall that $\mathrm{Diff}(T) = \{p \mid \eta_p^T = -1\}$.

Proposition 5.1. *Assume that $\frac{g}{4}$ is odd. Let $T \in \frac{1}{2}\mathcal{E}_g(\mathbb{Z}) \cap \mathrm{Sym}_g^+$.*

(1) *If $\chi_T = 1$, then $C_g(T) = 0$ unless $\mathrm{Diff}(T)$ is a singleton.*

(2) *If* $\chi_T = 1$ *and* $\mathrm{Diff}(T) = \{p\}$*, then*

$$C_g(T) = -\frac{2^{(g+2)/2} p^{-(g+e_p^T)/2} \log p}{\zeta\left(1-\frac{g}{2}\right)\prod_{i=1}^{(g-2)/2} \zeta(1-2i)} \frac{\partial F_p^T}{\partial X}(p^{-g/2}) \prod_{p \neq \ell | D_T} \ell^{-e_\ell^T/2} F_\ell^T(\ell^{-g/2}).$$

(3) *If* $\chi_T \neq 1$*, then*

$$C_g(T) = -\frac{2^{(g+2)/2} L(1,\chi_T)}{\zeta\left(1-\frac{g}{2}\right)\prod_{i=1}^{(g-2)/2} \zeta(1-2i)} \prod_{p | D_T} p^{-e_p^T/2} F_p^T(p^{-g/2}).$$

Proof. We have already proved (1) in Proposition 2.5. Taking

$$\zeta(2i) = (-1)^i \frac{(2\pi)^{2i}}{2(2i-1)!} \zeta(1-2i)$$

into account, we have

$$\zeta\left(\frac{g}{2}\right) \prod_{i=1}^{(g-2)/2} \zeta(2i) = \frac{(2\pi)^{g^2/4} \zeta\left(1-\frac{g}{2}\right)}{2^{g/2}\left(\frac{g}{2}-1\right)!} \prod_{i=1}^{(g-2)/2} \frac{\zeta(1-2i)}{(2i-1)!}$$

Recall that $a\left(T, Y, -\frac{1}{2}\right) = \frac{2^g \pi^{g^2/2}}{\Gamma_g(\frac{g}{2}) D_T^{1/2}}$ by (2.3). Since

$$\Gamma_g\left(\frac{g}{2}\right) = \frac{\pi^{g^2/4}}{2^{(g^2-2g)/4}} \prod_{i=1}^{(g-2)/2} (2i)!, \quad \zeta(0) = -\frac{1}{2}, \quad L'(0,\chi_T) = \frac{\sqrt{\mathfrak{d}^T}}{2} L(1,\chi_T),$$

we get (2) and (3). □

Hereafter we let $g = 4$. By a quaternion algebra over a field k we mean a central simple algebra over k of dimension 4. Let $\mathbb{B}_p$ denote the definite quaternion algebra over $k = \mathbb{Q}$ that ramifies only at a prime number p. The reduced norm Nrd on $\mathbb{B}_p$ defines a positive definite quadratic space $\mathcal{V}_p$. Fix a maximal order $\mathcal{O}_p$ of $\mathbb{B}_p$. Let $\varphi_\ell \in \mathcal{S}(\mathcal{C}_\ell^g)$ be the characteristic function of $\mathrm{M}_2(\mathbb{Z}_\ell)^g$ and $\varphi'_p \in \mathcal{S}(\mathcal{V}_p^g(\mathbb{Q}_p))$ the characteristic function of $\mathcal{O}_p^g \otimes \mathbb{Z}_p$. We regard $\varphi' = \varphi'_p \otimes (\otimes_{\ell \neq p} \varphi_\ell)$ as the characteristic function of $\mathcal{O}_p^g \otimes \hat{\mathbb{Z}}$. We write S_p for the matrix representation of $\mathcal{V}_p$ with respect to a $\mathbb{Z}$-basis of $\mathcal{O}_p$. Put

$$S_0 = \mathrm{diag}\left[\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right].$$

Lemma 5.2. *Let* $T \in \mathrm{Sym}_g(\mathbb{Q}_p)$*.*

(1) *If* $T \notin \frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$*, then* $W_T\left(f_{\varphi_p}^{(s)}\right)$ *is identically zero.*

(2) *If* $T \in \frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$ *with* $\det T \neq 0$*,* $\chi^T = 1$ *and* $\eta_p^T = -1$*, then*

$$\lim_{s \to -1/2} \frac{W_{S_p}\left(f_{\varphi'_p}^{(s)}\right)}{W_T\left(f_{\varphi'_p}^{(s)}\right)} \frac{\frac{\partial}{\partial s} W_T\left(f_{\varphi_p}^{(s)}\right)}{p W_{S_0}\left(f_{\varphi_p}^{(s)}\right)} = \left(p^{-2} \frac{\partial F_p^{H'}}{\partial X}(p^{-2}) - \frac{p^{-e_p^T/2}}{p-1} F_p^{H'}(p^{-1})\right) \log p,$$

where we put $H' = \mathrm{EGK}_p(T)'$*.*

Proof. The first part is trivial. Since

$$\alpha_p(S_p,T)=p^{(e_p^T-2)/2}\alpha_p(S_p,S_p)$$

by Hilfssatz 17 of [23], it follows from Proposition 3.1 that

$$\lim_{s\to-1/2}\frac{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}{W_T\left(f^{(s)}_{\varphi'_p}\right)}=p^{-(e_p^T-2)/2}.$$

On the other hand, Proposition 4.1 and Theorem 4.3 give

$$\lim_{s\to-1/2}\frac{\frac{\partial}{\partial s}W_T\left(f^{(s)}_{\varphi_p}\right)}{W_{S_0}\left(f^{(s)}_{\varphi_p}\right)}=\left(p^{(e_p^T-4)/2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2})-\frac{F_p^{H'}(p^{-1})}{p-1}\right)\log p.$$

These complete our proof. □

Let $\bar{\mathbb{F}}_p$ be an algebraic closure of a finite field $\mathbb{F}_p$ with p elements. For two supersingular elliptic curves E,E' over $\bar{\mathbb{F}}_p$ we consider the free $\mathbb{Z}$-module $\mathrm{Hom}(E',E)$ of homomorphisms $E'\to E$ over $\bar{\mathbb{F}}_p$ together with the quadratic form given by the degree. As E and E' are supersingular, $\mathrm{Hom}(E',E)$ has rank 4 as a $\mathbb{Z}$-module. For two quadratic spaces over $\mathbb{Z}$ we write $N(L,L')$ for the number of isometries $L'\to L$.

We are now ready to prove our main result.

Theorem 5.3. *If $T\in\frac{1}{2}\mathcal{E}_4(\mathbb{Z})$ is positive definite, $\chi_T=1$ and $\mathrm{Diff}(T)$ consists of a single prime p, then*

$$C_4(T)=2^6\cdot3^2\left(p^{-2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2})-\frac{F_p^{H'}(p^{-1})}{\sqrt{p}^{e_p^T}(p-1)}\right)\log p\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where we put $H'=\mathrm{EGK}_p(T)'$ and where (E',E) extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Proof. Proposition 3.5 and (3.2) applied to $L=\mathcal{O}_p$ gives

$$C_4(T)=R'(\mathcal{O}_p,T)c\lim_{s\to-1/2}\frac{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}{W_T\left(f^{(s)}_{\varphi'_p}\right)}\frac{\frac{\partial}{\partial s}W_T\left(f^{(s)}_{\varphi_p}\right)}{pW_{S_0}\left(f^{(s)}_{\varphi_p}\right)},$$

where

$$c=\frac{2p}{\mathfrak{m}'(\mathcal{O}_p)}\lim_{s\to-1/2}\frac{W_{S_0}\left(f^{(s)}_{\varphi_p}\right)}{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}.$$

If $T=S_p$, then we claim that $R'(\mathcal{O}_p,S_p)=1$. To prove this, it suffices to show that $N(\mathscr{L},S_p)=0$ if $\mathscr{L}$ is not isometric to $\mathcal{O}_p$ and $N(\mathcal{O}_p,S_p)=\sharp\mathrm{O}(\mathcal{O}_p)$, where $\mathscr{L}\in\Xi'(\mathcal{O}_p)$. If $N(\mathscr{L},S_p)\neq0$, then there is an injection $f:\mathcal{O}_p\to\mathscr{L}$ as a lattice preserving the associated quadratic forms. Thus we only need to show that f is surjective. If it is not surjective, then $\mathscr{L}$ and

$\mathcal{O}_p$ have different discriminant, which is a contradiction to the assumption that $\mathscr{L}$ and $\mathcal{O}_p$ are in the same genus.

Applying Proposition 3.4 and (3.2) to $T = S_p$, we get

$$\frac{2}{\mathfrak{m}'(\mathcal{O}_p)} = c_4 D_{S_p}^{-1/2} \lim_{s\to -1/2} W_{S_p}\left(f^{(s)}_{\varphi'_p}\right) \prod_{\ell\neq p} W_{S_p}\left(f^{(s)}_{\varphi_\ell}\right).$$

It follows that

$$\begin{aligned} c &= pc_4 D_{S_p}^{-1/2} \lim_{s\to -1/2} \prod_{\ell} W_{S_0}\left(f^{(s)}_{\varphi_\ell}\right) \\ &= c_4 \lim_{s\to -1/2} \prod_{\ell} \gamma^S_\ell(\ell^{-(5+2s)/2}) = \frac{c_4}{\zeta(2)^2} \lim_{s\to -1/2} \frac{\zeta\left(s+\frac{1}{2}\right)}{\zeta(2s+1)} = 2^7\cdot 3^2. \end{aligned}$$

Since $R(\mathcal{O}_p, T) = 2R'(\mathcal{O}_p, T)$ by (3.1) and (3.2), and

$$R(\mathcal{O}_p, T) = \sum_{\mathscr{L}\in\Xi(\mathcal{O}_p)} \frac{N(\mathscr{L}, T)}{\sharp\mathrm{SO}(\mathscr{L})} = \sum_{(E',E)} \frac{N(\mathrm{Hom}(E', E), T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')} \tag{5.1}$$

by Proposition 4.1 of [25], our statement follows from Lemma 5.2(2). □

Conjecture 5.4. Let $\mathcal{V}$ be a totally positive definite quadratic space over a totally real number field k of dimension g. Fix a maximal integral lattice L of $\mathcal{V}$. Let $T \in \frac{1}{2}\mathcal{E}_g(\mathfrak{o})$ be totally positive definite. If g is even and $\chi^{\mathcal{V}} = 1$, then there is a totally positive definite matrix $T' \in \frac{1}{2}\mathcal{E}_{g-1}(\mathfrak{o})$ such that

$$R(L, T) = 2R(L, T').$$

Proposition 5.5. *If $k = \mathbb{Q}$ and $g = 4$, then Conjecture 5.4 is true.*

Proof. Since $R(L, T) = 0$ unless $\mathrm{Diff}(T) = \mathrm{Diff}(\mathcal{V})$, we may assume that

$$\mathrm{Diff}(T) = \mathrm{Diff}(\mathcal{V}).$$

Lemma 4.4 gives $T'_p \in \frac{1}{2}\mathcal{E}_3(\mathbb{Z}_p)$ such that $F_p^{T'_p} = F_p^{\mathrm{EGK}_p(T)'}$ for every rational prime p. In addition, the proof of Lemma 4.4 yields that T'_p is unimodular for almost all primes p. Thus we can find a positive rational number $0 < \delta \in \mathbb{Q}^\times$ such that $\delta^{-1}\det T'_p \in \mathbb{Z}_p^\times$ for every $p \notin \mathrm{Diff}(\mathcal{V})$. For $p \in \mathrm{Diff}(\mathcal{V})$ we fix an arbitrary anisotropic ternary quadratic form T'_p over $\mathbb{Z}_p$. Recall that $\alpha_p(S_p, T'_p)$ is independent of the choice of T'_p.

Since $F_p^{uT'_p} = F_p^{T'_p}$ for $u \in \mathbb{Z}_p^\times$, there is no harm in assuming that $\delta = \det T'_p$. Since $\eta_p^{T'_p} = 1$ for $p \notin \mathrm{Diff}(\mathcal{V})$, the Minkowski-Hasse theorem gives $z \in \mathrm{Sym}_3(\mathbb{Q})$ which is positive definite and such that $z \in T'_p[\mathrm{GL}_3(\mathbb{Q}_p)]$ for every p. Take $A \in \mathrm{GL}_3(\mathbb{A}_\mathbf{f})$ so that $z = T'_p[A_p]$ for every p. We can take $D \in \mathrm{GL}_3(\mathbb{Q})$ in such a way that $AD^{-1} \in \mathrm{GL}_3(\mathbb{Z}_p)$ for every p. Put $T' = z[D^{-1}]$. Then $T' \in T'_p[\mathrm{GL}_3(\mathbb{Z}_p)]$ for every p. In particular, $T' \in \frac{1}{2}\mathcal{E}_3(\mathbb{Z})$.

In view of (3.2) it suffices to show that

$$\frac{R'(L, T)}{\mathfrak{m}'(L)} = 2\frac{R'(L, T')}{\mathfrak{m}'(L)}.$$

We see by the Siegel formula that

$$\frac{R'(L,T)}{\mathfrak{m}'(L)} = 2^{-1}d_\infty(L,T)2^4 \prod_{p\in\mathrm{Diff}(\mathcal{V})}\frac{\alpha_p(S_p,T)}{2}\prod_{q\notin\mathrm{Diff}(\mathcal{V})}(1-q^{-2})^2F_q^T(q^{-2}).$$

Recall that the archimedean densities are given by

$$d_\infty(L,T)=\frac{\prod_{i=1}^4\frac{\pi^{i/2}}{\Gamma(\frac{i}{2})}}{\det(2T)^{1/2}[L^*:L]^2}, \qquad d_\infty(L,T')=\frac{\prod_{i=2}^4\frac{\pi^{i/2}}{\Gamma(\frac{i}{2})}}{[L^*:L]^{3/2}}.$$

Since

$$\alpha_p(S_p,T')=2(p+1)(1+p^{-1}), \qquad \alpha_p(S_p,T)=4p^{e_p^T/2}(p+1)^2.$$

by [26, Theorem 1.1] and Proposition 6.5 of [1]. The latter result can be derived more generally from Shimura's exact mass formula. Since $[L^*:L]=\prod_{p\in\mathrm{Diff}(\mathcal{V})}p^2$ by assumption, we have

$$d_\infty(L,T)=[L^*:L]^{-2}\det(2T)^{-1/2}\prod_{i=1}^4\frac{\pi^{i/2}}{\Gamma\left(\frac{i}{2}\right)}=\frac{d_\infty(L,T')}{\det(2T)^{1/2}}\prod_{p\in\mathrm{Diff}(\mathcal{V})}p^{-1}.$$

We combine these with Theorem 4.3 to obtain

$$\frac{R'(L,T)}{\mathfrak{m}'(L)} = d_\infty(L,T')2^3\prod_{p\in\mathrm{Diff}(\mathcal{V})}\alpha_p(S_p,T')\prod_{q\notin\mathrm{Diff}(\mathcal{V})}(1-q^{-2})^2F_q^{T'}(q^{-2}).$$

The final expression equals $2\frac{R'(L,T')}{\mathfrak{m}'(L)}$ by the Siegel formula. □

Corollary 5.6. If T is a positive definite symmetric half-integral matrix of size 4 which satisfies $\chi^T=1$ and $\eta_\ell^T=1$ for $\ell\neq p$, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that

$$\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}=2\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Proof. Proposition 4.1 of [25] gives

$$R(\mathcal{O}_p,T')=\sum_{L\in\Xi(\mathcal{O}_p)}\frac{N(L,T')}{\sharp\mathrm{SO}(L)}=\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}.$$

We can derive Corollary 5.6 from (5.1) and Proposition 5.5. □

Let $T\in\frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$ be an anisotropic symmetric matrix with (naive) extended Gross-Keating invariant $(a_1,a_2,a_3,a_4;\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. Note that $\varepsilon_1=\varepsilon_4=1$ by definition. One can easily see that $\varepsilon_2\neq 1$ and $\varepsilon_3=-1$. Proposition 5.3 of [1] gives a partition $\{1,2,3,4\}=\{i,j\}\cup\{k,l\}$ such that

$$a_i\equiv a_j\not\equiv a_k\equiv a_l \pmod 2.$$

Lemma 5.7. (1) *If* $a_1 \not\equiv a_2 \pmod 2$, *then*

$$\begin{aligned}F_p^{T'}(p^{-1}) =& \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{\{a_1+3(a_2+1)\}/2} - \frac{p^{a_1+1}+1}{p+1}\right) \\ &- \frac{p^{(a_1+a_2+2a_3+1)/2}}{p-1}\left\{(a_1+1)p^{(a_1+a_2+1)/2} - \frac{p^{a_1+1}-1}{p-1}\right\}.\end{aligned}$$

(2) *If* $a_1 \equiv a_2 \pmod 2$, *then*

$$\begin{aligned}F_p^{T'}(p^{-1}) =& \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{(a_1+3a_2)/2} - \frac{p^{a_1+1}+1}{p+1}\right) \\ &- \frac{p^{(a_1+a_2+2a_3+2)/2}}{p-1}\left\{(a_1+1)p^{(a_1+a_2)/2} - \frac{p^{a_1+1}-1}{p-1}\right\} \\ &+ p^{(a_1+3a_2)/2}\frac{p^{a_1+1}-1}{p^2-1}(p^{a_1-a_2+1}+1).\end{aligned}$$

Proof. We write the naive extended Gross-Keating invariant of T as

$$\mathrm{EGK}_p(T) = (a_1, a_2, a_3, a_4; 1, \varepsilon_2, \varepsilon_3, 1).$$

Let σ be either 1 or 2 according as $a_1 - a_2$ is odd or even. Section 8 of [5] expresses $F_p^{\mathrm{EGK}_p(T)'}(X)$ in terms of $\mathrm{EGK}_p(T)' = (a_1, a_2, a_3; 1, \varepsilon_2, \varepsilon_3)$:

$$\begin{aligned}F_p^{\mathrm{EGK}_p(T)'}(p^{-2}X) = \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{i+j}X^{i+2j} \\ \varepsilon_3 \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{(a_1+a_2-\sigma)/2-j}X^{a_3+\sigma+i+2j} \\ + \varepsilon_2^2 p^{(a_1+a_2-\sigma+2)/2} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \varepsilon_2^j X^{a_2-\sigma+2+i+j}.\end{aligned}$$

We now specialize the formula to $X = p$ and $\varepsilon_3 = -1$. Then

$$\begin{aligned}F_p^{T'}(p^{-1}) =& \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{\{a_1+3(a_2-\sigma+2)\}/2} - \frac{p^{a_1+1}+1}{p+1}\right) \\ &- \frac{p^{(a_1+a_2+2a_3+\sigma)/2}}{p-1}\left((a_1+1)p^{(a_1+a_2-\sigma+2)/2} - \frac{p^{a_1+1}-1}{p-1}\right) \\ &+ \varepsilon_2^2 p^{\{a_1+3(a_2-\sigma+2)\}/2}\frac{(p^{a_1+1}-1)(1-(\varepsilon_2 p)^{a_1-a_2+2\sigma-3})}{(p-1)(1-\varepsilon_2 p)}.\end{aligned}$$

Since $\varepsilon_2 = 0$ or -1 according as $a_1 - a_2$ is odd or even by Proposition 2.2 of [4] and Proposition 5.4 of [1], we obtain the stated formulas. □

The degree $\deg \mathscr{Z}(B)$ is defined in (1.2) for positive definite symmetric half-integral 3×3 matrices B such that $\mathrm{Diff}(B)$ is a singleton.

Corollary 5.8. Let T be a positive definite symmetric half-integral 4×4 matrix such that $\chi_T = 1$ and $\mathrm{Diff}(T) = \{p\}$. Let σ be either 1 or 2 according as $a_1 - a_2$ is odd or even. If $\deg \mathscr{Z}(T') \neq 0$, then

$$\left|\frac{C_4(T)}{-2^8 \cdot 3^2 \cdot \deg \mathscr{Z}(T')} - 1\right| < \frac{4}{p\sqrt{p}}\left(p^{-(a_4-3+\sigma)/2} + \frac{4p^{-(a_4-a_1)/2}}{a_1+1}\right),$$

where $\mathrm{GK}_p(T) = (a_1, a_2, a_3, a_4)$. In particular,

$$\left|\frac{C_4(T)}{-2^8 \cdot 3^2 \cdot \deg \mathscr{Z}(T')} - 1\right| < \frac{20}{p\sqrt{p}}, \qquad \lim_{e_p^T \to \infty} \frac{C_4(T)}{-2^9 \cdot 3^2 \cdot \deg \mathscr{Z}(T')} = 1.$$

Proof. By (2.12) and (2.13) of [26]

$$\begin{aligned} -p^{-2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2}) &\geq (a_1+1)p^{(a_1+a_2)/2}\left(\frac{a_3-a_2+2\sigma}{\sqrt{p}^{\sigma}} + \varepsilon_2^2\frac{a_3-a_2+1}{2}\right) \\ &\geq (a_1+1)p^{(a_1+a_2-(2-\sigma))/2}. \end{aligned}$$

Recall that if $\sigma = 1$, then $a_1 < a_2 \leq a_3 \leq a_4$ while if $\sigma = 2$, then $a_1 \leq a_2 < a_3 \leq a_4$. An examination of the proof of Lemma 5.7 confirms that

$$\begin{aligned} \left|\frac{F_p^{H'}(p^{-1})}{\sqrt{p}^{e_p^T}(p-1)}\right| &\leq \frac{a_1+1}{(p-1)^2}p^{(a_1+a_2-a_4+2)/2} + \frac{p^{a_1+a_2-(a_3+a_4+3\sigma)/2+4}}{(p-1)^2(p^3-1)} \\ &\quad + \varepsilon_2^2\frac{p^{2a_1+2-(a_3+a_4)/2}}{(p-1)^2(p+1)} + \varepsilon_2^2\frac{p^{a_1+a_2+1-(a_3+a_4)/2}}{(p-1)^2(p+1)} \\ &< 4p^{(a_1+a_2)/2-1}\{(a_1+1)p^{-a_4/2} + 2p^{-(a_4-a_1+3\sigma)/2} + 2\varepsilon_2^2 p^{-(a_4-a_1+1)/2}\}. \end{aligned}$$

Now our proof is completed by Theorem 1.3. □

References

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