Description and complexity of Non-markovian open quantum dynamics

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In open quantum system theory, unitary groups over system-environment Hilbert space corresponding to delta-function memory kernels, and thus generating Markovian system dynamics, are specified as the solution of a quantum stochastic differential equation. In this paper, we identify a larger class of non-Markovian memory kernels, described by complex-valued radon measures, and rigorously define their dynamics by constructing system-environment unitary groups corresponding to the memory kernels. We then consider $k-$local many-body non-Markovian systems and show that under physically reasonable assumptions on the total variation and smoothness of the memory kernels, their dynamics can be efficiently approximated on quantum computers thus providing a rigorous verification of the Extended Church-Turing thesis for non-Markovian open quantum systems.

I. INTRODUCTION

Quantum systems invariably interact with their environment, and any model describing their behaviour needs to capture this interaction. Often such systems are assumed to approximately be Markovian, wherein the environment does not retain any memory of the system. Markovian open quantum systems have been extensively studied in quantum information theory and quantum optics — a mathematically rigorous description of a unitary group over the system-environment Hilbert space which generates Markovian open quantum dynamics is provided in the theory of quantum stochastic calculus [1, 2] as the solution of a quantum stochastic differential equation. When evolved according to this unitary group, the system's reduced state satisfies the well-known Lindbladian master equation [3].

However, a number of quantum systems arising in solid-state physics [4–7], quantum optics [8–12] as well as quantum biology and chemistry [13–16] are not Markovian and the environment's memory needs to be explicitly taken into account. This opens up the question of formulating a mathematically rigorous description of non-Markovian dynamics. While it is generically expected that non-Markovian open quantum systems satisfy a generalized Nakajima-Zwazig [15, 17] or time-convolutionless [18–20] master equation, it is usually hard to obtain such a master equation explicitly except for when the system only weakly couples to its environment [21–24]. Given the difficulty of generalizing the Markovian master equation to the non-Markovian setting, a natural question to ask is if this generalization can be done with the unitary groups described by quantum stochastic differential equations.

This question has been addressed in several specific settings in the quantum optics literature — using standard theory of second quantization, unitary groups have been constructed for non-Markovian spin boson models with normalizable [25] as well as weakly interacting spin boson models with non-normalizable [26] form factors. Furthermore, for time-delayed feedback systems, the non-Markovian dynamics can be rigorously defined by expanding the system Hilbert space with time [27, 28]. In this paper, we identify a general class of non-perturbative models with memory kernels described by tempered radon measures [29], which are a natural generalization of delta functions arising in classical probability and distribution theory. We rigorously define a unitary group, which in general generates non-Markovian system dynamics, associated with a tempered radon measure. Our generalization ties together disparate models used for open-system dynamics within the same mathematical framework — as special cases, we recover Markovian dynamics, quantum systems with time-delay and feedback [8, 27, 30] and spin-boson models described by spectral density functions with vanishing high-frequency response [25]. A major difficulty in dealing with memory kernels in this class is that the Schroedinger's equation for the system-environment state of the resulting non-Markovian model is not guaranteed to have a solution. Our key technical contribution is to construct this unitary group via a regularization procedure — we use standard mollifiers to regularize the radon measure to obtain a non-Markovian model where the Schroedinger's equation has a guaranteed solution, show that the limit of this solution as the regularization is removed exists and hence defines the dynamics associated with the radon measure.

We next consider the simulatability of non-Markovian many-body dynamics, thus defined, on quantum computers. While the quantum simulatability of Markovian many-body open quantum systems [31–34], and many-body closed quantum systems[35–38] have been extensively studied, non-Markovian open quantum systems have remain relatively unexplored. A recent work developed an efficient quantum algorithm for non-Markovian dynamics where a chain Markovian dilation is known [39]. For spin-boson models with rapidly decaying spectral density functions, a Markovian dilation with finite Lieb-Robinson velocity has previously been established [40, 41] which implies quantum simulatability of the model. Here, we establish that the general class of non-Markovian many-body models with memory kernels described by tempered radon measures can be efficiently simulated on a quantum computer under physically motivated assumptions on (the growth of) the total variation of the memory kernel and its smoothness. Consequently, we show that the non-Markovian

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generalization of Markovian dynamics established in this paper is a model of physical system consistent with the Extended Church Turing thesis, according to which any physically reasonable model of a many-body system should be efficiently simulatable on quantum computers. The quantum algorithm for simulating non-Markovian dynamics relies on a Markovian dilation of non-Markovian dynamics using a Lanczos iteration, also referred to the star-to-chain transformation [42, 43], which has been previously analyzed for the Pauli-Fierz Hamiltonians [40, 41, 44], as well as for general models with distributional memory kernels but under the assumption of a finite particle emission rate into the environment [45]. The key technical contribution in our work that allows us to prove that the quantum algorithm is efficient (i.e. with run-time polynomial in system size *and* the evolution time) is to establish error bounds between the non-Markovian dynamics and its Markovian dilation which grows polynomial with system size and evolution time, as opposed to the previously known bounds which in the worst case grow exponentially with evolution time [45, 46].

II. SUMMARY OF RESULTS

We consider non-Markovian open quantum system models where the quantum system, with a finite-dimensional Hilbert space $\mathcal{H}_S \cong \mathbb{C}^d$, interacts with M bosonic baths, which individually are symmetric Fock spaces over $L^2(\mathbb{R})$. We will consider a Hamiltonian between the system and the environment that can be formally be written as

$$H = H_S(t) + \sum_{\alpha=1}^{M}\left(L_\alpha^\dagger A_\alpha(t) + L_\alpha A_\alpha^\dagger(t)\right), \qquad (1)$$

where $H_S(t)$ is the time-dependent Hamiltonian describing the system dynamics in the absence of its interaction with the environment, $\{L_\alpha \in \mathfrak{L}(\mathcal{H}_S)\}_{\alpha\in\{1,2\ldots M\}}$ [1] are system operators (which we will refer to as 'jump operators' to be consistent with the terminology for Markovian environments) and for $\alpha \in \{1, 2\ldots M\}$, $A_\alpha(t)$ is an operator that acts on the α^{th} bath can be formally expressed as

$$A_\alpha(t) = \int_{\mathbb{R}} \hat{v}_\alpha(\omega) a_{\alpha,\omega} e^{-i\omega t}\frac{d\omega}{\sqrt{2\pi}},$$

where $a_{\alpha,\omega}$ can be interpreted as the annihilation operator for frequency ω in the α^{th} bath, and $\hat{v}_\alpha(\omega)$, referred to as a 'coupling function' throughout this paper, describes the frequency-dependence of the interaction between the system and the α^{th} bath.

Of particular importance to the physics of non-Markovian systems described by the Hamiltonian in Eq. 1 is the kernel

$$\mu_\alpha \cong [A_\alpha(t), A_\alpha^\dagger(t')] = \int_{\mathbb{R}} |\hat{v}_\alpha(\omega)|^2 e^{-i\omega(t-t')}\frac{d\omega}{2\pi}.$$

For Markovian dynamics, $|\hat{v}_\alpha(\omega)| = \text{const}$ and $\mu_\alpha \cong \delta(t-t')$. In order to generalize this description to the non-Markovian setting, it is necessary to describe the family of kernels for which a dynamical group can be associated with the Hamiltonian in Eq. 1. A natural class of kernels which also contains the delta function, but will in general describe non-Markovian dynamics, is the set of *radon measures* $\mathcal{M}(\mathbb{R})$[29]. A radon measure μ is a map from the space of continuous and compactly supported functions ($\mathrm{C}_c^0(\mathbb{R})$) to complex numbers which is bounded in the sense that $\forall f \in \mathrm{C}_c^0(\mathbb{R})$ with support $\text{supp}(f) \subseteq \Omega \subseteq \mathbb{R}$

$$|\langle \mu, f\rangle| \leq \mathrm{TV}_\Omega(\mu)\sup_{x\in\Omega}|f(x)|,$$

with $\mathrm{TV}_\Omega(\mu)$ is defined to be the total variation of μ within the compact set Ω. Furthermore, we additionally assume that the radon measures that we consider have a fourier transform which is of at-most polynomial growth in frequency — this assumption stems from the physical intuition that the fourier transform of μ_α describes the spectral density function of the α^{th} bath, and will not grow extremely rapidly with ω in all problems of interest.

Several commonly occuring kernels fall into the class of radon measures — for instance, it is evident that the single delta function kernel is a tempered radon measure. Furthermore, kernels expressed as sum of multiple delta functions ($\mu \cong \sum_{i=1}^{P}\delta(t-t'-\tau_i)$ for some $\{\tau_1, \tau_2\ldots\tau_P\}$), which arise in the study of time-delay feedback systems [8, 27, 30] also fall into this class. As a final example, consider the kernels corresponding to square integrable coupling functions ($\hat{v} \in L^2(\mathbb{R})$) — these models arise in the study of cavity QED systems in quantum optics [47–52], or the non-Markovian spin-boson model [25]. Such kernels are also tempered radon measures with $\mathrm{TV}_\Omega(\mu) \leq \|\hat{v}\|_{L^2}^2\,\text{diam}(\Omega)$[2]. A detailed analysis of the properties of these specific radon measures is provided in section IV B.

A non-Markovian model can thus be specified by the system Hamiltonian, jump operators and the coupling functions in between system and the baths. These coupling functions are provided as a tempered radon transform, which specifies the magnitude of the coupling function, and the phase of the coupling function.

Definition 1 (Non-Markovian model)**.** *A non-Markovian open system model for a quantum system with Hilbert space $\mathcal{H}_S$ is specified by*

(a) A time-dependent system Hamiltonian $H_S(t) \in \mathfrak{L}(\mathcal{H}_S)$ which is Hermitian, norm continuous and differentiable in t,

[1] The space of linear bounded operators from $\mathcal{H}_S$ to $\mathcal{H}_S$ will be denoted by $\mathfrak{L}(\mathcal{H}_S)$.

[2] For $\Omega \subseteq \mathbb{R}$, $\text{diam}(\Omega) = \sup_{x,y\in\Omega}|x-y|$

(b) *A set of coupling functions $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha\in\{1,2\ldots M\}}$, where $\mu_1, \mu_2 \ldots \mu_M$ are tempered radon measures and $\varphi_1, \varphi_2 \ldots \varphi_M : \mathbb{R} \to \mathbb{C}$ specify the phase of the coupling functions,*

(c) *A set of bounded jump operators $\{L_\alpha \in \mathfrak{L}(\mathcal{H}_S)\}_{\alpha\in\{1,2\ldots M\}}$.*

We now turn to the question of defining the quantum dynamics corresponding to a non-Markovian model. In general, this cannot be done simply through the Schroedinger's equation, since for general radon measure kernel it isnt clear if a meaningful solution with the Hamiltonian in Eq. 1 exists. For the Markovian case, this is circumvented in the theory of quantum stochastic calculus by formulating $A_\alpha(t)dt$ and $A_\alpha^\dagger(t)dt$ as operator-valued Ito increments, and the rewriting Schroedinger's equation with the Hamiltonian in Eq. 1 a quantum stochastic differential equation whose solution can be shown to exist [1, 2]. For non-Markovian cases, we approach the problem of defining the associated quantum dynamics through a regularization procedure. An elementary but important observation that enables this regularization is that if the coupling functions are square integrable ($v_\alpha \in L^2(\mathbb{R})$), then the solution to Schroedinger's equation with Hamiltonian in Eq. 1 can be shown to exist using standard tools from the theory of non-autonomous differential equations on Banach spaces [53, 54] (for completeness, we provide a proof of this in section IV A). Now, a coupling function (μ, φ) can be approximated by a square integrable function by using a mollifier[3](smoothing function) ρ.

Definition 2 (Regularization)**.** *For $\varepsilon > 0$ and given a symmetric mollifier $\rho \in \mathrm{C}_c^\infty(\mathbb{R})$, an $\varepsilon, \rho-$regularization of a distributional coupling function (μ, φ) is a square integrable function $v_\varepsilon \in L^2(\mathbb{R})$ whose fourier transform[4] is given by $\hat{v}_\varepsilon \in L^2(\mathbb{R})$ is given by*

$$\hat{v}_\varepsilon(\omega) = \sqrt{\hat{\mu}(\omega)}\hat{\rho}(\omega\varepsilon)e^{i\varphi(\omega)}\ \forall\omega \in \mathbb{R},$$

where $\hat{\rho}$ is the fourier transform of ρ and $\hat{\mu}$ is the fourier transform of μ [5].

We note that since ρ is smooth and compact, its fourier transform $\hat{\rho}$ falls off faster than any polynomial in ω as $\omega \to \infty$ and thus v_ε is indeed a square integrable function. Equivalently, this regularization step can be considered as approximating the radon transform μ with another radon transform μ_ε whose action on a continuous compact function f is given by

$$\langle \mu_\varepsilon, f\rangle = \langle \mu, f \star \rho_\varepsilon \star \rho_\varepsilon\rangle,$$

where $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\varepsilon^{-1}x)$ and $\star$ denotes a convolution operation. It can be seen that as $\varepsilon \to 0$, $f \star \rho_\varepsilon \star \rho_\varepsilon$ becomes an increasingly better approximation to f, and thus μ_ε becomes an increasingly better approximation of μ.

Since the regularized coupling functions are square integrable, their associated dynamics can be computed by solving the Schroedinger's equation. We can now study the limit of this dynamics on removing the regularization ($\varepsilon \to 0$) — our first result shows that this limit exists, and is independent of the choice of the mollifier. The proof of this result, provided in section IV B, relies on an upper bound bound on the rate of change of two-point correlation functions of system observables that is uniform in the regularization parameter ε.

Theorem 1 (Informal, Non-markovian dynamics)**.** *Given a non-Markovian open system model (definition 1) with $U_{\varepsilon,\rho}(t, s)$ for $t, s \in \mathbb{R}$ being the propagator corresponding to an $\varepsilon, \rho-$regularization of its coupling functions, $\lim_{\varepsilon\to 0} U_{\varepsilon,\rho}(t, s)$ exists weakly[6] as an isometry from a dense subspace $\mathcal{D}$ of the system-environment Hilbert space and is independent of the choice of the mollifier ρ.*

This result thus establishes the well definition of dynamics of a non-Markovian system as specified in definition 1. Our next result considers the simultability of non-Markovian dynamics of many-body systems on quantum computers. The motivation for analyzing this problem is two fold — first, a number of non-Markovian physical systems are many-body in nature and their dynamics are expected to be hard to simulate on classical computers. Second, it is of interest to understand the relationship of the non-Markovian model in definition 1 with the Extended Church-Turing thesis, which posits that the dynamics of any physical system should be efficiently simulatable on a quantum computer.

To make further progress, we need two additional assumptions — one on the radon measures describing the coupling functions between the many-body system and the bath, and the second on the initial state in the environment.

Assumption 1 (Polynomial growth of Radon measure)**.** *The radon measure μ corresponding to the coupling function should satisfy:*

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[3] A mollifier ρ is a smooth compact function ($\rho \in \mathrm{C}_c^\infty(\mathbb{R})$) which is positive and with support $\mathrm{supp}(\rho) \subseteq [-1, 1]$ with

$$\int_{[-1,1]} \rho(x)dx = 1.$$

Unless otherwise mentioned, we will assume ρ to be a symmetric (even) function.

[4] We will assume the following convention for the fourier transform $\hat{v}$ of $v \in L^2(\mathbb{R})$:

$$\hat{v}(t) = \int_{\mathbb{R}} v(\omega)e^{-i\omega t}\frac{d\omega}{\sqrt{2\pi}}.$$

[5] For a tempered distribution $\mu \in \mathcal{S}'(\mathbb{R})$, the fourier transform $\hat{\mu}$ (if it exists as a function from $\mathbb{R}$ to $\mathbb{C}$) is defined by demanding that

$$\langle \mu, f\rangle = \int_{\mathbb{R}} \hat{\mu}(\omega)\hat{f}(\omega)\frac{d\omega}{\sqrt{2\pi}}$$

for all smooth compact functions $f \in \mathrm{C}_c^\infty(\mathbb{R})$.

[6] A single parameter family of operators $\{O_x : \mathcal{D} \to \mathcal{H}\}_{x\in[0,\infty)}$ is said to converge weakly (or converge in the weak topology) to $O : \mathcal{D} \to \mathcal{H}$ as $x \to 0$ if $\forall |\psi\rangle \in \mathcal{D}$, $\lim_{x\to 0} O_x |\psi\rangle = O |\psi\rangle$.

(a) *For any compact interval* $[a,b] \subseteq \mathbb{R}$, $\mathrm{TV}_{[a,b]}(\mu) \leq \mathrm{poly}(|a|,|b|)$ *and*

(b) *Given a compact interval* $[a,b] \subseteq \mathbb{R}$, $\exists \Delta^0_{\mu;[a,b]}(\varepsilon)$, $\Delta^1_{\mu;[a,b]}(\varepsilon)$ *which are locally integrable and of polynomial growth with respect to* a, b, *vanish polynomially as* $\varepsilon \to 0$ *and for any differentiable function* $f \in \mathrm{C}^1(\mathbb{R})$

$$\left|\langle \mu, f_{[a,b]} \star \rho_\varepsilon\rangle - \lim_{\varepsilon\to 0}\langle \mu, f_{[a,b]} \star \rho_\varepsilon\rangle\right| \leq \Delta^0_{\mu;[a,b]}(\varepsilon) \sup_{t\in[a,b]} |f(t)| + \Delta^1_{\mu;[a,b]}(\varepsilon) \sup_{t\in[a,b]} |f'(t)|$$

where $f_{[a,b]}(t) = f(t)$ *if* $t \in [a,b]$ *and* 0 *otherwise.*

Assumption 1(a), which constrains the growth of the total variation of the memory kernel, can be physically interpreted as limiting the amount of "memory" that the non-Markovian system can accumulate. Assumption 1(b) is a constraint on the smoothness of the memory kernel (i.e. the error incurred on smoothing the memory kernel with a mollifier), and it limits how rapidly the memory kernel diverges from its smooth approximations. We show in section IV B (Examples 1-3) that these assumptions are satisfied by a large class of coupling functions encountered in experimentally relevant physical systems.

Assumption 2 (Initial environment state)**.** *The initial environment state* $|\phi_1\rangle \otimes |\phi_2\rangle \dots |\phi_M\rangle$ *where for* $\alpha \in \{1,2\dots M\}$, $|\phi_\alpha\rangle \in \mathrm{Fock}[L^2(\mathbb{R})]$ *and*

(a) *for its* $n-$*particle wavefunctions* $\phi_{\alpha,n} \in L^2(\mathbb{R}^n)$, *and any* $j, k \geq 0$, $\exists \mathcal{N}_{j,k} > 0$ *such that*

$$\sum_{n=0}^{\infty} n^j \int_{\mathbb{R}^n} (1+\omega_1^2)^k \left|\phi_{\alpha,n}(\omega)\right|^2 d\omega < \mathcal{N}_{j,k}.$$

(b) *for* $v_1, v_2 \dots v_m \in L^2(\mathbb{R})$ *and* $P \in \mathbb{Z}_{>0}$, *all the amplitudes*

$$\langle \mathrm{vac}| \prod_{i=1}^{m} \left(\int_{\mathbb{R}} v_i(\omega) a_\omega d\omega\right)^{n_i} |\phi_\alpha\rangle$$

with $n_1 + n_2 \dots n_m \leq P$ *can be computed in* $\mathrm{poly}(m, P)$ *time on a classical or quantum computer.*

Assumption 2(a) demands that both high particle number or high frequency amplitude of the initial wavefunction vanishes superpolynomially. This assumption is reminiscent of assumption on particle number and energies of initial states made in studying the simulatability of quantum field theories [55, 56]. A number of commonly used initial environment states (such as thermal states) in physically relevant open systems have exponentially vanishing high energy amplitudes, and satisfy this assumption. Assumption 2(b) formalizes the expectation that the it should be efficient to represent a reasonable initial state computationally.

The computational problem of simulating $k-$local many-body non-Markovian dynamics can now be stated as

Problem 1 ($k-$local non-Markovian dynamics)**.** *Consider a system of* n *qudits* ($\mathcal{H}_S = \left(\mathbb{C}^d\right)^{\otimes n}$) *interacting with* $M = \mathrm{poly}(n)$ *baths with*

(a) *System Hamiltonian* $H_S(t)$ *is* $k-$*local i.e.* $H_S(t) = \sum_{i=1}^{N} H_i(t)$, *where* $N = \mathrm{poly}(n)$, *and for* $i \in \{1,2\dots N\}$, $H_i(t)$ *is an operator acting on atmost* k *qudits and* $\|H_i(t)\| \leq 1$.

(b) *Jump operators* $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ *such that for* $\alpha \in \{1,2\dots M\}$, L_α *acts on at-most* k *qudits and* $\|L_\alpha\| \leq 1$.

(c) *Coupling functions* $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha\in\{1,2\dots M\}}$ *such that for* $\alpha \in \{1,2\dots M\}$, μ_α *satisfies the polynomial growth conditions (assumption 1).*

(d) *An initial state* $|\Psi\rangle = |0\rangle^{\otimes n} \otimes |\Phi\rangle$, *where* $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots |\phi_M\rangle$ *is an initial state which satisfies assumption 2.*

Denoting by $\rho_S(t)$ *the reduced state of the system at time* t *for this non-Markovian model, then for* $t = \mathrm{poly}(n)$, *prepare a quantum state* $\hat{\rho}$ *such that* $\|\hat{\rho} - \rho_S(t)\|_{\mathrm{tr}} \leq 1/\mathrm{poly}(n)$.

The key ingredient to analyzing this problem is a Markovian dilation of the non-Markovian model, which identifies a finite number of modes in the environment's Hilbert space and then approximates the non-Markovian model by a Hamiltonian simulation of the system only interacting with these modes. We make this precise in the definition below.

Definition 3 (Markovian Dilation)**.** *Given a non-Markovian open system model (definition 1) with* M *baths, a Markovian dilation with* N_m *modes and bandwidth* B *has a Hilbert space* $\mathcal{H}_S \otimes \mathrm{Fock}[L^2(\mathbb{R})]^{\otimes M}$ *and Hamiltonian*

$$H(t) = H_S(t) + \sum_{\alpha=1}^{M}\left(g_\alpha a_{\alpha,1} L_\alpha^\dagger + \sum_{i,j=1}^{N_m} t_{\alpha,i,j} a^\dagger_{\alpha,i} a_{\alpha,j} + \mathrm{h.c.}\right),$$

where for $\alpha \in \{1,2\dots M\}, j \in \{1,2\dots N_m\}$

(a) $a_{\alpha,j} = \int_{\mathbb{R}} \varphi^*_{\alpha,j}(\omega) a_{\alpha,\omega} d\omega$ *is the annihilation operator corresponding to the* j^{th} *mode of the* α^{th} *bath described by orthonormal mode functions* $\varphi_{\alpha,j}$ ($\langle \varphi_{\alpha,i}, \varphi_{\alpha,j}\rangle = \delta_{i,j}$),

(b) *The parameters* $|g_\alpha|, |t_{\alpha,i,j}| \leq B$.

Our next lemma, which is used in the analysis of problem 1, uses the well-known star-to-chain transformation [42, 43] to systematically construct a Markovian dilation to the non-Markovian system. We analyze the error between the dynamics of the non-Markovian system and its Markovian dilation and estimate the number of modes and bandwidth of a Markovian dilation needed to approximate the non-Markovian model.

Lemma 1 (Markovian dilation)**.** *Consider a non-Markovian model specified by a system Hamiltonian* $H_S(t)$, *jump operators* $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ *and coupling* *functions* $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha\in\{1,2\ldots M\}}$ *where* μ_α *satisfy assumption 1 with* $\hat{\mu}_\alpha(\omega) < O(\omega^{2k})$ *for some* $k > 0$. *For* $|\Psi_0\rangle := |\sigma\rangle \otimes |\Phi_0\rangle \in \mathcal{H}_S \otimes \text{Fock}[L^2(\mathbb{R})]^{\otimes M}$, *where* $|\sigma\rangle \in \mathcal{H}_S$ *and* $|\Phi_0\rangle$ *is an initial environment state that satisfies assumption 2, then* $\exists$ *a Markovian dilation of the non-Markovian model with*

$$N_m, B \leq O\bigg(\text{poly}\bigg(\frac{1}{\epsilon}, t, M, \sup_\alpha \|L_\alpha\|, \sup_{\alpha, s\in[0,t]} \|[H_S(s), L_\alpha]\|, \mathcal{N}_{1,k+1}, \mathcal{N}_{1,k+2}, \mathcal{N}_{1,0}\bigg)\bigg)$$

whose system-environment state at time t *is within* ϵ *norm distance of the exact state.*

Our analysis of the Markovian dilation, detailed in section V A, is done in three steps. First, we analyze the error incurred in regularizing the non-Markovian model with a mollifier ρ, then we introduce a sharp frequency cutoff on the resulting regularized square-integrable functions. One the key technical contribution in our analysis of the mollification and frequency cutoff is to prove bounds on error that grow only polynomially with time, which improves previous bounds that, in the worst case, grow exponentially with time [45, 46]. After introduction of this cut-off, we perform a star-to-chain transformation — the analysis of this step closely follows previous works that have studied the convergence of the star-to-chain transformation for the spin-boson models with a hard frequency cutoff [40, 41, 44].

Using this lemma, we can map problem 1 into a Hamiltonian simulation problem with a finite number of modes. This problem is still infinite-dimensional — however, we can easily show that the moments of the particle number operator for the environment can grow at-most polynomially with the problem size n. Therefore, we can truncate the Hilbert space of this model and obtain a finite-dimensional Hilbert space — an application of the sparse Hamiltonian lemma [57] then yields the the second main result of our paper. A detailed proof of this theorem is provided in section V B

Theorem 2 ($k-$local Non-Markovian dynamics $\in$ BQP)**.** *Problem 1 can be solved in* $\text{poly}(n)$ *time on a quantum computer.*

The remainder of this paper is organized as follows — Section IV is devoted to establishing the well definition of a non-Markovian model associated with a memory kernel that is a tempered radon transform. In section V, we rigorously develop and analyze a Markovian dilation of these models based on the star-to-chain transformation, and then use it to study the simulatability of many-body non-Markovian dynamics on quantum computers.

III. NOTATION AND PRELIMINARIES

This section describes the notation used throughout this paper. For the convenience of the reader, and for the sake of completeness, we also collect some basic definitions and facts from the theory of function spaces and analysis that we use in this paper — the interested reader can refer to Refs. [58, 59] for more detailed discussion.

General: For $x := (x_1, x_2 \ldots x_n) \in \mathbb{R}^n, y := (y_1, y_2 \ldots y_m) \in \mathbb{R}^m$ we will denote by $(x, y) \in \mathbb{R}^{n+m}$ defined by $(x, y) = (x_1, x_2 \ldots x_n, y_1, y_2 \ldots y_m)$. For an ordered subset B of $\{1, 2 \ldots n\}$ and $x \in \mathbb{R}^n$, $Bx = (x_{B(1)}, x_{B(2)} \ldots)$. We will denote by α^n the $n-$element constant vector $(\alpha, \alpha \ldots \alpha)$, and by α^∞ the constant sequence $(\alpha, \alpha, \alpha \ldots)$.

Function spaces and analysis: Throughout this paper, all integrals over $\mathbb{R}^n$ will be Lesbesgue integrals with respect to the Lesbesgue measure over $\mathbb{R}^n$. Two measurable functions $f, g : \mathbb{R}^n \to \mathbb{C}$ are equal almost everywhere, denoted by $f =_{a.e.} g$, if the set $\{x | f(x) \neq g(x)\}$ is a zero measure set. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and a measurable set $\Omega \subseteq \mathbb{R}^n$, $\text{ess sup}_{x\in\Omega} f(x) = c$ if the set $\{x \in \Omega | f(x) > c\}$ is a zero measure set. For $p \geq 1$, we will denote by

$$L^p(\mathbb{R}^n) = \bigg\{f : \mathbb{R} \to \mathbb{C} \bigg| \|f\|_{L^p} < \infty\bigg\} \bigg/ =_{a.e.} \text{ where } \|f\|_{L^p} := \bigg(\int_{\mathbb{R}^n} |f(x)|^p dx\bigg)^{1/p},$$

and

$$L^\infty(\mathbb{R}^n) = \bigg\{f : \mathbb{R} \to \mathbb{C} \bigg| \|f\|_{L^\infty} < \infty\bigg\} \text{ where } \|f\|_{L^\infty} = \text{ess sup}_{x\in\mathbb{R}^n} |f(x)|.$$

A map $f : \mathbb{R} \to \mathbb{C}$ is said to be a compactly supported function with support $\text{supp}(f) \subseteq \mathbb{R}$ if $\overline{\text{supp}(f)}$ is compact, and the set $\{x \in \mathbb{R} \setminus \text{supp}(f) | f(x) \neq 0\}$ is a zero measure set. For $k \in \mathbb{Z}_{\geq 0}$, we will denote by $\text{C}^k(\mathbb{R})$ the set of $k-$differentiable functions (with $k = 0$ being continuous functions, and $k = \infty$ being smooth) from $\mathbb{R}$ to $\mathbb{C}$, and by $\text{C}^k_c(\mathbb{R})$ the set of such functions with compact support.

A function $\rho \in \text{C}^\infty_c(\mathbb{R})$ is said to be a mollifier if $\rho(x) \geq 0\ \forall x \in \mathbb{R}$ and $\int_{\mathbb{R}} \rho(x)dx = 1$ — unless otherwise mentioned, we will assume that $\text{supp}(\rho) = [-1, 1]$. The mollifier is symmetric if $\rho(x) = \rho(-x)\ \forall x \in \mathbb{R}$. Given a mollifier ρ and $\varepsilon > 0$, we will denote by $\rho_\varepsilon \in \mathrm{C}_c^\infty(\mathbb{R})$ the map

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)\ \forall\, x \in \mathbb{R}.$$

Note that ρ_ε is also a mollifier with $\text{supp}(\rho_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$. Given a subset $\Omega \subseteq \mathbb{R}$, its indicator function $\mathcal{I}_\Omega : \mathbb{R} \to \mathbb{C}$ is defined by

$$\mathcal{I}_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

A linear map $\mu : \mathrm{C}_c^0(\mathbb{R}) \to \mathbb{C}$ is a radon measure if $\forall \Omega \subset \mathbb{R}$ which are compact, $\exists u_\Omega > 0$ such that

$$\langle \mu, f\rangle \leq u_\Omega \sup_{x\in\mathbb{R}} |f(x)|\ \forall f \in \mathrm{C}_c^0(\mathbb{R}) \text{ with } \text{supp}(f) \subseteq \Omega.$$

The smallest such u_Ω is defined to be the total variation of μ in Ω and will be denoted by $\mathrm{TV}_\Omega(\mu)$. The set of all radon measures will be denoted by $\mathcal{M}(\mathbb{R})$. By the Lesbesgue decomposition theorem [60], any $\mu \in \mathcal{M}(\mathbb{R})$ can be uniquely expressed as

$$\mu = \mu_c + \mu_d,$$

where $\mu_c \in \mathcal{M}(\mathbb{R})$ is called the continuous part of μ and μ_a is called its atomic part. The continuous part can be characterized by a function $\phi_c \in \mathrm{C}^0(\mathbb{R})$ where $\forall f \in \mathrm{C}_c^1(\mathbb{R})$,

$$\langle \mu_c, f\rangle = -\int_{\mathbb{R}} f'(x)\phi_c(x)dx.$$

The function ϕ_c is also often denoted by $\phi_c(x) = \mu_c((-\infty, x])$ to be consistent with the 'cumulative function' of μ_c in the measure-theoretic definition of the radon measure. μ_c can further be decomposed into an absolutely continuous part, which can be described by a density function, and a Cantor part — we will not require this decomposition in this paper. The atomic part, μ_a, which can be expressed as

$$\mu_a \cong \sum_{i\in I} a_i\delta(x - x_i),$$

for some finite or countably infinite sequence $\{a_i \in \mathbb{C}\}_{i\in I}$ and $\{x_i \in \mathbb{R}\}_{i\in I}$ such that for any compact $\Omega \subseteq \mathbb{R}$,

$$\sum_{i\in I|x_i\in\Omega} |a_i| < \infty.$$

For any compact $\Omega \subseteq \mathbb{R}$, it can be shown that

$$\mathrm{TV}_\Omega(\mu) = \mathrm{TV}_\Omega(\mu_c) + \mathrm{TV}_\Omega(\mu_a) \text{ where } \mathrm{TV}_\Omega(\mu_c) = \sup_{\substack{f\in\mathrm{C}_c^1(\mathbb{R})\\ \|f\|_{L^\infty}=1}} \left|\int_{\mathbb{R}} \phi_c(x)f'(x)dx\right| \text{ and } \mathrm{TV}_\Omega(\mu_d) = \sum_{i\in I|x_i\in\Omega} |a_i|.$$

We will use standard notation for Schwartz space, $\mathcal{S}(\mathbb{R})$ and tempered distributions by $\mathcal{S}'(\mathbb{R})$. Note that every radon measure is a distribution (i.e. a continuous map from compact smooth function to complex numbers) — a radon measure $\mu \in \mathcal{M}(\mathbb{R})$ which is additionally a tempered distribution will be called a tempered radon measure. From the Schwartz representation theorem [61], it follows that any tempered distribution can be expressed as

$$\langle \mu, f\rangle = \sum_{\alpha=0}^{m} \int_{\mathbb{R}} u_\alpha(\omega)\frac{\partial^\alpha}{\partial\omega^\alpha}\hat{f}(\omega)\frac{d\omega}{\sqrt{2\pi}}\ \forall\ f \in \mathcal{S}(\mathbb{R}),$$

where $\hat{f}$ is the fourier transform of f, and $u_0, u_1 \ldots u_m$ are continuous functions of at-most polynomial growth. Of particular interest will be distributions which contain only the term corresponding to $\alpha = 0$ i.e.

$$\langle \mu, f\rangle = \int_{\mathbb{R}} \hat{\mu}(\omega)\hat{f}(\omega)\frac{d\omega}{\sqrt{2\pi}}\ \forall\ f \in \mathcal{S}(\mathbb{R}),$$

where $\hat{\mu}$ is a continuous function of at-most polynomial growth. Such a distribution will be called a tempered distribution with a fourier transform being a function of at-most polynomial growth, and $\hat{\mu}$ will be referred to as the fourier transform of μ.

Given a Banach space X and an operator $O : X \to X$, the operator norm will be denoted by $\|O\| = \sup_{x\in X} \|Ox\| / \|x\|$. The space of bounded linear operators on a Banach space X will be denoted by $\mathfrak{L}(X)$ i.e. $\mathfrak{L}(X) = \{O : X \to X | \, \|O\| < \infty\}$. A map $F : \mathbb{R} \to \mathfrak{L}(X)$ is

$$\text{norm continuous at } t \text{ if } \lim_{s\to t} \|F(t) - F(s)\| = 0 \text{ and strongly continuous at } t \text{ if } \forall x \in X, \lim_{s\to t} \|F(t)x - F(s)x\| = 0.$$

and

$$\text{norm differentiable at } t \text{ if } \exists F'(t) : \lim_{s\to t} \left\| F'(t) - \frac{F(t) - F(s)}{t - s} \right\| = 0 \text{ and}$$
$$\text{strongly differentiable at } \text{ if } \exists F'(t) : \forall x \in X \lim_{s\to t} \left\| F'(t)x - \frac{F(t)x - F(s)x}{t - s} \right\| = 0.$$

Note that if X is finite dimensional, then the notion of norm continuity/differentiability and strong continuity/differentiability are equivalent. A sequence $\{\mu_n : X \to \mathbb{C}\}_{n\in\mathbb{N}}$ weakly converges to $\mu^* : X \to \mathbb{C}$, denoted by $\mu^* = \text{wlim}_{n\to\infty}\mu_n$, if $\forall x \in X, \mu^* x = \lim_{n\to\infty} \mu_n x$.

Given a Hilbert space $\mathcal{H}$, a densely defined operator $O : \text{dom}[O] \to \mathcal{H}$ is said to be closed if $\forall \psi \in \text{dom}[O]$, such that $\forall$ sequences $\{\psi_n \in \text{dom}[O]\}_{n\in\mathbb{N}}$ which converge to 0 such that the sequence $\{O\psi_n \in \mathcal{H}\}_{n\in\mathbb{N}}$ also converges, $\lim_{n\to\infty} O\psi_n = 0$. A densely defined operator O is said to be closable if it has a closed extension, called the closure of the operator and denoted by $\overline{O}$. We will use the following property of the domain of the closure, $\text{dom}[\overline{O}]$: $\psi \in \text{dom}[\overline{O}]$ if and only if $\exists$ a sequence $\{\psi_n \in \text{dom}[O]\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \psi_n = \psi$, and the sequence $\{O\psi_n \in \mathcal{H}\}_{n\in\mathbb{N}}$ also converges. Furthermore if O is closable, the limit of the sequence $\{O\psi_n \in \mathcal{H}\}_{n\in\mathbb{N}}$ is independent of the sequence $\{\psi_n \in \mathcal{H}\}_{n\in\mathbb{N}}$, and is equal to $\overline{O}\psi$. The adjoint of a densely defined operator $O : \text{dom}[O] \to \mathcal{H}$ is an operator $O^\dagger : \text{dom}[O^\dagger] \to \mathcal{H}$ where $\text{dom}[O^\dagger] = \{\psi \in \mathcal{H} | \langle \psi, O\cdot\rangle : \text{dom}[O] \to \mathbb{C} \text{ is bounded}\}$ and by the Riesz' representation theorem, $\forall \psi \in \mathcal{H}, O^\dagger\psi$ is identified as the unique vector which satisfies $\langle O^\dagger\psi, \phi\rangle = \langle \psi, O\phi\rangle \; \forall\phi \in \text{dom}[O]$. An operator is self adjoint if $\text{dom}[O] = \text{dom}[O^\dagger]$. A closable operator is essentially self adjoint if it has a self adjoint extension, which then coincides with its closure.

Fock Spaces: For a separable Hilbert space $\mathcal{H}$, and $n \in \mathbb{Z}_{\geq 1}$, we will denote by $\text{Sym}_n(\mathcal{H}) \subseteq \mathcal{H}^{\otimes n}$ the set of symmetric (permutationally invariant) states in $\mathcal{H}^{\otimes n}$. We will denote by $\text{Fock}[\mathcal{H}] := \mathbb{C} \oplus \bigoplus_{n\in\mathbb{Z}_{\geq 1}} \text{Sym}_n(\mathcal{H})$ the symmetric (bosonic) Fock space generated by $\mathcal{H}$. We will denote by $\Pi_n : \text{Fock}[\mathcal{H}] \to \text{Fock}[\mathcal{H}]$ the projector onto $\text{Sym}_n(\mathcal{H})$ (i.e. the n particle sector), by $\Pi_{\leq n} := \sum_{i=0}^{n} \Pi_i$ and by $\Pi_{>n} = \text{id} - \Pi_{\leq n}$.

We will denote by $\text{F}_\infty[\mathcal{H}] \subseteq \text{Fock}[\mathcal{H}]$ the space of all states with a finite number of particles i.e.

$$\text{F}_\infty[\mathcal{H}] = \left\{ |\Psi\rangle \in \text{Fock}[\mathcal{H}] \middle| \exists N_0 \in \mathbb{N} \text{ such that } \Pi_n |\Psi\rangle = 0 \; \forall n > N_0 \right\},$$

by $\text{F}_k(\mathcal{H}) \subseteq \text{Fock}[\mathcal{H}]$ for $k \in \mathbb{Z}_{\geq 1}$

$$\text{F}_k[\mathcal{H}] = \left\{ |\Psi\rangle \in \text{Fock}[\mathcal{H}] \middle| \sum_{n=0}^{\infty} n^k \langle\Psi| \Pi_n |\Psi\rangle < \infty \right\},$$

and by $\text{F}_{\mathcal{S}}[\mathcal{H}]$ the space

$$\text{F}_{\mathcal{S}}[\mathcal{H}] := \bigcap_{k=1}^{\infty} \text{F}_k[\mathcal{H}] = \left\{ |\Psi\rangle \in \text{Fock}[\mathcal{H}] \middle| \sum_{n=0}^{\infty} n^k \langle\Psi| \Pi_n |\Psi\rangle < \infty \; \forall k \in \mathbb{Z}_{\geq 1} \right\}.$$

We remark that $\text{F}_\infty[\mathcal{H}]$, $\text{F}_{\mathcal{S}}[\mathcal{H}]$ and $\text{F}_k[\mathcal{H}]$ (for any $k \in \mathbb{Z}_{\geq 1}$) are dense in $\text{Fock}[\mathcal{H}]$. For $|\Psi\rangle \in \text{F}^k[\mathcal{H}]$, we will denote by $\mu^{(k)}_{|\Psi\rangle}$ the k^{th} moment of the photon number operator i.e.

$$\mu^{(k)}_{|\Psi\rangle} = \sum_{n=0}^{\infty} n^k \langle\Psi| \Pi_n |\Psi\rangle \, .$$

For any $v \in \mathcal{H}$, we will denote by a_v^- and a_v^+ the corresponding annhilation and creation operator. These operators can be explicitly defined over the domain $\mathrm{F}_\infty[\mathcal{H}]$ — $a_v^-:\mathrm{F}_\infty[\mathcal{H}]\to\mathrm{Fock}[\mathcal{H}]$ is an operator defined by

$$a_v^-(\alpha,0^\infty)=0\;\forall\,\alpha\in\mathbb{C},$$
$$a_v^-(0^n,u^{\otimes n},0^\infty)=(0^{n-1},\sqrt{n}\langle v,u\rangle u^{\otimes n-1},0^\infty)\;\forall\,u\in\mathcal{H},n\in\mathbb{Z}_{\geq 1}.$$

Since for every $n\in\mathbb{Z}_{\geq 1}$, the set $\mathrm{span}\{u^{\otimes n}|u\in\mathcal{H}\}$ is dense in $\mathrm{Sym}_n(\mathcal{H})$, and when domain-restricted to $\mathrm{span}\{u^{\otimes n}|u\in\mathcal{H}\}$, a_v^- as defined above is a bounded operator, it can be uniquely extended to $\mathrm{Sym}_n(\mathcal{H})$ as a consequence of the bounded linear transformation theorem, and then extended to $\mathrm{F}_\infty[\mathcal{H}]$ by linearity. Similarly, $a_v^+:\mathrm{F}_\infty[\mathcal{H}]\to\mathrm{Fock}[\mathcal{H}]$ is defined via

$$a_v^+(\alpha,0^\infty)=(0,\alpha v,0^\infty)\;\forall\,\alpha\in\mathbb{C},$$
$$a_v^+(0^n,u^{\otimes n},0^\infty)=\left(0^{n+1},\frac{1}{\sqrt{n+1}}\sum_{i=0}^{n}u^{\otimes i}\otimes v\otimes u^{\otimes(n-i)},0^\infty\right)\;\forall\,u\in\mathcal{H},n\in\mathbb{Z}_{\geq 1}.$$

As with a_v^-, this definition of a_v^+ can be extended uniquely to F_0.

In this paper, we will encounter finite tensor products of Fock spaces. Given a Hilbert space $\mathcal{H}$, $\mathrm{Fock}[\mathcal{H}]^{\otimes M}\simeq\mathrm{Fock}\big[\mathcal{H}^{\oplus M}\big]$, where the tensor product is taken as a tensor product over Hilbert spaces. We will use the notation $\mathrm{F}_\infty^M[\mathcal{H}]=\mathrm{F}_\infty[\mathcal{H}^{\oplus M}]$ and $\mathrm{F}_k^M[\mathcal{H}]=\mathrm{F}_k[\mathcal{H}^{\oplus M}]$ for $k\in\mathbb{Z}_{\geq 1}$. For $\alpha\in\{1,2\dots M\}$ and $v\in\mathcal{H}$, we define $a_{\alpha,v}^-:\mathrm{F}_\infty^M[\mathcal{H}]\to\mathrm{Fock}[\mathcal{H}]^{\otimes M}$ via

$$a_{\alpha,v}^-(\alpha,0^\infty)=0\;\forall\,\alpha\in\mathbb{C}$$
$$a_{\alpha,v}^-(0^n,u^{\otimes n},0^\infty)=(0^{n-1},\sqrt{n}\langle v_\alpha,u\rangle u^{\otimes(n-1)},0^\infty)\;\forall u\in\mathcal{H}^{\oplus M},n\in\mathbb{Z}_{\geq 1}.$$

where $v_\alpha=0^{\oplus(\alpha-1)}\oplus v\oplus 0^{\oplus(M-\alpha)}$. Similarly, we define $a_{\alpha,v}^+:F_\infty^M[\mathcal{H}]\to\mathrm{Fock}[\mathcal{H}]^{\otimes M}$ via

$$a_{\alpha,v}^+(c,0^\infty)=(0,cv_\alpha,0^\infty)\;\forall\,c\in\mathbb{C},$$
$$a_{\alpha,v}^+(0^n,u^{\otimes n},0^\infty)=\left(0^{n+1},\frac{1}{\sqrt{n+1}}\sum_{i=0}^{n}u^{\otimes i}\otimes v_\alpha\otimes u^{\otimes(n-i)},0^\infty\right)\;\forall\,u\in\mathcal{H},n\in\mathbb{Z}_{\geq 1}.$$

We will denote by $\Pi_n:\mathrm{Fock}[\mathcal{H}]^{\otimes M}\to\mathrm{Fock}[\mathcal{H}]^{\otimes M}$ the projector onto $\mathrm{Sym}_n(\mathcal{H}^{\oplus M})$, by $\Pi_{\leq n}:=\sum_{i=0}^n\Pi_i$ and by $\Pi_{>n}:=\mathrm{id}-\Pi_{\leq n}$.

IV. WELL-DEFINITION OF NON-MARKOVIAN MODELS

A. Square integrable coupling functions

In this section, we consider the simple case of square integrable coupling functions, and show that the solution to the (time-dependent) Schroedinger's equation corresponding to the non-Markovian model exists. We will consider an environment with M baths, which are individually bosonic Fock spaces over $L^2(\mathbb{R})$. We will assume the system to be finite-dimensional. Denoting the Hilbert space of the system by $\mathcal{H}_S$, the Hilbert space of the system and environment will be $\mathcal{H}=\mathcal{H}_S\otimes\mathrm{Fock}[L^2(\mathbb{R})]^{\otimes M}$. The basic data needed to specify a non-Markovian model with square-integrable coupling functions is provided in the definition below.

Definition 4. *A non-Markovian open system model for a quantum system with Hilbert space $\mathcal{H}_S$ with square integrable system-environment coupling functions is specified by*

(a) A time-dependent system Hamiltonian $H_S(t)\in\mathfrak{L}(\mathcal{H}_S)$ which is Hermitian, norm continuous and differentiable in t.

(b) M square integrable functions $\{v_\alpha\in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$,

(c) M bounded operators on the system Hilbert space $\{L_\alpha\in\mathfrak{L}(\mathcal{H}_S)\}_{\alpha\in\{1,2\dots M\}}$.

(d) M strongly continuous single-parameter unitary groups on $L^2(\mathbb{R})$, $\{\tau_{\alpha,t}:L^2(\mathbb{R})\to L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$.

Definition 5. *For a non-Markovian model with square integrable function as specified by definition 4 and for $t\in\mathbb{R}$, $H(t):\mathcal{H}_S\otimes\mathrm{F}_\infty^M[L^2(\mathbb{R})]\to\mathcal{H}$ via*

$$H(t)=H_S(t)+\sum_{\alpha=1}^{M}\big(L_\alpha^\dagger a_{\alpha,\tau_{\alpha,t}v_\alpha}^-+L_\alpha a_{\alpha,\tau_{\alpha,t}v_\alpha}^+\big).$$

For convenience throughout this section, we will define ℓ to be the constant

$$\ell = \sum_{\alpha=1}^{M} \|v_\alpha\|_{L^2} \|L_\alpha\| . \tag{2}$$

Lemma 2. *For all $t \in \mathbb{R}$,*

(a) $H(t) : \mathcal{H}_S \otimes \mathrm{F}_\infty^M[L^2(\mathbb{R})] \to \mathcal{H}$ is essentially self adjoint.

(b) $H(t)$ is closable, and if $\overline{H(t)} : \mathrm{dom}[\overline{H(t)}] \to \mathcal{H}$ is its closure then $\mathcal{H}_S \otimes \mathrm{F}_1^M[L^2(\mathbb{R})] \subseteq \mathrm{dom}[\overline{H(t)}]$ and $\forall \, |\Psi\rangle \in \mathrm{dom}[\overline{H(t)}]$, $\overline{H(t)} \, |\Psi\rangle = \sum_{n=0}^{\infty} H(t)\big(\Pi_n \, |\Psi\rangle\big)$.

Proof:
(a) is shown using Nelson's analytic vector theorem (Theorem X.39 of Ref. [59]), and showing that all the vectors in $\mathcal{H}_S \otimes \mathrm{F}_0[L^2(\mathbb{R})]^{\otimes M}$ are analytic vectors of $H_S(t)$. We note that it follows easily from the definition of $H(t)$ that for every $n \in \mathbb{Z}_{\geq 0}$, $H(t)(\mathrm{id} \otimes \Pi_{\leq n})$ is a bounded operator and

$$\big\| H(t)\big(\mathrm{id} \otimes \Pi_{\leq n}\big) \big\| \leq \|H_S(t)\| + 2\ell\sqrt{n+1}.$$

Recall that given an operator $O : \mathrm{dom}[O] \to \mathcal{H}$, a vector x is an analytic vector of O if $\sum_{n=0}^{\infty} t^n \|O^n x\| / n! < \infty \; \forall \; t \in \mathbb{R}$. Let $|\Psi\rangle \in \mathcal{H}_S \otimes \mathrm{F}_0[L^2(\mathbb{R})]^{\otimes M}$ and let $N_0 \in \mathbb{Z}_{\geq 1}$ be the number of particles in the environment (i.e. $\Pi_{>N_0} |\Psi\rangle = 0$). It then immediately follows that for any $k \in \mathbb{Z}_{\geq 0}$, $H^k(t) \, |\Psi_0\rangle$ has at most $N_0 + k$ particles, and thus

$$\big\| H^k(t) \, |\Psi_0\rangle \big\| \leq \big(\|H_S(t)\| + 2\sqrt{N_0 + k + 1}\ell \big)^k \, \||\Psi_0\rangle\| \leq 2^k \big(\|H_S(t)\|^k + (N_0 + k + 1)^{k/2} \ell^k \big) \, \||\Psi_0\rangle\| ,$$

and thus for $t \geq 0$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t^k}{k!} \big\| H_S^k(t) \, |\Psi_0\rangle \big\| &\leq e^{2t\|H_S(t)\|} \, \||\Psi_0\rangle\| + \sum_{k=0}^{\infty} \frac{(2\ell t)^k}{k!} (N_0 + k + 1)^{k/2} \, \||\Psi_0\rangle\| , \\ &\leq e^{2t\|H_S(t)\|} \, \||\Psi_0\rangle\| + \||\Psi_0\rangle\| + \sum_{k=1}^{\infty} \frac{(2e\ell t)^k}{k^{k/2}} \left(1 + \frac{N_0 + k}{k} \right)^{k/2} \, \||\Psi_0\rangle\| . \end{aligned}$$

Wherein we have used the Stirling's approximation in the last estimate. The summation can now be seen to converge for any t and hence $|\Psi_0\rangle$ is an analytic vector of $H(t)$.

(b) We first consider a sequence $\{|\Psi_i\rangle \in \mathcal{H}_S \otimes \mathrm{F}_\infty^M(L^2(\mathbb{R}))\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} |\Psi_i\rangle = 0$ and the sequence $\{H(t) \, |\Psi_i\rangle\}_{i \in \mathbb{N}}$ converges. We first show that under these conditions, $\lim_{i \to \infty} H(t) \, |\Psi_i\rangle = 0$ — to see this, assume the contrary i.e. $\lim_{i \to \infty} H(t) \, |\Psi_i\rangle = |\Phi\rangle \neq 0$. Since $|\Phi\rangle \neq 0$, $\exists N > 0$, $\big(\mathrm{id} \otimes \Pi_{\leq N}\big) \, |\Phi\rangle \neq 0$. Note that

$$\big\| \big(\mathrm{id} \otimes \Pi_{\leq N}\big) H(t) \big\| \leq \|H_S(t)\| + 2\ell\sqrt{N+1},$$

and therefore $\Pi_{\leq N} \, |\Phi\rangle = \lim_{i \to \infty} \Pi_{\leq N} H(t) \, |\Psi_i\rangle = \Pi_{\leq N} H(t) \lim_{i \to \infty} |\Psi_i\rangle = 0$, where we have used that $\Pi_{\leq N}, \Pi_{\leq N} H(t)$ are bounded operators to swap the order of limits. Thus, we contradict our original assumption of $|\Phi\rangle \neq 0$ and hence $|\Phi\rangle = 0$. This shows that the operator $H(t)$ is closable.

Next, we consider $|\Psi\rangle \in \mathcal{H}_S \otimes \mathrm{F}_1^M[L^2(\mathbb{R})]$ — we consider the sequence $\{|\Psi_n\rangle := \big(\mathrm{id} \otimes \Pi_{\leq n}\big) \, |\Psi\rangle\}_{n \in \mathbb{N}}$ which converges to $|\Psi\rangle$. Furthermore, we note that the sequence $\{H(t) \, |\Psi_n\rangle\}_{n \in \mathbb{N}}$ also converges, and converges to $\sum_{m=0}^{\infty} H(t)\Pi_m \, |\Psi\rangle$ since

$$\left\| H(t) \, |\Psi_n\rangle - \sum_{m=0}^{\infty} H(t)\Pi_m \, |\Psi\rangle \right\| \leq \|H_S(t)\| \left(\sum_{m=n+1}^{\infty} \|\Pi_m \, |\Psi\rangle\|^2 \right)^{1/2} + 2\ell \left(\sum_{m=n+1}^{\infty} (m+1) \, \|\Pi_m \, |\Psi\rangle\|^2 \right)^{1/2},$$

and since $\||\Psi\rangle\| < \infty$ and $\mu^{(1)}_{|\Psi\rangle} < \infty$, $\sum_{m=n}^{\infty} \|\Pi_m \, |\Psi\rangle\|^2 , \sum_{m=n}^{\infty} m \, \|\Pi_m \, |\Psi\rangle\|^2 \to 0$ as $n \to \infty$. Consequently, we obtain that $|\Psi\rangle \in \mathrm{dom}[\overline{H(t)}]$ and $\overline{H(t)} \, |\Psi\rangle = \sum_{n=0}^{\infty} H(t)\Pi_n \, |\Psi\rangle$. □

We are now poised to first investigate the existence of solution of the Schrödinger's equation for the time-dependent Hamiltonian defined in Definition 5. We restrict ourselves to initial states with only a finite number of particles in the environment i.e. $|\Psi(0)\rangle \in \mathcal{H}_S \otimes \mathrm{F}_\infty^M[L^2(\mathbb{R})]$, and use the density of $|\Psi(0)\rangle \in \mathcal{H}_S \otimes \mathrm{F}_\infty^M[L^2(\mathbb{R})]$ to extend it to the entire system-environment

Hilbert space. While $H(t)$, for every t, admits a self adjoint extension, since the equation under consideration is non-autonomous, this by itself does not imply that the solution of this equation exists. Instead, we analyze this equation by first truncating the number of particles in the environment, and analyzing the convergence of the obtained solution with the truncation.

Definition 6. *For $p \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}$, define $H^p(t) : \mathcal{H} \to \mathcal{H}$ via $H^p(t) = \Pi_{\leq p} H(t) \Pi_{\leq p}$.*

Several properties of $H^p(t)$ follows trivially from its definition.

Lemma 3. *$H^p(t)$ has the following properties*

(a) $H^p(t)$ is a bounded operator for all $t \in \mathbb{R}$.

(b) $H^p(t)$ is norm continuous with respect to t.

Proof:
(a) This follows straightforwardly by noting that $\forall t \in \mathbb{R}, \alpha \in \{1, 2 \dots M\}$, $\left\| a^-_{\alpha, \tau_t v_\alpha} \Pi_{\leq p} \right\| \leq \sqrt{p} \left\| v_\alpha \right\|_{L^2}$ and $\left\| a^+_{\alpha, \tau_t v_\alpha} \Pi_{\leq p} \right\| \leq \sqrt{p+1} \left\| v_\alpha \right\|_{L^2}$.

(b) For any $\delta > 0$, note that

$$\left\| H^p(t+\delta) - H^p(t) \right\| \leq \left\| H_S(t+\delta) - H_S(t) \right\| + \sum_{\alpha=1}^{M} \sqrt{p} \left\| L_\alpha \right\| \left\| (\tau_{t+\delta} - \tau_t) v_\alpha \right\|_{L^2}.$$

Since $H_S(t)$ is norm continuous, $\left\| H_S(t+\delta) - H_S(t) \right\| \to 0$ as $\delta \to 0$, and since τ_t is strongly continuous in t, $\left\| (\tau_{t+\delta} - \tau_t) v_\alpha \right\|_{L^2} \to 0$ as $\delta \to 0$, thus showing from the above estimate that $H^p(t)$ is norm continuous. □

Lemma 4. *For any $p \in \mathbb{N}$ and $\tau, s \in \mathbb{R}$, there exists a unitary operator $U^p(\tau, s) : \mathcal{H} \to \mathcal{H}$ which is norm continuous and differentiable with respect to both s and τ and which satisfies*

$$i \frac{d}{d\tau} U^p(\tau, s) = H^p(\tau) U^p(\tau, s) \text{ with } U^p(s, s) = \text{id}.$$

Furthermore, let $|\Phi\rangle \in \mathcal{H}_S \otimes \mathrm{F}_{\mathcal{S}}[L^2(\mathbb{R})]$, and for $t > 0$, consider $M^{(k)}_{|\Phi\rangle}(t)$ defined by

$$M^{(0)}_{|\Phi\rangle}(t) = \left\| |\Phi\rangle \right\|^2, M^{(k)}_{|\Phi\rangle}(t) = 2\mu^{(k)}_{|\Phi\rangle} + 2^{2k-3} \ell^2 t^2 \bigg(\left\| |\Phi\rangle \right\|^2 + M^{(k-1)}_{|\Phi\rangle}(t) \bigg)^2 \text{ for } k \geq 1,$$

then $\forall \tau, s \in [0, t]$, $\mu^{(k)}_{U^p(\tau,s)|\Phi\rangle} \leq M^{(k)}(t)$ $\forall p \in \mathbb{Z}_{\geq 0}$.

Proof: Since $H^p(\tau)$ is both norm continuous in τ and bounded, the existence, norm continuity and differentiability of $U^p(\tau, s)$ follows follows directly from Dyson expansion (see theorem X.69 of Ref. [59]). For part (b), we use the Schödinger equation. Note that

$$U^p(\tau, s) |\Phi\rangle = \Pi_{>p} |\Phi\rangle + U^p(\tau, s) \Pi_{\leq p} |\Phi\rangle,$$

and furthermore, $\mu^{(k)}_{\Pi_{\leq p}|\Phi\rangle} \leq \mu^{(k)}_{|\Phi\rangle}$ $\forall k \in \mathbb{Z}_{\geq 1}$. For convenience of notation, we set $|\Psi^p(\tau, s)\rangle = U^p(\tau, s) \Pi_{\leq p} |\Phi\rangle$. From the Schroedinger's equation, it follows that

$$\frac{d}{d\tau} \mu^{(k)}_{|\Psi^p(\tau,s)\rangle} = \sum_{\alpha=1}^{M} \sum_{n=0}^{p-1} \big((n+1)^k - n^k \big) \mathrm{Im} \left\langle \Psi^p(\tau, s) \right| \Pi_{n+1} a^+_{\alpha, \tau_\tau v_\alpha} L_\alpha \Pi_n \left| \Psi^p(\tau, s) \right\rangle.$$

and therefore

$$\left| \frac{d}{d\tau} \mu^{(k)}_{|\Psi^p(\tau,s)\rangle} \right| \leq 2\ell \sum_{n=0}^{p-1} \sqrt{n+1} \big((n+1)^l - n^l \big) \left\| \Pi_{n+1} |\Psi^p(\tau, s)\rangle \right\| \left\| \Pi_n |\Psi^p(\tau, s)\rangle \right\|,$$

Since $(n+1)^l - n^l = (n+1)^{l-1} + n(n+1)^{l-2} + n^2(n+1)^{l-3} \dots n^{l-1}$ for $l \in \mathbb{Z}_{\geq 1}$, we obtain that

$$\left| \frac{d}{d\tau} \mu^{(k)}_{|\Psi^p(\tau,s)\rangle} \right| \leq 2\ell \sum_{n=0}^{p-1} \sum_{s=1}^{k} (n+1)^{k-s+1/2} n^{s-1} \left\| \Pi_{n+1} |\Psi^p(\tau, s)\rangle \right\| \left\| \Pi_n |\Psi^p(\tau, s)\rangle \right\|$$

An application of the Cauchy-Schwarz inequality yields that $\forall s \in \{1, 2 \dots k\}$

$$\sum_{n=0}^{p-1}(n+1)^{k-s+1/2}n^{s-1}\left\|\Pi_{n+1}\left|\Psi^p(\tau,s)\right\rangle\right\|\left\|\Pi_n\left|\Psi^p(\tau,s)\right\rangle\right\|$$
$$\leq \bigg(\sum_{n=0}^{p-1}(n+1)^k\left\|\Pi_{n+1}\left|\Psi^p(\tau,s)\right\rangle\right\|^2\bigg)^{1/2}\bigg(\sum_{n=0}^{p-1}n^{2s-2}(n+1)^{k+1-2s}\left\|\Pi_n\left|\psi^p(\tau,s)\right\rangle\right\|^2\bigg)^{1/2}$$
$$\leq \big(\mu^{(k)}_{|\Psi^p(t,s)\rangle}\big)^{1/2}\bigg(\sum_{n=0}^{p-1}(n+1)^{k-1}\left\|\Pi_n\left|\Psi^p(\tau,s)\right\rangle\right\|^2\bigg)^{1/2}$$

Noting that

$$\sum_{n=0}^{p-1}(n+1)^{k-1}\left\|\Pi_n\left|\Psi^p(\tau,s)\right\rangle\right\|^2 \leq 2^{k-2}\sum_{n=0}^{p-1}(n^{k-1}+1)\left\|\Pi_n\left|\Psi^p(\tau,s)\right\rangle\right\|^2 = 2^{k-2}\bigg(\mu^{(0)}_{|\Psi^p(\tau,s)\rangle}+\mu^{(k-1)}_{|\Psi^p(\tau,s)\rangle}\bigg).$$

Noting that $\mu^{(0)}_{|\Psi^p(\tau,s)\rangle} = \mu^{(0)}_{\Pi_{\leq p}|\Phi\rangle} \leq \mu^{(0)}_{|\Phi\rangle}$, we thus obtain

$$\left|\frac{d}{dt}\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}\right| \leq 2^{k-1}\ell\big(\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}\big)^{1/2}\bigg(\mu^{(0)}_{|\Phi\rangle}+\mu^{(k-1)}_{|\Psi^p(\tau,s)\rangle}\bigg),$$

Integrating which we obtain that for $\tau \geq s$

$$\big(\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}\big)^{1/2}-\big(\mu^{(k)}_{\Pi_{\leq p}|\Phi\rangle}\big)^{1/2} \leq 2^{k-2}\ell\bigg((\tau-s)\mu^{(0)}_{|\Phi\rangle}+\int_s^\tau \mu^{(k-1)}_{|\Psi^p(\tau',s)\rangle}d\tau'\bigg) \leq 2^{k-2}\ell\bigg(t\mu^{(0)}_{|\Phi\rangle}+\int_s^\tau \mu^{(k-1)}_{|\Psi^p(\tau',s)\rangle}d\tau'\bigg),$$

and for $\tau < s$

$$\big(\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}\big)^{1/2}-\big(\mu^{(k)}_{\Pi_{\leq p}|\Phi\rangle}\big)^{1/2} \leq 2^{k-2}\ell\bigg((s-\tau)\mu^{(0)}_{|\Phi\rangle}+\int_\tau^s \mu^{(k-1)}_{|\Psi^p(\tau',s)\rangle}d\tau'\bigg) \leq 2^{k-2}\ell\bigg(t\mu^{(0)}_{|\Phi\rangle}+\int_\tau^s \mu^{(k-1)}_{|\Psi^p(\tau',s)\rangle}d\tau'\bigg),$$

Since $\forall k \in \mathbb{Z}_{\geq 0},\ \mu^{(k)}_{U^p(\tau,s)|\Phi\rangle} = \mu^{(k)}_{\Pi_{>p}|\Phi\rangle}+\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}$These equations can be recursively solved to obtain the functions $M^{(k)}_{|\Phi_0\rangle}(t)$. Note that $\mu^{(0)}_{|\Psi^p(\tau,s)\rangle} = \mu^{(0)}_{\Pi_{\leq p}|\Phi\rangle} \leq \langle\Phi|\,\Phi\rangle$, and hence $M^{(0)}_{|\Phi\rangle}(t)$ can be set to $\||\Phi\rangle\|^2$. Assuming that $\mu^{(k-1)}_{U^p(\tau,s)|\Phi\rangle} \leq M^{(k-1)}_{|\Phi\rangle}(t)\ \forall \tau, s \in [0,t]$, we then obtain that

$$\mu^{(k)}_{|\Psi^p(\tau,s)\rangle} \leq \bigg(\big(\mu^{(k)}_{\Pi_{\leq p}|\Phi\rangle}\big)^{1/2}+2^{k-2}\ell t\bigg(\langle\Phi|\,\Phi\rangle+M^{(k-1)}_{|\Phi\rangle}(t)\bigg)\bigg)^2 \leq 2\mu^{(k)}_{\Pi_{\leq p}|\Phi\rangle}+2^{2k-3}\ell^2t^2\bigg(\langle\Phi|\,\Phi\rangle+M^{(k-1)}_{|\Phi\rangle}(t)\bigg)^2,$$

and since $\mu^{(k)}_{U^p(\tau,s)|\Phi\rangle} \leq \mu^{(k)}_{|\Phi\rangle}+\mu^{(k)}_{|\Psi^p(\tau,s)\rangle}$, we can choose $M^{(k)}_{|\Phi\rangle} = 2\mu^{(k)}_{|\Phi\rangle}+2^{2k-3}\ell^2t^2\big(\langle\Phi|\,\Phi\rangle+M^{(k-1)}_{|\Phi\rangle}(t)\big)^2$, which proves the lemma statement. □

It is important to note that the bounds on $\mu^p_l(t)$ are uniform in p — we will exploit this in the following proofs to show the existence and differentiability of $|\Psi^p(t)\rangle$.

Lemma 5. *Let* $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^\infty_{\mathcal{S}}[L^2(\mathbb{R})]$*, then*

(a) $\forall t, s \geq 0$, $\lim_{p\to\infty} U^p(t,s)\left|\Psi_0\right\rangle$ *exists and* $\in \mathcal{H}_S \otimes \mathrm{F}^\infty_{\mathcal{S}}[L^2(\mathbb{R})]$.

(b) $\forall t, s \geq 0$, $\lim_{p\to\infty} H^p(t)U^p(t,s)\left|\Psi_0\right\rangle = \overline{H(t)}\lim_{p\to\infty} U^p(t,s)\left|\Psi_0\right\rangle$ *and* $\overline{H(t)}\lim_{p\to\infty} U^p(t,s)\left|\Psi_0\right\rangle$ *is strongly continuous in* t.

(c) $\forall t, s \geq 0$, $\lim_{p\to\infty} U^p(t,s)H^p(s)\left|\Psi_0\right\rangle = \lim_{p\to\infty} U^p(t,s)\overline{H(s)}\left|\Psi_0\right\rangle$ *and* $\lim_{p\to\infty} U^p(t,s)\overline{H(s)}\left|\Psi_0\right\rangle$ *is strongly continuous in* s.

(d) $\exists g, h \in \mathrm{C}^0(\mathbb{R})$ *such that* $\left\|H^p(t)U^p(t,s)\left|\Psi_0\right\rangle\right\| \leq g(t)$ *and* $\left\|H^p(t)\left|\Psi_0\right\rangle\right\| \leq h(t)\ \forall\ t \geq 0$ *and* $p \in \mathbb{Z}_{\geq 0}$.

Proof:
(a) To prove the existence of limit, we appeal to the completeness of $\mathcal{H}$ and show that the sequence $\{U^p(t,s)\left|\Psi_0\right\rangle\}_{p\in\mathbb{Z}_{\geq 0}}$ is

Cauchy. Consider $p, q \in \mathbb{N}$ with $p > q$ — we note that

$$\|U^p(t,s)\,|\Psi_0\rangle - U^q(t,s)\,|\Psi_0\rangle\| \leq \|U^p(t,s)\Pi_{\leq q}\,|\Psi_0\rangle - U^q(t,s)\Pi_{\leq q}\,|\Psi_0\rangle\| + 2\,\|\Pi_{>q}\,|\Psi_0\rangle\|\,.$$

Furthermore, since both $U^p(t,s)$ and $U^q(t,s)$ are norm (and thus strongly) differentiable with respect to t and s, we obtain that

$$\begin{aligned}\|U^p(t,s)\Pi_{\leq q}\,|\Psi_0\rangle - U^q(t,s)\Pi_{\leq q}\,|\Psi_0\rangle\| &= \left\|\int_s^t \frac{d}{d\tau}\big(U^p(s,\tau)U^q(\tau,s)\big)\Pi_{\leq q}\,|\Psi_0\rangle\, d\tau\right\| \\ &\leq \int_s^t \left\|\big(H^p(\tau) - H^q(\tau)\big)U^q(\tau,s)\Pi_{\leq q}\,|\Psi_0\rangle\right\| d\tau. \end{aligned} \tag{3}$$

Furthermore, since $p > q$, we obtain that

$$|\Phi_{p,q}(\tau,s)\rangle := \big(H^p(\tau) - H^q(\tau)\big)U^q(\tau,s)\Pi_{\leq q}\,|\Psi_0\rangle = \big(H^p(s) - H^q(s)\big)\Pi_q U^q(\tau,s)\Pi_{\leq q}\,|\Psi_0\rangle\,, \tag{4}$$

and thus

$$\||\Phi_{p,q}(s)\rangle\| \leq \ell\sqrt{q+1}\,\|\Pi_q U^q(\tau,s)\Pi_{\leq q}\,|\Psi_0\rangle\|\,. \tag{5}$$

Using the bound from lemma 4, we obtain that $\|\Pi_q U^q(\tau,s)\Pi_{\leq q}\,|\Psi_0\rangle\| \leq \sqrt{M^{(2)}_{\Pi_{\leq q}|\Psi(0)\rangle}(\max(s,t))/q} \leq \sqrt{M^{(2)}_{|\Psi_0\rangle}(\max(s,t))/q}$ and thus

$$\int_s^t \||\Phi_{p,q}(\tau,s)\rangle\|\, d\tau \leq \frac{\ell\,|t-s|\,\sqrt{q+1}}{q}\sqrt{M^{(2)}_{\Pi_{\leq q}|\Psi_0\rangle}(\max(s,t))} \leq \frac{\ell\,|t-s|\,\sqrt{q+1}}{q}\sqrt{M^{(2)}_{|\Psi_0\rangle}(\max(s,t))},$$

and hence

$$\|U^p(t,s)\,|\Psi_0\rangle - U^q(t,s)\,|\Psi_0\rangle\| \leq \frac{\ell\,|t-s|\,\sqrt{q+1}}{q}\sqrt{M^{(2)}_{|\Psi_0\rangle}(\max(s,t))} + 2\,\|\Pi_{\leq q}\,|\Psi_0\rangle\|\,.$$

Thus, $\|U^p(t,s)\,|\Psi_0\rangle - U^q(t,s)\,|\Psi_0\rangle\| \to 0$ as $p, q \to \infty$, thus implying that the sequence $\{U^p(t,s)\,|\Psi_0\rangle\}_{p\in\mathbb{N}}$ is Cauchy, and hence converges. Furthermore, from lemma 4, the moments $\mu^{(k)}_{U^p(t,s)|\Psi_0\rangle}$ are bounded uniformly in p for all $k \in \mathbb{Z}_{\geq 0}$, and hence from the dominated convergence theorem it follows that all the particle number moments of $\lim_{p\to\infty} U^p(t,s)\,|\Psi_0\rangle$ are also bounded — this shows that $\lim_{p\to\infty} U^p(t,s)\,|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$.

(b) For $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$, since $\|\lim_{p\to\infty} U^p(t,s)\Pi_{>p}\,|\Psi_0\rangle\| = \lim_{p\to\infty}\|\Pi_{>p}\,|\Psi_0\rangle\| = 0$, we obtain that

$$\overline{H(t)}\lim_{p\to\infty} U^p(t,s)\,|\Psi_0\rangle = \overline{H(t)}\lim_{p\to\infty} U^p(t,s)\Pi_{\leq p}\,|\Psi_0\rangle\,.$$

We already established in part (a) that the sequence $\{U^p(t,s)\,|\Psi_0\rangle\}_{p\in\mathbb{Z}_{\geq 0}}$, and hence the sequence $\{U^p(t,s)\Pi_{\leq p}\,|\Psi_0\rangle\}_{p\in\mathbb{Z}_{\geq 0}}$, converges. We now show that the sequence $\{H(t)U^p(t,s)\Pi_{\leq p}\,|\Psi_0\rangle\}_{p\in\mathbb{Z}_{\geq 0}}$ also converges. To see this, we note that for $p, q \in \mathbb{Z}_{\geq 0}$ with $q \leq p$,

$$\|H(t)U^p(t,s)\Pi_{\leq p}\,|\Psi_0\rangle - H(t)U^q(t,s)\Pi_{\leq q}\,|\Psi_0\rangle\| \leq \int_s^t \|H(t)U^p(t,\tau)\,|\Phi_{p,q}(\tau,s)\rangle\|\, d\tau + \|H(t)U^p(t,s)\,|\Gamma_{p,q}(t,s)\rangle\|\,, \tag{6}$$

where $|\Phi_{p,q}(\tau,s)\rangle$ is defined in Eq. 4 and $|\Gamma_{p,q}(t,s)\rangle = \big(\Pi_{\leq p} - \Pi_{\leq q}\big)\,|\Psi_0\rangle$. Now, from the definition of $H(t)$, it follows that

$$\|H(t)U^p(t,s)\,|\Gamma_{p,q}(t,s)\rangle\| \leq \|H_S(t)\|\,\||\Gamma_{p,q}(t,s)\rangle\| + 2\ell\bigg(\||\Gamma_{p,q}(t,s)\rangle\|^2 + \mu^{(1)}_{U^p(t,s)|\Gamma_{p,q}(t,s)\rangle}\bigg)^{1/2}.$$

Furthermore, using lemma 4, we obtain that

$$\mu^{(1)}_{U^p(t,s)|\Gamma_{p,q}(t,s)\rangle} \leq 2\bigg(\mu^{(1)}_{\Gamma_{p,q}(t,s)} + \ell^2\max{}^2(s,t)\,\||\Gamma_{p,q}(t,s)\rangle\|^4\bigg).$$

Since $|\Psi_0\rangle \in \mathcal{H}_S \otimes \text{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$, we obtain that

$$\| |\Gamma_{p,q}(t,s)\rangle \| \leq \|\Pi_{>q} |\Psi_0\rangle\| + \|\Pi_{>p} |\Psi_0\rangle\| \to 0 \text{ as } p, q \to \infty \text{ and,}$$
$$\mu^{(1)}_{|\Gamma_{p,q}(t,s)\rangle} \leq \mu^{(1)}_{|\Pi_{>q}|\Psi_0\rangle\rangle} + \mu^{(1)}_{|\Pi_{>p}|\Psi_0\rangle\rangle} \to 0 \text{ as } p, q \to \infty,$$

and therefore it follows from the previous estimates that $\|H(t)U^p(t,s) |\Gamma_{p,q}(t,s)\rangle\| \to 0$ as $p, q \to \infty$. Consider now the second term in Eq. 6 — since $U^p(t,\tau) |\Phi_{p,q}(\tau,s)\rangle \in \mathcal{H}_S \otimes \text{F}^M_\infty[L^2(\mathbb{R})]$, we obtain that

$$\int_s^t \|H(\tau)U^p(t,\tau) |\Phi_{p,q}(\tau,s)\rangle\| \, d\tau \leq \int_s^t \|H_S(\tau)\| \, \| |\Phi_{p,q}(\tau,s)\rangle \| \, d\tau + 2\ell \int_s^t \left(\| |\Phi_{p,q}(\tau,s)\rangle\|^2 + \mu^{(1)}_{U^p(t,\tau)|\Phi_{p,q}(\tau,s)\rangle} \right)^{1/2} d\tau.$$

Using Eq. 5 and the bound from lemma 4, we obtain that

$$\int_s^t \|H_S(\tau)\| \, \| |\Phi_{p,q}(\tau,s)\rangle\| \, d\tau \leq \frac{\ell\sqrt{q+1}}{q} \sqrt{M^{(2)}_{|\Psi_0\rangle}(t_{\max})} \int_s^t \|H_S(\tau)\| \, d\tau.$$

It follows from lemma 4 that for $\tau \in [\min(t,s), \max(t,s)]$,

$$\mu^{(1)}_{U^p(t,\tau)|\Phi_{p,q}(\tau,s)\rangle} \leq 2\left(\mu^{(1)}_{|\Phi_{p,q}(\tau,s)\rangle} + \ell^2 \max{}^2(s,t) \, \| |\Phi_{p,q}(\tau,s)\rangle\|^4 \right),$$

and using Eq. 5 it follows that for $\tau \in [\min(t,s), \max(t,s)]$,

$$\mu^{(1)}_{|\Phi_{p,q}(\tau,s)\rangle} \leq \frac{(q+1)^2}{q^3} \ell^2 M^{(3)}_{\Pi_{\leq q}|\Psi_0\rangle}(\max(s,t)) \leq \frac{(q+1)^2}{q^3} \ell^2 M^{(3)}_{|\Psi_0\rangle}(\max(s,t)).$$

From these estimates, it thus follows that

$$\int_s^t \|H(\tau)U^p(t,\tau) |\Phi_{p,q}(\tau,s)\rangle\| \, d\tau \to 0 \text{ as } p, q \to \infty.$$

Therefore, from Eq. 6, it follows that the sequence $\{H(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle\}_{p\in\mathbb{N}}$ converges — since $H(t)$ is a closable operator, it then follows that

$$\lim_{p\to\infty} H(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle = \overline{H(t)} \lim_{p\to\infty} U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle = \overline{H(t)} \lim_{p\to\infty} U^p(t,s) |\Psi_0\rangle .$$

Finally, we show that $\lim_{p\to\infty} H^p(t)U^p(t,s) |\Psi_0\rangle = \lim_{p\to\infty} H(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle$. We begin by noting that

$$H^p(t)U^p(t,s) |\Psi_0\rangle = H^p(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle + \Pi_{>p} |\Psi_0\rangle \implies \lim_{p\to\infty} H^p(t)U^p(t,s) |\Psi_0\rangle = \lim_{p\to\infty} H^p(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle .$$

Furthermore,

$$\|(H(t) - H^p(t))U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle\|^2 \leq$$
$$(p+1)\ell^2 \, \|\Pi_p U^p(t,s) |\Psi_0\rangle\|^2 \leq \frac{p+1}{p^2} \ell^2 M^{(2)}_{\Pi_{\leq p}|\Psi_0\rangle}(\max(t,s)) \leq \frac{p+1}{p^2} \ell^2 M^{(2)}_{|\Psi_0\rangle}(\max(t,s)).$$

and thus $\lim_{p\to\infty} H^p(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle = \lim_{p\to\infty} H(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle$. Hence, we obtain that $\overline{H(t)} \lim_{p\to\infty} U^p(t,s) |\Psi_0\rangle = \lim_{p\to\infty} H^p(t)U^p(t,s) |\Psi_0\rangle$.

Now, we investigate the continuity of $\overline{H(t)} \lim_{p\to\infty} U^p(t,s) |\Psi_0\rangle = \lim_{p\to\infty} H^p(t)U^p(t,s)\Pi_{\leq p} |\Psi_0\rangle$ with respect to t. For $p \in \mathbb{Z}_{\geq 0}$ and $\delta > 0$, define $\Delta^p(\delta)$ va

$$\Delta^p(\delta) = \left\| \left(H^p(t+\delta)U^p(t+\delta,s) - H^p(t)U^p(t,s) \right) \Pi_{\leq p} |\Psi_0\rangle \right\| .$$

We need to show that $\lim_{\delta\to 0}\lim_{p\to\infty}\Delta^p(\delta)=0$. To see this, we note that

$$\Delta^p(\delta)\le \left\|\Big(H^p(t+\delta)-H^p(t)\Big)U^p(t,s)\Pi_{\le p}\left|\Psi_0\right\rangle\right\| + \left\|H^p(t+\delta)\Big(U^p(t+\delta,s)-U^p(t,s)\Big)\Pi_{\le p}\left|\Psi_0\right\rangle\right\|.$$

Now

$$\begin{aligned}&\left\|\Big(H^p(t+\delta)-H^p(t)\Big)U^p(t,s)\left|\Psi_0\right\rangle\right\|\\ &\qquad\le \|H_S(t+\delta)-H_S(t)\|\,\|\left|\Psi_0\right\rangle\| + \bigg(\|\left|\Psi_0\right\rangle\|^2+\mu^{(1)}_{U^p(t,s)\Pi_{\le p}\left|\Psi_0\right\rangle}\bigg)^{1/2}\sum_{\alpha=1}^{M}\|L_\alpha\|\left\|\Big(\tau_{\alpha,t+\delta}-\tau_{\alpha,t}\Big)v_\alpha\right\|_{L^2},\\ &\qquad\le \|H_S(t+\delta)-H_S(t)\|\,\|\left|\Psi_0\right\rangle\| + \bigg(\|\left|\Psi_0\right\rangle\|^2+M^{(1)}_{\left|\Psi_0\right\rangle}\bigg)^{1/2}\sum_{\alpha=1}^{M}\|L_\alpha\|\left\|\Big(\tau_{\alpha,t+\delta}-\tau_{\alpha,t}\Big)v_\alpha\right\|_{L^2},\end{aligned}$$

and consequently by the strong continuity of $\tau_{\alpha,t}$,

$$\lim_{\delta\to 0}\lim_{p\to\infty}\left\|\Big(H^p(t+\delta)-H^p(t)\Big)U^p(t,s)\left|\Psi_0\right\rangle\right\|=0.$$

Furthermore,

$$\left\|H^p(t+\delta)\Big(U^p(t+\delta,s)-U^p(t,s)\Big)\Pi_{\le p}\left|\Psi_0\right\rangle\right\|\le\int_t^{t+\delta}\|H^p(\tau+\delta)H^p(\tau)U^p(\tau,s)\Pi_{\le p}\left|\Psi_0\right\rangle\|\,d\tau.$$

It follows from lemma 4 that $\|H^p(\tau+\delta)H^p(\tau)U^p(\tau,s)\left|\Psi_0\right\rangle\|$ is bounded above by a constant independent of p and continuous in τ. Thus, we obtain that

$$\lim_{\delta\to 0}\lim_{p\to\infty}\left\|H^p(t+\delta)\Big(U^p(t+\delta,s)-U^p(t,s)\Big)\Pi_{\le p}\left|\Psi_0\right\rangle\right\|=0.$$

Thus, we obtain that $\lim_{\delta\to 0}\lim_{p\to\infty}\Delta^p(\delta)=0$.

(c) The proof of this part closely follows that of part (b), with only minor modifications which we outline here. We can show that $\lim_{p\to\infty}\left\|H^p(s)\left|\Psi_0\right\rangle-\overline{H(s)}\left|\Psi_0\right\rangle\right\|=0$, which would imply that $\lim_{p\to\infty}U^p(t,s)H^p(s)\left|\Psi_0\right\rangle=\lim_{p\to\infty}U^p(t,s)\overline{H(s)}\left|\Psi_0\right\rangle$, in two steps — first, we establish that $\lim_{p\to\infty}H^p(s)\left|\Psi_0\right\rangle=\lim_{p\to\infty}H^p(s)\Pi_{\le p}\left|\Psi_0\right\rangle$ using the fact that all the particle-number moments of $\left|\Psi_0\right\rangle$ are finite. Then, we can show that $\lim_{p\to\infty}H^p(s)\Pi_{\le p}\left|\Psi_0\right\rangle=\lim_{p\to\infty}H(s)\Pi_{\le p}\left|\Psi_0\right\rangle$ by analyzing the norm $\|(H(s)-H^p(s))\Pi_{\le p}\left|\Psi_0\right\rangle\|$. By showing that the sequence $\{H(s)\Pi_{\le p}\left|\Psi_0\right\rangle\}_{p\in\mathbb{N}}$ converges, and using the closability of $H(t)$, we then obtain that $\lim_{p\to\infty}H(s)\Pi_{\le p}\left|\Psi_0\right\rangle=\overline{H(s)}\lim_{p\to\infty}\Pi_{\le p}\left|\Psi_0\right\rangle$. Finally, using lemma 2b, you obtain that $\overline{H(s)}\lim_{p\to\infty}\Pi_{\le p}\left|\Psi_0\right\rangle=\overline{H(s)}\left|\Psi_0\right\rangle$. To prove that $\lim_{p\to\infty}U^p(t,s)\overline{H(s)}\left|\Psi_0\right\rangle=\lim_{p\to\infty}U^p(t,s)H^p(s)\left|\Psi_0\right\rangle$ is strongly continuous, we can again analyze $\Delta^p(\delta)$, where

$$\begin{aligned}\Delta^p(\delta)&=\left\|\Big(U^p(t,s+\delta)H^p(s+\delta)-U^p(t,s)H^p(s)\Big)\left|\Psi_0\right\rangle\right\|,\\ &\le\left\|\Big(U^p(t,s+\delta)-U^p(t,s)\Big)H^p(s+\delta)\left|\Psi_0\right\rangle\right\|+\left\|U^p(t,s)\Big(H^p(t,s+\delta)-H^p(s)\Big)\left|\Psi_0\right\rangle\right\|.\end{aligned}$$

Using lemma 4, we can show that $\lim_{p\to\infty}\lim_{\delta\to 0}\Delta^p(\delta)=0$.

(d) This follows straightforwardly from lemma 4, and noting that

$$\|H^p(t)U^p(t,s)\left|\Psi_0\right\rangle)\|\le\|H_S(t)\|\,\|\left|\Psi_0\right\rangle\|+2\ell\bigg(\|\left|\Psi_0\right\rangle\|^2+M^{(1)}_{\left|\Psi_0\right\rangle}(\max(t,s))\bigg)^{1/2}.$$

which yields the upper bound $g(t)$. Similarly,

$$\|H^p(t)\,|\Psi_0\rangle\| \leq \|H_S(t)\|\,\||\Psi_0\rangle\| + 2\ell\bigg(\||\Psi_0\rangle\|^2 + \mu^{(1)}_{|\Psi_0\rangle}\bigg)^{1/2},$$

which yields the upper bound $h(t)$. □

Lemma 6. *Given a non-Markovian model with square integrable coupling functions and for $t, s \in \mathbb{R}$, there exists a unique isometry $U(t,s) : \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})] \to \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})] \subseteq \mathcal{H}$ which is strongly continuous and differentiable in both t, s and satisfies*

$$\frac{d}{dt}U(t,s) = -i\overline{H(t)}U(t,s) \text{ and } \frac{d}{ds}U(t,s) = iU(t,s)\overline{H(s)}, \tag{7}$$

with $U(s,s) = \mathrm{id}\ \forall s \in \mathbb{R}$.

Proof: We first construct the unitary group $U(t,s)$ — given a state $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$, we let

$$U(t,s)\,|\Psi_0\rangle = \lim_{p\to\infty} U^p(t,s)\,|\Psi_0\rangle\,.$$

It follows from lemma 5 that $U(t,s)$ is well defined, that $U(t,s)\,|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ and that $U(t,s)$ is an isometry. Now, we note that since $U^p(t,s)$ is the propagator corresponding to $H^p(t)$,

$$U(t,s)\,|\Psi_0\rangle = \lim_{p\to\infty} U^p(t,s)\,|\Psi_0\rangle = |\Psi_0\rangle - i\lim_{p\to\infty}\int_s^t H^p(\tau)U^p(\tau,s)\,|\Psi_0\rangle\,d\tau.$$

From lemma 5(d), it follows that $\|H^p(\tau)U^p(\tau,s)\,|\Psi_0\rangle\|$ is bounded above by a continuous (and thus integrable) function of τ, and hence from the dominated convergence theorem, we obtain that

$$U(t,s)\,|\Psi_0\rangle = |\Psi_0\rangle - i\int_s^t \lim_{p\to\infty} H^p(\tau)U^p(\tau,s)\,|\Psi_0\rangle\,d\tau.$$

Finally, using lemma 5(b), we obtain that $\lim_{p\to\infty} H^p(\tau)U^p(\tau,s)\,|\Psi_0\rangle = \overline{H(\tau)}U(\tau,s)\,|\Psi_0\rangle$, and since $\overline{H(\tau)}U(\tau,s)\,|\Psi_0\rangle$ is strongly continuous in τ, $U(t,s)\,|\Psi_0\rangle$ is strongly differentiable in t. Thus,

$$U(t,s)\,|\Psi_0\rangle = |\Psi_0\rangle - i\int_s^t \overline{H(\tau)}U(\tau,s)\,|\Psi_0\rangle\,d\tau \implies \frac{d}{dt}U(t,s)\,|\Psi_0\rangle = -i\overline{H(t)}U(t,s)\,|\Psi_0\rangle\,.$$

which shows that $dU(t,s)/dt = -i\overline{H(t)}U(t,s)$ (where derivatives are understood as strong derivatives) with $U(s,s) = \mathrm{id}$. A similar argument can be made using lemmas 5(c) and 5(d) to show that $dU(t,s)\,|\Psi_0\rangle\,/ds = iU(t,s)\overline{H(s)}$.

Since $\forall t \in \mathbb{R}$, $H(t)$ is essentially self adjoint, $\overline{H(t)}$ is self adjoint — from this, it immediately follows that if the solution to Eq. 7 exists, the it must be unique. To see this, we simply note that the self-adjointness of $\overline{H(t)}$ implies that $\|U(t,s)\,|\Psi_0\rangle\| = \|U(s,s)\,|\Psi_0\rangle\|$, and hence $|\Psi_0\rangle = 0 \implies |\Psi_0\rangle = 0\ \forall t \geq 0$. Now if there were two distinct solutions $U_1(t,s), U_2(t,s)$ to Eq. 7, then $\big(U_1(t,s) - U_2(t,s)\big)\,|\Psi_0\rangle$ would be a non-zero vector, which leads to a contradiction since by essential self adjointness of $H(t)$, $\|\big(U_1(t,s) - U_2(t,s)\big)\,|\Psi_0\rangle\| = 0$. □

In the following lemmas, we provide some further properties of dynamics of non-Markovian models with square integrable functions which will be useful for the analysis of distributional coupling functions.

Lemma 7. *Given $u \in L^2(\mathbb{R})$ and a non-Markovian model with square integrable coupling functions, $\forall \alpha \in \{1, 2 \ldots M\}, s, t \in [0, \infty)$,*

$$a^-_{\alpha,u}U(t,s) = U(t,s)a^-_{\alpha,u} - i\int_s^t \langle u, \tau_{\alpha,\tau} v_\alpha\rangle U(t,\tau)L_\alpha U(\tau,s)d\tau$$

over the domain $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$, where $U(t,s)$ is the propagator corresponding to the non-Markovian model as defined in lemma 6.

Proof: Throughout this proof, all operators are considered to be over the domain $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ — in particular, we extend

$a^{\pm}_{\alpha,u}$ from the domain $\mathcal{H}_S \otimes \mathrm{F}^M_\infty[L^2(\mathbb{R})]$ to $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ via

$$a^{\pm}_{\alpha,u} |\Psi\rangle = \sum_{n=0}^{\infty} a^{\pm}_{\alpha,u}\Pi_n |\Psi\rangle \ \forall |\Psi\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})].$$

Note that since $U(t,s)$ is strongly differentiable with respect to t, and it maps $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ to $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$, it follows that the operator $a^-_{\alpha,u}(t,s) = U(s,t)a^-_{\alpha,u}U(t,s)$ is strongly differentiable in both t and s. Differentiating it with respect to t, and using the characterization of $\overline{H(t)}$ when acting on $\mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ as provided in lemma 2, we obtain

$$\frac{d}{dt}a^-_{\alpha,u}(t,s) = -i\langle u, \tau_{\alpha,t}v_\alpha\rangle U(s,t)L_\alpha U(t,s).$$

Noting that since $U(t,s)$ is strongly continuous in both of its arguments and since L_α is a bounded operator, the right hand side in the above equation is strongly continuous in t and thus the equation can be integrated to obtain

$$a^-_{\alpha,u}U(t,s) = U(t,s)a^-_{\alpha,u} - i\int_s^t \langle u, \tau_{\alpha,\tau}v_\alpha\rangle U(t,\tau)L_\alpha U(\tau,s)d\tau,$$

which proves the lemma. □

Definition 7 (System Green's functions)**.** *Consider $k+1$ non-Markovian models specified by coupling functions $v_i = \{v_{i,\alpha} \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$ for $i \in \{1,2\dots k+1\}$, but with the same time-dependent system Hamiltonian $H_S(t)$ and jump operators $\{L_\alpha \in \mathfrak{L}(\mathcal{H}_S)\}_{\alpha\in\{1,2\dots M\}}$. For $O_1, O_2 \dots O_k \in \mathfrak{L}(\mathcal{H}_S)$, $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}$ and $t_1, t_2 \dots t_k \in [0,\infty)$, then the Green's function $G^{v_1,v_2\dots v_{k+1}}_{O_1,O_2\dots O_k}(t_1,t_2\dots t_k)$ is defined by*

$$G^{v_1,v_2\dots v_{k+1}}_{O_1,O_2\dots O_k;|\Psi_1\rangle,|\Psi_2\rangle}(t_1,t_2\dots t_k) = \langle\Psi_1| U_{v_{k+1}}(0,t_{k+1})O_k U_{v_k}(t_k,t_{k-1})O_{k-1}\dots O_1 U_{v_1}(t_1,0)|\Psi_2\rangle,$$

where $U_{v_i}(\cdot,\cdot)$ is the propagator, as defined in lemma 6, for the i^{th} model for $i \in \{1,2\dots k+1\}$.

Lemma 8. *Consider two non-Markovian models described by coupling functions $v = \{v_\alpha \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$, $u = \{u_\alpha \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$, but with the same system Hamiltonian $H_S(t)$, jump operators $\{L_\alpha \in \mathfrak{L}(\mathcal{H}_S)\}_{\alpha\in\{1,2\dots M\}}$ and environment single-particle unitary group $\{\tau_{\alpha,t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\}$. For $A, B \in \mathfrak{L}(\mathcal{H}_S)$, $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\mathcal{S}}[L^2(\mathbb{R})]$ and $t, s \in [0,\infty)$,*

$$\begin{aligned}
&\frac{d}{ds}G^{v,v,u}_{A,B;|\Psi\rangle,|\Phi\rangle}(s,t) = iG^{v,v,u}_{[H_S(s),A],B;|\Psi\rangle,|\Phi\rangle}(s,t) + \\
&\quad i\sum_{\alpha=1}^M\bigg(G^{v,v,u}_{[L^\dagger_\alpha,A],B;|\Psi\rangle,a^-_{\alpha,\tau_s v_\alpha}|\Phi\rangle}(s,t) + G^{v,v,u}_{[L_\alpha,A],B;a^-_{\alpha,\tau_s v_\alpha}|\Psi\rangle,|\Phi\rangle}(s,t)\bigg) - \\
&\quad i\sum_{\alpha=1}^M\bigg(\int_0^s \langle\tau_s v_\alpha, \tau_\tau v_\alpha\rangle G^{v,v,v,u}_{L_\alpha,[L^\dagger_\alpha,A],B;|\Psi\rangle,|\Phi\rangle}(s,\tau,t)d\tau + \int_0^s \langle\tau_\tau u_\alpha, \tau_s v_\alpha\rangle G^{v,v,u,u}_{[L_\alpha,A],B,L^\dagger_\alpha;|\Psi\rangle,|\Phi\rangle}(s,t,\tau)d\tau - \\
&\qquad \int_s^t \langle\tau_\tau u_\alpha, \tau_s v_\alpha\rangle G^{v,v,u,u}_{[L_\alpha,A],L^\dagger_\alpha,B;|\Psi\rangle,|\Phi\rangle}(s,\tau,t)d\tau\bigg)
\end{aligned}$$

Proof: Differentiating $G^{v,v,u}_{A,B;|\Psi\rangle,|\Phi\rangle}(s,t)$ with respect to s and using lemma 2(b), we obtain that

$$\begin{aligned}
\frac{d}{ds}G^{v,v,u}_{A,B;|\Psi\rangle,|\Phi\rangle}(s,t) &= i\langle\Psi| U_u(0,t)BU_v(t,s)[H_S(s),A]U_v(s,0)|\Phi\rangle + \\
&\quad i\sum_{\alpha=1}^M\bigg(\langle\Psi| U_u(0,t)BU_v(t,s)a^+_{\alpha,\tau_s v_\alpha}[L_\alpha,A]U_v(s,0)|\Phi\rangle + \\
&\qquad \langle\Psi| U_u(0,t)BU_v(t,s)[L^\dagger_\alpha,A]a^-_{\alpha,\tau_s v_\alpha}U_v(s,0)|\Phi\rangle\bigg).
\end{aligned}$$

We note that

$$\langle\Psi| U_u(0,t)BU_v(t,s)[H_S(s),A]U_v(s,0)|\Phi\rangle = G^{v,v,u}_{[H_S(s),A],B;|\Psi\rangle,|\Phi\rangle}(s,t).$$

Using lemma 7, we obtain that

$$\langle\Psi|\, U_u(0,t)BU_v(t,s)[L_\alpha^\dagger, A]a^-_{\alpha,\tau_s v_\alpha}U_v(s,0)\,|\Phi\rangle = \\ G^{v,v,u}_{[L_\alpha^\dagger,A],B;|\Psi\rangle,a^-_{\alpha,\tau_s v_\alpha}|\Phi\rangle}(s,t) - i\int_0^s \langle\tau_s v_\alpha, \tau_\tau v_\alpha\rangle G^{v,v,v,u}_{L_\alpha,[L_\alpha^\dagger,A],B;|\Psi\rangle,|\Phi\rangle}(s,\tau,t)d\tau,$$

and

$$\langle\Psi|\, U_u(0,t)BU_v(t,s)a^+_{\alpha,\tau_s v_\alpha}[L_\alpha, A]U_v(s,0)\,|\Phi\rangle = \\ G^{v,v,u}_{[L_\alpha,A],B;a^-_{\alpha,\tau_s v_\alpha}|\Psi\rangle,|\Phi\rangle}(s,t) + i\int_s^t \langle\tau_\tau u_\alpha, \tau_s v_\alpha\rangle G^{v,v,u,u}_{[L_\alpha,A],L_\alpha^\dagger,B;|\Psi\rangle,|\Phi\rangle}(s,\tau,t)d\tau - \\ i\int_0^s \langle\tau_\tau u_\alpha, \tau_s v_\alpha\rangle G^{v,v,u,u}_{[L_\alpha,A],B,L_\alpha^\dagger;|\Psi\rangle,|\Phi\rangle}(s,t,\tau)d\tau,$$

which completes the proof of the lemma. □

B. Extension to general radon measures

Definition 8 (Distributional coupling function)**.** *A distributional coupling function is specified by a tuple* (μ,φ) *where* $\mu \in \mathcal{M}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ *is a radon measure and a tempered distribution whose Fourier transform is a positive continuous function of at-most polynomial growth and* $\varphi \in \mathrm{C}^\infty(\mathbb{R})$*.*

An important consideration is how to apply the radon measure μ on a discontinuous function, since by definition it is only specified as acting on continuous functions. The approach to extend μ to a space of discontinuous functions is to use a mollifier i.e. first smoothen the discontinuous function to a continuous function using a mollifier, and then applying μ.

Definition 9 (Function space $\mathrm{PWC}^1(\mathbb{R})$)**.** $\mathrm{PWC}^1(\mathbb{R})$ *is space of all functions which are expressible as* $g \cdot \mathcal{I}_{[a,b]}$ *for some* $g \in \mathrm{C}^1(\mathbb{R})$ *and* $[a,b] \subseteq \mathbb{R}$*.*

Definition 10. *Given a* $\mu \in \mathcal{M}(\mathbb{R})$*, consider the map* $\mu_\varepsilon : \mathrm{PWC}^1(\mathbb{R}) \to \mathbb{C}$ *given by* $\langle\mu_\varepsilon, \cdot\rangle = \langle\mu, \rho_\varepsilon \star (\cdot)\rangle$ *with* ρ *being a symmetric mollifier, then* $\mu^* : \mathrm{PWC}^1(\mathbb{R}) \to \mathbb{C}$ *is defined by* $\mu^* := \mathrm{wlim}_{\varepsilon\to 0}\mu_\varepsilon$*.*

In the following lemmas, we establish that μ^* is well defined (i.e. the limit defining μ^* exists), is independent of the precise choice of the mollifier and that when restricted to the function space $\mathrm{C}^1_c(\mathbb{R})$ (i.e. the space of continuously differentiable compactly supported functions), its action coincides with that of the radon measure μ. We also derive certain properties of the map μ^* which will be useful in the following subsection. We first present a technical lemma.

Lemma 9. *Consider* $\mu \in \mathcal{M}(\mathbb{R})$ *with the Lesbesgue decomposition* $\mu = \mu_c + \mu_d$ *with* $\phi_c \in \mathrm{C}^0(\mathbb{R})$ *given by* $\phi_c(x) = \mu_c((-\infty, x])\ \forall\, x \in \mathbb{R}$*, and* $\mu_d \cong \sum_{i\in I}\alpha_i\delta(x-y_i)$ *for some* $\{\alpha_i \in \mathbb{C}\}_{i\in I}, \{y_i \in \mathbb{R}\}_{i\in I}$ *and finite and countably infinite index set* I*. Given a compact interval* $[a,b] \subseteq \mathbb{R}$ *and* $f \in \mathrm{C}^1(\mathbb{R})$*, define* $\langle\mu^*_{[a,b]}, f\rangle$ *by*

$$\langle\mu^*_{[a,b]}, f\rangle = \langle\mu^*_{c,[a,b]}, f\rangle + \langle\mu^*_{d,[a,b]}, f\rangle \text{ where} \\ \langle\mu^*_{c,[a,b]}, f\rangle = f(b)\phi_c(b) - f(a)\phi_c(a) - \int_a^b \phi_c(x)f'(x)dx \text{ and} \\ \langle\mu^*_{d,[a,b]}, f\rangle = \frac{1}{2}\sum_{i\in I | y_i\in\{a,b\}}\alpha_i f(y_i) + \sum_{i\in I | y_i \in (a,b)}\alpha_i f(y_i).$$

Then, for every compact intervals $[a,b] \subseteq \mathbb{R}$*,* $\exists\Delta^0_{\mu;[a,b]}(\varepsilon), \Delta^1_{\mu;[a,b]}(\varepsilon) > 0$ *where* $\Delta^0_{\mu;[a,b]}(\varepsilon), \Delta^1_{\mu;[a,b]}(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *such that* $\forall\varepsilon \in (0,(b-a)/2)$ *and for any mollifier* $\rho \in \mathrm{C}^\infty_c(\mathbb{R})$ *with* $\mathrm{supp}(\rho) \subseteq [-\varepsilon, \varepsilon]$

$$\left|\langle\mu^*_{[a,b]}, f\rangle - \langle\mu, \rho \star (f\cdot\mathcal{I}_{[a,b]})\rangle\right| \leq \Delta^0_{\mu;[a,b]}(\varepsilon)\sup_{x\in[a,b]}|f(x)| + \Delta^1_{\mu;[a,b]}(\varepsilon)\sup_{x\in[a,b]}|f'(x)|\,.$$

Proof: Let $\rho \in \mathrm{C}^\infty_c(\mathbb{R})$ be a symmetric and positive-valued function with $\mathrm{supp}(\rho) \subseteq [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Let $f \in \mathrm{C}^1(\mathbb{R})$. For $[a,b] \subseteq \mathbb{R}$, define $f^\rho_{[a,b]} = \left(f\cdot\mathcal{I}_{[a,b]}\right) \star \rho$.

Analysis of the continuous part. Now, we consider $\langle \mu_c, f^{\rho}_{[a,b]} \rangle$ — since $f^{\rho}_{[a,b]} \in \mathrm{C}^1_c(\mathbb{R})$, we note that

$$\langle \mu_c, f^{\rho}_{[a,b]} \rangle = -\int_{\mathbb{R}} \phi_c(x) f^{\rho'}_{[a,b]}(x) dx.$$

We note that $\forall x \in \mathbb{R}$,

$$f^{\rho'}_{[a,b]}(x) = \int_a^b \rho'(x-y) f(y) dy = f(a)\rho(x-a) - f(b)\rho(x-b) + \int_a^b f'(y)\rho(x-y)dy.$$

Therefore,

$$\langle \mu_c, f^{\rho}_{[a,b]} \rangle = \int_{\mathbb{R}} f(b)\phi_c(y)\rho(y-b)dy - \int_{\mathbb{R}} f(a)\phi_c(y)\rho(y-a)dy - \int_{\substack{y\in\mathbb{R}\\ x\in[a,b]}} \phi_c(y) f'(x)\rho(y-x) dx dy.$$

Since $\varepsilon < (b-a)/2 \implies \mathrm{supp}(\rho) \subseteq [-(b-a)/2, (b-a)/2]$, this can be rewritten with integrals being only over compact intervals,

$$\langle \mu_c, f^{\rho}_{[a,b]} \rangle = \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(b)\phi_c(y)\rho(y-b)dy - \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(a)\phi_c(y)\rho(y-a)dy - \int_{y=\frac{3a-b}{2}}^{\frac{3b-a}{2}} \int_{x=a}^{b} \phi_c(y) f'(x)\rho(y-x) dx dy.$$

Now, since ϕ_c is continuous, it is uniformly continuous over the compact interval $[(3a-b)/2, (3b-a)/2]$. Thus, $\exists \delta_{\mu_c;a,b}(\varepsilon)$ where $\delta_{\mu_c;[a,b]}(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\forall y, y' \in [(3a-b)/2, (3b-a)/2]$ with $|y-y'| < \varepsilon$, $|\phi_c(y) - \phi_c(y')| < \delta_{\mu_c;[a,b]}(\varepsilon)$. Using this, we obtain that $\forall x \in [a,b]$

$$\left|\phi_c(x) - \int_{y\in[(3a-b)/2,(3b-a)/2]} \phi_c(y)\rho(x-y)dy\right| = \int_{y\in[x-\varepsilon,x+\varepsilon]} \left|\left(\phi_c(x) - \phi_c(y)\right)\right| \rho(x-y)dy \le \delta_{\mu_c;[a,b]}.$$

It then follows that

$$\left|\langle \mu_c^*, f\rangle - \langle \mu_c, f^{\rho}_{[a,b]} \rangle\right| \le \sum_{x\in\{a,b\}} |f(x)| \left|\phi_c(x) - \int_{[(3a-b)/2,(3b-a)/2]} \phi_c(y)\rho(y-x)dy\right| +$$
$$\int_{x\in[a,b]} \left|\phi_c(y) - \int_{y\in[(3a-b)/2,(3b-a)/2]} \phi_c(y)\rho(x-y)dy\right| |f'(x)|\, dx,$$

and consequently,

$$\left|\langle \mu_c^*, f\rangle - \langle \mu_c, f^{\rho}_{[a,b]} \rangle\right| \le \delta_{\mu_c;[a,b]}(\varepsilon)\left(2\sup_{x\in[a,b]}|f(x)| + \int_a^b |f'(x)|\, dx\right) \le \Delta^0_{\mu_c;[a,b]}(\varepsilon) \sup_{x\in[a,b]}|f(x)| + \Delta^1_{\mu_c;[a,b]} \sup_{x\in[a,b]}|f'(x)|, \tag{8a}$$

where

$$\Delta^0_{\mu_c;[a,b]}(\varepsilon) = 2\delta_{\mu_c;[a,b]}(\varepsilon) \text{ and } \Delta^1_{\mu_c;[a,b]}(\varepsilon) = (b-a)\delta_{\mu_c;[a,b]}(\varepsilon), \tag{8b}$$

both of which $\to 0$ as $\varepsilon \to 0$.

Analysis of the atomic part. We now consider $\langle \mu_d, f^{\rho}_{[a,b]} \rangle$ — since $f^{\rho}_{[a,b]} \in \mathrm{C}^0_c(\mathbb{R})$, we obtain that

$$\langle \mu_d, f^{\rho}_{[a,b]} \rangle = \sum_{i\in I} \alpha_i f^{\rho}_{[a,b]}(y_i).$$

Therefore,

$$\left|\langle \mu_d^*, f\rangle - \langle \mu_d, f^{\rho}_{[a,b]} \rangle\right| \le \sum_{i\in I|y_i\notin[a,b]} |\alpha_i| \left|f^{\rho}_{[a,b]}(y_i)\right| + \sum_{i\in I|y_i\in(a,b)} |\alpha_i| \left|f(y_i) - f^{\rho}_{[a,b]}(y_i)\right| + \sum_{i\in I|y_i\in\{a,b\}} |\alpha_i| \left|\frac{f(y_i)}{2} - f^{\rho}_{[a,b]}(y_i)\right|.$$

Since $\text{supp}(f^{\rho}_{[a,b]}) \subseteq [a-\varepsilon, a+\varepsilon]$, and $\left\|f^{\rho}_{[a,b]}\right\|_{L^\infty} \leq \sup_{x\in[a,b]}|f(x)|$, we obtain that

$$\sum_{i\in I|y_i\notin[a,b]} |\alpha_i| \left|f^{\rho}_{[a,b]}(y_i)\right| \leq \left(\sup_{x\in[a,b]}|f(x)|\right)\left(\sum_{i\in I|y_i\in[a-\varepsilon,a)}|\alpha_i| + \sum_{i\in I|y_i\in(b,b+\varepsilon]}|\alpha_i|\right).$$

Furthermore, we note that for $x\in[a+\varepsilon, b-\varepsilon]$, we obtain that

$$\left|f(x)-f^{\rho}_{[a,b]}(x)\right| \leq \int_{[-\varepsilon,\varepsilon]} |f(x)-f(x-y)|\,\rho(y)dy \leq \sup_{y\in[a,b]}|f'(y)| \int_{[-\varepsilon,\varepsilon]} |y|\rho(y)dy \leq \varepsilon \sup_{y\in[a,b]}|f'(y)|\,,$$

and thus we obtain that

$$\sum_{i\in I|y_i\in(a,b)} |\alpha_i| \left|f(y_i)-f^{\rho}_{[a,b]}(y_i)\right| \leq 2\sup_{y\in[a,b]}|f(y)| \sum_{\substack{i\in I|y_i\in(a,a+\varepsilon] \\ \text{or } y_i\in(b-\varepsilon,b]}} |\alpha_i| + \varepsilon \sup_{y\in[a,b]}|f'(y)| \sum_{i\in I|y_i\in(a+\varepsilon,b-\varepsilon)} |\alpha_i|\,.$$

Similarly, we can note that

$$\left|\frac{1}{2}f(a)-f^{\rho}_{[a,b]}(a)\right| \leq \int_{[0,\varepsilon]} |f(a)-f(a+y)|\,\rho(y)dy \leq \sup_{y\in[a,b]}|f'(y)| \int_{[0,\varepsilon]} y\rho(y) \leq \frac{\varepsilon}{2}\sup_{y\in[a,b]}|f'(y)| \text{ and,}$$
$$\left|\frac{1}{2}f(b)-f^{\rho}_{[a,b]}(b)\right| \leq \int_{[0,\varepsilon]} |f(b)-f(b-y)|\,\rho(y)dy \leq \sup_{y\in[a,b]}|f'(y)| \int_{[0,\varepsilon]} y\rho(y) \leq \frac{\varepsilon}{2}\sup_{y\in[a,b]}|f'(y)|\,,$$

and thus we obtain that

$$\sum_{i\in I|y_i\in\{a,b\}} |\alpha_i| \left|\frac{f(y_i)}{2}-f^{\rho}_{[a,b]}(y_i)\right| \leq \frac{\varepsilon}{2}\sup_{y\in[a,b]}|f'(y)| \sum_{i\in I|y_i\in\{a,b\}} |\alpha_i|\,.$$

Therefore, we obtain that

$$\left|\langle\mu_d^*, f\rangle - \langle\mu_d, f^{\rho}_{[a,b]}\rangle\right| \leq \Delta^0_{\mu_d;[a,b]}(\varepsilon)\sup_{y\in[a,b]}|f(y)| + \Delta^1_{\mu_d;[a,b]}(\varepsilon)\sup_{y\in[a,b]}|f'(y)|\,, \tag{9a}$$

where

$$\Delta^0_{\mu_d;[a,b]}(\varepsilon) = \sum_{\substack{i\in I|y_i\in[a-\varepsilon,a) \text{ or} \\ y_i\in(b,b+\varepsilon]}} |\alpha_i| + 2\sum_{\substack{i\in I|y_i\in(a,a+\varepsilon] \\ \text{or } y_i\in(b-\varepsilon,b]}} |\alpha_i| \text{ and } \Delta^1_{\mu_d;[a,b]}(\varepsilon) = \varepsilon\left(\sum_{i\in I|y_i\in(a+\varepsilon,b-\varepsilon)}|\alpha_i| + \frac{1}{2}\sum_{i\in I|y_i\in\{a,b\}}|\alpha_i|\right). \tag{9b}$$

We can note that, by construction, $\Delta^0_{\mu_d;[a,b]}(\varepsilon), \Delta^1_{\mu_d;[a,b]}(\varepsilon) \to 0$ as $\varepsilon\to 0$. Using Eqs. 8 and 9, we obtain the lemma statement. □

Lemma 10 (Existence of μ^*)**.** *Given a $\mu\in\mathcal{M}(\mathbb{R})$, consider the map $\mu_\varepsilon : \mathrm{PWC}^1(\mathbb{R}) \to \mathbb{C}$ given by $\langle\mu_\varepsilon,\cdot\rangle = \langle\mu, \rho_\varepsilon \star (\cdot)\rangle$ with ρ being a symmetric mollifier, then*

(a) $\forall f\in\mathrm{PWC}^1(\mathbb{R})$, $\langle\mu^*, f\rangle := \lim_{\varepsilon\to 0}\langle\mu_\varepsilon, f\rangle$ *exists and is independent of the choice of the mollifier,*

(b) $\forall f\in\mathrm{C}^1_c(\mathbb{R})$, $\langle\mu^*, f\rangle = \langle\mu, f\rangle$.

Proof: Suppose that $f\in\mathrm{PWC}^1(\mathbb{R})$ has the representation $f = g\cdot\mathcal{I}_{[a,b]}$ for some $g\in\mathrm{C}^1(\mathbb{R})$ and compact $[a,b]\subseteq\mathbb{R}$. Part *(a)* of the lemma follows directly from lemma 9 from which it follows that $\lim_{\varepsilon\to 0}\langle\mu, \rho_\varepsilon\star f\rangle = \langle\mu^*_{[a,b]}, g\rangle$, which can be identified as $\langle\mu^*, f\rangle$. Furthermore, we note that by construction, μ^* is independent of the choice of the mollifier.

For part *(b)*, we note that for $f = g\cdot\mathrm{I}_{[a,b]} \in \mathrm{C}^1_c(\mathbb{R}) \subseteq \mathrm{PWC}^1(\mathbb{R})$, $g(a) = g(b) = 0$, and therefore from the definition of $\langle\mu^*_{[a,b]}, g\rangle$ in lemma 9

$$\langle\mu^*, f\rangle = \langle\mu^*_{[a,b]}, g\rangle = -\int_a^b \phi_c(x)g'(x)dx + \sum_{i\in I}\alpha_i g(y_i) = -\int_a^b \phi_c(x)f'(x)dx + \sum_{i\in I}\alpha_i f(y_i) = \langle\mu, f\rangle,$$

which proves the lemma statement. □

Finally, lemma 9 also straightforwardly yields a convergence estimate for the limit that defines μ^* — the functions appearing in the convergence estimate ($\Delta^0_{\mu;[a,b]}, \Delta^1_{\mu;[a,b]}$ in lemma 9) will be key to our analysis in the following sections, and we collect them into the following definition.

Definition 11 (Error functions for μ)**.** *Given $\mu \in \mathcal{M}(\mathbb{R})$ and its associated $\mu^* : \text{PWC}^1(\mathbb{R}) \to \mathbb{C}$ (Definition 10), the error functions of μ for a compact interval $[a,b] \subseteq \mathbb{R}$, $\Delta^0_{\mu;[a,b]}, \Delta^1_{\mu;[a,b]} : (0,(b-a)/2) \to \mathbb{R}^+$ are functions such that $\Delta^0_{\mu;[a,b]}(\varepsilon), \Delta^1_{\mu;[a,b]}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and $\forall f \in \text{PWC}^1(\mathbb{R})$ expressible as $g \cdot \mathcal{I}_{[a,b]}$ for some $g \in \text{C}^1(\mathbb{R})$ and $\forall$ symmetric mollifiers $\rho \in \text{C}^\infty_c(\mathbb{R})$ with $\text{supp}(\rho) \subseteq [-\varepsilon, \varepsilon]$ with $\varepsilon < (b-a)/2$,*

$$|\langle \mu^*, f\rangle - \langle \mu, \rho \star f\rangle| \leq \Delta^0_{\mu;[a,b]}(\varepsilon) \sup_{x\in[a,b]} |f(x)| + \Delta^1_{\mu;[a,b]}(\varepsilon) \sup_{x\in[a,b]} |f'(x)| .$$

Example 1 (Square integrable coupling functions)**.** *Although we analyzed square-integrable coupling functions separately in the previous section, they can also be represented as, and thus are a special case of, distributional coupling functions. In particular, for $v \in L^2(\mathbb{R})$, with Fourier transform $\hat{v} \in L^2(\mathbb{R})$, note that*

$$\kappa(t) = \int_{-\infty}^{\infty} |\hat{v}(\omega)|^2 e^{-i\omega t} d\omega,$$

is a continuous function with $\|\kappa\|_{L^\infty} \leq \|v\|^2_{L^2}$, and thus can be described by the radon measure μ defined by

$$\langle \mu, f\rangle = \int_{-\infty}^{\infty} f(t)\kappa(t)dt \;\forall\; f \in \text{C}^0_c(\mathbb{R}) \tag{10}$$

is a Radon measure. Furthermore, the map $\mu^* : \text{PWC}^1(\mathbb{R}) \to \mathbb{C}$ as defined in definition 10 is given by

$$\langle \mu^*, f\rangle = \int_{-\infty}^{\infty} f(t)\kappa(t)dt \;\forall\; f \in \text{PWC}^1(\mathbb{R}),$$

Consider now $f = g \cdot \mathcal{I}_{[a,b]} \in \text{PWC}^1(\mathbb{R})$ for some compact interval $[a,b] \subseteq \mathbb{R}$ and $g \in \text{C}^1(\mathbb{R})$. Since $\kappa \in \text{C}^1(\mathbb{R})$, $\exists \delta_{\kappa,[a,b]}(\varepsilon) > 0$, where $\delta_{\kappa;[a,b]}(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $\forall x, x' \in [(3a-b)/2, (3b-a)/2]$, with $|x - x'| < \varepsilon$, $|f(x) - f(x')| \leq \delta_{\kappa;[a,b]}(\varepsilon)$. Consequently,

$$|\langle \mu^*, f\rangle - \langle \mu, f \star \rho\rangle| \leq \delta_{\kappa;[a,b]}(\varepsilon) \sup_{x\in[a,b]} |f(x)| ,$$

and thus the error functions (definition 11) for μ defined in Eq. 10 are $\Delta^0_{\mu;[a,b]} = \delta_{\kappa;[a,b]}, \Delta^1_{\mu;[a,b]} = 0$. As a specific example, we consider coupling functions which in frequency-domain are expressible as sum of lorentzians i.e.

$$|\hat{v}(\omega)|^2 = \sum_{i=1}^M \frac{\alpha_i}{(\omega - \omega_i)^2 + \gamma_i^2} \;\forall \omega \in \mathbb{R}, \text{ or equivalently } \kappa(t) = \sum_{j=1}^M \frac{\alpha_j}{2\gamma_j} e^{-\gamma_j |t|} e^{-i\omega_j t}$$

for some $\{\alpha_i \in \mathbb{R}_{>0}\}_{i\in\{1,2\ldots M\}}, \{\omega_i \in \mathbb{R}\}_{i\in\{1,2\ldots M\}}$ and $\{\gamma_i \in \mathbb{R}_{>0}\}_{i\in\{1,2\ldots M\}}$. Such coupling functions arise commonly in modelling resonant light-matter interactions in quantum optics []. For this model, since κ is differentiable almost everywhere and $\|\kappa'\|_{L^\infty} \leq \sum_{j=1}^M \alpha_j \sqrt{\gamma_j^2 + \omega_j^2}/2\gamma_j$ and hence $\delta_{\kappa;[a,b]}(\varepsilon) = \varepsilon \sum_{j=1}^M \alpha_j \sqrt{\gamma_j^2 + \omega_j^2}/2\gamma_j$.

Example 2 (Delta trains)**.** *This class of coupling functions arise frequently in models studying quantum systems with time-delay and feedback. Consider the coupling function specified by radon measure $\mu \in \mathcal{M}(\mathbb{R})$*

$$\langle \mu, f\rangle = \sum_{i=1}^M \alpha_i f(x_i) \;\forall f \in \text{C}^0_c(\mathbb{R}) \left(\text{equivalently } \mu \cong \sum_{i=1}^M \alpha_i \delta(x - x_i)\right)$$

for some $\{x_i \in \mathbb{R}\}_{i\in\{1,2\ldots M\}}, \{\alpha_i \in \mathbb{C}\}_{i\in\{1,2\ldots M\}}$ with $x_1 < x_2 \cdots < x_M$. Furthermore, the map $\mu^* : \text{PWC}^1(\mathbb{R}) \to \mathbb{C}$, as *defined in definition 10 is given by*

$$\langle \mu^*, f \rangle = \sum_{i=1}^{M} \alpha_i \frac{1}{2}\bigg(\lim_{x \to x_i^+} f(x) + \lim_{x \to x_i^-} f(x) \bigg).$$

The error functions (definition 11) for μ can be chosen to be

$$\Delta^0_{\mu,[a,b]}(\varepsilon) = \sum_{\substack{i \in \{1,2\dots M\} | \\ y_i \in [a-\varepsilon, a) \cup (b, b+\varepsilon]}} |\alpha_i| + \sum_{\substack{i \in \{1,2\dots M\} | \\ y_i \in (a, a+\varepsilon] \cup (b-\varepsilon, b]}} 2\,|\alpha_i|\,, \Delta^1_{\mu,[a,b]}(\varepsilon) = \varepsilon \bigg(\sum_{\substack{i \in \{1,2\dots M\} | \\ y_i \in (a+\varepsilon, b-\varepsilon)}} |\alpha_i| + \frac{1}{2} \sum_{\substack{i \in \{1,2\dots M\} | \\ y_i \in \{a,b\}}} |\alpha_i| \bigg).$$

We refer the reader to the proof of lemma 9 for a derivation of these error functions in a more general setting of a delta train with a countably finite number of delta functions.

Example 3 (Complex gaussian)**.** *Consider the coupling function specified by the radon measure $\mu \in \mathcal{M}(\mathbb{R})$*

$$\langle \mu, f \rangle = \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{ik_j x^2} f(x) dx \;\forall\, f \in \mathrm{C}_c^0(\mathbb{R}),$$

where $k_j \in \mathbb{R}, c_j \in \mathbb{C}$ for $j \in \{1, 2 \dots M\}$. Such a coupling function arises frequently in the study of quantum optical systems where the bath is a wire or channel with group velocity dispersion. Furthermore, the map $\mu^* : \mathrm{PWC}^1(\mathbb{R}) \to \mathbb{C}$, as defined in definition 10 is given by

$$\langle \mu^*, f \rangle = \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{ik_j x^2} f(x) dx \;\forall\, f \in \mathrm{C}_c^0(\mathbb{R}),$$

independent of the mollifier ρ. The error functions (definition 11) for μ can be chosen to be

$$\Delta^0_{\mu,[a,b]}(\varepsilon) = \varepsilon \sum_{j=1}^{M} |c_j k_j| \max \left(\left| \frac{3b-a}{2} \right|, \left| \frac{3a-b}{2} \right| \right) \;\textit{and}\; \Delta^1_{\mu,[a,b]}(\varepsilon) = 0.$$

Definition 1, repeated (Non-Markovian model)**.** *A non-Markovian open system model for a quantum system with Hilbert space $\mathcal{H}_S$ is specified by*

(a) A time-dependent system Hamiltonian $H_S(t) \in \mathfrak{L}(\mathcal{H}_S)$ which is Hermitian, norm continuous and differentiable in t,

(b) A set of distributional coupling functions $\{(\mu_i, \varphi_i)\}_{i \in \{1,2\dots M\}}$ as defined in Definition 8,

(c) A set of bounded coupling operators $\{L_i \in \mathfrak{L}(\mathcal{H}_S)\}_{i \in \{1,2\dots M\}}$.

Definition 2, repeated (Regularization)**.** *For $\varepsilon > 0$ and given a symmetric mollifier $\rho \in \mathrm{C}_c^\infty(\mathbb{R})$, an $\varepsilon, \rho-$regularization of a distributional coupling function (μ, φ) is a square integrable function $v_\varepsilon \in L^2(\mathbb{R})$ whose fourier transform $\hat{v}_\varepsilon \in L^2(\mathbb{R})$ is given by*

$$\hat{v}_\varepsilon(\omega) = \sqrt{\hat{\mu}(\omega)}\hat{\rho}(\omega\varepsilon)e^{i\varphi(\omega)} \;\forall \omega \in \mathbb{R}.$$

It is easily seen that $\hat{v} \in L^2(\mathbb{R})$, since by assumption $\hat{\mu}(\omega)$ has atmost polynomial growth in ω, and $\hat{\rho} \in \mathcal{S}(\mathbb{R})$ decays faster than any polynomial. For square integrable coupling functions, lemma 6 guarantees the existence of the solution to the Schroedinger's equation — we can then study whether the solution to the Schroedinger's equation converges as $\varepsilon \to 0$ and define the limit as the dynamics associated with the coupling function specified by (μ, ρ, φ).

Lemma 11. *Consider two non-Markovian models with coupling functions $v = \{v_\alpha \in L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$ and $u = \{u_\alpha \in L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$ respectively but with the same system Hamiltonian $H_S(t)$, jump operators $\{L_\alpha\}_{\alpha \in \{1,2\dots M\}}$ and single-particle environment dynamics described by the time-translation unitary group. Let $|\Psi_v(t)\rangle = U_v(t,0)\,|\Psi_0\rangle\,, |\Psi_u(t)\rangle = U_u(t,0)\,|\Psi_0\rangle$, where $U_v(t,s), U_u(t,s)$ are the propagators corresponding to the two non-Markovian models, and $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}_\infty^M[L^2(\mathbb{R})]$, then*

$$\| |\Psi_u(t)\rangle - |\Psi_v(t)\rangle \|^2 \leq \sum_{\alpha=1}^{M} \bigg(\int_0^t \mathcal{D}_\alpha^{u,v}(\tau) d\tau + \int_0^t \mathcal{E}_\alpha^{u,v}(\tau) d\tau \bigg),$$

where for $\tau \in [0, t]$ and $\alpha \in \{1, 2 \dots M\}$,

$$\mathcal{D}_\alpha^{u,v}(\tau) = 4 \left\| L \right\|_\alpha \left\| a^-_{\alpha, \tau_\tau(u_\alpha - v_\alpha)} |\Psi_0\rangle \right\|, \text{ and}$$
$$\mathcal{E}_\alpha^{u,v}(\tau) = 2 \bigg| \int_0^\tau \langle \tau_\tau(u_\alpha - v_\alpha), \tau_s v_\alpha \rangle G^{v,v,u}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(s, \tau) ds \bigg| + 2 \bigg| \int_0^\tau \langle \tau_s u_\alpha, \tau_\tau(u_\alpha - v_\alpha) \rangle G^{u,u,v}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(\tau, s) ds \bigg|,$$

where the Green's functions $G^{v,v,u}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(s, \tau)$ and $G^{u,u,v}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(\tau, s)$ are defined in definition 7.

Proof: Note that $\left\| |\Psi_u(t)\rangle - |\Psi_v(t)\rangle \right\|^2 = 2 - 2\,\text{Re}[\langle \Psi_u(t) | \Psi_v(t) \rangle]$. Consider now the inner product $\langle \Psi_u(t) | \Psi_v(t) \rangle$ — differentiating this with respect to t, we obtain that

$$\frac{d}{dt} \langle \Psi_u(t) | \Psi_v(t) \rangle = i \sum_{\alpha=1}^M \bigg(\langle \Psi_u(t) | L_\alpha^\dagger a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_v(t)\rangle + \langle \Psi_u(t) | L_\alpha a^+_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_v(t)\rangle \bigg).$$

We note that from lemma 7, it follows that $\forall \alpha \in \{1, 2 \dots M\}$,

$$\begin{aligned} \langle \Psi_u(t) | L_\alpha^\dagger a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_v(t)\rangle &= \langle \Psi_0 | U_u(0, t) L_\alpha^\dagger a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} U_v(t, 0) |\Psi_0\rangle, \\ &= G^{v,u}_{L_\alpha^\dagger; a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle, |\Psi_0\rangle} - i \int_0^t \langle \tau_t(u_\alpha - v_\alpha), \tau_s v_\alpha \rangle G^{v,v,u}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(s, t) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} \langle \Psi_u(t) | L_\alpha a^+_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_v(t)\rangle &= \langle \Psi_0 | U_u(0, t) L_\alpha a^+_{\alpha, \tau_t(u_\alpha - v_\alpha)} U_v(t, 0) |\Psi_0\rangle, \\ &= G^{v,u}_{L_\alpha; |\Psi_0\rangle, a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle} + i \int_0^t \langle \tau_s u_\alpha, \tau_t(u_\alpha - v_\alpha) \rangle G^{u,u,v}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(t, s) ds. \end{aligned}$$

Here, we have used the notation for Green's functions defined in definition 7. Furthermore, we can note that $\forall \alpha \in \{1, 2 \dots M\}$,

$$\left| G^{v,u}_{L_\alpha^\dagger; a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle, |\Psi_0\rangle} \right|, \left| G^{v,u}_{L_\alpha; |\Psi_0\rangle, a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle} \right| \leq \| L_\alpha \| \left\| a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle \right\|,$$

and thus we obtain

$$\begin{aligned} \left| \frac{d}{dt} \langle \Psi_u(t) | \Psi_v(t) \rangle \right| &\leq 2 \sum_{\alpha=1}^M \| L_\alpha \| \left\| a^-_{\alpha, \tau_t(u_\alpha - v_\alpha)} |\Psi_0\rangle \right\| + \\ &\sum_{\alpha=1}^M \bigg(\bigg| \int_0^t \langle \tau_t(u_\alpha - v_\alpha), \tau_s v_\alpha \rangle G^{v,v,u}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(s, t) ds \bigg| + \bigg| \int_0^t \langle \tau_s v_\alpha, \tau_t(u_\alpha - v_\alpha) \rangle G^{u,u,v}_{L_\alpha, L_\alpha^\dagger; |\Psi_0\rangle, |\Psi_0\rangle}(t, s) ds \bigg| \bigg). \end{aligned} \tag{11}$$

Finally, noting that

$$\left\| |\Psi_u(t)\rangle - |\Psi_v(t)\rangle \right\|^2 \leq 2 \big| 1 - \langle \Psi_u(t) | \Psi_v(t) \rangle \big| \leq 2 \int_0^t \left| \frac{d}{d\tau} \langle \Psi_u(\tau) | \Psi_v(\tau) \rangle \right| d\tau. \tag{12}$$

Combining the estimates in Eq. 11 and 12, we obtain the lemma statement. □.

Definition 12. *For $M \in \mathbb{Z}_{\geq 1}$, define $\mathrm{F}^M_{\infty, \mathcal{S}} \subset \mathrm{F}^M_\infty[L^2(\mathbb{R})]$ as the set of vectors $|\Phi\rangle$ such that*

$$\forall n \in \mathbb{Z}_{\geq 1} : \Pi_n |\Phi\rangle \in \text{span}\bigg(\bigg\{ u^{\otimes n} \bigg| u = \bigoplus_{\alpha=1}^M u_\alpha, u_\alpha \in \mathcal{S}(\mathbb{R}) \bigg\} \bigg).$$

Lemma 12. *Let (μ, φ) be a distributional coupling function. Given two mollifiers $\rho, \sigma \in \mathrm{C}_c^\infty(\mathbb{R})$ and $\varepsilon, \delta > 0$, let v_ε and $v_\delta \in L^2(\mathbb{R})$ be the $\varepsilon, \rho-$ and $\delta, \sigma-$regularization of (μ, φ) respectively. Let $\mathcal{H}_S$ be Hilbert space, then,*

(a) $\forall |\Phi\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\infty, \mathcal{S}}$, $\exists c_{\mu, |\Phi\rangle} > 0$, $\forall \tau \geq 0$, $\forall \alpha \in \{1, 2 \dots M\}, \varepsilon > 0$ such that $\left\| a^-_{\alpha, \tau_\tau v_\varepsilon} |\Phi\rangle \right\| \leq c_{\mu, |\Phi\rangle}$.

(b) $\forall |\Phi\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\infty, \mathcal{S}}$, $\exists c_{\mu, \rho, |\Phi\rangle}, d_{\mu, \sigma, |\Phi\rangle} > 0$, $\forall \tau \geq 0, \forall \alpha \in \{1, 2 \dots M\}$, $\varepsilon, \delta > 0$ such that $\left\| a^-_{\alpha, \tau_\tau(v_\varepsilon - v_\delta)} |\Phi\rangle \right\| \leq$

$c_{\mu,\rho,|\Phi\rangle}\varepsilon + c_{\mu,\sigma,|\Phi\rangle}\delta.$

Proof: Any state $|\Psi\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\infty,\mathcal{S}}$ can be expressed as

$$|\Psi\rangle = \sum_{j=1}^{N} |\sigma_j\rangle \otimes |u_j\rangle^{\otimes n_j},$$

for some $N \in \mathbb{Z}_{\geq 1}$, and

$$\{|\sigma_j\rangle \in \mathcal{H}_S\}_{j\in\{1,2\ldots N\}}, \left\{ |u_j\rangle = \bigoplus_{\alpha=1}^{M} |u_{\alpha,j}\rangle, u_{\alpha,j} \in \mathcal{S}(\mathbb{R})\ \forall\ \alpha \in \{1,2\ldots M\}\right\}_{j\in\{1,2\ldots N\}} \text{ and } \{n_j \in \mathbb{Z}_{\geq 0}\}_{j\in\{1,2\ldots N\}}.$$

(a) We obtain that

$$\begin{aligned}\left\|a^-_{\alpha,\tau_\tau v_\varepsilon}|\Phi\rangle\right\| &\leq \sum_{j=1}^{N}\sqrt{n_j}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\left|\int_{\mathbb{R}}\sqrt{\hat{\mu}(\omega)}\hat{\rho}(\omega\varepsilon)u_{\alpha,j}(\omega)d\omega\right|,\\ &\leq \sum_{j=1}^{N}\sqrt{n_j}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\int_{\mathbb{R}}\left|\sqrt{\hat{\mu}(\omega)}\hat{\rho}(\omega\varepsilon)u_{\alpha,j}(\omega)\right|d\omega.\end{aligned}$$

Note that by assumption, $|\sqrt{\hat{\mu}(\omega)}|$ is a continuous function of at-most polynomial growth. Since $\forall j \in \{1,2\ldots N\}, \alpha \in \{1,2\ldots M\}$, $u_{\alpha,j} \in \mathcal{S}(\mathbb{R})$,

$$\left\|\sqrt{\hat{\mu}(\omega)}u_{\alpha,j}(\omega)(1+\omega^2)\right\|_{L^\infty} < \infty.$$

Furthermore, note that since ρ is a mollifier,

$$\|\hat{\rho}\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\rho(s)|ds = \frac{1}{\sqrt{2\pi}}.$$

Therefore,

$$\begin{aligned}\left\|a^-_{\alpha,\tau_\tau v_\varepsilon}|\Phi\rangle\right\| &\leq \sum_{j=1}^{N}\sqrt{\frac{n_j}{2\pi}}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\left\|\sqrt{\hat{\mu}(\omega)}u_{\alpha,j}(\omega)(1+\omega^2)\right\|_{L^\infty}\int_{\mathbb{R}}\frac{d\omega}{1+\omega^2},\\ &\leq \sup_{\alpha\in\{1,2\ldots M\}}\left(\sum_{j=1}^{N}\sqrt{\frac{\pi n_j}{2}}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\left\|\sqrt{\hat{\mu}(\omega)}u_{\alpha,j}(\omega)(1+\omega^2)\right\|_{L^\infty}\right),\end{aligned}$$

where the bound can be identified as the constant $c_{\mu,|\Psi\rangle}$ in the lemma statement.

(b) We obtain that

$$\begin{aligned}\left\|a^-_{\alpha,\tau_\tau(v_\varepsilon-v_\delta)}|\Phi\rangle\right\| &\leq \sum_{j=1}^{N}\sqrt{n_j}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\left|\int_{\mathbb{R}}u_{\alpha,j}(\omega)\sqrt{\hat{\mu}(\omega)}\big(\hat{\rho}(\omega\varepsilon)-\hat{\sigma}(\omega\delta)\big)d\omega\right|\\ &\leq \sum_{j=1}^{N}\sqrt{n_j}\,\|\sigma_j\|\,\|u_j\|^{n_j-1}\int_{\mathbb{R}}\left|\sqrt{\hat{\mu}(\omega)}u_{\alpha,j}(\omega)\big(\hat{\rho}(\omega\varepsilon)-\hat{\sigma}(\omega\delta)\big)\right|d\omega.\end{aligned}$$

Again, since by assumption $|\sqrt{\hat{\mu}(\omega)}|$ is a function of at-most polynomial growth, and $\forall\alpha \in \{1,2\ldots M\}, j \in \{1,2\ldots N\}$, $u_{\alpha,j} \in \mathcal{S}(\mathbb{R})$ and therefore,

$$\left\|\sqrt{\hat{\mu}(\omega)}u_{\alpha,j}(\omega)(1+\omega^2)^2\right\|_{L^\infty} < \infty.$$

Furthermore, since $\rho, \sigma \in \mathrm{C}_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, $\hat{\rho}, \hat{\sigma} \in \mathcal{S}(\mathbb{R})$. In particular, $\|\hat{\rho}'\|_{L^\infty}, \|\hat{\sigma}'\|_{L^\infty} < \infty$. Furthermore,

$$\hat{\rho}(0) - \hat{\sigma}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \rho(s)ds - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \sigma(s)ds = 0,$$

since the mollifiers are, by definition, normalized to have unit area. Thus, using the Taylor's remainder theorem, we obtain that

$$\left|\hat{\rho}(\omega\varepsilon) - \hat{\sigma}(\omega\delta)\right| \leq \|\hat{\rho}'\|_{L^\infty} |\omega|\varepsilon + \|\hat{\sigma}'\|_{L^\infty} |\omega|\delta \;\forall\, \omega \in \mathbb{R}.$$

We thus obtain that

$$\left\|a^-_{\alpha, \tau_\tau(v_\varepsilon - v_\delta)} |\Phi\rangle\right\| \leq \max_{\alpha \in \{1,2\dots M\}} \sum_{j=1}^N \sqrt{n_j}\, \|\sigma_j\| \, \|u_j\|^{n_j - 1} \left\|\sqrt{\hat{\mu}(\omega)} u_{\alpha,j}(\omega)(1+\omega^2)^2\right\|_{L^\infty} \int_{\mathbb{R}} \frac{\|\hat{\rho}'\|_{L^\infty} |\omega|\varepsilon + \|\hat{\sigma}'\|_{L^\infty}|\omega|\delta}{(1+\omega^2)^2} d\omega.$$

Noting that $\int_{\mathbb{R}} |\omega|/(1+\omega^2)^2 d\omega < \infty$, we obtain the lemma statement with the constants

$$c_{\mu,\rho,|\Phi\rangle} = \max_{\alpha \in \{1,2\dots M\}} \sum_{j=1}^N \sqrt{n_j}\, \|\sigma_j\| \, \|u_j\|^{n_j - 1} \left\|\sqrt{\hat{\mu}(\omega)} u_{\alpha,j}(\omega)(1+\omega^2)^2\right\|_{L^\infty} \|\hat{\rho}'\|_{L^\infty} \int_{\mathbb{R}} \frac{|\omega|}{(1+\omega^2)^2} d\omega. \qquad \square$$

Lemma 13. *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, M jump operators $\{L_\alpha\}_{\alpha \in \{1,2\dots M\}}$ and coupling functions, $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha \in \{1,2\dots M\}}$. Given two mollifiers $\rho, \sigma \in \mathrm{C}_c^\infty(\mathbb{R})$ and $\varepsilon, \delta \in (0, 1/2)$, consider two non-Markovian models with the same system Hamiltonian and jump operators, but with square integrable coupling functions given by $v_\varepsilon = \{v_{\alpha,\varepsilon} \in L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$ and $v_\delta = \{v_{\alpha,\delta} \in L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$, where $v_{\alpha,\varepsilon}$ and $v_{\alpha,\delta}$ are $\varepsilon, \rho-$ and $\delta, \sigma-$regularizations of (μ_α, φ) respectively. Given an initial state $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^M_{\infty,\mathcal{S}}[L^2(\mathbb{R})]$, then the errors $\mathcal{E}^{v_\delta, v_\varepsilon}_{\alpha, |\Psi_0\rangle}(t)$ and $\mathcal{D}^{v_\delta, v_\varepsilon}_{\alpha, |\Psi_0\rangle}(t)$ defined in lemma 11 satisfy the estimates*

(a) For all $\alpha \in \{1, 2\dots M\}$ and $t > 0$,

$$\mathcal{E}^{v_\delta, v_\varepsilon}_{\alpha, |\Psi_0\rangle}(t) \leq 4\|L_\alpha\|^2 \,\mathrm{TV}_{[-1,t+1]}(\mu_\alpha).$$

(b) For all $\alpha \in \{1, 2\dots M\}$ and $t > 0$,

$$\begin{aligned}\mathcal{E}^{v_\delta, v_\varepsilon}_{\alpha, |\Psi_0\rangle}(t) \leq 2&\bigg(2\Delta^1_{\mu_\alpha;[0,t]}(\varepsilon+\delta) + \Delta^1_{\mu_\alpha;[0,t]}(2\varepsilon) + \Delta^1_{\mu_\alpha;[0,t]}(2\delta)\bigg) \times \\ &\bigg(\|L_\alpha\| \sup_{s\in[0,t]} \|[H_S(s), L_\alpha]\| + 4\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L'_\alpha\| c_{\mu_{\alpha'}, |\Psi_0\rangle} + 6\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L_{\alpha'}\|^2 \,\mathrm{TV}_{[-1,t+1]}(\mu_{\alpha'}).\bigg) + \\ &2\bigg(2\Delta^0_{\mu_\alpha;[0,t]}(\varepsilon+\delta) + \Delta^0_{\mu_\alpha;[0,t]}(2\varepsilon) + \Delta^0_{\mu_\alpha;[0,t]}(2\delta)\bigg)\|L_\alpha\|^2,\end{aligned}$$

where $\Delta^0_{\mu_\alpha;[-t,0]}, \Delta^1_{\mu_\alpha;[-t,0]}$ are the error functions corresponding to μ_α (definition 11) and $c_{\mu_\alpha,|\Psi_0\rangle}$ is the constant introduced in lemma 12(a).

(c) For all $\alpha \in \{1, 2\dots M\}$ and $t > 0$,

$$\mathcal{D}^{v_\delta, v_\varepsilon}_{\alpha, |\Psi_0\rangle}(t) \leq 4\|L_\alpha\| \left(c_{\mu_\alpha,\rho,|\Psi_0\rangle}\varepsilon + c_{\mu_\alpha,\sigma,|\Psi_0\rangle}\delta\right),$$

where $c_{\mu_\alpha,\rho,|\Psi_0\rangle}, c_{\mu_\alpha,\sigma,|\Psi_0\rangle}$ are constants introduced in lemma 12(b).

Proof: For this proof, it is convenient to note that for $\alpha \in \{1, 2\dots M\}$ and any $s, \in [0, t]$,

$$\langle \tau_\tau v_{\alpha,\delta}, \tau_s v_{\alpha,\varepsilon}\rangle = 2\pi \int_{\mathbb{R}} \hat{\mu}_\alpha(\omega)\hat{\rho}(\varepsilon\omega)\hat{\sigma}^*(\delta\omega) e^{-i\omega(s-t)} d\omega.$$

Since if $\varepsilon, \delta \in (0, 1/2)$, $\mathrm{supp}(\rho_\varepsilon \star \rho_\varepsilon), \mathrm{supp}(\sigma_\delta \star \sigma_\delta), \mathrm{supp}(\sigma_\delta \star \rho_\varepsilon) \subseteq [-1, 1]$. Consequently, $\forall f \in \mathrm{C}^0(\mathbb{R})$ and $t_1, t_2 \in (0, t]$,

$$\left|\int_{t_1}^{t_2} \langle \tau_t v_{\alpha,\delta}, \tau_s v_{\alpha,\varepsilon}\rangle f(s) ds\right| = \left|\langle \mu_\alpha, \sigma_\delta \star \rho_\varepsilon \star \tau_t\big(f \cdot \mathcal{I}_{(t_1,t_2]}\big)\rangle\right| \leq \mathrm{TV}_{[-1,t+1]}(\mu_\alpha) \sup_{s\in[0,t]} |f(s)|. \tag{13a}$$

Similarly,

$$\left|\int_{t_1}^{t_2}\langle \tau_t v_{\alpha,\varepsilon}, \tau_s v_{\alpha,\delta}\rangle f(s)ds\right|, \left|\int_{t_1}^{t_2}\langle \tau_t v_{\alpha,\varepsilon}, \tau_s v_{\alpha,\varepsilon}\rangle f(s)ds\right|, \left|\int_{t_1}^{t_2}\langle \tau_t v_{\alpha,\delta}, \tau_s v_{\alpha,\delta}\rangle f(s)ds\right| \leq \text{TV}_{[-1,t+1]}(\mu_\alpha) \sup_{s\in[0,t]} |f(s)|\,. \tag{13b}$$

(a) We note that

$$\left|\int_0^t \langle \tau_t(v_{\alpha,\delta} - v_{\alpha,\varepsilon}), \tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta}_{L_\alpha,L_\alpha^\dagger}(s,t)ds\right| \leq \left|\int_0^t \langle \tau_t v_{\alpha,\delta}, \tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta}(s,\tau)ds\right| + \left|\int_0^t \langle \tau_t v_{\alpha,\delta}, \tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta}(s,\tau)ds\right|$$

Since $\forall s \in [0,t] : \left|G^{v_\varepsilon,v_\varepsilon,v_\delta}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s,\tau)\right|, \leq \|L_\alpha\|^2$, and using Eq. 13, we obtain that

$$\left|\int_0^t \langle \tau_t(v_{\alpha,\delta} - v_{\alpha,\varepsilon}), \tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta}_{L_\alpha,L_\alpha^\dagger}(s,t)ds\right| \leq 2\,\|L_\alpha\|^2\, \text{TV}_{[-1,t+1]}(\mu_\alpha).$$

Similarly, we can obtain that

$$\left|\int_0^t \langle \tau_s v_{\alpha,\delta}, \tau_t(v_{\alpha,\delta} - v_{\alpha,\varepsilon})\rangle G^{v_\varepsilon,v_\delta,v_\delta}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}\right| \leq 2\,\|L_\alpha\|^2\, \text{TV}_{[-1,t+1]}(\mu_\alpha).$$

Combining these two estimates, we obtain the part *(a)* of the lemma statement.

(b) We begin by noting that

$$\langle \tau_t(v_{\alpha,\delta} - v_{\alpha,\varepsilon}), \tau_s v_{\alpha,\varepsilon}\rangle = 2\pi\int_{-\infty}^{\infty} \hat{\mu}_\alpha(\omega)\hat{\rho}(\varepsilon\omega)\big(\hat{\rho}^*(\varepsilon\omega) - \hat{\sigma}^*(\delta\omega)\big)e^{-i\omega(s-t)}d\omega,$$

and therefore,

$$\int_0^t \langle \tau_t(v_{\alpha,\delta} - v_{\alpha,\varepsilon}), \tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,u_\delta}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s,t)ds = \big\langle \mu_\alpha, \rho_\varepsilon \star \rho_\varepsilon \star g^{\varepsilon,\delta}_{\alpha,t} - \rho_\varepsilon \star \sigma_\delta \star g^{\alpha,t}_{\varepsilon,\delta}\big\rangle,$$

where

$$g^{\varepsilon,\delta}_{\alpha,t} = \tau_t\bigg(G^{v_\varepsilon,v_\varepsilon,u_\delta}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(\cdot,t)\cdot \mathcal{I}_{[0,t]}\bigg).$$

We note that $g^{\varepsilon,\delta}_{\alpha,t}$ is a continuous and differentiable function when restricted to $[-t,0]$ and hence $\in \text{PWC}^1(\mathbb{R})$. Consequently,

$$\begin{aligned}\left|\big\langle \mu_\alpha, \rho_\varepsilon \star \rho_\varepsilon \star g^{\varepsilon,\delta}_{\alpha,t} - \rho_\varepsilon \star \sigma_\delta \star g^{\alpha,t}_{\varepsilon,\delta}\big\rangle\right| &\leq \left|\big\langle \mu_\alpha^*, g^{\varepsilon,\delta}_{\alpha,t}\big\rangle - \big\langle \mu_\alpha, \rho_\varepsilon \star \rho_\varepsilon \star g^{\varepsilon,\delta}_{\alpha,t}\big\rangle\right| + \left|\big\langle \mu_\alpha^*, g^{\varepsilon,\delta}_{\alpha,t}\big\rangle - \big\langle \mu_\alpha, \rho_\varepsilon \star \sigma_\delta \star g^{\alpha,t}_{\varepsilon,\delta}\big\rangle\right| \\ &\leq \bigg(\Delta^0_{\mu_\alpha;[-t,0]}(2\varepsilon) + \Delta^0_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\bigg)\sup_{s\in[0,t]}\left|g^{\varepsilon,\delta}_{\alpha,t}(s)\right| + \bigg(\Delta^1_{\mu_\alpha;[-t,0]}(2\varepsilon) + \Delta^1_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\bigg)\sup_{s\in[0,t]}\left|g^{\varepsilon,\delta'}_{\alpha,t}(s)\right|\end{aligned} \tag{14}$$

Furthermore, note that

$$\sup_{s\in[-t,0]}\left|g^{\varepsilon,\delta}_{\alpha,t}(s)\right| \leq \|L_\alpha\|^2\,, \tag{15}$$

We next provide a bound on the derivative ($\sup_{s\in[-t,0]} |g_{\alpha,t}^{\varepsilon,\delta'}(s)|$) which is uniform in ε,δ. An application of lemma 8, yields

$$\begin{aligned}\left|g_{\alpha,t}^{\varepsilon,\delta'}(s)\right| &\leq \left|G^{v_\varepsilon,v_\varepsilon,v_\delta}_{[H_S(s+t),L_\alpha],L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s+t,t)\right| + \\ &\sum_{\alpha'=1}^M \left(\left|G^{v_\varepsilon,v_\varepsilon,v_\delta}_{[L_{\alpha'}^\dagger,L_\alpha],L_\alpha^\dagger;|\Psi_0\rangle,a^-_{\alpha',\tau_{(s+t)}v_\alpha}|\Psi_0\rangle}(s+t,t)\right| + \left|G^{v_\varepsilon,v_\delta,v_\delta}_{[L_{\alpha'},L_\alpha],L_\alpha^\dagger;|\Psi_0\rangle,a^-_{\alpha',\tau_{(s+t)}v_\alpha}|\Psi_0\rangle}(s+t,t)\right|\right) + \\ &\sum_{\alpha'=1}^M \left(\left|\int_0^{s+\tau} \langle \tau_{s+\tau}v_{\alpha',\varepsilon}, \tau_{s'}v_{\alpha',\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\varepsilon,v_\delta}_{L_{\alpha'},[L_{\alpha'}^\dagger,L_\alpha],L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s',s+\tau,\tau)ds'\right| + \right. \\ &\left|\int_0^{\tau} \langle \tau_{s'}v_{\alpha',\delta}, \tau_{s+\tau}v_{\alpha',\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta,v_\delta}_{[L_{\alpha'},L_\alpha],L_\alpha^\dagger,L_{\alpha'}^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s+\tau,\tau,s')ds'\right| + \\ &\left.\left|\int_{s+\tau}^{\tau} \langle \tau_{s'}v_{\alpha',\delta}, \tau_{s+\tau}v_{\alpha',\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,v_\delta,v_\delta}_{[L_{\alpha'},L_\alpha],L_{\alpha'}^\dagger,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s+\tau,s',\tau)ds'\right|\right).\end{aligned}$$

Using Eq. 13, we thus obtain that

$$\sup_{s\in(-t,0)}\left|g_{\alpha,t}^{\varepsilon,\delta'}(s)\right| \leq \|L_\alpha\| \sup_{s\in[0,t]} \|[H_S(s),L_\alpha]\| + 4\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L_{\alpha'}\| \sup_{s\in[0,t]} \left\|a^-_{\alpha',\tau_s v_{\alpha'}}|\Psi_0\rangle\right\| + 6\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L_\alpha'\|^2 \mathrm{TV}_{[-1,t+1]}(\mu_\alpha)$$

From lemma 12(a), it follows that $\forall \alpha' \in \{1,2\ldots M\}, s\in[0,t]$, $\left\|a^-_{\alpha',\tau_s v_{\alpha'}}|\Psi_0\rangle\right\| \leq c_{\mu_{\alpha'},|\Psi_0\rangle}$ and therefore, we obtain that

$$\sup_{s\in(-t,0)}\left|g_{\alpha,t}^{\varepsilon,\delta'}(s)\right| \leq \sup_{s\in[0,\tau]} \|[H_S(s),L_\alpha]\|\,\|L_\alpha\| + 4\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L_\alpha'\|\, c_{\mu_{\alpha'},|\Psi_0\rangle} + 6\|L_\alpha\|^2 \sum_{\alpha'=1}^M \|L_{\alpha'}\|^2 \mathrm{TV}_{[-1,t+1]}(\mu_{\alpha'}). \tag{16}$$

From Eqs. 14, 15 and 16, we obtain that

$$\begin{aligned}\left|\int_0^t \langle \tau_t(v_{\alpha,\delta}-v_{\alpha,\varepsilon}),\tau_s v_{\alpha,\varepsilon}\rangle G^{v_\varepsilon,v_\varepsilon,u_\delta}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(s,t)ds\right| &\leq \left(\Delta^0_{\mu_\alpha;[-t,0]}(2\varepsilon)+\Delta^0_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\right)\|L_\alpha\|^2 + \Big(\Delta^1_{\mu_\alpha;[-t,0]}(2\varepsilon)+ \\ \Delta^1_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\Big)\Big(\sup_{s\in[0,\tau]}\|[H_S(s),L_\alpha]\|\,\|L_\alpha\| &+ 4\|L_\alpha\|^2\sum_{\alpha'=1}^M\|L_{\alpha'}\|\,c_{\mu_{\alpha'},|\Psi_0\rangle} + 6\|L_\alpha\|^2\sum_{\alpha'=1}^M\|L_{\alpha'}\|^2\mathrm{TV}_{[-1,t+1]}(\mu_{\alpha'})\Big).\end{aligned}$$

Similary,

$$\begin{aligned}\left|\int_0^t \langle \tau_s v_{\alpha,\delta},\tau_t(v_{\alpha,\delta}-v_{\alpha,\varepsilon})\rangle G^{v_\delta,v_\delta,v_\varepsilon}_{L_\alpha,L_\alpha^\dagger;|\Psi_0\rangle,|\Psi_0\rangle}(t,s)ds\right| &\leq \left(\Delta^0_{\mu_\alpha;[-t,0]}(2\delta)+\Delta^0_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\right)\|L_\alpha\|^2 + \Big(\Delta^1_{\mu_\alpha;[-t,0]}(2\delta)+ \\ \Delta^1_{\mu_\alpha;[-t,0]}(\delta+\varepsilon)\Big)\Big(\sup_{s\in[0,\tau]}\|[H_S(s),L_\alpha]\|\,\|L_\alpha\| &+ 4\|L_\alpha\|^2\sum_{\alpha'=1}^M\|L_{\alpha'}\|\,c_{\mu_{\alpha'},|\Psi_0\rangle} + 6\|L_\alpha\|^2\sum_{\alpha'=1}^M\|L_{\alpha'}\|^2\mathrm{TV}_{[-1,t+1]}(\mu_{\alpha'})\Big).\end{aligned}$$

(c) This follows from a direct application of lemma 12.

Theorem 1, repeated (Formal, Non-markovian dynamics)**.** *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, M jump operators $\{L_\alpha\}_{\alpha\in\{1,2\ldots M\}}$ and distributional coupling functions, $\{(\mu_\alpha,\varphi_\alpha)\}_{\alpha\in\{1,2\ldots M\}}$. Construct a square-integral non-Markovian model with the same system Hamiltonian and jump operators, but with coupling functions $v_\varepsilon := \{v_{\alpha,\varepsilon} \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\ldots M\}}$, where for $\alpha\in\{1,2\ldots M\}$, $v_{\alpha,\varepsilon}$ is an $\varepsilon,\rho-$regularization of $(\mu_\alpha,\varphi_\alpha)$ for a symmetric mollifier $\rho\in \mathrm{C}_c^\infty(\mathbb{R})$, $\varepsilon>0$ and let $U_{v_\varepsilon}(\cdot,\cdot)$ be its propagator. Then, for $t>0$, $U(t): \mathrm{F}^M_{\infty,\mathcal{S}}\otimes\mathcal{H}_S\to\mathcal{H}$ defined via $U(t)|\Psi_0\rangle = \lim_{\varepsilon\to 0}U_{v_\varepsilon}(t,0)|\Psi_0\rangle$ exists, is an isometry and is independent of the choice of mollifier ρ.*

Proof: For simplicity, we will assume that $\varepsilon,\delta\in(0,1)$. Consider two symmetric mollifiers $\rho,\sigma\in\mathrm{C}_c^\infty(\mathbb{R})$ — let $v_\varepsilon := \{v_{\alpha,\varepsilon}\in L^2(\mathbb{R})\}_{\alpha\in\{1,2\ldots M\}}$ and $v_\delta := \{v_{\alpha,\delta}\in L^2(\mathbb{R})\}_{\alpha\in\{1,2\ldots M\}}$ be the $\varepsilon,\rho-$ and $\delta,\sigma-$regularizations of the distributional coupling functions. For $|\Psi_0\rangle\in\mathrm{F}^M_{\infty,\mathcal{S}}\otimes\mathcal{H}_S$, let $|\Psi_{v_\varepsilon}(t)\rangle = U_{v_\varepsilon}(t,0)|\Psi_0\rangle$ and $|\Psi_{v_\delta}(t)\rangle = U_{v_\delta}(t,0)|\Psi_0\rangle$, where $U_{v_\varepsilon}(\cdot,\cdot),U_{v_\delta}(\cdot,\cdot)$ are the propagators corresponding to the two models. We note that from lemma 13(c) that $\forall\alpha\in\{1,2\ldots M\}$ and $\tau\in[0,t]$

$$\lim_{\varepsilon,\delta\to 0}\mathcal{D}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)=0 \text{ and } \mathcal{D}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)\leq 4\|L_\alpha\|\left(c_{\mu_\alpha,\rho,|\Psi_0\rangle}+c_{\mu_\alpha,\sigma,|\Psi_0\rangle}\right).$$

From the dominated convergence theorem, we then obtain that

$$\lim_{\varepsilon,\delta\to 0}\int_0^t \mathcal{D}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)d\tau = \int_0^t \lim_{\varepsilon,\delta\to 0}\mathcal{D}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)d\tau = 0.$$

Similarly, we note from lemma 13(b) that $\forall \alpha \in \{1, 2 \dots M\}$ and $\tau \in [0, t]$

$$\lim_{\varepsilon,\delta\to 0}\mathcal{E}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau) = 0.$$

From lemma 13(a), we obtain that $\forall \alpha \in \{1, 2 \dots M\}$ and $\tau \in [0, t]$

$$\mathcal{E}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau) \leq 4\left\|L_\alpha\right\|^2 \text{TV}_{[-1,\tau+1]}(\mu_\alpha) \leq 4\left\|L_\alpha\right\|^2 \text{TV}_{[-1,t+1]}(\mu_\alpha).$$

Hence, again by dominated convergence theorem, we obtain that

$$\lim_{\varepsilon,\delta\to 0}\int_0^t \mathcal{E}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)d\tau = \int_0^t \lim_{\varepsilon,\delta\to 0}\mathcal{E}^{v_\delta,v_\varepsilon}_{\alpha,|\Psi_0\rangle}(\tau)d\tau = 0.$$

We thus obtain from lemma 11 that

$$\lim_{\varepsilon,\delta\to 0}\left\||\Psi_{v_\varepsilon}(t)\rangle - |\Psi_{v_\delta}(t)\rangle\right\| = 0, \tag{17}$$

for all symmetric mollifiers ρ, σ. From this condition, using $\rho = \sigma$, we obtain that $\lim_{\varepsilon\to 0}|\Psi_{v_\varepsilon}(t)\rangle$ exists. Furthermore, $\|\lim_{\varepsilon\to 0}|\Psi_{v_\varepsilon}(t)\rangle\| = \lim_{\varepsilon\to 0}\||\Psi_{v_\varepsilon}(t)\rangle\| = \||\Psi_0\rangle\|$, and hence the operator mapping $|\Psi_0\rangle$ to $\lim_{\varepsilon\to 0}|\Psi_{v_\varepsilon}(t)\rangle$ is an isometry. Furthermore, since the limit exists, Eq. 17 additionally implies that the limit is independent of the choice of the mollifier. □

V. COMPLEXITY OF NON-MARKOVIAN DYNAMICS

A. Certifiable Markovian dilations

In this section, we develop a certifiable Markovian dilation of a non-Markovian model. We use the well-known star-to-chain transformation for mapping the non-Markovian problem to a Hamiltonian simulation problem, and provide error bounds on this dilation.

Definition 13 (Chain unitary group on $L^2(\mathbb{R})$). *Given $v \in L^2(\mathbb{R})$ with* supp$(\hat{v}) \in [-\omega_c, \omega_c]$ *for some $\omega_c > 0$, a chain unitary group with N_m modes generated by v is the strongly continuous single parameter unitary group $\mathsf{v}_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by*

$$\mathsf{v}_t f = \sum_{\beta=1}^{N_m} c_\beta(t)\varphi_\beta + \bigg(f - \sum_{\beta=1}^{N}\langle\varphi_\beta, f\rangle\varphi_\beta\bigg),$$

where

(a) $\{\varphi_\beta \in L^2(\mathbb{R})\}_{\beta\in\{1,2\dots N_m\}}$, called the mode functions, are a set of orthonormal functions (i.e. $\langle\varphi_\alpha, \varphi_\beta\rangle = \delta_{\alpha,\beta}$ that are given by

$$\hat{\varphi}_\alpha(\omega) = \frac{p_\alpha(\omega)\hat{v}(\omega)}{\big[\int_{-\omega_c}^{\omega_c} p_\alpha^2(\omega)|\hat{v}(\omega)|^2 d\omega\big]^{1/2}}\ \forall\ \alpha \in \{1, 2 \dots N_m\},$$

where p_α is a degree $\alpha - 1$ polynomial generated by the following recursion starting from $p_1(\omega) = 1, B_1 = 0$,

$$a_\alpha = \frac{\int_{-\omega_c}^{\omega_c}\omega p_\alpha^2(\omega)|\hat{v}(\omega)|^2 d\omega}{\int_{-\omega_c}^{\omega_c} p_\alpha^2(\omega)|\hat{v}(\omega)|^2 d\omega}, p_{\alpha+1}(\omega) = (\omega - A_\alpha)p_\alpha(\omega) - B_{\alpha-1}p_{\alpha-1}(\omega), B_\alpha = \frac{\int_{-\omega_c}^{\omega_c} p_{\alpha-1}^2(\omega)|\hat{v}(\omega)|^2 d\omega}{\int_{-\omega_c}^{\omega_c} p_\alpha^2(\omega)|\hat{v}(\omega)|^2 d\omega}. \tag{18}$$

(b) The coefficients $\{c_\beta(t) \in \mathbb{C}\}_{\beta\in\{1,2\dots M\}}$ are given by the dynamical law: $c_\beta(0) = \langle\varphi_\beta, f\rangle$ for $\beta \in \{1, 2 \dots M\}$, together

with

$$i\frac{d}{dt}\begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ \vdots \\ c_M(t) \end{bmatrix} = \begin{bmatrix} \omega_1 & t_1 & 0 & 0 & \dots & 0 \\ t_1 & \omega_2 & t_2 & 0 & \dots & 0 \\ 0 & t_2 & \omega_3 & t_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \omega_M \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ \vdots \\ c_M(t) \end{bmatrix} \tag{19}$$

where $\omega_\alpha = A_\alpha \, \forall \alpha \in \{1, 2 \dots M\}$ *and* $t_\alpha = \sqrt{B_{\alpha+1}} \, \forall \, \alpha \in \{1, 2 \dots N_m\}$,

For completeness, we provide a simple and well-known upper bound on the coefficients $\{\omega_\alpha\}_{\alpha \in \{1,2 \dots N_m\}}$ and $\{t_\alpha\}_{\alpha \in \{1,2 \dots N_m - 1\}}$ which will be useful in the following sections.

Lemma 14 (Upper bound on ω_α, t_α (Ref.))**.** *Given a chain unitary group with N_m modes generated by $v \in L^2(\mathbb{R})$ with* $\text{supp}(\hat{v}) \subseteq [-\omega_c, \omega_c]$ *for $\omega_c \geq 0$, then*

$$|\omega_\alpha| \leq \omega_c \text{ and } t_\alpha \leq \omega_c \, \forall \, \alpha \in \{1, 2 \dots N_m\},$$

where $\{\omega_\alpha\}_{\alpha \in \{1,2 \dots N_m\}}$ and $\{t_\alpha\}_{\alpha \in \{1,2 \dots N_m\}}$ are the parameters of the chain unitary group defined in definition 13.

Lemma 15. *Given $v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with* $\text{supp}(f) \in [-\omega_c, \omega_c]$*, let $\nu_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the chain unitary group with N_m modes generated by v (definition 13), then $\forall t \geq 0$,*

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 \leq \|v\|_{L^2}^2 \, N_m^2 e^{N_m} \left(\frac{2\omega_c t}{N_m}\right)^{N_m},$$

where $\tau_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the translation group $[(\tau_t f)(x) = f(x+t) \, \forall \, f \in L^2(\mathbb{R})]$.

Proof: We define the polynomial π_α of degree $\alpha - 1$ via

$$\pi_\alpha = \frac{\|v\|_{L^2}}{\|p_i \hat{v}\|_{L^2}} p_\alpha \text{ for } \alpha \in \{1, 2 \dots N_m\}.$$

We will denote by $A \in \mathbb{R}^{N_m \times N_m}$ the matrix

$$A = \begin{bmatrix} \omega_1 & t_1 & 0 & 0 & \dots & 0 \\ t_1 & \omega_2 & t_2 & 0 & \dots & 0 \\ 0 & t_2 & \omega_3 & t_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \omega_{N_m} \end{bmatrix},$$

We will denote by $\lambda_i \in \mathbb{R}$ and $u^i \in \mathbb{R}^{N_m}, i \in \{1, 2 \dots N_m\}$ the eigenvalues and eigenvectors of the matrix A. We note that if $(\lambda \in \mathbb{R}, u \in \mathbb{R}^{N_m})$ is an eigenvalue, eigenvector pair of A, then

$$\begin{aligned} &\big(\omega_1 - \lambda\big)u_1 + t_1 u_2 = 0, \\ &\big(\omega_i - \lambda\big)u_i + t_{i-1}u_{i-1} + t_i u_{i+1} = 0 \text{ for } i \in \{2, 3 \dots N_m - 1\}, \\ &\big(\omega_{N_m} - \lambda\big)u_{N_m} + t_{N_m - 1}u_{N_m - 1} = 0. \end{aligned}$$

Solving these recursions, we obtain that

$$u_i = u_1 \pi_i(\lambda) \text{ and } \pi_{N_m+1}(\Omega) = 0 \text{ (or } p_{N_m+1}(\Omega) = 0).$$

Therefore, the eigenvalues $\lambda_1, \lambda_2 \dots \lambda_{N_m}$ are the roots of the polynomial p_{N_m+1}, and the eigenvectors are given by

$$u_j^i = \frac{\pi_j(\lambda_i)}{N_i} \text{ where } N_i = \left(\sum_{j=1}^{N_m} \pi_j^2(\lambda_i)\right)^{1/2}$$

It can be noted that the matrix A is hermitian, and consequently, its eigenvectors for an orthonormal basis for $\mathbb{R}^{N_m}$, which implies that

$$\sum_{j=1}^{N_m} \pi_j(\lambda_i)\pi_j(\lambda_{i'}) = N_i^2 \delta_{i,i'} \text{ and } \sum_{i=1}^{N_m} \pi_j(\lambda_i)\pi_{j'}(\lambda_i) = N_i^2 \delta_{j,j'}. \tag{20}$$

We next compute $\nu_t v$ — noting that $v \propto \varphi_1$, we obtain that

$$\nu_t v = \sum_{\beta=1}^{N_m} c_\beta(t)\varphi_\beta, \text{ where } \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_{N_m}(t) \end{bmatrix} = \|v\| \, e^{-iAt} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which can be rewritten as

$$e^{-iAt} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{i=1}^{N_m} \frac{\pi_1(\lambda_i)}{N_i} e^{-i\lambda_i t} u^i \implies c_j(t) = \|v\|_{L^2} \sum_{i=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} e^{-i\lambda_i t}.$$

We now consider

$$\begin{aligned} \frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 &= \|v\|_{L^2}^2 - \sum_{j=1}^{N_m} \text{Re}\bigg(c_j^*(t) \int_{\mathbb{R}} \hat{\varphi}_j^*(\omega)\hat{v}(\omega) e^{-i\omega t} d\omega \bigg) \\ &= \|v\|_{L^2}^2 - \sum_{i,j=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 \cos((\omega - \lambda_i)t) d\omega. \end{aligned}$$

We next use the Gauss quadrature theorem [] — we note that the polynomials $p_1, p_2 \dots p_{N_m}$ are the polynomials that would be used to approximate the integral of $f(\omega)|\hat{v}(\omega)|^2$ in the interval $[-\omega_c, \omega_c]$ with Gaussian quadrature with N_m interpolating points. In particular, for every $N_m \in \mathbb{Z}_{>1}$, $\exists w \in [0, \infty)^{N_m}$ with $\|w\|_1 = 1$ such that for all polynomials q of degree $\leq 2N_m - 1$,

$$\frac{1}{\|v\|_{L_2}^2} \int_{-\omega_c}^{\omega_c} q(\omega)|\hat{v}(\omega)|^2 d\omega = \sum_{i=1}^{N_m} w_i q(\lambda_i).$$

Note that from the Taylor's remainder theorem, it follows that,

$$\forall \omega \in [-\omega_c, \omega_c], \ \cos(\omega t) = q_{N_m}(\omega) + r_{N_m}(\omega),$$

where q_{N_m} is a polynomial of degree N_m with $q_{N_m}(0) = 1$, and

$$\sup_{\omega \in [-2\omega_c, 2\omega_c]} |r_{N_m}(\omega)| \leq (2\omega_c t)^{N_m+1}/(N_m+1)!. \tag{21}$$

We thus obtain that

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 =$$
$$\|v\|_{L^2}^2 - \sum_{i,j=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 q_{N_m}(\omega - \lambda_i) d\omega - \sum_{i,j=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 r_{N_m}(\omega - \lambda_i) d\omega.$$

Now, from the Gauss quadrature theorem, it follows that since for $j \in \{1, 2 \dots N_m\}$ degree of $\pi_j(\omega) q_{N_m}(\omega - \lambda_j) \leq 2N_m - 1$

$$\int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 q_{N_m}(\omega - \lambda_i) d\omega = \|v\|_{L_2}^2 \sum_{k=1}^{N_m} w_k \pi_j(\lambda_k) q_{N_m}(\lambda_k - \lambda_i),$$

and therefore

$$\sum_{i,j=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 q_{N_m}(\omega - \lambda_i) d\omega = \|v\|_{L^2}^2 \sum_{i,j,k=1}^{N_m} w_k \frac{\pi_j(\lambda_i)\pi_j(\lambda_k)}{N_i^2} q_{N_m}(\lambda_k - \lambda_i).$$

where we have used that $\pi_1(\omega) = 1 \; \forall \omega \in \mathbb{R}$. Furthermore, using Eq. 20, we obtain that

$$\sum_{i,j=1}^{N_m} \frac{\pi_1(\lambda_i)\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 q_{N_m}(\omega - \lambda_i) d\omega = \|v\|_{L^2}^2 \|w\|_1 = \|v\|_{L_2}^2 ,$$

and therefore

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 = -\sum_{i,j=1}^{N_m} \frac{\pi_j(\lambda_i)}{N_i^2} \int_{-\omega_c}^{\omega_c} \pi_j(\omega)|\hat{v}(\omega)|^2 r_{N_m}(\omega - \lambda_i) d\omega.$$

Since for $i \in \{1, 2 \dots N_m\}$, $\lambda_i \in [-\omega_c, \omega_c]$ and therefore

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 \leq \sum_{i,j=1}^{N_m} \left| \frac{\pi_j(\lambda_i)}{N_i^2} \right| \sup_{\omega \in [-2\omega_c, 2\omega_c]} |r_{N_m}(\omega)| \int_{-\omega_c}^{\omega_c} |\pi_j(\omega)||\hat{v}(\omega)|^2 d\omega$$

Note that $\forall i, j \in \{1, 2 \dots N_m\}$, $|\pi_j(\lambda_i)| \leq N_i$ and $N_i \geq 1$. Using this and the estimate in Eq. 21, we obtain that

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 \leq \frac{(2\omega_c t)^{N_m+1}}{(N_m+1)!} \sum_{i,j=1}^{N_m} \int_{-\omega_c}^{\omega_c} |\pi_j(\omega)||\hat{v}(\omega)|^2 d\omega \leq \frac{(2\omega_c t)^{N_m+1}}{(N_m+1)!} \sum_{i,j=1}^{N_m} \|\hat{v}\pi_j\|_{L^2} \|v\|_{L^2} = \frac{(2\omega_c t)^{N_m+1}}{(N_m+1)!} N_m^2 \|v\|_{L^2}^2 .$$

Finally, using Stirling's approximation to estimate $(N_m+1)! \geq (N_m+1)^{N_m+1} e^{-N_m} \geq N_m^{N_m+1} e^{-N_m}$, we obtain that

$$\frac{1}{2} \|\tau_t v - \nu_t v\|_{L^2}^2 \leq \|v\|_{L^2}^2 N_m^2 \left(\frac{2e\omega_c t}{N_m} \right)^{N_m},$$

which proves the lemma statement. □

Definition 14 (Chain-approximation)**.** *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, functions $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha \in \{1,2\dots M\}}$, jump operators $\{L_\alpha\}_{\alpha \in \{1,2\dots M\}}$ and single-particle environment dynamics described by the time-translation unitary group. A chain approximation to this non-Markovian model, with frequency cutoff $\omega_c > 0$ and $N_m \in \mathbb{Z}_{>0}$ modes is a non-Markovian model specified by the same system Hamiltonian and jump operartors, but with square integrable coupling functions $\{v_\alpha \in L^2(\mathbb{R}) | \text{supp}(\hat{v}_\alpha) \subseteq [-\omega_c, \omega_c]\}_{\alpha \in \{1,2\dots M\}}$ and single-particle environment dynamics described by the unitary groups $\{\nu_{\alpha,t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$ where $\nu_{\alpha,t}$ is the chain-unitary group with N_m modes generated by v_α.*

Next, we analyze the error incurred on approximating a non-Markovian model with its chain approximation. There are two sources of error in this approximation — the first is in introducing a frequency cutoff into the model, and the next is in approximating environment in the resulting model by its chain representation. We analyze both of these errors separately — for this analysis, we restrict ourselves to coupling functions whose Fourier transforms fall off sufficiently fast with frequency. Then, we consider models specified by a distributional coupling functions where an additional regularization step (as described in the previous section) is needed to map them to coupling functions in this class.

Lemma 16. *Consider a non-Markovian model described by coupling functions $v = \{v_\alpha \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) | \, \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty} < \infty\}_{\alpha \in \{1,2\dots M\}}$, jump operators $\{L_\alpha\}_{\alpha \in \{1,2\dots M\}}$ and system Hamiltonian $H_S(t)$. For $\omega_c > 0$, consider a non-Markovian model described by coupling functions $v_{\omega_c} = \{v_{\omega_c} \in L^2(\mathbb{R}) | \hat{v}_{\alpha,\omega_c} := \hat{v}_\alpha \mathcal{I}_{[-\omega_c,\omega_c]}\}_{\alpha \in \mathbb{Z}_{\geq 0}}$ but with the same jump operators and system Hamiltonian. Let $|\Psi(t)\rangle$ and $|\Psi_{\omega_c}(t)\rangle$ be the state at time t for both of these models starting with initial state $|\Psi_0\rangle \in \mathcal{H}_S \otimes F_\infty^M(L^2(\mathbb{R}))$, then $\forall \, k \in \mathbb{Z}_{\geq 1}$,*

$$\| |\Psi(t)\rangle - |\Psi_{\omega_c}(t)\rangle \|^2 \leq \frac{2}{\omega_c^{1/2}} \sum_{\alpha=1}^{M} \|L_\alpha\| \, \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty} \left(\|L_\alpha\| \, \|v_\alpha\|_{L^2} t^2 + 2\mu_{|\Psi_0\rangle}^{(1)} t \right).$$

Proof: It is useful to note that since $\forall\, \alpha \in \{1, 2 \dots M\}$, $|\hat{v}_\alpha(\omega)| \leq \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}/\omega$. Consequently, we obtain that

$$\|v_\alpha - v_{\alpha,\omega_c}\|^2 = \int_{|\omega|\geq\omega_c} |\hat{v}_\alpha(\omega)|^2 d\omega \leq \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}^2 \int_{|\omega|\geq\omega_c} \frac{d\omega}{\omega^2} = \frac{\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}^2}{\omega_c}. \tag{22}$$

The proof of this lemma follows from lemma 11. Consider the two error terms defined in lemma 11. First, note that for $\tau \in [0, t]$ and $\alpha \in \{1, 2 \dots M\}$,

$$\mathcal{D}_\alpha^{v,v_{\omega_c}}(\tau) = 4\,\|L_\alpha\| \left\|a_{\alpha,\tau_\tau(v_\alpha - v_{\alpha,\omega_c})}\,|\Psi_0\rangle\right\| \leq 4\mu^{(1)}_{|\Psi_0\rangle}\,\|L_\alpha\|\,\|v_\alpha - v_{\alpha,\omega_c}\|_{L^2} \leq 4\mu^{(1)}_{|\Psi_0\rangle}\frac{\|L_\alpha\|\,\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}}{\omega_c^{1/2}}.$$

Furthermore, since $\forall\alpha \in \{1, 2 \dots M\}$, $v_\alpha, v_{\alpha,\omega_c} \in L^2(\mathbb{R})$, we obtain that $\forall\tau, s \in [0, t]$

$$|\langle \tau_\tau(v_\alpha - v_{\alpha,\omega_c}), \tau_s v_\alpha\rangle| \leq \left|\int_{\mathbb{R}} (\hat{v}_\alpha^*(\omega) - \hat{v}_{\alpha,\omega_c}^*(\omega))\hat{v}_\alpha(\omega)e^{-i\omega(s-\tau)}d\omega\right| \leq \|v_\alpha\|_{L^2}\,\|v_\alpha - v_{\alpha,\omega_c}\|_{L^2} \leq \frac{\|v_\alpha\|_{L_2}\,\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}}{\omega_c^{1/2}},$$

$$|\langle \tau_s v_{\alpha,\omega_c}, \tau_\tau(v_\alpha - v_{\alpha,\omega_c})\rangle| \leq \left|\int_{\mathbb{R}} (\hat{v}_\alpha(\omega) - \hat{v}_{\alpha,\omega_c}(\omega))\hat{v}_{\alpha,\omega_c}^*(\omega)e^{i\omega(s-\tau)}d\omega\right| \leq \|v_{\alpha,\omega_c}\|_{L^2}\,\|v_\alpha - v_{\alpha,\omega_c}\|_{L^2} \leq \frac{\|v_\alpha\|_{L_2}\,\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}}{\omega_c^{1/2}}.$$

Therefore, we obtain that $\forall\tau \in [0, t]$,

$$\mathcal{E}_\alpha^{v,v_{\omega_c}}(\tau) \leq 4\tau\frac{\|L_\alpha\|^2\,\|v_\alpha\|_{L_2}\,\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}}{\omega_c^{1/2}}$$

From lemma 11, we then obtain

$$\||\Psi(t)\rangle - |\Psi_{\omega_c}(t)\rangle\|^2 \leq \frac{2}{\omega_c^{1/2}}\sum_{\alpha=1}^M \|L_\alpha\|\,\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty}\left(\|L_\alpha\|\,\|v_\alpha\|_{L^2}\,t^2 + 2\mu^{(1)}_{|\Psi_0\rangle}t\right),$$

which proves the lemma statement. □

Lemma 17. *Consider two non-Markovian models with system Hamiltonian $H_S(t)$, square-integrable coupling functions $v = \{v_\alpha \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$ and jump operators $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ but with two different single-parameter strongly continuous unitary groups $\{\tau_{\alpha,t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$, $\{\nu_{\alpha,t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$ specifying the single-particle environment dynamics. Denoting by $|\Psi_\tau(t)\rangle$ and $|\Psi_\nu(t)\rangle$ the system-environment state for the two models at time $t \geq 0$ with $|\Psi_\tau(0)\rangle = |\Psi_\nu(0)\rangle = |\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}_\infty^M(L^2(\mathbb{R}))$, then*

$$\||\Psi_\tau(t)\rangle - |\Psi_\nu(t)\rangle\| \leq \left(t + 2t\sqrt{\mu^{(1)}_{|\Psi_0\rangle}} + \frac{1}{2}\sum_{\alpha=1}^M \|L_\alpha\|\,\|v_\alpha\|_{L^2}\,t^2\right)\sum_{\alpha=1}^M \|L_\alpha\| \sup_{s\in[0,t]} \|\nu_{\alpha,s}v_\alpha - \tau_{\alpha,s}v_\alpha\|_{L^2}.$$

Proof: Let $U_\tau(t, s)$ and $U_\nu(t, s)$ be the propagators corresponding to the two models — we note that both $U_\tau(t, s)\,|\Psi\rangle$ and $U_\nu(t, s)\,|\Psi\rangle$ are strongly differentiable with respect to t and s if $|\Psi\rangle \in \mathcal{H}_S \otimes \mathrm{F}_1^M[L^2(\mathbb{R})]$. Consider now,

$$\||\Psi_\tau(t)\rangle - |\Psi_\nu(t)\rangle\| = \||\Psi_0\rangle - U_\nu(0, t)U_\tau(t, 0)\,|\Psi_0\rangle\|.$$

Now,

$$\begin{aligned}\frac{d}{dt}\Big(U_\nu(0, t)U_\tau(t, 0)\,|\Psi_0\rangle\Big) &= iU_\nu(0, t)\big(H_\nu(t) - H_\tau(t)\big)U_\tau(t, 0)\,|\Psi_0\rangle \\ &= i\sum_{\alpha=1}^M U_\nu(0, t)\big(L_\alpha a^+_{\alpha,\nu_{\alpha,t}v_\alpha - \tau_{\alpha,t}v_\alpha} + L_\alpha^\dagger a^-_{\alpha,\nu_{\alpha,t}v_\alpha - \tau_{\alpha,t}v_\alpha}\big)\,|\Psi_\tau(t)\rangle\,.\end{aligned}$$

We can thus obtain the estimate,

$$\left\|\frac{d}{dt}\Big(U_\nu(0, t)U_\tau(t, 0)\,|\Psi_0\rangle\Big)\right\| \leq \sum_{\alpha=1}^M \|L_\alpha\|\left(\left\|a^+_{\alpha,\nu_{\alpha,t}v_\alpha - \tau_{\alpha,t}v_\alpha}\,|\Psi_\tau(t)\rangle\right\| + \left\|a^-_{\alpha,\nu_{\alpha,t}v_\alpha - \tau_{\alpha,t}v_\alpha}\,|\Psi_\tau(t)\rangle\right\|\right).$$

Moreover,

$$\left\|a^{+}_{\alpha,\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha}\left|\Psi_\tau(t)\right\rangle\right\|^2 \leq \left\|\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha\right\|^2_{L^2}\sum_{n=0}^{\infty}(n+1)\left\|\Pi_n\left|\Psi_\tau(t)\right\rangle\right\|^2 = \left\|\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha\right\|^2_{L^2}\left(1+\mu^{(1)}_{|\Psi_\tau(t)\rangle}\right) \text{ and}$$

$$\left\|a^{-}_{\alpha,\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha}\left|\Psi_\tau(t)\right\rangle\right\|^2 \leq \left\|\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha\right\|^2_{L^2}\sum_{n=0}^{\infty}n\left\|\Pi_n\left|\Psi_\tau(t)\right\rangle\right\|^2 = \left\|\nu_{\alpha,t}v_\alpha-\tau_{\alpha,t}v_\alpha\right\|^2_{L^2}\mu^{(1)}_{|\Psi_\tau(t)\rangle}$$

Finally, using lemma 4, we obtain that

$$\mu^{(1)}_{|\Psi_\tau(t)\rangle} \leq 2\mu^{(1)}_{|\Psi_0\rangle} + 2t^2\left(\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\left\|v_\alpha\right\|_{L^2}\right)^2.$$

From these estimates, we thus obtain

$$\left\|\left|\Psi_\tau(t)\right\rangle - \left|\Psi_\nu(t)\right\rangle\right\| \leq 2t\left(1+2\mu^{(1)}_{|\Psi_0\rangle}+2t^2\left(\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\left\|v_\alpha\right\|_{L^2}\right)^2\right)^{1/2}\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\sup_{s\in[0,t]}\left\|\nu_{\alpha,s}v_\alpha-\tau_{\alpha,s}v_\alpha\right\|_{L^2}$$

Lemma 18. *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, coupling functions $\{v_\alpha \in L^2(\mathbb{R}) \cap \mathrm{C}^\infty(\mathbb{R})\big|\ \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty} < \infty\}_{\alpha\in\{1,2\ldots M\}}$, jump operators $\{L_\alpha\}_{\alpha\in\{1,2\ldots M\}}$ and single particle environment dynamics specified by time-translation unitary group. For $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^\infty_M[L^2(\mathbb{R})]$, let $|\Psi(t)\rangle$ be the system-environment state at time t, and let $|\Psi_{\omega_c,N_m}(t)\rangle$ be the state obtained from a chain approximation to the non-Markovian model (definition 14), then,*

$$\left\|\left|\Psi(t)\right\rangle - \left|\Psi_{\omega_c,N_m}(t)\right\rangle\right\| \leq \sqrt{2}tN_m\left(\frac{2e\omega_c t}{N_m}\right)^{\frac{N_m}{2}}\left(1+2\mu^{(1)}_{|\Psi_0\rangle}+2t^2\left(\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\left\|v_\alpha\right\|_{L^2}\right)^2\right)^{1/2}\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\left\|v_\alpha\right\|_{L^2} + \left[\frac{2}{\omega_c^{1/2}}\sum_{\alpha=1}^{M}\left\|L_\alpha\right\|\left\|(\cdot)\hat{v}_\alpha(\cdot)\right\|_{L^\infty}\left(\left\|L_\alpha\right\|\left\|v_\alpha\right\|_{L^2}t^2+2\mu^{(1)}_{|\Psi_0\rangle}t\right)\right]^{1/2}$$

Proof: Let $|\Psi_{\omega_c}(t)\rangle$ be the state corresponding to a non-Markovian model wherein the environment state is restricted to the frequency interval $[-\omega_c,\omega_c]$ i.e. the model with system Hamiltonian $H_S(t)$, coupling functions $\{v_{\alpha,\omega_c} \in L^2(\mathbb{R}) | \hat{v}_{\alpha,\omega_c} = \hat{v}_\alpha \cdot \mathcal{I}_{[-\omega_c,\omega_c]}\}_{\alpha\in\{1,2\ldots M\}}$ and jump operators $\{L_\alpha\}_{\alpha\in\{1,2\ldots M\}}$. Using lemma 16, we can obtain an upper bound on the error $\||\Psi(t)\rangle - |\Psi_{\omega_c}(t)\rangle\|$, and using lemmas 17 and 15 we can obtain an upper bound on $\|\Psi_{\omega_c}(t) - |\Psi_{\omega_c,N_m}(t)\rangle\|$. Using triangle inequality together with these two bounds, we obtain the lemma statement. □

Lemma 19. *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, coupling functions $\{v_\alpha \in L^2(\mathbb{R}) \cap \mathrm{C}^\infty(\mathbb{R})\big|\ \|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty} < \infty\}_{\alpha\in\{1,2\ldots M\}}$, jump operators $\{L_\alpha\}_{\alpha\in\{1,2\ldots M\}}$ and single particle environment dynamics specified by time-translation unitary group. Suppose that $\|L_\alpha\|\|v_\alpha\| \leq \mathcal{V}_{L^2}$ and $\|L_\alpha\|\|(\cdot)\hat{v}_\alpha(\cdot)\|_{L^\infty} \leq \mathcal{V}_\mathcal{S}$ for all $\alpha \in \{1,2\ldots M\}$. For $|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathrm{F}^\infty_M[L^2(\mathbb{R})]$, then $\exists$ a chain approximation of the non-Markovian model with $\omega_c < O(\mathrm{poly}(M,t,\mathcal{V}_{L^2},\mathcal{V}_\mathcal{S},\mu^{(1)}_{|\Psi_0\rangle},1/\epsilon))$ and $N_m < O(\mathrm{poly}(t,M,\mathcal{V}_{L^2},\mathcal{V}_\mathcal{S},\mu^{(1)}_{|\Psi_0\rangle},1/\epsilon))$ whose system-environment state at time t is within ϵ norm distance of the exact state.*

Proof: Using lemma 18 with $k=1$, we obtain that

$$\left\|\left|\Psi(t)\right\rangle - \left|\Psi_{\omega_c,N_m}(t)\right\rangle\right\| \leq N_m\left(\frac{2e\omega_c t}{N_m}\right)^{N_m/2}\mathrm{poly}(M,t,\mathcal{V}_{L^2},\mu^{(1)}_{|\Psi_0\rangle}) + \frac{1}{\sqrt{\omega_c}}\mathrm{poly}(M,t,\mathcal{V}_{L^2},\mathcal{V}_\mathcal{S},\mu^{(1)}_{|\Psi_0\rangle}).$$

Therefore, to ensure that $\||\Psi(t)\rangle - |\Psi_{\omega_c,N_m}(t)\rangle\| < \varepsilon$, we can choose $\omega_c < O(\mathrm{poly}(M,t,\mathcal{V}_{L^2},\mathcal{V}_\mathcal{S},\mu^{(1)}_{|\Psi_0\rangle},1/\epsilon))$ and $N_m < O(\mathrm{poly}(\omega_c t,M,t,\mathcal{V}_{L^2},\mu^{(1)}_{|\Psi_0\rangle},1/\epsilon))$, which yields the estimates in the lemma statement. □

Now, we consider distributional models. We need some additional assumptions on the coupling functions to show that they can be simulated efficiently.

Assumption 1, repeated (Polynomial growth of Radon measure)**.** *The radon measure μ corresponding to the coupling function should satisfy:*

(a) For any interval $[a,b] \subseteq \mathbb{R}$, $\mathrm{TV}_{[a,b]}(\mu) \leq \mathrm{poly}(|a|,|b|)$ and

(b) *The error functions corresponding to μ, $\Delta^0_{\mu;[a,b]}(\varepsilon)$ and $\Delta^1_{\mu;[a,b]}(\varepsilon)$ as specified in definition 11 are individually locally integrable with respect to a, b and grow at most polynomially with $|a|, |b|$, and fall off polynomially with ε i.e.*

$$\Delta^0_{\mu;[a,b]}(\varepsilon), \Delta^1_{\mu;[a,b]}(\varepsilon) \leq \text{poly}(|a|, |b|, \varepsilon).$$

We will consider initial states in the environment which are product state over the different baths, and that the individual product state have a sufficiently rapidly decaying high frequency response. Furthermore, we assume that the initial states are efficiently representable i.e. their projection onto a finite set of environment modes, specified by square integrable modal functions, can be efficiently computed.

Assumption 2, repeated (Initial environment state)**.** *The initial environment state $|\phi_1\rangle \otimes |\phi_2\rangle \dots |\phi_M\rangle$ where for $\alpha \in \{1, 2 \dots M\}$, $|\phi_\alpha\rangle \in \text{Fock}[L^2(\mathbb{R})]$ and*

(a) *for its $n-$particle wavefunctions $\phi_{\alpha,n} \in L^2(\mathbb{R}^n)$, and any $j, k \geq 0$, $\exists \mathcal{N}_{j,k} > 0$ such that*

$$\sum_{n=0}^{\infty} n^j \int_{\mathbb{R}^n} (1 + \omega_1^2)^k \left|\phi_{\alpha,n}(\omega)\right|^2 d\omega < \mathcal{N}_{j,k}.$$

(b) *for $v_1, v_2 \dots v_m \in L^2(\mathbb{R})$ and $P \in \mathbb{Z}_{>0}$, all the amplitudes*

$$\langle \text{vac}| \prod_{i=1}^{m} \left(\int_{\mathbb{R}} v_i(\omega) a_\omega d\omega \right)^{n_i} |\phi_\alpha\rangle$$

with $n_1 + n_2 \dots n_m \leq P$ can be computed in $\text{poly}(m, P)$ *time on a classical or quantum computer.*

From a physical standpoint, this assumption ensures that the initial state does not induce an infinitely large 'field' in the environment that impacts the dynamics of the system. We make this formal in the next lemma, which can be considered as providing concrete estimates for the constants $c_{\mu,|\Phi\rangle}, \gamma_{\mu,|\Phi\rangle}$ defined in lemma 12.

Lemma 20. *Let $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots |\phi_M\rangle \in \text{Fock}[L^2(\mathbb{R})]^{\otimes M}$ be an initial state satisfying assumption 2 and let (μ, φ) be a distributional coupling function such that $\hat{\mu}(\omega) \leq O(\omega^{2k})$ for some $k > 0$. For $\varepsilon > 0$, and a compact mollifier $\rho \in \text{C}_c^\infty(\mathbb{R})$, let v_ε be a $\varepsilon, \rho-$regularization of μ then*

(a) $\forall \alpha \in \{1, 2 \dots M\}$ *and* $t \geq 0$,

$$\|a_{\alpha,\tau_t v_\varepsilon} |\Phi\rangle\| \leq \left(\mathcal{N}_{1,k+1}\right)^{1/2} \left\|(1 + (\cdot)^2)^{-(k+1)} \hat{\mu}(\cdot)\right\|_1^{1/2},$$

(b) $\forall \alpha \in \{1, 2 \dots M\}$ *and* $t \geq 0$,

$$\lim_{\varepsilon' \to 0} \left\|\left(a_{\alpha,\tau_t v_{\varepsilon'}} - a_{\alpha,\tau_t v_\varepsilon}\right) |\Phi\rangle\right\| \leq \left(\mathcal{N}_{1.k+2}\right)^{1/2} \left\|(1 + (\cdot)^2)^{-(k+2)} \hat{\mu}(\cdot)\right\|_1^{1/2} \varepsilon.$$

Proof:
(a) We note that for $t \geq 0$ and $\alpha \in \{1, 2 \dots M\}$

$$\begin{aligned}
\|a_{\alpha,\tau_t v_\varepsilon} |\Phi\rangle\|^2 &= \sum_{n=0}^{\infty} n \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} \hat{v}_\varepsilon^*(\omega_1) e^{i\omega_1 t} \phi_{\alpha,n}((\omega_1, \omega)) d\omega_1 \right|^2 d\omega, \\
&\leq \left(\int_{\mathbb{R}} (1 + \omega_1^2)^{-(k+1)} \left|\hat{v}_\varepsilon(\omega_1)\right|^2 d\omega_1 \right) \left(\sum_{n=0}^{\infty} n \int_{\mathbb{R}^n} (1 + \omega_1^2)^{k+1} \left|\phi_{\alpha,n}(\omega)\right|^2 d\omega \right), \\
&\leq \mathcal{N}_{1,k+1} \left(\int_{\mathbb{R}} (1 + \omega_1^2)^{-(k+1)} \left|\hat{v}_\varepsilon(\omega_1)\right|^2 d\omega_1 \right).
\end{aligned}$$

Since v_ε is an $\varepsilon, \rho-$regularization of (μ, φ) and since $\|\hat{\rho}\|_{L^\infty} < \|\rho\|_1 = 1$, we obtain that

$$\int_{\mathbb{R}} (1 + \omega_1^2)^{-(k+1)} \left|\hat{v}_\varepsilon(\omega_1)\right|^2 d\omega_1 \leq \int_{\mathbb{R}} (1 + \omega_1^2)^{-(k+1)} \hat{\mu}(\omega_1) d\omega_1 = \left\|(1 + (\cdot)^2)^{-(k+1)} \hat{\mu}(\cdot)\right\|_1 < \infty,$$

and thus

$$\|a_{\alpha,\tau_t v_\varepsilon}|\Phi\rangle\| \leq \left(\mathcal{N}_{1,k+1}\right)^{1/2}\left\|(1+(\cdot)^2)^{-(k+1)}\hat{\mu}(\cdot)\right\|_1^{1/2}.$$

(b) For $\varepsilon, \varepsilon' > 0$, $t \geq 0$ and $\alpha \in \{1, 2 \dots M\}$, we note that

$$\begin{aligned}\left\|\left(a_{\alpha,\tau_t v_\varepsilon} - a_{\alpha,\tau_t v_{\varepsilon'}}\right)|\Phi\rangle\right\|^2 &= \sum_{n=0}^{\infty} n \int_{\mathbb{R}^{n-1}} \left|\int_{\mathbb{R}} \left(\hat{v}^*_{\varepsilon'}(\omega) - \hat{v}^*_\varepsilon(\omega_1)\right) e^{i\omega_1 t}\phi_{\alpha,n}((\omega_1,\omega))d\omega_1\right|^2 d\omega \\ &\leq \left(\int_{\mathbb{R}} (1+\omega_1^2)^{-(k+2)}\left|\hat{v}_\varepsilon(\omega_1) - \hat{v}_{\varepsilon'}(\omega_1)\right|^2 d\omega_1\right)\left(\sum_{n=0}^{\infty} n\int_{\mathbb{R}^n}(1+\omega_1^2)^{k+2}|\phi_{\alpha,n}(\omega)|^2 d\omega\right), \\ &\leq \mathcal{N}_{1,k+2}\int_{\mathbb{R}}(1+\omega_1^2)^{-(k+2)}\left|\hat{v}_\varepsilon(\omega_1) - \hat{v}_{\varepsilon'}(\omega_1)\right|^2 d\omega\end{aligned}$$

Now, we note that

$$\left|\hat{v}_\varepsilon(\omega_1) - \hat{v}_{\varepsilon'}(\omega_1)\right|^2 = \hat{\mu}(\omega)\left|\hat{\rho}(\omega_1\varepsilon) - \hat{\rho}(\omega_1\varepsilon')\right|^2 \leq \hat{\mu}(\omega_1)\omega_1^2(\varepsilon-\varepsilon')^2\left\|\hat{\rho}'\right\|_{L^\infty}.$$

Noting that $\|\hat{\rho}'\|_{L^\infty} \leq \|(\cdot)\rho(\cdot)\|_{L^1} \leq 1$, and thus

$$\int_{\mathbb{R}}(1+\omega_1^2)^{-(k+2)}\left|\hat{v}_\varepsilon(\omega_1) - \hat{v}_{\varepsilon'}(\omega_1)\right|^2 d\omega \leq (\varepsilon-\varepsilon')^2\left\|(1+(\cdot)^2)^{-(k+1)}\hat{\mu}(\cdot)\right\|_{L^1}$$

and thus

$$\left\|\left(a_{\alpha,\tau_t v_\varepsilon} - a_{\alpha,\tau_t v_{\varepsilon'}}\right)|\Phi\rangle\right\| \leq \left(\mathcal{N}_{1,k+2}\right)^{1/2}\left\|(1+(\cdot)^2)^{-(k+1)}\hat{\mu}(\cdot)\right\|_{L^1}^{1/2}|\varepsilon-\varepsilon'|.$$

Taking the limit of $\varepsilon' \to 0$ in this estimate, we obtain the lemma statement. □

Theorem 3. *Consider a non-Markovian model specified by a system Hamiltonian $H_S(t)$, jump operators $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ and coupling functions $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha\in\{1,2\dots M\}}$ where μ_α satisfy assumption 1 with $\hat{\mu}_\alpha(\omega) < O(\omega^{2k})$ for some $k > 0$. For $|\Psi_0\rangle := |\sigma\rangle \otimes |\Phi_0\rangle \in \mathcal{H}_S \otimes \text{Fock}[L^2(\mathbb{R})]^{\otimes M}$, where $|\Phi_0\rangle$ is an initial environment state that satisfies assumption 2, then $\exists$ a chain approximation of the non-Markovian model with $\omega_c, N_m \leq O\big(\text{poly}(\epsilon^{-1}, t, M, \sup_\alpha\|L_\alpha\|, \sup_{\alpha,s\in[0,t]}\|[H_S(t), L_\alpha]\|, \mathcal{N}_{1,k+1}, \mathcal{N}_{1,k+2}, \mathcal{N}_{1,0})\big)$ whose system-environment state at time t is within ϵ norm distance of the exact state.*

Proof: Suppose $\rho \in C_c^\infty(\mathbb{R})$ is a symmetric mollifier. Consider the non-Markovian model which has the system Hamiltonian $H_S(t)$, jump operators $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ and coupling functions $v_\varepsilon = \{v_{\alpha,\varepsilon} \in L^2(\mathbb{R})\}_{\alpha\in\{1,2\dots M\}}$ where $v_{\alpha,\varepsilon}$ is the $\varepsilon,\rho-$regularization of $(\mu_\alpha, \varphi_\alpha)$. Denoting the system-environment state at time t corresponding to the distributional model by $|\Psi(t)\rangle$, and the regularized model by $|\Psi_\varepsilon(t)\rangle$, we obtain from lemmas 11 and 13 that for $\varepsilon \in (0, 1/2)$,

$$\||\Psi(t)\rangle - |\Psi_\varepsilon(t)\rangle\| \leq \lim_{\varepsilon'\to 0}\sum_{\alpha=1}^{M}\left(\int_0^t \mathcal{E}^{v_\varepsilon, v_{\varepsilon'}}_{\alpha,|\Psi_0\rangle}(\tau)d\tau + \int_0^t \mathcal{D}^{v_\varepsilon, v_{\varepsilon'}}_{\alpha,|\Psi_0\rangle}(\tau)d\tau\right).$$

Now, since by assumption 1, the error functions $\Delta^0_{\mu_\alpha;[a,b]}$ and $\Delta^1_{\mu_\alpha;[a,b]}$ are integrable in a and b. Since they are of polynomial growth in a, b and of polynomial decrease in ε, we obtain that

$$\int_0^t \Delta^0_{\mu_\alpha;[-\tau,0]}(\varepsilon)d\tau, \int_0^t \Delta^1_{\mu_\alpha;[-\tau,0]}(\varepsilon)d\tau \leq O\big(\varepsilon^p \text{poly}(t)\big),$$

for some $p > 0$. Furthermore, the upper bound in lemma 13(a) and (b) allows us to use the dominated convergence theorem to obtain

$$\lim_{\varepsilon'\to 0}\sum_{\alpha=1}^{M}\int_0^t \mathcal{E}^{v_\varepsilon, v_{\varepsilon'}}_{\alpha,|\Psi_0\rangle}(\tau)d\tau = \sum_{\alpha=1}^{M}\int_0^t \lim_{\varepsilon'\to 0}\mathcal{E}^{v_\varepsilon, v_{\varepsilon'}}_{\alpha,|\Psi_0\rangle}(\tau)d\tau \leq O\left(\varepsilon^p \text{poly}\left(t, M, \sup_\alpha\|L_\alpha\|, \sup_{\alpha,s\in[0,t]}\|[H_S(s), L_\alpha]\|, \mathcal{N}_{1,k+1}\right)\right).$$

where we have used lemma 20 to estimate $c_{\mu_\alpha,|\Psi_0\rangle}$ in lemma 13(b). Similarly, using lemma 13(c) together with lemma 13(b), we obtain that

$$\lim_{\varepsilon'\to 0}\sum_{\alpha=1}^{M}\int_0^t \mathcal{D}_{\alpha,|\Psi_0\rangle}^{v_\varepsilon,v_{\varepsilon'}}(\tau)d\tau \leq O\bigg(\varepsilon \,\text{poly}\bigg(t, M, \sup_\alpha \|L_\alpha\|, \mathcal{N}_{1,k+2}\bigg)\bigg).$$

and thus

$$\||\Psi(t)\rangle - |\Psi_\varepsilon(t)\rangle\| \leq O\bigg(\varepsilon^p \text{poly}\bigg(t, M, \sup_\alpha \|L_\alpha\|, \sup_{\alpha, s\in[0,t]} \|[H_S(s), L_\alpha]\|, \mathcal{N}_{k+1}\bigg)\bigg) + O\bigg(\varepsilon \,\text{poly}\bigg(t, M, \sup_\alpha \|L_\alpha\|, \mathcal{N}_{k+2}\bigg)\bigg).$$

Consequently, to ensure that $\||\Psi(t)\rangle - |\Psi_\varepsilon(t)\rangle\| \leq \epsilon/2$, we need to choose. regularization parameter

$$\varepsilon^{-1} = \text{poly}\bigg(\epsilon^{-1}, t, M, \sup_\alpha \|L_\alpha\|, \sup_{\alpha, s\in[0,t]} \|[H_S(t), L_\alpha]\|, \mathcal{N}_{1,k+1}, \mathcal{N}_{1,k+2}\bigg). \tag{23}$$

Next, we analyze the chain approximation of the regularized model. We note that if $\hat{\mu}_\alpha$ is a function of polynomial growth, since $\hat{\rho} \in \mathcal{S}(\mathbb{R})$, $\left\|(\cdot)^j \hat{v}_{\alpha,\varepsilon}(\cdot)\right\|_{L^\infty} < \infty$ for any $j > 0$. In particular, since $\hat{\mu}_\alpha(\omega) < O(\omega^{2k})$ for some $k \geq 0$, then

$$\|v_{\alpha,\varepsilon}\|_{L^2}^2 = \int_{\mathbb{R}} \hat{\mu}_\alpha(\omega)\, |\hat{\rho}(\varepsilon\omega)|^2\, d\omega \leq O(\varepsilon^{-(k+1)}) \text{ and}$$

$$\|(\cdot)\hat{v}_{\alpha,\varepsilon}(\cdot)\|_{L^\infty} = \sup_{\omega\in\mathbb{R}} |\omega|\, \hat{\mu}_\alpha(\omega)\, |\rho(\varepsilon\omega)| \leq O(\varepsilon^{-(k+1)}).$$

It thus follows from lemma 18 that there is a chain approximation of this regularized model with

$$\omega_c, N_m < O(\text{poly}(\epsilon^{-1}, t, M, \sup_\alpha \|L_\alpha\|, \varepsilon^{-2(k+1)}, \mathcal{N}_{1,0})), \tag{24}$$

such that the error between the system-environment state at time t is within $\epsilon/2$ norm distance from the state obtained from the chain approximation. Combining the estimate of the regularizing parameter (Eq. 23) with this estimate, we obtain the theorem statement. □

B. $k-$local Non-Markovian open system dynamics is in BQP

We next consider the $k-$local Non-Markovian open system problem.

Problem 1, repeated ($k-$local non-Markovian dynamics)**.** *Consider a system of n qudits ($\mathcal{H}_S = \left(\mathbb{C}^d\right)^{\otimes n}$) interacting with $M = \text{poly}(n)$ baths with*

(a) System Hamiltonian $H_S(t)$ is $k-$local i.e. $H_S(t) = \sum_{i=1}^N H_i(t)$, where $N = \text{poly}(n)$, and for $i \in \{1, 2 \dots N\}$, $H_i(t)$ is an operator acting on atmost k qudits and $\|H_i(t)\| \leq 1$.

(b) Jump operators $\{L_\alpha\}_{\alpha\in\{1,2\dots M\}}$ such that for $\alpha \in \{1, 2 \dots M\}$, L_α acts on at-most k qudits and $\|L_\alpha\| \leq 1$.

(c) Coupling functions $\{(\mu_\alpha, \varphi_\alpha)\}_{\alpha\in\{1,2\dots M\}}$ such that for $\alpha \in \{1, 2 \dots M\}$, μ_α satisfies the polynomial growth conditions (assumption 1).

(d) Initial state $|\Psi\rangle = |0\rangle^{\otimes n} \otimes |\Phi\rangle$, where $|\Phi\rangle$ satisfies assumption 2.

Denoting by $\rho_S(t)$ the reduced state of the system at time t for this non-Markovian model, then for $\varepsilon > 0$ and $t = \text{poly}(n)$, prepare a quantum state $\hat{\rho}$ such that $\|\hat{\rho} - \rho_S(t)\|_{\text{tr}} \leq \varepsilon$.

To prove that this problem can be efficiently solved on a quantum computer, we proceed in three steps. First, we compute a Markovian dilation of the non-Markovian system with N_m modes — from theorem 3 it follows that $N_m = \text{poly}(n, \varepsilon^{-1})$ modes are needed to ensure that the error between the dynamics of the non-Markovian and its Markovian dilation is $< \varepsilon$. Next, we simulate the Markovian dilation on a quantum computer — we thus consider a different problem, defined below.

Problem 2 ($k-$local non-Markovian chain model)**.** *Consider a system of n qudits ($\mathcal{H}_S = \left(\mathbb{C}^d\right)^{\otimes n}$) interacting with $M = \text{poly}(n)$ baths with* *(a) System Hamiltonian $H_S(t)$ is $k-$local i.e. $H_S(t) = \sum_{i=1}^{N} H_i(t)$, where $N = \text{poly}(n)$, and for $i \in \{1, 2 \dots N\}$, $H_i(t)$ is an operator acting on atmost k qudits and $\|H_i(t)\| \leq 1$.*

(b) Jump operators $\{L_\alpha\}_{\alpha \in \{1,2\dots M\}}$ such that for $\alpha \in \{1, 2 \dots M\}$, L_α acts on at-most k qudits and $\|L_\alpha\| \leq 1$.

(c) Square-integrable coupling functions $\{v_\alpha \in L^2(\mathbb{R}) | \, \|v_\alpha\|_{L^2} = \text{poly}(n), \ \text{supp}(\hat{v}_\alpha) \subseteq [-\omega_c, \omega_c]\}_{\alpha \in \{1,2\dots M\}}$ where $\omega_c = \text{poly}(n)$.

(d) Single-particle environment dynamics specified by $\{\nu_{\alpha,t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\}_{\alpha \in \{1,2\dots M\}}$ where $\nu_{\alpha,t}$ is the chain unitary group with $N_m = \text{poly}(n)$ modes generated by v_α.

(e) Initial state $|\Psi\rangle = |0\rangle^{\otimes n} \otimes |\Phi\rangle$, where $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots |\phi_M\rangle$ satisfies assumption 2.

Denoting by $\rho_S(t)$ the reduced state of the system at time t for this non-Markovian model, then $t = \text{poly}(n)$, prepare a quantum state $\hat{\rho}$ such that $\|\hat{\rho} - \rho_S(t)\|_{\text{tr}} \leq 1/\text{poly}(n)$.

Since the Markovian dilation is still an infinite dimensional system (with an effective Hilbert space $\mathcal{H}_S \otimes \text{Fock}[\mathbb{C}^{N_m}]^{\otimes M}$), this requires a truncation of Hilbert space to a finite-dimensional one, followed by simulating the finite-dimensional quantum dynamics. To show that the approximating finite-dimensional quantum system can be efficiently simulated, we use the Hamiltonian simulatability lemma from Ref. [57], which we restate below

Lemma 21 (Hamiltonian simulatability, Ref. [57])**.** *Given a Hamiltonian $H(t)$ over a system of n qudits such that for every $t \geq 0$*

(a) $\forall$ computational basis element $|a\rangle$, the the set of computational basis elements $|b\rangle$ such that $\langle a| H(t) |b\rangle \neq 0$ together with the elements $\langle a| H(t) |b\rangle$ can be computed in $\text{poly}(n)$ time, and

(b) $\int_0^t \|H(t')\| \, dt' = \text{poly}(n)$,

then $\exists$ a quantum circuit over n qudits of depth $\text{poly}(n)$ which implements a unitary $\hat{U}$ such that $\left\| \hat{U} - U(t, 0) \right\| \leq 1/\text{poly}(n)$, where $U(\cdot, \cdot)$ is the propagator corresponding to $H(t)$.

Lemma 22. *Problem 2 can be solved on a quantum computer in run time $\text{poly}(n, 1/\epsilon)$.*

Proof: For notational simplicity, we will denote by $a_{\alpha,j}$ for $\alpha \in \{1, 2 \dots M\}$ and $j \in \{1, 2 \dots N_m\}$ the annihilation operator corresponding to the j^{th} chain mode of the α^{th} bath, which have the commutation relations $[a_{\alpha,j}, a^\dagger_{\alpha',j'}] = \delta_{\alpha,\alpha'}\delta_{j,j'}$. Problem 2 is then equivalent to the simulation of the Hamiltonian defined on the Hilbert space $\mathcal{H}_S \otimes \text{Fock}[\mathbb{C}^{N_m}]^{\otimes M}$

$$H(t) = H_S(t) + \sum_{\alpha=1}^{M} \|v_\alpha\|_{L^2} \left(L_\alpha a^\dagger_{\alpha,1} + L^\dagger_\alpha a_{\alpha,1}\right) + \sum_{\alpha=1}^{M}\sum_{j=1}^{N_m} \omega_{\alpha,j} a^\dagger_{\alpha,j} a_{\alpha,j} + \sum_{\alpha=1}^{M}\sum_{j=1}^{N_m-1} t_{\alpha,j}\left(a_{\alpha,j}a^\dagger_{\alpha,j+1} + a_{\alpha,j+1}a^\dagger_{\alpha,j}\right), \tag{25}$$

where $\{\omega_{\alpha,j}\}_{\alpha \in \{1,2\dots M\},\{1,2\dots N_m\}}, \{t_{\alpha,j}\}_{\alpha \in \{1,2\dots M\},\{1,2\dots N_m\}}$ are the chain parameters corresponding to the unitary group $\nu_{\alpha,t}$. We first truncate this model into a finite-dimensional model — to do so, we first derive a bound on the expectation number of particles in the M baths coupling to the system. Denoting by $\mu^{(1)}_\alpha = \langle \sum_{j=1}^{N_m} a^\dagger_{\alpha,j} a_{\alpha,j}\rangle$ and $\mu^{(2)}_\alpha = \langle \sum_{j=1}^{N_m} (a^\dagger_{\alpha,j} a_{\alpha,j})^2\rangle$, we obtain from Heisenberg's equations of motion that

$$\begin{aligned}
\frac{d}{dt}\mu^{(1)}_\alpha &= -i\|v_\alpha\|_{L^2}\left(\langle L_\alpha a^\dagger_{\alpha,1}\rangle - \text{c.c.}\right) \leq 2\|v_\alpha\|_{L^2}\|L_\alpha\|\sqrt{\langle a^\dagger_{\alpha,1}a_{\alpha,1}\rangle} \leq 2\|v_\alpha\|_{L^2}\|L_\alpha\|\sqrt{\mu^{(1)}_\alpha}, \\
\frac{d}{dt}\mu^{(2)}_\alpha &= -i\|v_\alpha\|_{L^2}\left(2\langle L_\alpha \big(a^\dagger_{\alpha,1}\big)^2 a_{\alpha,1}\rangle + \langle L_\alpha a^\dagger_{\alpha,1}\rangle - \text{c.c.}\right) \leq 2\|v_\alpha\|_{L^2}\|L_\alpha\|\sqrt{\mu^{(1)}_\alpha}\left(2\sqrt{\mu^{(2)}_\alpha} + 1\right)
\end{aligned}$$

integrating which yields

$$\mu^{(1)}_\alpha(t) \leq \left(\sqrt{\mu^{(1)}_\alpha(0)} + \|v_\alpha\|_{L^2}\|L_\alpha\|\, t\right)^2,$$

and

$$\sqrt{\mu^{(2)}_\alpha(t)} - \frac{1}{2}\log\left(1 + 2\sqrt{\mu^{(2)}_\alpha(t)}\right) \leq \sqrt{\mu^{(2)}_\alpha(0)} - \frac{1}{2}\log\left(1 + 2\sqrt{\mu^{(2)}_\alpha(0)}\right) + 2\|v_\alpha\|_{L^2}\|L_\alpha\|\left(\sqrt{\mu^{(1)}_\alpha(0)}t + \frac{1}{2}\|v_\alpha\|_{L^2}\|L_\alpha\|\, t^2\right),$$

or equivalently

$$\sqrt{\mu_\alpha^{(2)}(t)} - \left(\frac{\mu_\alpha^{(2)}(t)}{4}\right)^{1/4} \leq \sqrt{\mu_\alpha^{(2)}(0)} + 2\left\|v_\alpha\right\|_{L^2}\left\|L_\alpha\right\|\left(\sqrt{\mu_\alpha^{(1)}(0)}t + \frac{1}{2}\left\|v_\alpha\right\|_{L^2}\left\|L_\alpha\right\|t^2\right),$$

where we have used that for $x \geq 0$, $0 \leq \log(1+x) \leq \sqrt{x}$. We thus obtain that if $t = \text{poly}(n)$, $\mu_\alpha^{(2)}(t) = \text{poly}(n)$. With this bound, we consider truncating the Hamiltonian — given $p \in \mathbb{Z}_{>1}$, we consider the projector $\mathcal{P}_p = \text{id} \otimes \Pi_{\leq p}^{\otimes M}$, where $\Pi_{\leq p}$ is a projector onto the space with less than or equal to p particles defined on Fock$[\mathbb{C}^{N_m}]$. Also, we define $\mathcal{Q}_p = \text{id} - \mathcal{P}_p$. Denoting by $|\Psi(t)\rangle$ the state corresponding to the Hamiltonian under consideration (Eq. 25) at time t and by $U_{\mathcal{P}}(t, 0)$ the propagator corresponding to the Hamiltonian $\mathcal{P}H(t)\mathcal{P}$, then

$$\left\||\Psi(t)\rangle - U_{\mathcal{P}}(t,0)\mathcal{P}\left|\Psi_0\right\rangle\right\| \leq \left\|\mathcal{Q}_p\left|\Psi(t)\right\rangle\right\| + \int_0^t \left\|\mathcal{P}_p H(s)\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\| ds.$$

Both the terms in the above estimate can be easily bounded from above in terms of p — note that

$$\left\|\mathcal{Q}_p\left|\Psi(t)\right\rangle\right\|^2 \leq \sum_{\alpha=1}^M \left\langle\Psi(t)\right| \text{id} \otimes \left(\text{id}^{\otimes(\alpha-1)} \otimes \Pi_{>p} \otimes \text{id}^{M-\alpha}\right)\left|\Psi(t)\right\rangle \leq \frac{1}{p}\sum_{\alpha=1}^M \mu_\alpha^{(1)}(t) = \frac{1}{p}\text{poly}(n), \tag{26}$$

where we have used $M = \text{poly}(n)$. Furthermore, noting that $\mathcal{P}_p, \mathcal{Q}_p$ commute with any system operators, and for $\alpha \in \{1, 2 \dots M\}$, $i, j \in \{1, 2 \dots N_m\}$, $\mathcal{P}_p a_{\alpha,i}^\dagger a_{\alpha,j}\mathcal{Q}_p = 0$ and $\mathcal{P}_p a_{\alpha,i}^\dagger \mathcal{Q}_p = 0$, we obtain that for $s \in (0, t)$,

$$\left\|\mathcal{P}_p H(s)\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\| \leq \sum_{\alpha=1}^M \left\|v_\alpha\right\|_{L^2}\left\|L_\alpha\right\|\left\|\mathcal{P}_p a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\| \leq \sum_{\alpha=1}^M \left\|v_\alpha\right\|_{L^2}\left\|L_\alpha\right\|\left\|a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\|.$$

For $\alpha \in \{1, 2 \dots M\}$, we obtain that

$$\left\|a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\|^2 = \left\langle\Psi(s)\right|\mathcal{Q}_p a_{\alpha,1}^\dagger a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle = \left\langle\Psi(s)\right|a_{\alpha,1}^\dagger a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle \leq \left\langle\Psi(s)\right|\left(a_{\alpha,1}^\dagger a_{\alpha,1}\right)^2\left|\Psi(s)\right\rangle\left\langle\Psi(s)\right|\mathcal{Q}_p\left|\Psi(s)\right\rangle,$$

and consequently using Eq. 26,

$$\left\|a_{\alpha,1}\mathcal{Q}_p\left|\Psi(s)\right\rangle\right\|^2 \leq \frac{\mu_\alpha^{(2)}(s)}{p}\sum_{\alpha'=1}^M \mu_{\alpha'}^{(1)}(s).$$

Therefore, $\int_0^t \left\|\mathcal{P}_p H(s)\mathcal{Q}_p(s)\left|\Psi(s)\right\rangle\right\| ds \leq \text{poly}(n)/\sqrt{p}$. Thus, we obtain the estimate

$$\left\||\Psi(t)\rangle - U_{\mathcal{P}}(t,0)\mathcal{P}\left|\Psi_0\right\rangle\right\| \leq \frac{\text{poly}(n)}{\sqrt{p}}.$$

Hence, to ensure that the error is within the truncation is below $1/\text{poly}(n)$, we need to choose $p = \text{poly}(n)$.

Finally, we apply lemma 21 to prove the simulatability of the hamiltonan $\mathcal{P}_p H(t)\mathcal{P}_p$ — we need to show that on a (suitably chosen) basis, for any choice of computational basis $|b\rangle$, $H(t)|b\rangle$ can be efficiently computed as a sparse vector. We consider the basis set of the form $\mathcal{B} = \mathcal{B}_S \times \mathcal{B}_1 \times \mathcal{B}_2 \dots \times \mathcal{B}_M$, where $\mathcal{B}_S$ is the computational basis for the n qudit system, and for $\alpha \in \{1, 2 \dots M\}$, $\mathcal{B}_\alpha$ is the subset of Fock state basis for the α^{th} bath with number of particles less than p i.e.

$$\mathcal{B}_\alpha = \left\{\left(a_{\alpha,1}^\dagger\right)^{n_1}\left(a_{\alpha,2}^\dagger\right)^{n_2}\dots\left(a_{\alpha,N_m}^\dagger\right)^{n_{N_m}}\left|\text{vac}\right\rangle \middle| n_1, n_2 \dots n_{N_m} \in \mathbb{Z}_{\geq 0} \text{ with } n_1 + n_2 \dots n_{N_m} \leq p\right\}$$

Consider now the Hamiltonian $\mathcal{P}_p H(t)\mathcal{P}_p$ — it can be expressed as sum the following terms:

- $\mathcal{P}_p H_S(t)\mathcal{P}_p$ — Since by assumption is expressible only as poly(n) operators that act on at-most k qudits, it immediately follows that $\mathcal{P}_p H_S(t)\mathcal{P}_p|b\rangle = H_S(t)|b\rangle$ can be classically efficiently computed for $|b\rangle \in \mathcal{B}$.
- $\mathcal{P}_p \omega_{\alpha,j} a_{\alpha,j}^\dagger a_{\alpha,j}\mathcal{P}_p$ for $\alpha \in \{1, 2 \dots M\}, j \in \{1, 2 \dots N_m\}$ — this term is diagonal in the basis $\mathcal{B}$. Furthermore, since there are only $N_m M = \text{poly}(n)$ such terms, $\mathcal{P}_p \sum_{\alpha=1}^M \sum_{j=1}^{N_m} \omega_{\alpha,j} a_{\alpha,j}^\dagger a_{\alpha,j}\mathcal{P}_p|b\rangle$ can be efficiently computed $\forall\,|b\rangle \in \mathcal{B}$.
- $\mathcal{P}_p t_{\alpha,j}(a_{\alpha,j}a_{\alpha,j+1}^\dagger + a_{\alpha,j+1}a_{\alpha,j}^\dagger)\mathcal{P}_p$ for $\alpha \in \{1, 2 \dots M\}, j \in \{1, 2 \dots N_m - 1\}$ — applying this term on $|b\rangle \in \mathcal{B}$

produces a vector with at-most two non-zero elements when represented on the same basis. Furthermore, since there are only $(N_m - 1)M = \text{poly}(n)$ such terms, $\mathcal{P}_p \sum_{\alpha=1}^{M} \sum_{j=1}^{N_m-1} t_{\alpha,j}(a_{\alpha,j}a^\dagger_{\alpha,j+1} + a_{\alpha,j+1}a^\dagger_{\alpha,j})\mathcal{P}_p \left|b\right\rangle$ can be efficiently computed $\forall \left|b\right\rangle \in \mathcal{B}$.

- $\mathcal{P}_p\big(L_\alpha a^\dagger_{\alpha,1} + L^\dagger_\alpha a_{\alpha,1}\big)\mathcal{P}_p$ for $\alpha \in \{1, 2 \dots M\}$ — since L_α only acts on at-most k qudits, applying this term on $\left|b\right\rangle \in \mathcal{B}$ produces a vector with at-most $2d^k$ non-zero elements when represented on the same basis. Furthermore, since there are only $M = \text{poly}(n)$ such terms, $\mathcal{P}_p \sum_{\alpha=1}^{M} \big(L_\alpha a^\dagger_{\alpha,1} + L^\dagger_\alpha a_{\alpha,1}\big)\mathcal{P}_p \left|b\right\rangle$ can be efficiently computed $\forall \left|b\right\rangle \in \mathcal{B}$.

It thus follows that $\mathcal{P}_p H(t)\mathcal{P}_p \left|b\right\rangle$ can be efficiently computed $\forall \left|b\right\rangle \in \mathcal{B}$. Finally, we note from lemma 14 that $\left|\omega_{\alpha,j}\right|, t_{\alpha,j} \leq \omega_c$ for all $\alpha \in \{1, 2 \dots M\}, j \in \{1, 2 \dots N_m\}$

$$\left\|\mathcal{P}_p H(t)\mathcal{P}_p\right\| \leq \left\|H_S(t)\right\| + 2\sqrt{p+1}\sum_{\alpha=1}^{M} \left\|L_\alpha\right\| \left\|v_\alpha\right\|_{L^2} + pMN_m\omega_c + 2\omega_c(p+1)M(N_m - 1),$$

where we have used the estimates $\left\|\mathcal{P}_p a_{\alpha,j}\mathcal{P}_p\right\|, \left\|\mathcal{P}_p a^\dagger_{\alpha,j}\mathcal{P}_p\right\| \leq \sqrt{p+1}$, $\left\|\mathcal{P}_p a^\dagger_{\alpha,j}a_{\alpha,j}\mathcal{P}_p\right\| \leq p$. Noting that by assumption $\left\|L_\alpha\right\| \leq 1, \left\|H_S(t)\right\|, p, M, N_m, \omega_c, t = \text{poly}(n)$, we obtain that $\int_0^t \left\|\mathcal{P}_p H(s)\mathcal{P}_p\right\| ds \leq O(\text{poly}(n))$. Thus, from lemma 21, we can show that there is a circuit with depth $\text{poly}(n)$ that approximates the propagator corresponding to $\mathcal{P}_p H(s)\mathcal{P}_p$ with evolution time t within $1/\text{poly}(n)$ spectral norm error.Furthermore, the initial state can be efficiently represented on the basis $\mathcal{B}$ since it is efficiently projectable (assumption 2b), and hence the reduced system state $\rho_S(t)$ can be efficiently simulated on this quantum circuit. □

Theorem 2, repeated ($k-$local Non-Markovian dynamics $\in$ BQP)**.** *Problem 1 can be solved in* $\text{poly}(n)$ *time on a quantum computer.*

Proof: An application of lemma 3 approximates problem 1 to an instance of problem 2, and then the theorem statement follows from lemma 22. □

VI. CONCLUSION

Our work identifies the class of tempered radon measures as memory kernels for which a unitary group generating non-Markovian system dynamics can be constructed. We therefore generalize the unitary group for Markovian dynamics (i.e. with a delta function memory kernel) described in the theory of quantum stochastic calculus. We then consider the $k-$local many-body non-Markovian systems, and show that their dynamics can be efficiently simulated on quantum computers, thus establishing the consistency of this generalization with the Extended Church-Turing thesis.

Our work leaves open a few important open questions regarding non-Markovian dynamics. The first question is to further understand if the growth conditions on the radon measure describing the memory kernel (assumption 1) are necessary — while there are radon measures that violate these conditions, it is possible that these growth conditions hold for any *tempered* radon measure. Alternatively, perhaps violating these growth condition can lead to unphysical situations, such as “infinitely long” memory times in the non-Markovian system. Formalizing these ideas would allow us to further sharpen the class of memory kernels that describe physically reasonable non-Markovian models.

Second question would be to characterize the unitary group describing non-Markovian dynamics constructed in this paper. An important characterization would be to understand if this unitary group is generated by a self-adjoint Hamiltonian. Alternatively, is the unitary group strongly continuous? Similar questions have been previously answered for the unitary group for Markovian dynamics provided by a quantum stochastic differential equation [63–65].

Finally, it would also be of interest to develop quantum algorithms for non-Markovian dynamics with better dependence on the problem size as well as the incurred approximation error by exploiting further structure in the non-Markovian model (e.g. spatial locality, or availability of the Hamiltonian/jump operators as linear combination of unitaries) and using similar techniques that have been used in Hamiltonian or Lindbladian simulation problems [33, 36, 38, 66].

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