

BOST–CONNES SYSTEMS AND \mathbb{F}_1 -STRUCTURES IN GROTHENDIECK RINGS, SPECTRA, AND NORI MOTIVES

JOSHUA F. LIEBER, YURI I. MANIN, AND MATILDE MARCOLLI

ABSTRACT. We construct geometric lifts of the Bost–Connes algebra to Grothendieck rings and to the associated assembler categories and spectra, as well as to certain categories of Nori motives. These categorifications are related to the integral Bost–Connes algebra via suitable Euler characteristic type maps and zeta functions, and in the motivic case via fiber functors. We also discuss aspects of \mathbb{F}_1 -geometry, in the framework of torifications, that fit into this general setting.

1. INTRODUCTION AND SUMMARY

This survey/research paper interweaves many different strands that recently became visible in the fabric of algebraic geometry, arithmetics, (higher) category theory, quantum statistics, homotopical “brave new algebra” etc., see especially A. Connes and C. Consani [25] [26]; A. Huber, St. Müller–Stach [43], etc.

In this sense, our present paper can be considered as a continuation and further extension of [54], and we will be relying on much of the work in that paper for details and examples. The motivational starting point in [54] was coming from the interpretation given in [27] of the Bost–Connes quantum statistical mechanical system, and in particular the integral Bost–Connes algebra, as a form of \mathbb{F}_1 -structure, or “geometry below $\text{Spec}(\mathbb{Z})$ ”. The main theme of [54] is an exploration of how this structure manifests itself beyond the usual constructions of \mathbb{F}_1 -structures on certain classes of varieties over \mathbb{Z} . In particular, the results of [54] focus on lifts of the integral Bost–Connes algebra to certain Grothendieck rings and to associated homotopy-theoretic spectra obtained via assembler categories, and also on another form of \mathbb{F}_1 -structures arising through quasi-unipotent Morse–Smale dynamical systems.

The main difference between the present paper and [54] consists in a change of the categorical environment: the unifying vision we already considered in [54] was provided by I. Zakharevich’s notions of assemblers and scissors congruences: cf. [70], [71], [72], and [21]. In this paper, we continue to use the formalism of assemblers and the associated spectra, but we complement it with categories of Nori motives, [43].

As in [54], we focus primarily on various geometrizations of the Bost–Connes algebra(s). Some of these constructions take place in Grothendieck rings, like the previous cases considered in [54], and are aimed at lifting the Bost–Connes endomorphisms to the level of homotopy theoretic spectra through the use of Zakharevich’s formalism of assembler categories. We focus on the case of relative Grothendieck rings, endowed with appropriate equivariant Euler characteristic maps. For varieties that

admits torifications, we introduce zeta functions based on the counting of points over \mathbb{F}_1 and over extensions \mathbb{F}_{1^m} . We present a more general construction of Bost–Connes type systems associated to exponentiable motivic measures and the associated zeta functions with values in Witt rings, obtained using a lift of the Bost–Connes algebra to Witt rings via Frobenius and Verschiebung maps.

We then consider lifts of the Bost–Connes algebra to Nori motives, where we use a (slightly generalized) version of Nori motives, which may be of independent interest in view of possible versions of equivariant periods. In this categorical setting we show that the fiber functor from Nori motives maps to a categorification of the Bost–Connes algebra previously constructed by Tabuada and the third author, compatibly with the functors realizing the Bost–Connes structure.

1.1. Structure of the paper and main results. Below we will briefly describe the content of the subsequent Sections, and the main results of the paper, with pointers to the specific statements where these are proved.

1.1.1. Bost–Connes systems and relative equivariant Grothendieck rings. In §2, we show the existence of a lift of the Bost–Connes structure to the relative equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, extending similar results previously obtained in [54] for the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$. The main result in this part of the paper is Theorem 2.11, where the existence of this lift is proved. The rest of the section covers the preliminary results needed for this main result.

In particular, we first introduce the integral Bost–Connes algebra in §2.1, in the form in which it was introduced in [27]. We recall in §2.2 and 2.3 the relative and the equivariant relative Grothendieck rings, and in §2.4 the associated equivariant Euler characteristic map.

In §2.5 we recall from [54] the geometric form of the Verschiebung map that is used in the lifting of the Bost–Connes structure to varieties with suitable $\hat{\mathbb{Z}}$ -actions. In §2.6 we introduce the Bost–Connes maps σ_n and $\tilde{\rho}_n$ on classes in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ and Proposition 2.6 shows the way they transform the varieties and the base scheme with their respective $\hat{\mathbb{Z}}$ -action.

In §2.7 we recall from [27] the prime decomposition of the integral Bost–Connes algebra, which for a finite set of primes F separates out an F -part and an F -coprime part of the algebra. We then show in §2.8, and in particular Proposition 2.8, that, given a scheme S with a good effectively finite action of $\hat{\mathbb{Z}}$, there is an associated finite set of primes F such that the F -coprime part of the Bost–Connes algebra lifts to endomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$.

Finally in §2.9 we show how to lift the full Bost–Connes algebra to homomorphisms between Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ where the scheme S and the action α are also transformed by the Bost–Connes map. By an analysis of the structure of periodic points in Lemma 2.9 we show the compatibility with the equivariant Euler characteristic, so we can then prove the main result in Theorem 2.11, showing that the equivariant Euler characteristic intertwines the Bost–Connes maps σ_n and $\tilde{\rho}_n$ on

the $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ rings with the original σ_n and $\tilde{\rho}_n$ maps of the integral Bost–Connes algebra.

1.1.2. *Bost–Connes systems on assembler categories and spectra.* In §3 we further lift the Bost–Connes structure obtained at the level of Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ to assembler categories underlying these Grothendieck rings and to the homotopy-theoretic spectra defined by these categories. Again this extends to the equivariant relative case results that were obtained in [54] for the non-relative setting. The main result in this part of the paper is Theorem 3.15, where it is shown that the maps σ_n and $\tilde{\rho}_n$ on the Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ constructed in the previous section lift to functors of the underlying assembler categories, that induce these maps on K_0 .

In §3.1 we recall the formalism of assembler categories of [70], underlying scissor congruence relations and Grothendieck rings. In §3.2 we review Segal’s Γ -spaces formalism and how one obtains the homotopy-theoretic spectrum associated to an assembler. In §3.3 and §3.4 we then lift this formalism by endowing the main relevant objects with an action of a finite cyclic group, with appropriate compatibility conditions. It is this further structure that provides a framework for the respective lifts of the Bost–Connes algebras, as in the cases discussed in [54] and in the ones we will be discussing in the following sections. We give here a very general definition of Bost–Connes systems in categories, based on endofunctors of subcategories of the automorphism category. In the applications considered in this paper we will be using only the special case where the automorphisms are determined by an effectively finite action of $\hat{\mathbb{Z}}$, but we introduce the more general framework in anticipation of other possible applications.

In §3.5 we construct the assembler underlying the equivariant relative Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ and we prove the main result in Theorem 3.15 on the lift of the Bost–Connes structure to functors of these assemblers.

1.1.3. *Bost–Connes systems on Grothendieck rings and assemblers of torified varieties.* In §4 we consider the approach to \mathbb{F}_1 -geometry via torifications of varieties over \mathbb{Z} , introduced in [51]. The main results of this part of the paper are Proposition 4.2 and Proposition 4.5 where we construct assembler categories of torified varieties and we show the existence of a lift of the Bost–Connes algebra to these categories.

In §4.1 we recall the notion of torified varieties from [51] and the different versions of morphisms of torified varieties from [53], and we construct Grothendieck rings of torified varieties for each flavor of morphisms. In §4.1.1 we introduce a relative version of these Grothendieck rings of torified varieties. In §4.2 we describe \mathbb{Q}/\mathbb{Z} and $\hat{\mathbb{Z}}$ -actions on torifications.

In §4.3 we construct the assembler categories underlying these relative Grothendieck rings and in §4.4 we prove the first main result of this section by constructing the lift of the Bost–Connes structure.

1.1.4. *Torified varieties, \mathbb{F}_1 -points, and zeta functions.* This section continues the theme of torified varieties from the previous section but with main focus on some associated zeta functions. We consider two different kinds of zeta function: \mathbb{F}_1 -zeta functions that count \mathbb{F}_1 -points of torified varieties, in an appropriate sense that is discussed in §5.1, and dynamical zeta functions associated to endomorphisms of torified varieties that are compatible with the torification. The use of dynamical zeta functions is motivated by a proposal made in [54] for a notion of \mathbb{F}_1 -structures based on dynamical systems that induce quasi-unipotent maps in homology.

The two main results of this section are Proposition 5.4 and Proposition 5.8 where we show that the \mathbb{F}_1 -zeta function, respectively the dynamical zeta function, determine exponentiable motivic measures from the Grothendieck rings of torified varieties introduced in the previous section to the ring $W(\mathbb{Z})$ of Witt vectors.

We introduce in §5.1 and §5.2 the counting of \mathbb{F}_1 -points of a torified variety and its relation to the Grothendieck class. We show in §5.2 how the Bialynicki–Birula decomposition can be used to determine torifications and we give in §5.3 some explicit examples of computations of Grothendieck classes in simple cases that have physical significance in the context of BPS counting in string theory.

In §5.4 we introduce the \mathbb{F}_1 -zeta function and we prove Proposition 5.4. In §5.5 we explain how the \mathbb{F}_1 -zeta function can be obtained from the Hasse–Weil zeta function.

In §5.6 we consider torified varieties with dynamical systems compatible with the torification and the associated Lefschetz and Artin–Mazur dynamical zeta functions. We recall the definition and main properties of these zeta functions in §5.6.1 and we prove in Proposition 5.8 in §5.6.2.

1.1.5. *Spectrification of Witt vectors and lifts of zeta functions.* In the constructions described in §§ 3 and 4 of [54] and in §§ 2–6 of the present paper we obtain lifts of the integral Bost–Connes algebra to various assembler categories and associated spectra, starting from a ring homomorphism (motivic measure) from the relevant Grothendieck ring to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost–Connes algebra, that is equivariant with respect to the maps σ_n and $\tilde{\rho}_n$ of (2.4) and (2.5) of the Bost–Connes algebra and the maps (also denoted by σ_n and $\tilde{\rho}_n$) on the Grothendieck ring induced by a Bost–Connes system on the corresponding assembler category. The motivic measure provides in this way a map that lifts the Bost–Connes structure.

This part of the paper considers then a more general class of zeta functions ζ_μ associated to exponentiable motivic measures $\mu : K_0(\mathcal{V}) \rightarrow R$ with values in a commutative ring R , that admit a factorization into linear factors in the subring $W_0(R)$ of the Witt ring $W(R)$.

Our main results in this section are Proposition 6.9, showing that these zeta functions lift to the level of assemblers and spectra, and Proposition 6.14, which shows that the Frobenius and Verschiebung maps on the endomorphism category lift, through the lift of the zeta function, to a Bost–Connes system on the assembler category of the Grothendieck ring of varieties $K_0(\mathcal{V})$.

The main step toward establishing the main results of this section is the construction in §6.2 and §6.3 of a spectrification of the ring $W_0(R)$. This is obtained using its description in terms of the K_0 of the endomorphism category \mathcal{E}_R and of R , and the formalism of Segal Gamma-spaces. The spectrification we use here is not the same as the spectrification of the ring of Witt vectors introduced in [40]. The lifting of Bost–Connes systems via motivic measures is discussed in §6.4, where Proposition 6.14 is proved.

We also consider again in §6.5 the setting on dynamical \mathbb{F}_1 -structures proposed in [54], with a pair (X, f) of a variety and an endomorphism that induces a quasi-unipotent map in homology, and we associate to these data the operator-theoretic spectrum of the quasi-unipotent map, seen as an element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. This determines a spectral map $\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the properties of an Euler characteristic.

Another main result in this section is Proposition 6.16, showing that this spectral Euler characteristic lifts to a functor from the assembler category underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to the Tannakian category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ that categorifies the ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, and that the resulting functor lifts the Bost–Connes structure on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ described in [57] to a Bost–Connes structure on the assembler of $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$.

1.1.6. *Bost–Connes systems in categories of Nori motives.* When we replace the formalism of assembler categories and homotopy theoretic spectra underlying the Grothendieck rings with geometric diagrams and associated Tannakian categories of Nori motives, with the same notion of categorical Bost–Connes systems introduced in Definitions 3.9 and 3.11, we can lift the Euler characteristic type motivic measures to the level of categorifications, where, as in the previous section, the categorification of the Bost–Connes algebra is the one introduced in [57], given by a Tannakian category of \mathbb{Q}/\mathbb{Z} -graded vector spaces endowed with Frobenius and Verschiebung endofunctors.

In §7 in this paper we construct Bost–Connes systems in categories of Nori motives. The main result of this part of the paper is Theorem 7.7, which shows that there is a categorical Bost–Connes system on a category of equivariant Nori motives, and that the fiber functor to the categorification of the Bost–Connes algebra constructed in [57] intertwines the respective Bost–Connes endofunctors.

In §7.1 and §7.2 we review the construction of Nori motives from diagrams and their representations. In §7.3 we construct a category of equivariant Nori motives. In §7.4 we describe the endofunctors of this category that implement the Bost–Connes structure and we prove the main result in Theorem 7.7. In §7.6 we generalize this result to the relative case, using Arapura’s motivic sheaves version of Nori motives.

Finally, in §7.6 we consider Nori diagrams associated to assemblers and we formulate the question of their “universal cohomological representations”. This is a contemporary embodiment of the primordial Grothendieck’s dream that motives constitute a universal cohomology theory of algebraic varieties.

2. BOST-CONNES SYSTEMS IN GROTHENDIECK RINGS

In [54] it was shown that the integral Bost–Connes algebra of [27] admits lifts to certain Grothendieck rings, via corresponding equivariant Euler characteristic maps. The cases analyzed in [54] included the cases of the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ and the equivariant Konsevich–Tschinkel Burnside ring $\text{Burn}^{\hat{\mathbb{Z}}}(\mathbb{K})$. We treat here, in a similar way, the case of the relative equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. This case is more delicate than the other cases considered in [54], because when the Bost–Connes maps act on the classes in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ they also change the base scheme S with its $\hat{\mathbb{Z}}$ -action.

The main result in this section is the existence of a lifting of the Bost–Connes structure to $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, proved in Theorem 2.11.

We first review the definition of the integral Bost–Connes algebra in §2.1 and the equivariant relative Grothendieck ring in §2.3. The rest of the section then develops the intermediate steps leading to the proof of the main results of Theorem 2.11.

2.1. Bost–Connes algebra. The Bost–Connes algebra was introduced in [15] as a quantum statistical mechanical system that exhibit the Riemann zeta function as partition function, the generators of the cyclotomic extensions of \mathbb{Q} as values of zero-temperature KMS equilibrium states on arithmetic elements in the algebra, and the abelianized Galois group $\hat{\mathbb{Z}}^* \simeq \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab}$ as group of quantum symmetries. In particular, the arithmetic subalgebra of the Bost–Connes system is given by the semigroup crossed product

$$(2.1) \quad \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$$

of the multiplicative semigroup \mathbb{N} of positive integers acting on the group algebra of the group \mathbb{Q}/\mathbb{Z} .

The additive group \mathbb{Q}/\mathbb{Z} can be identified with the multiplicative group ν^* of *roots of unity embedded into \mathbb{C}^** : namely, $r \in \mathbb{Q}/\mathbb{Z}$ corresponds to $e(r) := \exp(2\pi i r)$. More generally, the choice of the embedding can be modified by an arbitrary choice of an element in $\hat{\mathbb{Z}}^* = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, as is usually done in representations of the Bost–Connes algebra, see [15]. Thus, we will use here interchangeably the notation ζ or r for elements of \mathbb{Q}/\mathbb{Z} assuming a choice of embedding as above. The group algebra $\mathbb{Q}[\nu^*]$ consists of formal finite linear combinations $\sum_{a_\zeta \in \mathbb{Q}} a_\zeta \zeta$ of roots of unity $\zeta \in \nu^*$. Formality means here that the sum is *not* related to the additive structure of \mathbb{C} .

The action of the semigroup \mathbb{N} on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ that defines the crossed product (2.1) is given by the endomorphisms

$$(2.2) \quad \rho_n \left(\sum a_\zeta \zeta \right) := \sum a_\zeta \frac{1}{n} \sum_{\zeta' n = \zeta} \zeta'.$$

Equivalently, the algebra (2.1) is generated by elements $e(r)$ with the relations $e(0) = 1$, $e(r + r') = e(r)e(r')$, and elements μ_n and μ_n^* satisfying the relations

$$(2.3) \quad \begin{aligned} \mu_n^* \mu_n &= 1, \forall n; & \mu_n \mu_n^* &= \pi_n, \forall n & \text{with } \pi_n &= \frac{1}{n} \sum_{nr=0} e(r); \\ \mu_{nm} &= \mu_n \mu_m, \forall n, m; & \mu_{nm}^* &= \mu_n^* \mu_m^*, \forall n, m; & \mu_n^* \mu_m^* &= \mu_m^* \mu_n^* \text{ if } (n, m) = 1. \end{aligned}$$

The semigroup action (2.2) is then equivalently written as $\rho_n(a) = \mu_n a \mu_n^*$, for all $a = \sum a_\zeta \zeta$ in $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. The element $\pi_n \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is an idempotent, hence the generators μ_n are isometries but not unitaries. See [15] and §3 of [28] for a detailed discussion of the Bost–Connes system and the role of the arithmetic subalgebra (2.1).

In [27] an integral model of the Bost–Connes algebra was constructed in order to develop a model of \mathbb{F}_1 -geometry in which the Bost–Connes system encodes the extensions \mathbb{F}_{1^m} , in the sense of [45], of the “field with one element” \mathbb{F}_1 .

The integral Bost–Connes algebra is obtained by considering the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, which we can again implicitly identify with $\mathbb{Z}[\nu^*]$ for a choice of embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$ as roots of unity.

Define its *ring endomorphisms* σ_n :

$$(2.4) \quad \sigma_n\left(\sum a_\zeta \zeta\right) := \sum a_\zeta \zeta^n.$$

Define *additive maps* $\tilde{\rho}_n: \mathbb{Z}[\nu^*] \rightarrow \mathbb{Z}[\nu^*]$:

$$(2.5) \quad \tilde{\rho}_n\left(\sum a_\zeta \zeta\right) := \sum a_\zeta \sum_{\zeta'^n = \zeta} \zeta'.$$

The maps σ_n and $\tilde{\rho}_n$ satisfy the relations

$$(2.6) \quad \sigma_n \circ \tilde{\rho}_n = n \text{ id}, \quad \tilde{\rho}_n \circ \sigma_n = n \pi_n.$$

The integral Bost–Connes algebra is then defined as the algebra generated by the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and generators $\tilde{\mu}_n$ and μ_n^* with the relations

$$(2.7) \quad \begin{aligned} \tilde{\mu}_n a \mu_n^* &= \tilde{\rho}_n(a), \forall n; & \mu_n^* a &= \sigma_n(a) \mu_n^*, \forall n; & a \tilde{\mu}_n &= \tilde{\mu}_n \sigma_n(a), \forall n; \\ \tilde{\mu}_{nm} &= \tilde{\mu}_n \tilde{\mu}_m, \forall n, m; & \mu_{nm}^* &= \mu_n^* \mu_m^*, \forall n, m; & \tilde{\mu}_n \mu_m^* &= \mu_m^* \tilde{\mu}_n \text{ if } (n, m) = 1. \end{aligned}$$

where the relations in the first line hold for all $a = \sum a_\zeta \zeta \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, with σ_n and $\tilde{\rho}_n$ as in (2.4) and (2.5).

The maps $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the semigroup action ρ_n in the rational Bost–Connes algebra (2.1) are related by

$$\rho_n = \frac{1}{n} \tilde{\rho}_n$$

with $\tilde{\rho}_n$ defined as in (2.5).

2.2. Relative Grothendieck ring. We describe here a variant of construction of [54], where we work with relative Grothendieck rings and with an Euler characteristic with values in a Grothendieck ring of locally constant sheaves. We show that this relative setting provides ways of lifting to the level of Grothendieck classes certain subalgebras of the integral Bost–Connes algebras associated to the choice of a finite set of non-archimedean places.

Definition 2.1. The relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety S over a field \mathbb{K} is generated by the isomorphism classes of data $f : X \rightarrow S$ of a variety X over S with the relations

$$[f : X \rightarrow S] = [f|_Y : Y \rightarrow S] + [f|_{X \setminus Y} : X \setminus Y \rightarrow S]$$

as in (7.1) for a closed embedding $Y \hookrightarrow X$ of varieties over S . The product is given by the fibered product $X \times_S Y$. We will write $[X]_S$ as shorthand notation for the class $[f : X \rightarrow S]$ in $K_0(\mathcal{V}_S)$.

A morphism $\phi : S \rightarrow S'$ induces a base change ring homomorphism $\phi^* : K_0(\mathcal{V}_{S'}) \rightarrow K_0(\mathcal{V}_S)$ and a direct image map $\phi_* : K_0(\mathcal{V}_S) \rightarrow K_0(\mathcal{V}_{S'})$ which is a group homomorphism and a morphism of $K_0(\mathcal{V}_{S'})$ -modules, but not a ring homomorphism. The class $[\phi : S \rightarrow S']$ as an element in $K_0(\mathcal{V}_{S'})$ is the image of $1 \in K_0(\mathcal{V}_S)$ under ϕ_* .

When $S = \text{Spec}(\mathbb{K})$ one recovers the ordinary Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

2.3. Equivariant relative Grothendieck ring. Let X be a variety with a good action $\alpha : G \times X \rightarrow X$ by a finite group G and X' a variety with a good action α' by G' . As morphisms we then consider pairs (ϕ, φ) of a morphism $\phi : X \rightarrow X'$ and a group homomorphism $\varphi : G \rightarrow G'$ such that $\phi(\alpha(g, x)) = \alpha'(\varphi(g), \phi(x))$, for all $g \in G$ and $x \in X$. Thus, isomorphisms of varieties with good G -actions are pairs of an isomorphism $\phi : X \rightarrow X'$ of varieties and a group automorphism $\varphi \in \text{Aut}(G)$ with the compatibility condition as above.

Given a base variety (or scheme) S with a given good action α_S of a finite group G , and varieties X, X' over S , with good G -actions $\alpha_X, \alpha_{X'}$ and G -equivariant maps $f : X \rightarrow S$ and $f' : X' \rightarrow S$, we consider morphisms given by a triple (ϕ, φ, ϕ_S) of a morphism $\phi : X \rightarrow X'$, a group homomorphism $\varphi : G \rightarrow G$ with the compatibility as above, and an endomorphism $\phi_S : S \rightarrow S$ such that $f' \circ \phi = \phi_S \circ f$. Then these maps also satisfy $\phi_S(\alpha_S(g, f(x))) = \alpha_S(\varphi(g), \phi_S(f(x)))$.

Definition 2.2. The relative equivariant Grothendieck ring $K_0^G(\mathcal{V}_S)$ is obtained as follows. Consider the abelian group generated by isomorphism classes $[f : X \rightarrow S]$ of varieties over S with compatible good G -actions, with respect to isomorphisms (ϕ, φ, ϕ_S) as above, with the inclusion-exclusion relations generated by equivariant embeddings with compatible G -equivariant maps

$$(2.8) \quad \begin{array}{ccccc} Y & \hookrightarrow & X & \longleftarrow & X \setminus Y \\ & \searrow & \downarrow f & \swarrow & \\ & & S & & \end{array} \begin{array}{l} f|_Y \\ f|_{X \setminus Y} \end{array}$$

and isomorphisms. This means that we have $[f : X \rightarrow S] = [f_Y : Y \rightarrow S] + [f_{X \setminus Y} : X \setminus Y \rightarrow S]$ if there are isomorphisms $(\phi_Y, \varphi_Y, \phi_{S,Y})$ and $(\phi_{X \setminus Y}, \varphi_{X \setminus Y}, \phi_{S, X \setminus Y})$, such that the diagram commutes

$$(2.9) \quad \begin{array}{ccccc} Y & \xrightarrow{\phi_Y} & Y \hookrightarrow & X & \xleftarrow{\quad} & X \setminus Y & \xleftarrow{\phi'_{X \setminus Y}} & X \setminus Y \\ \downarrow f_Y & & \searrow & \downarrow f & \swarrow & \downarrow f_{X \setminus Y} & & \downarrow f_{X \setminus Y} \\ S & \xrightarrow{\phi_{S,Y}} & & S & \xleftarrow{\phi_{S, X \setminus Y}} & S & & S \end{array}$$

The product $[f : X \rightarrow S] \cdot [f' : X' \rightarrow S]$ given by $[\tilde{f} : X \times_S X' \rightarrow S]$ with $\tilde{f} = f \circ \pi_X = f' \circ \pi_{X'}$ is well defined on isomorphism classes, with the diagonal action $\tilde{\alpha}(g, (x, x')) = (\alpha_X(g, x), \alpha_{X'}(g, x'))$ satisfying $f(\alpha_X(g, x)) = \alpha_S(g, f(x)) = \alpha_S(g, f'(x')) = f'(\alpha_{X'}(g, x'))$.

We will use the following terminology for the $\hat{\mathbb{Z}}$ -actions we consider.

Definition 2.3. A good effectively finite action of $\hat{\mathbb{Z}}$ on a variety X is a good action that factors through an action of some quotient $\mathbb{Z}/N\mathbb{Z}$. We will write $\mathbb{Z}/N\mathbb{Z}$ -effectively finite when we need to explicitly keep track of the level N .

In the case of the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ considered in [54], we can then also consider a relative version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, with S a variety with a good effectively finite $\hat{\mathbb{Z}}$ -action as above. We consider the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ given by the isomorphism classes of S -varieties $f : X \rightarrow S$ with good effectively finite $\hat{\mathbb{Z}}$ -actions with respect to which f is equivariant, with the notion of isomorphism described above. The product is given by the fibered product over S with the diagonal $\hat{\mathbb{Z}}$ -action. The inclusion-exclusion relations are as in (7.1) where $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are equivariant embeddings with compatible $\hat{\mathbb{Z}}$ -equivariant maps as in (2.9).

2.4. Equivariant Euler characteristic. There is an Euler characteristic map given by a ring homomorphism

$$(2.10) \quad \chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$$

to the Grothendieck ring of constructible sheaves over S with $\hat{\mathbb{Z}}$ -action, [39], [50], [59], [69].

Lemma 2.4. *Let S be a variety with a good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -action. Given a constructible sheaf $[\mathcal{F}]$ in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$, let $\mathcal{F}|_{S^g}$ denote the restrictions to the fixed point sets S^g , for $g \in \mathbb{Z}/N\mathbb{Z}$. These determine classes in $K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. One obtains in this way a map*

$$(2.11) \quad \chi : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow \bigoplus_{g \in \mathbb{Z}/N\mathbb{Z}} K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}].$$

Proof. The $\hat{\mathbb{Z}}$ action on S factors through some $\mathbb{Z}/N\mathbb{Z}$, hence the fixed point sets are given by S^g for $g \in \mathbb{Z}/N\mathbb{Z}$. Given a constructible sheaves \mathcal{F} over S with $\hat{\mathbb{Z}}$ -action, consider the restrictions $\mathcal{F}|_{S^g}$. The subgroup $\langle g \rangle$ generated by g acts trivially on S^g , hence for each $s \in S^g$ it acts on the stalk \mathcal{F}_s . Thus, these restrictions define classes $[\mathcal{F}|_{S^g}] \in K_0(\mathbb{Q}_{S^g}) \otimes R(\langle g \rangle) \subset K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. By precomposing with the Euler characteristic (2.10) one then obtains the map (2.11). \square

We will also consider the map $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ given by the Euler characteristic followed by restriction of sheaves to the fixed point set S^G of the group action.

2.4.1. *Fixed points and delocalized homology.* Equivariant characteristic classes from constructible sheaves to delocalized homology are constructed in [59].

For a variety S with a good action by a finite group G , and a (generalized) homology theory H , the associated delocalized equivariant theory is given by

$$H^G(S) = (\oplus_{g \in G} H(S^g))^G$$

where the disjoint union $\sqcup_g S^g$ of the fixed point sets S^g has an induced G -action $h : S^g \rightarrow S^{gh^{-1}}$. In the case of an abelian group we have $H^G(S) = (\oplus_{g \in G} H(S^g))^G$.

As an observation, we can see explicitly the relation of delocalized homology to the integral Bost–Connes algebra, by considering the following cases (see Remark 2.12). Let S be a variety with a good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -action. If S has the trivial $\mathbb{Z}/N\mathbb{Z}$ -action we have $H^{\mathbb{Z}/N\mathbb{Z}}(S) = H(S) \otimes \mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. In particular, if S is just a point, then this is $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. More generally, there is a morphism

$$\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}] \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \rightarrow H^{\mathbb{Z}/N\mathbb{Z}}(S)$$

induced by $H^{\mathbb{Z}/N\mathbb{Z}}(pt) \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \rightarrow H^{\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}}(pt \times S) \rightarrow H^{\mathbb{Z}/N\mathbb{Z}}(S)$ with the restriction to the diagonal subgroup as the last map.

2.5. **The geometric Verschiebung action.** We recall here how to construct the geometric Verschiebung action used in [54] to lift the Bost–Connes maps to the level of Grothendieck rings. This has the effect of transforming an action of $\hat{\mathbb{Z}}$ on X that factors through some $\mathbb{Z}/N\mathbb{Z}$ into an action of $\hat{\mathbb{Z}}$ on $X \times Z_n$, with $Z_n = \{1, \dots, n\}$, that factors through $\mathbb{Z}/Nn\mathbb{Z}$. For $x \in X$, let $\underline{x} = (x, a_i)_{a_i \in Z_n} = (x_i)_{i=1}^n$ be the subset $\{x\} \times Z_n$. For ζ_N a primitive N -th root of unity, we write in matrix form

$$V_n(\zeta_{Nn}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha(\zeta_N) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

so that we can write

$$(2.12) \quad V_n(\zeta_{Nn}) \cdot \underline{x} = \begin{cases} (x, a_{i+1}) & i = 1, \dots, n-1 \\ (\alpha(\zeta_N) \cdot x, a_1) & i = n \end{cases}$$

which satisfies $V_n(\zeta_{Nn})^n = \alpha(\zeta_N) \times \text{Id}_{Z_n}$. The resulting action $\Phi_n(\alpha)$ of $\hat{\mathbb{Z}}$ on $X \times Z_n$ that factors through $\mathbb{Z}/Nn\mathbb{Z}$ is specified by setting

$$(2.13) \quad \Phi_n(\alpha)(\zeta_{Nn}) \cdot (x, a) = (V_n(\alpha(\zeta_N)) \cdot \underline{x})_a.$$

2.6. Lifting the Bost–Connes endomorphisms. Consider a base scheme S with a good effectively finite action of $\hat{\mathbb{Z}}$. Let $f : X \rightarrow S$ be a variety over S with a good effectively finite $\hat{\mathbb{Z}}$ action such that the map is $\hat{\mathbb{Z}}$ -equivariant. We denote by $\alpha_S : \hat{\mathbb{Z}} \times S \rightarrow S$ the action on S and by $\alpha_X : \hat{\mathbb{Z}} \times X \rightarrow X$ the action on X . We write the equivariant relative Grothendieck ring as $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)})$ to explicitly remember the fixed (up to isomorphisms as in §2.3) action on S .

Definition 2.5. Let (S, α_S) be a scheme with a good effectively finite action of $\hat{\mathbb{Z}}$. Let $Z_n = \text{Spec}(\mathbb{Q}^n)$ and let $\Phi_n(\alpha_S)$ be the action of $\hat{\mathbb{Z}}$ on $S \times Z_n$ as in (2.12) and (2.13). Given a class $[f : (X, \alpha_X) \rightarrow (S, \alpha_S)]$ in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)})$, with α_X the compatible $\hat{\mathbb{Z}}$ -action on X , let

$$(2.14) \quad \sigma_n[f : (X, \alpha_X) \rightarrow (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \rightarrow (S, \alpha_S \circ \sigma_n)]$$

$$(2.15) \quad \tilde{\rho}_n[f : (X, \alpha_X) \rightarrow (S, \alpha_S)] = [f \times \text{id} : (X \times Z_n, \Phi_n(\alpha_X)) \rightarrow (S \times Z_n, \Phi_n(\alpha_S))].$$

Proposition 2.6. *For all $n \in \mathbb{N}$ the σ_n defined in (2.14) are ring homomorphisms*

$$(2.16) \quad \sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S \circ \sigma_n)})$$

and the $\tilde{\rho}_n$ defined in (2.15) are group homomorphisms

$$(2.17) \quad \tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha_S))}),$$

with compositions satisfying

$$\begin{aligned} \tilde{\rho}_n \circ \sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}) &\rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \alpha_S \times \alpha_n)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}) \\ \sigma_n \circ \tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}) &\rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)^{\oplus n}}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)}), \end{aligned}$$

with $\sigma_n \circ \tilde{\rho}_n = n \text{id}$ and $\tilde{\rho}_n \circ \sigma_n$ is the product by (Z_n, α_n) .

Proof. Consider the σ_n defined in (2.14). Since the group $\hat{\mathbb{Z}}$ is commutative and so is its endomorphism ring, these transformations σ_n respect isomorphism classes since for an isomorphism (ϕ, φ, ϕ_S) the actions satisfy

$$\phi_X(\alpha_X(\sigma_n(g), x)) = \alpha'_X(\varphi(\sigma_n(g)), \phi(x)) = \alpha'_X(\sigma_n(\varphi(g)), \phi(x)),$$

and similarly for the actions α_S, α'_S , so that (ϕ, φ, ϕ_S) is also an isomorphism of the images under σ_n . Similarly, the $\tilde{\rho}_n$ defined in (2.15) are well defined on the isomorphism classes.

As in [54] we see that $\sigma_n \circ \tilde{\rho}_n[f : (X, \alpha_X) \rightarrow (S, \alpha_S)] = [f : (X, \alpha_X) \rightarrow (S, \alpha_S)]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : (X, \alpha_X) \rightarrow (S, \alpha_S)] = [f \times \text{id} : (X \times Z_n, \alpha_X \times \alpha_n) \rightarrow (S \times Z_n, \alpha_S \times \alpha_n)]$. The Grothendieck groups $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \alpha_S \times \alpha_n)})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)^{\oplus n}})$ map to $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)})$ via the morphism induced by composition with the natural maps of the respective base varieties to (S, α_S) . \square

The fact that the ring homomorphisms (2.16) and (2.17) determine a lift of the ring endomorphism $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and group homomorphisms $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost–Connes algebra is discussed in Proposition 2.8 and §2.9.

We know from [54] that the integral Bost–Connes algebra lifts to the equivariant Grothendieck ring $K^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$ via maps σ_n and $\tilde{\rho}_n$ that, respectively, precompose the action with the Bost–Connes endomorphism σ_n and apply a geometric form of the Verschiebung map. The main difference with the relative case considered here lies in the fact that the lifts to the equivariant relative Grothendieck rings given by the maps (2.16) and (2.17) need to transform in a compatible way the actions on both X and S .

Remark 2.7. Because the maps σ_n and $\tilde{\rho}_n$ of (2.16) and (2.17) simultaneously modify the action on the varieties and on the base scheme S , they do not give endomorphisms of the same $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)})$. However, given (S, α_S) , it is possible to identify a subalgebra of the integral Bost–Connes algebra that lift to endomorphisms of a corresponding subring of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S, \alpha_S)})$, using the notion of “prime decomposition” of the Bost–Connes algebra. We discuss this more carefully in §2.7, §2.8 and §2.9.

2.7. Prime decomposition of the Bost–Connes algebra. As in [27], for each prime p , we can decompose the group \mathbb{Q}/\mathbb{Z} into a product $\mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is the Prüfer group, namely the subgroup of elements of \mathbb{Q}/\mathbb{Z} where the denominator is a power of p , isomorphic to $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, while $(\mathbb{Q}/\mathbb{Z})^{(p)}$ consists of the elements with denominator prime to p .

Similarly, given a finite set F of primes, we can decompose $\mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})_F \times (\mathbb{Q}/\mathbb{Z})^F$, where the first term $(\mathbb{Q}/\mathbb{Z})_F$ is identified with fractions in \mathbb{Q}/\mathbb{Z} whose denominator has prime factor decomposition consisting only of primes in F , while elements in $(\mathbb{Q}/\mathbb{Z})^F$ have denominators prime to all $p \in F$. The group ring decomposes accordingly as $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$.

The subsemigroup $\mathbb{N}_F \subset \mathbb{N}$ generated multiplicatively by the primes $p \in F$ acts on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})_F]$ by endomorphisms

$$\rho_n(e(r)) = \frac{1}{n} \sum_{nr'=r} e(r'), \quad n \in \mathbb{N}_F, \quad r \in (\mathbb{Q}/\mathbb{Z})_F.$$

The corresponding morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ act on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and we can consider the associated algebra $\mathcal{A}_{\mathbb{Z}, F}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and $\tilde{\mu}_n, \mu_n^*$ with $n \in \mathbb{N}_F$, with the relations

$$(2.18) \quad \tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \mu_n^* \tilde{\mu}_n = n, \quad \tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n,$$

where the first two relations hold for arbitrary $n, m \in \mathbb{N}$, the third for arbitrary $n \in \mathbb{N}$ and the fourth for $n, m \in \mathbb{N}$ satisfying $(n, m) = 1$, and the relations

$$(2.19) \quad x \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x) \quad \mu_n^* x = \sigma_n(x) \mu_n^*, \quad \tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x),$$

for any $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, where $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$, and with

$$\mathcal{A}_{\mathbb{Z},F} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})^F] \rtimes \mathbb{N}_F.$$

We refer to $\mathcal{A}_{\mathbb{Z},F}$ as the F -part of the integral Bost–Connes algebra.

The decomposition $\mathbb{N} = \mathbb{N}_F \times \mathbb{N}^{(F)}$, where $\mathbb{N}^{(F)}$ is generated by all primes $p \notin F$, gives also an algebra $\mathcal{A}_{\mathbb{Z}}^{(F)}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ and the $\tilde{\mu}_n$ and μ_n^* as in (2.19) with $p \notin F$ with

$$\mathcal{A}_{\mathbb{Z}}^{(F)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})^F] \rtimes \mathbb{N}^{(F)}.$$

We refer to $\mathcal{A}_{\mathbb{Z}}^{(F)}$ as the F -coprime part of the integral Bost–Connes algebra.

2.8. Lifting the F_N -coprime Bost–Connes algebra. Let $F = F_N$ be the set of prime factors of N and let $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ denote, as before, the part of the group ring of \mathbb{Q}/\mathbb{Z} involving only denominators relatively prime to N . The semigroup $\mathbb{N}^{(F)}$ is generated by primes $p \nmid N$ and we consider the morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ with $n \in \mathbb{N}^{(F)}$ and $r \in (\mathbb{Q}/\mathbb{Z})^F$ as discussed above.

Proposition 2.8. *Let S be a base scheme with a good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite action of $\hat{\mathbb{Z}}$. Let $\mathcal{Z}_{n,S}$ be defined as $\mathcal{Z}_{n,S} = S \times Z_n$, with $Z_n = \text{Spec}(\mathbb{Q}^n)$, with the action $\Phi_n(\alpha_S)$ obtained as in (2.12) and (2.13). The endomorphisms $\sigma_n : \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}] \rightarrow \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ with $n \in \mathbb{N}^{(F_N)}$ of the F_N -coprime part of the integral Bost–Connes algebra lift to endomorphisms $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, as in (2.14), which define a semigroup action of the multiplicative group $\mathbb{N}^{(F_N)}$ on the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The maps $\tilde{\rho}_n$, for $n \in \mathbb{N}^{(F_N)}$, lift to group homomorphisms $\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, as in (2.15), so that $\sigma_n \circ \tilde{\rho}_n[f : X \rightarrow S] = [f : X \rightarrow S]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : X \rightarrow S] = [f : X \rightarrow S] \cdot \mathcal{Z}_{n,S}$.*

Proof. Given the base variety S with a good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -action, let $F = F_N$ denote the set of prime factors of N . Let X be a variety over S , with a $\hat{\mathbb{Z}}$ -equivariant map $f : (X, \alpha_X) \rightarrow (S, \alpha_S)$, where we explicitly write the actions, satisfying $f(\alpha_X(\zeta, x)) = \alpha_S(\zeta, f(x))$. For $(N, n) = 1$, the maps $\sigma_n : [f : (X, \alpha_X) \rightarrow (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \rightarrow (S, \alpha_S \circ \sigma_n)]$, as in (2.14), satisfy $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$ with the notion of isomorphism discussed in §2.3, since $\zeta \mapsto \sigma_n(\zeta)$ is an automorphism of $\mathbb{Z}/N\mathbb{Z}$. Thus, the maps σ_n , for $n \in \mathbb{N}^{(F_N)}$ determine a semigroup action of $\mathbb{N}^{(F_N)}$ by endomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$.

Consider then $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S))$ as above, which we write equivalently as $\tilde{\rho}_n(S, \alpha_S)$ where $\tilde{\rho}_n$ is the lift of the Bost–Connes map to $K^{\hat{\mathbb{Z}}}(\mathcal{V})$ as in Proposition 3.5 of [54]. We know that $\tilde{\rho}_n \circ \sigma_n[S, \alpha_S] = [S, \alpha_S] \cdot [Z_n, \alpha_n]$ in $K^{\hat{\mathbb{Z}}}(\mathcal{V})$. Since for $(n, N) = 1$ we have $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$, this gives $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S)) \simeq (S \times Z_n, \alpha_S \times \gamma_n)$. Then setting $\tilde{\rho}_n(f : X \rightarrow S) = (\tilde{f} : X \times_S \mathcal{Z}_{n,S} \rightarrow S)$ with $\tilde{f} = f \circ \pi_X$ gives $X \times_S \mathcal{Z}_{n,S} \simeq X \times Z_n$, and the composition properties for $\tilde{\rho}_n \circ \sigma_n$ and $\sigma_n \circ \tilde{\rho}_n$ are satisfied.

Given a class $[f : X \rightarrow S]$, let $[\mathcal{F}_{X,S}]$ be the class in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ of the constructible sheaf given by the Euler characteristic (2.10) of $[f : X \rightarrow S]$. Let $[\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}]$ be the resulting class in $K_0(S^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ obtained by restriction to the fixed point set

$S^{\mathbb{Z}/N\mathbb{Z}}$ with the element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ specifying the representation of $\hat{\mathbb{Z}}$ on the stalks of the sheaf $\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}$. For $(N, n) = 1$, the action of σ_n by automorphisms of $\mathbb{Z}/N\mathbb{Z}$ with the resulting action by isomorphisms of S induces an action by isomorphisms on the $K_0(S^{\mathbb{Z}/N\mathbb{Z}})$ part and the usual Bost–Connes action on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The restriction of the semigroup action of $\mathbb{N}^{(F_N)}$ to the subring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ is then the image of the action of the maps σ_n and $\tilde{\rho}_n$ on the preimage of this subring under the morphism $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. \square

While this construction captures a lift of the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost–Connes algebra with the semigroup action of $\mathbb{N}^{(F_N)}$, the fact that the endomorphisms σ_n acting on the roots of unity in $\mathbb{Z}/N\mathbb{Z}$ are automorphisms when $(N, n) = 1$ loses some of the interesting structure of the Bost–Connes algebra, which stems from the partial invertibility of these morphisms. Thus, one also wants to recover the structure of the complementary part of the Bost–Connes algebra with the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_{F_N}]$ and the semigroup \mathbb{N}_{F_N} .

2.9. Lifting the full Bost–Connes algebra. Unlike the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost–Connes algebra described above, when one considers the full Bost–Connes algebra, including the F_N -part, the lift to the Grothendieck ring no longer consists of endomorphisms of a fixed $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$, but is given as in Proposition 2.6 by homomorphisms as in (2.14), (2.16) and (2.15), (2.17),

$$\begin{aligned} \sigma_n &: K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S \circ \sigma_n)}), \\ \tilde{\rho}_n &: K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha_S))}). \end{aligned}$$

For G a finite abelian group with a good action $\alpha : G \times S \rightarrow S$ on a variety S , let $(S, \alpha)_k^G = \{s \in S : \alpha(g^k, s) = s, \forall g \in G\}$ denote the set of periodic points of period k , with $(S, \alpha)_1^G = (S, \alpha)^G$ the set of fixed points. We always have $(S, \alpha)_k^G \subseteq (S, \alpha)_{km}^G$ for all $m \in \mathbb{N}$, hence in particular a copy of the fixed point set $(S, \alpha)^G$ is contained in all $(S, \alpha)_k^G$. For $G = \mathbb{Z}/N\mathbb{Z}$, with ζ_N a primitive N -th root of unity generator, the set of k -periodic points is given by $(S, \alpha)_k^{\mathbb{Z}/N\mathbb{Z}} = \{s \in S : \alpha(\zeta_N^k, s) = s\}$.

Lemma 2.9. *The sets of periodic points satisfy $(S, \alpha \circ \sigma_n)_k^G = (S, \alpha)_{nk}^G$. The sets $(S \times Z_n, \Phi_n(\alpha))_k^G$ can be non-empty only when $n|k$ with $(S \times Z_n, \Phi_n(\alpha))_k^G = ((S, \alpha)_{k/n}^G)^n$.*

Proof. Under the action $\alpha \circ \sigma_n$ the periodicity condition means $\alpha \circ \sigma_n(\zeta^k, s) = \alpha(\zeta^{nk}, s) = s$ for all $\zeta \in G$ hence the identification $(S, \alpha \circ \sigma_n)_k^G = (S, \alpha)_{nk}^G$. In the case of the geometric Verschiebung action $\Phi_n(\alpha)$ on $S \times Z_n$, the k -periodicity condition $\Phi_n(\alpha)(\zeta^k, (s, z)) = (s, z)$ implies that $n|k$ for the k -periodicity in the $z \in Z_n$ variable and that $\alpha(\zeta^{k/n}, s) = s$. \square

The identification $(S, \alpha \circ \sigma_n)_k^G = (S, \alpha)_{nk}^G$ implies the inclusion $(S, \alpha)_k^G \subseteq (S, \alpha \circ \sigma_n)_k^G$ and in particular the inclusion of the fixed point sets $(S, \alpha)^G \subseteq (S, \alpha \circ \sigma_n)^G$. Similarly, $(S \times Z_n, \Phi_n(\alpha))^G \subseteq ((S, \alpha)^G)^n$. Since these inclusions will in general be strict, due to the fact that the endomorphisms σ_n are not automorphisms, one cannot simply use

the map given by the equivariant Euler characteristic followed by the restriction to the fixed point set

$$K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0(\mathbb{Q}_{S^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

to lift the Bost–Connes endomorphisms to the maps (2.16) and (2.17) of Proposition 2.6. However, a simple variant of the same idea, where we consider sets of periodic points, gives the lift of the full Bost–Connes algebra to the equivariant relative Grothendieck rings $K_0^G(\mathcal{V}_{(S,\alpha)})$.

Consider the equivariant Euler characteristic map followed by the restrictions to the sets of periodic points

$$(2.20) \quad K_0^G(\mathcal{V}_{(S,\alpha)}) \xrightarrow{\chi_S^G} K_0^G(\mathbb{Q}_{(S,\alpha)}) \rightarrow \bigoplus_{k \geq 1} K_0^G(\mathbb{Q}_{(S,\alpha)_k^G}).$$

Also, for a given $n \in \mathbb{N}$, consider the same map composed with the projection to the summands with $n|k$

$$(2.21) \quad \chi_{S,n}^G : K_0^G(\mathcal{V}_{(S,\alpha)}) \xrightarrow{\chi_S^G} K_0^G(\mathbb{Q}_{(S,\alpha)}) \rightarrow \bigoplus_{k \geq 1 : n|k} K_0^G(\mathbb{Q}_{(S,\alpha)_k^G}).$$

For simplicity we consider the case where the fixed point set and periodic points sets of the action (S, α) are all finite sets.

Definition 2.10. Let (S, α) be a variety with a good effectively finite $\hat{\mathbb{Z}}$ -action. Consider data $(A_{(S,\alpha),n}, f_{(S,\alpha),n})$ and $(B_{(S,\alpha)}, h_{(S,\alpha)})$ of a family of rings $A_{(S,\alpha),n}$ with $n \in \mathbb{N}$ and $B_{(S,\alpha)}$ and ring homomorphisms $f_{(S,\alpha),n} : K_0^G(\mathcal{V}_{(S,\alpha)}) \rightarrow A_{(S,\alpha),n} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $h_{(S,\alpha)} : K_0^G(\mathcal{V}_{(S,\alpha)}) \rightarrow B_{(S,\alpha)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The maps $f_{(S,\alpha),n}$ and $h_{(S,\alpha)}$ are said to intertwine the Bost–Connes structure if there are ring isomorphisms $J_n : A_{(S,\alpha),n} \rightarrow B_{(S,\alpha \circ \sigma_n)}$ and isomorphisms of abelian groups $\tilde{J}_n : B_{(S,\alpha)} \rightarrow A_{(S \times \mathbb{Z}_n, \Phi_n(\alpha))}$, such that the following holds.

- (1) There is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) & \xrightarrow{f_{(S,\alpha),n}} & A_{(S,\alpha),n} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow \sigma_n & & \downarrow J_n \otimes \sigma_n \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha \circ \sigma_n)}) & \xrightarrow{h_{(S,\alpha)}} & B_{(S,\alpha \circ \sigma_n)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

where the maps $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha \circ \sigma_n)})$ are as in (2.16) and the maps $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are the endomorphisms of the integral Bost–Connes algebra.

(2) There is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) & \xrightarrow{h_{(S,\alpha)}} & B_{(S,\alpha)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow \tilde{\rho}_n & & \downarrow \tilde{J}_n \otimes \tilde{\rho}_n \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha))}) & \xrightarrow{f_{(S \times Z_n, \Phi_n(\alpha)), n}} & A_{(S \times Z_n, \Phi_n(\alpha))} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

where the maps $\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha))})$ are as in (2.17) and the $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are the maps (2.5) of the integral Bost–Connes algebra.

Theorem 2.11. *Let (S, α) be a variety with a good effectively finite $\hat{\mathbb{Z}}$ -action, such that the set $(S, \alpha)_k^{\hat{\mathbb{Z}}}$ of k -periodic points for this action is finite, for all $k \geq 1$. Then the maps (2.20) and (2.21) intertwine the Bost–Connes structure in the sense of Definition 2.10.*

Proof. Under the assumptions that all the $(S, \alpha)_k^G$ for $k \geq 0$ are finite sets, we can identify the target of the map with $\bigoplus_k K_0(\mathbb{Q}_{(S,\alpha)_k^G}) \otimes R(G)$. In the case of varieties with good effectively finite $\hat{\mathbb{Z}}$ actions, we obtain in this way ring homomorphisms

$$\begin{aligned} \chi_{(S,\alpha)}^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) &\rightarrow \bigoplus_{k \geq 1} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \chi_{(S,\alpha),n}^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) &\rightarrow \bigoplus_{k \geq 1 : n|k} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]. \end{aligned}$$

These maps fit in the following commutative diagrams of ring homomorphisms

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) & \xrightarrow{\chi_{(S,\alpha),n}^{\hat{\mathbb{Z}}}} & \bigoplus_{n|k} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow \sigma_n & & \downarrow J_n \otimes \sigma_n \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha \circ \sigma_n)}) & \xrightarrow{\tilde{\chi}_{(S,\alpha \circ \sigma_n)}^{\hat{\mathbb{Z}}}} & \bigoplus_{\ell} K_0(\mathbb{Q}_{(S,\alpha \circ \sigma_n)_\ell^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

where the map $(J_n)_{k,\ell}$ is non-trivial for $k = \ell n$ and identifies $K_0(\mathbb{Q}_{(S,\alpha)_\ell^{\hat{\mathbb{Z}}}})$ with $K_0(\mathbb{Q}_{(S,\alpha \circ \sigma_n)_k^{\hat{\mathbb{Z}}}})$, while the maps $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are the Bost–Connes endomorphisms. Similarly, we obtain commutative diagrams of group homomorphisms

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) & \xrightarrow{\chi_{(S,\alpha)}^{\hat{\mathbb{Z}}}} & \bigoplus_{\ell} K_0(\mathbb{Q}_{(S,\alpha)_\ell^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow \tilde{\rho}_n & & \downarrow \tilde{J}_n \otimes \tilde{\rho}_n \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha))}) & \xrightarrow{\chi_{(S \times Z_n, \Phi_n(\alpha)), n}^{\hat{\mathbb{Z}}}} & \bigoplus_{n|k} K_0(\mathbb{Q}_{(S \times Z_n, \Phi_n(\alpha))_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

where $(\tilde{J}_n)_{\ell,k}$ is non-trivial for $k = \ell n$ and maps $K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}})$ to $K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}})^{\oplus n}$ and identifies the latter with $K_0(\mathbb{Q}_{(S \times Z_n, \Phi_n(\alpha))_{\ell}^{\hat{\mathbb{Z}}}})$. \square

Remark 2.12. A similar argument can be given using a map obtained by composing the equivariant Euler characteristic considered here with values in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ with equivariant characteristic classes from constructible sheaves to delocalized equivariant homology as in [59], see §2.4.1.

3. FROM GROTHENDIECK RINGS TO SPECTRA

In this section we show that the Bost–Connes structure can be lifted further from the level of the relative Grothendieck ring to the level of spectra, using the assembler category construction of [70].

The results of this section are a natural continuation of the results in [54]. The general theme considered there consisted of the following steps:

- Appropriate equivariant Euler characteristic maps from certain $\hat{\mathbb{Z}}$ -equivariant Grothendieck rings to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are constructed.
- These Euler characteristic maps are then used to lift the Bost–Connes operations σ_n and $\tilde{\rho}_n$ from $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ to corresponding operations in the equivariant Grothendieck ring.
- Assembler categories with K_0 given by the equivariant Grothendieck ring are constructed.
- Endofunctors σ_n and $\tilde{\rho}_n$ of these assembler categories are constructed so that they induce the Bost–Connes structure in the Grothendieck ring.
- Induced maps of spectra are obtained from these endofunctors through the Gamma-space construction that associated a spectrum to an assembler category.

The construction of Bost–Connes operations σ_n and $\tilde{\rho}_n$ on the equivariant Grothendieck rings was generalized in the previous section to the case of relative Grothendieck rings. This section deals with the corresponding generalization of the remaining steps.

We start this section by a brief survey in §3.1 of Zakharevich’s formalism of *assemblers* which axiomatizes the “scissors congruence” relations (7.1).

A general framework for categorical Bost–Connes systems is introduced in §3.3 and §3.4 in terms of subcategories of the automorphism category (in our examples encoding the $\hat{\mathbb{Z}}$ -actions) and endofunctors σ_n and $\tilde{\rho}_n$ implementing the Bost–Connes structure.

In §3.5 we construct an assembler category for the equivariant relative Grothendieck ring, and we prove the main result of this section, Theorem 3.15, on the lifting of the Bost–Connes structure to this assembler category.

3.1. Assemblers. Below we will recall the basics of a general formalism for scissors congruence relations applicable in algebraic geometric contexts defined by I. Zakharevich in [70] and [71]. The abstract form of scissors congruences consists of categorical data called *assemblers*, which in turn determine a homotopy–theoretic *spectrum*, whose homotopy groups embody scissors congruence relations. This formalism is applied in [72] in the framework producing an assembler and a spectrum whose π_0 recovers the Grothendieck ring of varieties. This is used to obtain a characterisation of the kernel of multiplication by the Lefschetz motive, which provides a general explanation for the observations of [14], [58] on the fact that the Lefschetz motive is a zero divisor in the Grothendieck ring of varieties.

Consider a (small) category \mathcal{C} and an object X in \mathcal{C} .

Definition 3.1. A *sieve* \mathcal{S} over X in \mathcal{C} is a family of morphisms $f_i : X'_i \rightarrow X$ (also called “objects over X ”) satisfying the following conditions:

- a) Any isomorphism with target X belongs to \mathcal{S} (as a family with one element).
- b) If a morphism $X' \rightarrow X$ belongs to \mathcal{S} , then its precomposition with any other morphism in \mathcal{C} with target X'

$$X'' \rightarrow X' \rightarrow X$$

also belongs to \mathcal{S} .

It follows that composition of any two morphisms in \mathcal{S} composable in \mathcal{C} itself belongs to \mathcal{S} so that any sieve is a category in its own right.

Definition 3.2. A Grothendieck topology on a category \mathcal{C} consists of the assignment of a collection of sieves $\mathcal{J}(X)$ given for all objects X in \mathcal{C} , with the following properties:

- a) the total overcategory \mathcal{C}/X of morphisms with target X is a member of the collection $\mathcal{J}(X)$.
- b) The pullback of any sieve in $\mathcal{J}(X)$ under a morphism $f : Y \rightarrow X$ exists and is a sieve in $\mathcal{J}(Y)$. Here pullback of a sieve is defined as the family of pullbacks of its objects, $X' \rightarrow X$, whereas pullback of such an object w.r.t. $Y \rightarrow X$ is defined as $pr_Y : Y \times_X X' \rightarrow Y$.
- c) given $\mathcal{C}' \in \mathcal{J}(X)$ and a sieve \mathcal{T} in \mathcal{C}/X , if for every $f : Y \rightarrow X$ in \mathcal{C}' the pullback $f^*\mathcal{T}$ is in $\mathcal{J}(Y)$ then \mathcal{T} is in $\mathcal{J}(X)$.

For more details, see [47], Chapters 16 and 17, or [43], pp. 20–22.

Let \mathcal{C} be a category with a Grothendieck topology. Zakharevich’s notion of an assembler category is then defined as follows.

Definition 3.3. A collection of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ in \mathcal{C} is a *covering family* if the full subcategory of \mathcal{C}/X that contains all the morphisms of \mathcal{C} that factor through the f_i ,

$$\{g : Y \rightarrow X \mid \exists i \in I \ h : Y \rightarrow X_i \text{ such that } f_i \circ h = g\},$$

belongs to the sieve collection $\mathcal{J}(X)$.

In a category \mathcal{C} with an initial object \emptyset two morphisms $f : Y \rightarrow X$ and $g : W \rightarrow X$ are called *disjoint* if the pullback $Y \times_X W$ exists and is equal to \emptyset . A collection $\{f_i : X_i \rightarrow X\}_{i \in I}$ in \mathcal{C} is disjoint if f_i and f_j are disjoint for all $i \neq j \in I$.

Definition 3.4. An assembler category \mathcal{C} is a small category endowed with a Grothendieck topology, which has an initial object \emptyset (with the empty family as covering family), and where all morphisms are monomorphisms, with the property that any two finite disjoint covering families of X in \mathcal{C} have a common refinement that is also a finite disjoint covering family.

A morphism of assemblers is a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ that is continuous for the Grothendieck topologies and preserves the initial object and the disjointness property, that is, if two morphisms are disjoint in \mathcal{C} their images are disjoint in \mathcal{C}' .

For X a finite set, the coproduct of assemblers $\bigvee_{x \in X} \mathcal{C}_x$ is a category whose objects are the initial object \emptyset and all the non-initial objects of the assemblers \mathcal{C}_x . Morphisms of non-initial objects are induced by those of \mathcal{C}_x .

Consider a pair $(\mathcal{C}, \mathcal{D})$ where \mathcal{C} is an assembler category, and \mathcal{D} is a sieve in \mathcal{C} .

One has then an associated assembler $\mathcal{C} \setminus \mathcal{D}$ defined as the full subcategory of \mathcal{C} containing all the objects that are not initial objects of \mathcal{D} . The assembler structure on $\mathcal{C} \setminus \mathcal{D}$ is determined by taking as covering families in $\mathcal{C} \setminus \mathcal{D}$ those collections $\{f_i : X_i \rightarrow X\}_{i \in I}$ with X_i, X objects in $\mathcal{C} \setminus \mathcal{D}$ that can be completed to a covering family in \mathcal{C} , namely such that there exists $\{f_j : X_j \rightarrow X\}_{j \in J}$ with X_j in \mathcal{D} such that $\{f_i : X_i \rightarrow X\}_{i \in I} \cup \{f_j : X_j \rightarrow X\}_{j \in J}$ is a covering family in \mathcal{C} .

Moreover, there is a morphism of assemblers $\mathcal{C} \rightarrow \mathcal{C} \setminus \mathcal{D}$ that maps objects of \mathcal{D} to \emptyset and objects of $\mathcal{C} \setminus \mathcal{D}$ to themselves and morphisms with source in $\mathcal{C} \setminus \mathcal{D}$ to themselves and morphisms with source in \mathcal{D} to the unique morphism to the same target with source \emptyset . The data $(\mathcal{C}, \mathcal{D}, \mathcal{C} \setminus \mathcal{D})$ are called the abstract scissors congruences.

The construction of Γ -spaces, which we review more in detail in §3.2, then provides the homotopy theoretic spectra associated to assembler categories as in [70]. This construction of assembler categories and spectra provides the formalism we use here and in the previous paper [54] to lift Bost–Connes type algebras to the level of Grothendieck rings and spectra.

3.2. From categories to Γ -spaces and spectra. The Segal construction [65] associates a Γ -space (hence a spectrum) to a category \mathcal{C} with a zero object and a categorical sum. Let Γ^0 be the category of finite pointed sets with objects $\underline{n} = \{0, 1, \dots, n\}$ and morphisms $f : \underline{n} \rightarrow \underline{m}$ the functions with $f(0) = 0$. Let Δ_* denote the category of pointed simplicial sets. The construction can be generalized to symmetric monoidal categories, [68]. The associated Γ -space $F_{\mathcal{C}} : \Gamma^0 \rightarrow \Delta_*$ is constructed as follows. First assign to a finite pointed set X the category $P(X)$ with objects all the pointed subsets of X with morphisms given by inclusions. A functor $\Phi_X : P(X) \rightarrow \mathcal{C}$ is summing if it maps $\emptyset \in P(X)$ to the zero object of \mathcal{C} and given $S, S' \in P(X)$ with $S \cap S' = \{\star\}$ the base point of X , the morphism $\Phi_X(S) \oplus \Phi_X(S') \rightarrow \Phi_X(S \cup S')$ is an isomorphism. Let $\Sigma_{\mathcal{C}}(X)$ be the category whose objects are the summing functors

Φ_X with morphisms the natural transformations that are isomorphisms on all objects of $P(X)$. Setting

$$\Sigma_{\mathcal{C}}(f)(\Phi_X)(S) = \Phi_X(f^{-1}(S)),$$

for a morphisms $f : X \rightarrow Y$ of pointed sets and $S \in P(Y)$ gives a functor $\Sigma_{\mathcal{C}} : \Gamma^0 \rightarrow \text{Cat}$ to the category of small categories. Composing with the nerve \mathcal{N} gives a functor

$$F_{\mathcal{C}} = \mathcal{N} \circ \Sigma_{\mathcal{C}} : \Gamma^0 \rightarrow \Delta_*$$

which is the Γ -space associated to the category \mathcal{C} . The functor $F_{\mathcal{C}} : \Gamma^0 \rightarrow \Delta_*$ obtained in this way is extended to an endofunctor $F_{\mathcal{C}} : \Delta_* \rightarrow \Delta_*$ via the coend

$$F_{\mathcal{C}}(K) = \int^{\underline{n}} K^n \wedge F_{\mathcal{C}}(\underline{n}).$$

One obtains the spectrum $\mathbb{X} = F_{\mathcal{C}}(\mathbb{S})$ associated to the Γ -space $F_{\mathcal{C}}$ by setting $\mathbb{X}_n = F_{\mathcal{C}}(S^n)$ with maps $S^1 \wedge F_{\mathcal{C}}(S^n) \rightarrow F_{\mathcal{C}}(S^{n+1})$. The construction is functorial in \mathcal{C} , with respect to functors preserving sums and the zero object.

When \mathcal{C} is the category of finite sets, $F_{\mathcal{C}}(\mathbb{S})$ is the sphere spectrum \mathbb{S} , and when $\mathcal{C} = \mathcal{P}_R$ is the category of finite projective modules over a commutative ring R , the spectrum $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$ is the K -theory spectrum of R .

The Segal construction determines a functor from the category of small symmetric monoidal categories to the category of -1 -connected spectra. It is shown in [68] that this functor determines an equivalence of categories between the stable homotopy category of -1 -connected spectra and a localization of the category of small symmetric monoidal categories, obtained by inverting morphisms sent to weak homotopy equivalences by the functor.

Given an assembler category \mathcal{C} , one considers a category $\mathcal{W}(\mathcal{C})$ with objects $\{A_i\}_{i \in I}$ given by collections of non-initial objects A_i in \mathcal{C} indexed by finite sets and morphisms $\phi : \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J}$ consisting of a map of finite sets $f : I \rightarrow J$ and morphisms $\phi_i : A_i \rightarrow B_{f(i)}$ that form disjoint covering families $\{\phi_i \mid i \in f^{-1}(j)\}$, for all $j \in J$. One then obtains a Γ -space as the functor that assigns to a finite pointed set (X, x_0) the simplicial set $\mathcal{NW}(X \wedge \mathcal{C})$, the nerve of the category $\mathcal{W}(X \wedge \mathcal{C})$ where $X \wedge \mathcal{C}$ is the assembler $X \wedge \mathcal{C} = \bigvee_{x \in X \setminus \{x_0\}} \mathcal{C}$. The spectrum associated to the assembler \mathcal{C} is the spectrum defined by this Γ space, namely $X_n = \mathcal{NW}(S^n \wedge \mathcal{C})$.

For another occurrence of Γ -spaces in the context of \mathbb{F}_1 -geometry, see [26].

3.3. Automorphism category and enhanced assemblers. We describe in this and the next subsection a general formalism of “enhanced assemblers” underlying all the explicit cases of Bost–Connes structures in Grothendieck rings discussed in [54] and in some of the later sections of this paper.

We first recall the definition of the automorphism category.

Definition 3.5. The automorphism category $\text{Aut}(C)$ of a category C is given by:

- (i) Objects of $\text{Aut}(C)$ are pairs $\hat{X} = (X, v_X)$ where $X \in \text{Obj}(C)$ and $v_X : X \rightarrow X$ is an automorphism of X .

- (ii) Morphisms $\hat{f} : (X, v_X) \rightarrow (Y, v_Y)$ in $\text{Aut}(C)$ are morphisms $f : X \rightarrow Y$ such that $f \circ v_X = v_Y \circ f : X \rightarrow Y$ in C .
- (iii) The forgetful functor sends \hat{X} to X and \hat{f} to f .

We use here a standard categorical notation according to which, say, $f \circ v_X$ is the precomposition of f with v_X .

Thus, we can make the following general definition. In the following we will be especially interested in the case where the chosen subcategory is determined by a group action, see Remark 3.7.

Definition 3.6. Let C be a category. We will call here an *enhancement* of C a pair consisting of a choice of a subcategory \hat{C} of the automorphism category $\text{Aut}(C)$ and the forgetful functor $\hat{C} \rightarrow C$, where objects (X, v_X) of \hat{C} have automorphisms $v_X : X \rightarrow X$ of finite order, .

The main idea here is that a subcategory category \hat{C} of the automorphism category of C is where the endofunctors defining the lifts of the Bost–Connes structure are defined, as we make more precise in Definitions 3.9 and 3.11.

Remark 3.7. In the cases considered in [54] and in this paper, the subcategory of \hat{C} of $\text{Aut}(C)$ is usually determined by a finite group action, so that elements of \hat{C} are of the form $(X, \alpha_X(g))$ with $\alpha_X : G \times X \rightarrow X$ the group action. However, one expects other interesting examples that are not necessarily given by group actions, hence it is worth considering this more general formulation.

Remark 3.8. Assume that C is endowed with a structure of assembler. Then a series of constructions presented in §§ 3 and 4 of [54] and in §§ 2–6 of this paper, and restricted there to various categories of schemes, show in fact how this structure of assembler can be lifted from C to \hat{C} .

In particular the Bost–Connes type structures we are investigating can be formulated broadly in this setting of enhanced assemblers as follows.

3.4. Bost–Connes systems on categories. Let \hat{C} be an enhancement of a category C , in the sense of Definition 3.6.

Definition 3.9. We assume here that C is an additive (symmetric) monoidal category and that the enhancement \hat{C} is compatible with this structure. A Bost–Connes system in an enhancement \hat{C} of C consists of two families of endofunctors $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n \in \mathbb{N}}$ of \hat{C} with the following properties:

- (1) The functors σ_n are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.
- (2) For all $n, m \in \mathbb{N}$ these endofunctors satisfy

$$\sigma_{nm} = \sigma_n \circ \sigma_m, \quad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$

(3) The compositions satisfy

$$(3.1) \quad \sigma_n \circ \tilde{\rho}_n(X, v_X) = (X, v_X)^{\oplus n} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n(X, v_X) = (X, v_X) \otimes (Z_n, v_n),$$

for an object (Z_n, v_n) in \hat{C} that depends on n but not on (X, v_X) , and similarly on morphisms.

Here \oplus refers to the additive structure of C and \otimes to the monoidal structure.

Remark 3.10. In all the explicit cases considered in [54] and in this paper, the endofunctors σ_n and $\tilde{\rho}_n$ of Definition 3.9 have the form

$$\sigma_n(X, v_X) = (X, v_X \circ \sigma_n) \quad \text{and} \quad \tilde{\rho}_n(X, v_X) = (X \times Z_n, \Phi_n(v_X)),$$

where the endomorphism v_X is the action of a generator of some finite cyclic group $\mathbb{Z}/N\mathbb{Z}$ quotient of $\hat{\mathbb{Z}}$ and the action satisfies $v_X \circ \sigma_n(\zeta, x) = v_X(\sigma_n(\zeta), x)$, where $\sigma_n(\zeta) = \zeta^n$ is the Bost–Connes map of (2.4), while the action $\Phi_n(v_X)$ on $X \times Z_n$ is a geometric form of the Verschiebung, as will be discussed more explicitly in §2.5. The object (Z_n, v_n) in Definition 3.9 plays the role of the element $n\pi_n$ in the integral Bost–Connes algebra and the relations (3.1) play the role of the relations (2.6).

This definition covers the main examples considered in §§ 3 and 4 of [54] obtained using the assembler categories associated to the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ of varieties with a good $\hat{\mathbb{Z}}$ -action factoring through some finite cyclic quotient and to the equivariant version $\text{Burn}^{\hat{\mathbb{Z}}}$ of the Kontsevich–Tschinkel Burnside ring. This same definition also accounts for the construction we will discuss in §4 of this paper, based on assembler categories associated to torified varieties (see Remark 4.6).

The more general formulation given in Definition 3.9 is motivated by the fact that one expects other significant examples of categorical Bost–Connes structures where the choice of the subcategory \hat{C} of the automorphism category $\text{Aut}(C)$ is not determined by the action of a cyclic group as in the cases discussed here. Such more general classes of categorical Bost–Connes systems are not discussed in the present paper, but they are a motivation for future work, for which we just set the general framework in this section.

A generalization of Definition 3.9 is needed when considering relative cases, in particular the lift to assemblers of the construction presented in §2 for relative equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The reason why we need the following modification of Definition 3.9 is the fact that, in the relative setting, the base scheme S itself has its enhancement structure (the group action, in the specific examples) modified by the endofunctors implementing the Bost–Connes structure and this needs to be taken into account. We will see this additional structure more explicitly applied in §3.5, in the specific case where the automorphisms are determined by a group action (see Remark 3.16).

Definition 3.11. Let $\hat{\mathcal{I}}$ be an enhancement of an additive (symmetric) monoidal category \mathcal{I} as above, endowed with a Bost–Connes system given by endofunctors $\{\sigma_n^{\mathcal{I}}\}$ and $\{\tilde{\rho}_n^{\mathcal{I}}\}$ of $\hat{\mathcal{I}}$ as in Definition 3.9, with α_n the object in $\hat{\mathcal{I}}$ with $\tilde{\rho}_n \circ \sigma_n(\alpha) = \alpha \otimes \alpha_n$. Let

$\{\hat{C}_\alpha\}_{\alpha \in \hat{\mathcal{I}}}$ be a collection of enhancements of additive (symmetric) monoidal categories C_α , indexed by the objects of the auxiliary category $\hat{\mathcal{I}}$, endowed with functors $f_n : \hat{C}_{\alpha^{\oplus n}} \rightarrow \hat{C}_\alpha$ and $h_n : \hat{C}_{\alpha \times \beta} \rightarrow \hat{C}_\alpha$. Let $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n \in \mathbb{N}}$ be two collections of functors

$$\sigma_n : \hat{C}_\alpha \rightarrow \hat{C}_{\sigma_n^{\mathcal{I}}(\alpha)} \quad \text{and} \quad \tilde{\rho}_n : \hat{C}_\alpha \rightarrow \hat{C}_{\tilde{\rho}_n^{\mathcal{I}}(\alpha)}$$

satisfying the properties:

- (1) The functors σ_n are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.
- (2) For all $n, m \in \mathbb{N}$ these functors satisfy

$$\sigma_{nm} = \sigma_n \circ \sigma_m, \quad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$

- (3) The compositions

$$\sigma_n \circ \tilde{\rho}_n : \hat{C}_\alpha \rightarrow \hat{C}_{\alpha^{\oplus n}} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n : \hat{C}_\alpha \rightarrow \hat{C}_{\alpha \otimes \alpha_n}$$

satisfy

$$(3.2) \quad \begin{aligned} f_n \circ \sigma_n \circ \tilde{\rho}_n(X, v_X)_\alpha &= (X, v_X)_\alpha^{\oplus n} \quad \text{and} \\ h_n \circ \tilde{\rho}_n \circ \sigma_n(X, v_X)_\alpha &= (X, v_X)_\alpha \otimes (Z_n, v_n)_\alpha, \end{aligned}$$

for an object $(Z_n, v_n)_\alpha$ in \hat{C}_α that depends on n and α , but not on (X, v_X) .

We will first focus on the case of assembler categories, as those were at the basis of our constructions of Bost–Connes systems in [54], but we will also consider in §7 a different categorical setting that will allow us to identify analogous structures at a motivic level, following the formalism of geometric diagrams and Nori motives.

3.5. Assemblers for the relative Grothendieck ring. We consider the relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety S over a field \mathbb{K} , as in Definition 2.1.

An assembler \mathcal{C}_S such that the associated spectrum $K(\mathcal{C}_S)$ has $K_0(\mathcal{C}_S) = \pi_0 K(\mathcal{C}_S)$ given by the relative Grothendieck ring $K_0(\mathcal{V}_S)$ can be obtained as a slight modification of the construction given in [72] for the ordinary Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

Definition 3.12. The assembler \mathcal{C}_S for the relative Grothendieck ring $K_0(\mathcal{V}_S)$ has objects $f : X \rightarrow S$ that are varieties over S and morphisms that are locally closed embeddings of varieties over S .

Lemma 3.13. *The category \mathcal{C}_S of Definition 3.12 is indeed as assembler, with the Grothendieck topology on \mathcal{C}_S is generated by the covering families*

$$\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$$

with compatible maps (2.8)

$$(3.3) \quad \begin{array}{ccccc} Y & \hookrightarrow & X & \longleftarrow & X \setminus Y \\ & \searrow & \downarrow f & \swarrow & \\ & f|_Y & S & f|_{X \setminus Y} & \end{array}$$

Proof. The argument is the same as in [70], [72] and in [54]. In this setting finite disjoint covering families are maps

$$\begin{array}{ccc} X_i & \hookrightarrow & X \\ & \searrow & \downarrow f \\ & f_i & S \end{array}$$

where $X_i = Y_i \setminus Y_{i-1}$ with commutative diagrams

$$\begin{array}{ccccccc} Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & \dots & \hookrightarrow & Y_n = X \\ & \searrow & \searrow & \searrow & \searrow & \searrow & \downarrow f \\ & & & & & f_1 & S \\ & & & & & f_0 & \end{array}$$

The category has pullbacks, hence as shown in [70] (Remark after Definition 2.4) this suffices to obtain that any two finite disjoint covering families have a common refinement. Morphisms are embeddings compatible with the structure maps as in (3.3) hence in particular monomorphisms. Theorem 2.3 of [70] then shows that the spectrum $K(\mathcal{C}_S)$ associated to this assembler category has $\pi_0 K(\mathcal{C}_S) = K_0(\mathcal{V}_S)$. \square

In a similar way we obtain an assembler category and spectrum for the equivariant version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The argument is as in the previous case and in Lemma 4.5.1 of [54], using the inclusion-exclusion relations (2.9).

Corollary 3.14. *An assembler category $\mathcal{C}_{(S,\alpha)}^{\hat{\mathbb{Z}}}$ for $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ is constructed as in Lemma 3.13 with objects the $\hat{\mathbb{Z}}$ -equivariant $f : X \rightarrow S$, morphisms given by $\hat{\mathbb{Z}}$ -equivariant locally closed embeddings of varieties over S and with Grothendieck topology generated by the covering families given by $\hat{\mathbb{Z}}$ -equivariant maps as in (2.8) and (2.9).*

As in Proposition 4.2 of [54], we show that the lifting of the integral Bost–Connes algebra obtained in Proposition 2.6 and Theorem 2.11 further lifts to functors of the associated assembler categories, with the σ_n compatible with the monoidal structure, but not the $\tilde{\rho}_n$.

Theorem 3.15. *The maps $\sigma_n : (f : (X, \alpha_X) \rightarrow (S, \alpha)) \mapsto (f : (X, \alpha_X \circ \sigma_n) \rightarrow (S, \alpha \circ \sigma_n))$ and $\tilde{\rho}_n : (f : (X, \alpha_X) \rightarrow (S, \alpha)) \mapsto (f \times \text{id} : (X \times Z_n, \Phi_n(\alpha_X)) \rightarrow (S \times Z_n, \Phi_n(\alpha)))$ define functors of the assembler categories $\sigma_n : \mathcal{C}_{(S,\alpha)}^{\hat{\mathbb{Z}}} \rightarrow \mathcal{C}_{(S,\alpha \circ \sigma_n)}^{\hat{\mathbb{Z}}}$ and $\tilde{\rho}_n : \mathcal{C}_{(S,\alpha)}^{\hat{\mathbb{Z}}} \rightarrow \mathcal{C}_{(S \times Z_n, \Phi_n(\alpha))}^{\hat{\mathbb{Z}}}$. The functors σ_n are compatible with the monoidal structure.*

Proof. The functors σ_n defined as above on objects are compatibly defined on morphisms by assigning to a locally closed embedding

$$\sigma_n : \begin{array}{ccc} (Y, \alpha_Y) & \xrightarrow{j} & (X, \alpha_X) \\ & \searrow f_Y & \downarrow f_X \\ & & (S, \alpha) \end{array} \mapsto \begin{array}{ccc} (Y, \alpha_Y \circ \sigma_n) & \xrightarrow{j} & (X, \alpha_X \circ \sigma_n) \\ & \searrow f_Y & \downarrow f_X \\ & & (S, \alpha \circ \sigma_n) \end{array}$$

Similarly, we define the $\tilde{\rho}_n$ on morphisms by

$$\tilde{\rho}_n : \begin{array}{ccc} (Y, \alpha_Y) & \xrightarrow{j} & (X, \alpha_X) \\ & \searrow f_Y & \downarrow f_X \\ & & (S, \alpha) \end{array} \mapsto \begin{array}{ccc} (Y \times Z_n, \Phi_n(\alpha_Y)) & \xrightarrow{j} & (X \times Z_n, \Phi_n(\alpha_X)) \\ & \searrow f_Y & \downarrow f_X \\ & & (S \times Z_n, \Phi_n(\alpha)) \end{array}$$

The functors σ_n are compatible with the monoidal structure since $\sigma_n(X, \alpha_X) \times \sigma_n(X', \alpha_{X'}) = (X \times X', (\alpha \times \alpha') \circ \Delta \circ \sigma_n) = \sigma_n((X, \alpha_X) \times (X', \alpha_{X'}))$. \square

The functor of assembler categories determines an induced map of spectra and in turn an induced map of homotopy groups. By construction the induced maps on the π_0 homotopy agree with the maps (2.16) and (2.17) of Proposition 2.6.

Remark 3.16. We can associate to the assembler category $\mathcal{C}_{(S, \alpha)}^{\hat{\mathbb{Z}}}$ of Corollary 3.14 with the endofunctors σ_n and $\tilde{\rho}_n$ a categorical Bost–Connes structure in the sense of Definition 3.11, where the objects are $f : X \rightarrow S$ as above with the automorphisms given by elements $g \in \hat{\mathbb{Z}}$ acting on $f : X \rightarrow S$ through the action by $\alpha_X(g)$ on X and by $\alpha_S(g)$ on S , intertwined by the equivariant map f .

4. TORIFICATIONS, \mathbb{F}_1 -POINTS, ZETA FUNCTIONS, AND SPECTRA

In this section we relate the point of view developed in [54], with lifts of the Bost–Connes system to Grothendieck rings and spectra, to the approach to \mathbb{F}_1 -geometry based on torifications. This was first introduced in [51]. Weaker forms of torification were also considered in [53], which allow for the development of a form of \mathbb{F}_1 -geometry suitable for the treatment of certain classical moduli spaces.

The approach we follow here, in order to relate the case of torified geometry with the Bost–Connes systems on Grothendieck rings, assemblers, and spectra discussed in [54], is based on the following simple setting. Instead of working with the equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, where one assumes the varieties have a good effectively finite $\hat{\mathbb{Z}}$ -action, we consider here a variant that connects to the torifications point of view on \mathbb{F}_1 -geometry of [51]. We replace varieties with $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -actions with varieties with a \mathbb{Q}/\mathbb{Z} -action induced by a torification, where the group schemes \mathfrak{m}_n of n -th roots of unity, given by the kernels

$$1 \rightarrow \mathfrak{m}_n \rightarrow \mathbb{G}_m \xrightarrow{\lambda \mapsto \lambda^n} \mathbb{G}_m \rightarrow 1$$

determine a diagonal embedding in each torus and an action by multiplication. This is a very restrictive class of varieties, because the existence of a torification on a variety implies that the Grothendieck class is a sum of classes of tori with non-negative coefficients. The resulting construction will be more restrictive than the one considered in [54]. We will see, however, that one can still see in this context several interesting phenomena, especially in connection with the “dynamical” approach to \mathbb{F}_1 -geometry proposed in [54].

4.1. Torifications. A torification of an algebraic variety X defined over \mathbb{Z} is a decomposition $X = \sqcup_{i \in \mathcal{I}} T_i$ into algebraic tori $T_i = \mathbb{G}_m^{d_i}$. Weaker to stronger forms of torification [53] include

- (1) *torification of the Grothendieck class*: $[X] = \sum_{i \in \mathcal{I}} (\mathbb{L} - 1)^{d_i}$ with \mathbb{L} the Lefschetz motive;
- (2) *geometric torification*: $X = \sqcup_{i \in \mathcal{I}} T_i$ with $T_i = \mathbb{G}_m^{d_i}$;
- (3) *affine torification*: the existence of an affine covering compatible with the geometric torification, [51];
- (4) *regular torification*: the closure of each torus in the geometric torification is also a union of tori of the torification, [51].

Similarly, there are different possibilities when one considers morphisms of torified varieties, see [53]. In view of describing associated Grothendieck rings, we review the different notions of morphisms. The Grothendieck classes are then defined with respect to the corresponding type of isomorphism.

A torified morphism of geometric torifications in the sense of [51] between torified varieties $f : (X, T) \rightarrow (Y, T')$ is a morphism $f : X \rightarrow Y$ of varieties together with a map $h : I \rightarrow J$ of the indexing sets of the torifications $X = \sqcup_{i \in I} T_i$ and $Y = \sqcup_{j \in J} T'_j$ such that the restriction of f to tori T_i is a morphism of algebraic groups $f_i : T_i \rightarrow T'_{h(i)}$. There are then three different classes of morphisms of torified varieties that were introduced in [53]: strong, ordinary, and weak morphisms. To describe them, one first defines strong, ordinary, and weak equivalences of torifications, and one then uses these to define the respective class of morphisms.

Let T and T' be two geometric torifications of a variety X .

- (1) The torifications (X, T) and (X, T') are *strongly equivalent* if the identity map on X is a torified morphism as above.
- (2) The torifications (X, T) and (X, T') are *ordinarily equivalent* if there exists an automorphism $\phi : X \rightarrow X$ that is a torified morphism.
- (3) The torifications (X, T) and (X, T') are *weakly equivalent* if X has two decompositions $X = \cup_i X_i$ and $X = \cup_j X'_j$ into a disjoint union of subvarieties, compatible with the torifications, such that there are isomorphisms of varieties $\phi_i : X_i \rightarrow X'_{j(i)}$ that are torified.

In the weak case a “decomposition compatible with torifications” means that the intersections $T_i \cap X_j$ of the tori of T with the pieces of the decomposition (when non-empty) are tori of the torification of X_j , and similarly for $T'_i \cap X'_j$. In general

weakly equivalent torification are not ordinarily equivalent because the maps ϕ_i need not glue together to define a single map ϕ on all of X .

We then have the following classes of morphisms of torified varieties from [53]:

- (1) *strong morphisms*: these are torified morphisms in the sense of [51], namely morphisms that restrict to morphisms of tori of the respective torifications.
- (2) *ordinary morphisms*: an ordinary morphism of torified varieties (X, T) and (Y, T') is a morphism $f : X \rightarrow Y$ such that becomes a torified morphism after composing with strong isomorphisms, that is, $\phi_Y \circ f \circ \phi_X : (X, T) \rightarrow (Y, T')$ is a strong morphism of torified varieties, for some isomorphisms $\phi_X : X \rightarrow X$ and $\phi_Y : Y \rightarrow Y$. In other words, if we denote by T_ϕ and T'_ϕ the torifications such that $\phi_X : (X, T) \rightarrow (X, T_\phi)$ and $\phi_Y : (Y, T'_\phi) \rightarrow (Y, T')$ are torified, then $f : (X, T_\phi) \rightarrow (Y, T'_\phi)$ is torified.
- (3) *weak morphisms*: the torified varieties (X, T) and (Y, T') admit decompositions $X = \sqcup_i X_i$ and $Y = \sqcup_j Y_j$, compatible with the torifications, such that there exist ordinary morphisms $f_i : (X_i, T_i) \rightarrow (Y_{f(i)}, T'_{f(i)})$ of these subvarieties.

Note that the strong isomorphisms $\phi_X : (X, T) \rightarrow (X, T_\phi)$ and $\phi_Y : (Y, T'_\phi) \rightarrow (Y, T')$ used in the definition of ordinary morphisms are ordinary equivalences of the torifications T and T_ϕ , respectively T' and T'_ϕ .

Given these notions of morphisms, we can correspondingly construct Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ in the following way.

As an abelian group $K_0(\mathcal{T})^s$ is generated by isomorphism classes $[X, T]_s$ of pairs of a torifiable variety X and a torification T modulo strong isomorphisms, with the inclusion-exclusion relations $[X, T]_s = [Y, T_Y]_s + [X \setminus Y, T_{X \setminus Y}]_s$ whenever $(Y, T_Y) \hookrightarrow (X, T)$ is a strong morphism (that is, the inclusion of Y in X is compatible with the torification: Y is a union of tori of the torification of X) and (Y, T_Y) is a *complemented subvariety* in (X, T) , which means that the complement $X \setminus Y$ is also a union of tori of the torification so that the inclusion of $(X \setminus Y, T_{X \setminus Y})$ in (X, T) is also a strong morphism. This complemented condition is very strong. Indeed, one can see that, for example, there are in general very few complemented points in a torified variety. The product operation is $[X, T]_s \cdot [Y, T']_s = [X \times Y, T \times T']_s$ with the torification $T \times T'$ given by the product tori $T_{ij} = T_i \times T'_j = \mathbb{G}_m^{d_i + d_j}$.

The abelian group $K_0(\mathcal{T})^o$ is generated by isomorphism classes $[X]_o$ varieties that admit a torification with respect to ordinary isomorphisms, with the inclusion-exclusion relations $[X]_o = [Y]_o + [X \setminus Y]_o$ whenever the inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are ordinary morphisms. The product is the class of the Cartesian product $[X]_o \cdot [Y]_o = [X \times Y]_o$.

The abelian group $K_0(\mathcal{T})^w$ is generated by the isomorphism classes $[X]_w$ of torifiable varieties X with respect to weak morphisms, with the inclusion-exclusion relations $[X]_w = [Y]_w + [X \setminus Y]_w$ whenever the inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are weak morphisms. The product structure is again given by $[X]_w \cdot [Y]_w = [X \times Y]_w$.

The reader can consult the explicit examples given in [53] to see how these notions (and the resulting Grothendieck rings) can be different. For example, as mentioned in §2.2 of [53], consider the variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ and consider on it two torifications T and T' , where T is the standard torification given by the decomposition of each \mathbb{P}^1 into cells $\mathbb{A}^0 \cup \mathbb{A}^1$, with the cell \mathbb{A}^1 torified as $\mathbb{A}^0 \cup \mathbb{G}_m$, while T' is the torification where in the big cell \mathbb{A}^2 of $\mathbb{P}^1 \times \mathbb{P}^1$ we take a torification of the diagonal \mathbb{A}^1 and a torification of the complement of the diagonal, and we use the same torification of the lower dimensional cells as in T . These two torifications are related by a weak isomorphism, hence the elements $(\mathbb{P}^1 \times \mathbb{P}^1, T)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, T')$ define the same class in $K_0(\mathcal{T})^w$, but they are not related by an ordinary isomorphism so they define different classes in $K_0(\mathcal{T})^o$.

Note however that, in all these cases, the Grothendieck classes $[X]_a$ with $a = s, o, w$ have the form $[X]_a = \sum_{n \geq 0} a_n \mathbb{T}^n$ with $a_n \in \mathbb{Z}_+$ and $\mathbb{T}^n = [\mathbb{G}_m^n]$.

In the following, whenever we simply write $a = s, o, w$ without specifying one of the three choices of morphisms, it means that the stated property holds for all of these choices.

4.1.1. *Relative case.* In a similar way, we can construct relative Grothendieck rings $K_S(\mathcal{T})^a$ with $a = s, o, w$ where in the strong case $S = (S, T_S)$ is a choice of a variety with an assigned torification, with $K_S(\mathcal{T})^s$ generated as an abelian group by isomorphisms classes $[f : (X, T) \rightarrow (S, T_S)]$ where f is a strong morphism of torified varieties and the isomorphism class is taken with respect to strong isomorphisms ϕ, ϕ_S such that the diagram commutes

$$\begin{array}{ccc} (X, T) & \xrightarrow{\phi} & (X', T') \\ f \downarrow & & \downarrow f' \\ (S, T_S) & \xrightarrow{\phi_S} & (S, T_S) \end{array}$$

with inclusion-exclusion relations

$$[f : (X, T) \rightarrow (S, T_S)] =$$

$$[f|_{(Y, T_Y)} : (Y, T_Y) \rightarrow (S, T_S)] + [f|_{(X \setminus Y, T_{X \setminus Y})} : (X \setminus Y, T_{X \setminus Y}) \rightarrow (S, T_S)]$$

where $\iota_Y : (Y, T_Y) \hookrightarrow (X, T)$ is a strong morphism and (Y, T_Y) is complemented with $\iota_{X \setminus Y} : (X \setminus Y, T_{X \setminus Y}) \hookrightarrow (X, T)$ also a strong morphism and both these inclusions are compatible with the map $f : (X, T) \rightarrow (S, T_S)$, so that $f_Y = f \circ \iota_Y$ and $f|_{(X \setminus Y, T_{X \setminus Y})} = f \circ \iota_{X \setminus Y}$ are strong morphisms. The construction for ordinary and weak morphism is similar, with the appropriate changes in the definition.

4.2. **Group actions.** In order to operate on Grothendieck classes with Bost–Connes type endomorphisms, we introduce appropriate group actions.

Torified varieties carry natural \mathbb{Q}/\mathbb{Z} actions, since the roots of unity embed diagonally in each torus of the torification and act on it by multiplication. However, we

will also be interested in considering good effectively finite $\hat{\mathbb{Z}}$ -actions, in the sense already discussed in [54], that is, actions of $\hat{\mathbb{Z}}$ as in Definition 2.3.

Remark 4.1. The main reason for working with $\hat{\mathbb{Z}}$ -actions rather than with \mathbb{Q}/\mathbb{Z} actions lies in the fact that, in the construction of the geometric Verschiebung action discussed in §2.5 we need to be able to describe the cyclic permutation action of $\mathbb{Z}/n\mathbb{Z}$ on the finite set Z_n as an action factoring through $\mathbb{Z}/n\mathbb{Z}$. This cannot be done in the case of \mathbb{Q}/\mathbb{Z} -actions because there are no nontrivial group homomorphisms $\mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ since \mathbb{Q}/\mathbb{Z} is infinitely divisible.

In the case of the natural \mathbb{Q}/\mathbb{Z} -actions on torifications, we consider objects of the form (X, T, α) where X is a torifiable variety, T a choice of a torification, and $\alpha : \mathbb{Q}/\mathbb{Z} \times X \rightarrow X$ an action of \mathbb{Q}/\mathbb{Z} determined by an embedding of \mathbb{Q}/\mathbb{Z} as roots of unity in $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$, which act on each torus $T_i = \mathbb{G}_m^{k_i}$ diagonally by multiplication. An embedding of \mathbb{Q}/\mathbb{Z} in \mathbb{G}_m is determined by an invertible element in $\text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{G}_m) = \hat{\mathbb{Z}}$, hence the action α is uniquely determined by the torification T and by the choice of an element in $\hat{\mathbb{Z}}^*$.

The corresponding morphisms are, respectively, strong, ordinary, or weak morphisms of torified varieties compatible with the \mathbb{Q}/\mathbb{Z} -actions, in the sense that the resulting torified morphism (after composing with isomorphisms or with local isomorphisms in the ordinary and weak case) are \mathbb{Q}/\mathbb{Z} -equivariant. We can then proceed as above and obtain equivariant Grothendieck rings $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, and $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$ of torified varieties.

In the case of good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -actions, the setting is essentially the same. We consider objects of the form (X, T, α) where X is a torifiable variety, T a choice of a torification, and $\alpha : \mathbb{Z}/N\mathbb{Z} \times X \rightarrow X$ is given by the action of the N -th roots of unity on the tori $T_i = \mathbb{G}_m^{k_i}$ by multiplication. Thus, a good $\hat{\mathbb{Z}}$ -action is determined by T , by the choice of an embedding of roots of unity in \mathbb{G}_m (an element of $\hat{\mathbb{Z}}^*$) as above, and by the choice of $N \in \mathbb{N}$ that determines which subgroup of roots of unity is acting.

This choice of good $\mathbb{Z}/N\mathbb{Z}$ -effectively finite $\hat{\mathbb{Z}}$ -actions, with strong, ordinary, or weak morphisms whose associated torified morphisms are $\mathbb{Z}/N\mathbb{Z}$ -equivariant, determine equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ of torified varieties with good effectively finite $\hat{\mathbb{Z}}$ -actions.

4.3. Assembler and spectrum of torified varieties. As in the previous cases of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ of [54] and in the case of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ discussed above, we consider the Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ and their corresponding equivariant versions $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ from the point of view of assemblers and spectra developed in [70], [71], [72].

Proposition 4.2. *For $a = s, o, w$, the category \mathcal{C}_T^a has objects that are pairs (X, T) of a torifiable variety and a torification, with morphisms the locally closed embeddings*

that are, respectively, strong, ordinary, or weak morphisms of torified varieties. The Grothendieck topology is generated by the covering families

$$(4.1) \quad \{(Y, T_Y) \hookrightarrow (X, T_X), (X \setminus Y, T_{X \setminus Y}) \hookrightarrow (X, T_X)\}$$

where both embeddings are strong, ordinary, or weak morphisms, respectively. The category $\mathcal{C}_{\mathcal{T}}^a$ is an assembler with spectrum $K(\mathcal{C}_{\mathcal{T}}^a)$ satisfying $\pi_0 K(\mathcal{C}_{\mathcal{T}}^a) = K_0(\mathcal{T})^a$. Similarly, for $G = \mathbb{Q}/\mathbb{Z}$ or $G = \hat{\mathbb{Z}}$ let $\mathcal{C}_{\mathcal{T}}^{G,a}$ be the category with objects (X, T, α) given by a torifiable variety X with a torification T and a G -action α of the kind discussed in §4.2 and morphisms the locally closed embeddings that are G -equivariant strong, ordinary, or weak morphisms. The Grothendieck topology is generated by covering families (4.1) with G -equivariant embeddings. The category $\mathcal{C}_{\mathcal{T}}^{G,a}$ is also an assembler, whose associated spectrum $K(\mathcal{C}_{\mathcal{T}}^{G,a})$ satisfies $\pi_0 K(\mathcal{C}_{\mathcal{T}}^{G,a}) = K_0^G(\mathcal{T})^a$.

Proof. The argument is again as in [70], see Lemma 3.13. We check that the category admits pullbacks. In the strong case, if (Y, T_Y) and $(Y', T_{Y'})$ are objects with morphisms $f : (Y, T_Y) \hookrightarrow (X, T_X)$ and $f' : (Y', T_{Y'}) \hookrightarrow (X, T_X)$ given by embeddings that are strong morphisms of torified varieties. This means that the tori of the torification T_Y are restrictions to Y of tori of the torification T_X of X . Thus, both Y and Y' are unions of subcollections of tori of T_X . Their intersection $Y \cap Y'$ will then also inherit a torification consisting of a subcollection of tori of T_X and the resulting embedding $(Y \cap Y', T_{Y \cap Y'}) \hookrightarrow (X, T_X)$ is a strong morphism of torified varieties. In the ordinary case, we consider embeddings $f : Y \hookrightarrow X$ and $f' : Y' \hookrightarrow X$ that are ordinary morphisms of torified varieties, which means that, for isomorphisms $\phi_X, \phi'_X, \phi_Y, \phi_{Y'}$, the compositions

$$\phi_X \circ f \circ \phi_Y : (Y, T_Y) \hookrightarrow (X, T_X), \quad \phi'_X \circ f' \circ \phi_{Y'} : (Y', T_{Y'}) \hookrightarrow (X, T_X)$$

are (strong) torified morphisms. Thus, the tori of the torifications T_Y and $T_{Y'}$ are subcollections of tori of X , under the embeddings $\phi_X \circ f \circ \phi_Y$ and $\phi'_X \circ f' \circ \phi_{Y'}$. The intersection $\phi_X \circ f \circ \phi_Y(Y) \cap \phi'_X \circ f' \circ \phi_{Y'}(Y') \subset X$ is isomorphic to a copy of $Y \cap Y'$ and has an induced torification $T_{Y \cap Y'}$ by a subcollection of tori of T_X . The embedding of $Y \cap Y'$ in X with this image is an ordinary morphism with respect to this torification. The weak case is constructed similarly to the ordinary case on the pieces of the decomposition. The equivariant cases are constructed analogously, as discussed in the case of equivariant Grothendieck rings of varieties in [54]. \square

Corollary 4.3. *There are inclusions of assemblers $\mathcal{C}_{\mathcal{T}}^s \hookrightarrow \mathcal{C}_{\mathcal{T}}^o \hookrightarrow \mathcal{C}_{\mathcal{T}}^w$ that induce maps of K -theory, in particular $K_0(\mathcal{T})^s \rightarrow K_0(\mathcal{T})^o$ and $K_0(\mathcal{T})^o \rightarrow K_0(\mathcal{T})^w$. Similarly, for the G -equivariant cases of Proposition 4.2.*

Proof. Since for morphisms strong implies ordinary and ordinary implies weak, one obtains inclusions of assemblers as stated. \square

4.4. Lifting of the Bost–Connes system for torifications. We consider here lifts of the integral Bost–Connes algebra to the Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ and to the assemblers and spectra $K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^s)$, $K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^o)$, and $K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^w)$.

Definition 4.4. We regard the zero-dimensional variety Z_n as a torified variety with the torification consisting of n zero dimensional tori and with a good $\hat{\mathbb{Z}}$ action factoring through $\mathbb{Z}/n\mathbb{Z}$ that cyclically permutes the points of Z_n . We write (Z_n, T_0, γ) for this object. For (X, T, α) a triple of a torifiable variety X , a given torification T , and an effectively finite action α of $\hat{\mathbb{Z}}$, we then set, for all $n \in \mathbb{N}$,

$$(4.2) \quad \sigma_n(X, T, \alpha) = (X, T, \alpha \circ \sigma_n) \quad \text{and} \quad \tilde{\rho}_n(X, T, \alpha) = (X \times Z_n, \sqcup_{a \in Z_n} T, \Phi_n(\alpha)).$$

Proposition 4.5. *The σ_n and $\tilde{\rho}_n$ defined as in (4.2) determine endofunctors of the assembler categories $\mathcal{C}_{\mathcal{T}}^{\hat{\mathbb{Z}}, a}$ that induce, respectively, ring homomorphisms $\sigma_n : K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^a) \rightarrow K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^a)$ and group homomorphisms $\tilde{\rho}_n : K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^a) \rightarrow K^{\hat{\mathbb{Z}}}(\mathcal{C}_{\mathcal{T}}^a)$ with the Bost–Connes relations*

$$\tilde{\rho}_n \circ \sigma_n(X, T, \alpha) = (X, T, \alpha) \times (Z_n, T_0, \gamma) \quad \sigma_n \circ \tilde{\rho}_n(X, T, \alpha) = (X, T, \alpha)^{\oplus n}.$$

Proof. The proof is completely analogous to the case discussed in Theorem 3.15 and to the similar cases discussed in [54]. \square

Remark 4.6. The σ_n and $\tilde{\rho}_n$ defined as in (4.2) determine a categorical Bost–Connes system as in Definition 3.9, where the objects are pairs (X, T) and the automorphisms are elements $g \in \hat{\mathbb{Z}}$ acting through the effectively finite action $\alpha(g)$.

Remark 4.7. Bost–Connes type quantum statistical mechanical systems associated to individual toric varieties (and more generally to varieties admitting torifications) were constructed in [44]. Here instead of Bost–Connes endomorphisms of individual varieties we are interested in a Bost–Connes system over the entire Grothendieck ring and its associated spectrum.

Remark 4.8. Variants of the construction above can be obtained by considering the multivariable versions of the Bost–Connes system discussed in [55], with actions of subsemigroups of $M_N(\mathbb{Z})^+$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$, that is, subalgebras of the crossed product algebra

$$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N} \rtimes_{\rho} M_N(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and $\mu_{\alpha}, \mu_{\alpha}^*$ with

$$\rho_{\alpha}(e(\underline{r})) = \mu_{\alpha} e(\underline{r}) \mu_{\alpha}^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s}) = \underline{r}} e(\underline{s})$$

$$\sigma_{\alpha}(e(\underline{r})) = \mu_{\alpha}^* e(\underline{r}) \mu_{\alpha} = e(\alpha(\underline{r})).$$

The relevance of this more general setting to \mathbb{F}_1 -geometries lies in a result of Borger and de Smit [13] showing that every torsion free finite rank Λ -ring embeds in some $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$ with the action of \mathbb{N} determined by the Λ -ring structure compatible with the diagonal subsemigroup of $M_N(\mathbb{Z})^+$.

5. TORIFIED VARIETIES AND ZETA FUNCTIONS

We discuss in this section the connection between the dynamical point of view on \mathbb{F}_1 -geometry proposed in [54] and the point of view based on torifications.

We first discuss in §5.1 and §5.2 the notion of \mathbb{F}_1 -points of a torified variety and its relation to the torification of the Grothendieck class, with some explicit examples. We then introduce the \mathbb{F}_1 -zeta function in §5.4 and we show its main properties in Proposition 5.4, while in §5.5 we explain the relation between the \mathbb{F}_1 -zeta function and the Hasse–Weil zeta function.

In §5.6 we consider the point of view on \mathbb{F}_1 -structures proposed in [54] based on dynamical systems inducing quasi-unipotent endomorphisms on homology, in the particular case of torified varieties with dynamical systems compatible with the torification. We focus on the associated dynamical zeta functions, the Lefschetz zeta function and the Artin–Mazur zeta function, whose properties we recall in §5.6.1. We then prove in Proposition 5.8 that the resulting dynamical zeta function have similar properties to the \mathbb{F}_1 -zeta function in the sense that both define exponentiable motivic measures from the Grothendieck rings of torified varieties to the Witt ring.

5.1. Counting \mathbb{F}_1 -points. Assuming that a variety X over \mathbb{Z} admits an \mathbb{F}_1 -structure, regarded here as one of several possible forms of torified structure recalled above, [51], [53], the number of points of X over \mathbb{F}_1 is computed as the $q \rightarrow 1$ limit of the counting function $N_X(q)$ of points over \mathbb{F}_q of the mod p reduction of X , for q a power of p . Any form of torified structure in particular implies that the variety is polynomially countable, hence that the counting function $N_X(q)$ is a polynomial in q with \mathbb{Z} -coefficients. The limit $\lim_{q \rightarrow 1} N_X(q)$, possibly normalized by a power of $q - 1$, is interpreted as the number of \mathbb{F}_1 -points of X , see [67]. Similarly, one can define “extensions” \mathbb{F}_{1^m} of \mathbb{F}_1 , in the sense of [45] (see also [27]). These corresponds to actions of the groups \mathfrak{m}_m of m -th roots of unity. In terms of a torified structure, the points over \mathbb{F}_{1^m} count m -th roots of unity in each torus of the decomposition. In terms of the counting function $N_X(q)$ the counting of points of X over the extension \mathbb{F}_{1^m} is obtained as the value $N_X(m + 1)$, see Theorem 4.10 of [24] and Theorem 1 of [29]). Summarizing, we have the following.

Lemma 5.1. *Let X be a variety over \mathbb{Z} with torified Grothendieck class*

$$(5.1) \quad [X] = \sum_{i=0}^N a_i \mathbb{T}^i$$

with coefficients $a_i \in \mathbb{Z}_+$ and $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$. Then the number of points over \mathbb{F}_{1^m} of X is given by

$$(5.2) \quad \#X(\mathbb{F}_{1^m}) = \sum_{i=0}^N a_i m^i.$$

In particular, $\#X(\mathbb{F}_1) = a_0 = \chi(X)$ the Euler characteristic.

5.2. Bialynicki-Birula decompositions and torified geometries. As shown in [5], [18], the motive of a smooth projective variety with action of the multiplicative group admits a decomposition, obtained via the method of Bialynicki-Birula, [9], [10], [11]. We recall the result here, in a particular case which gives rise to examples of torified varieties.

Lemma 5.2. *Let X be a smooth projective k -variety X endowed with a \mathbb{G}_m action such that the fixed point locus $X^{\mathbb{G}_m}$ admits a torification of the Grothendieck class. Then X also admits a torification of the Grothendieck class. Consider the filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$ over the components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, associated to the Bialynicki-Birula decomposition. If the fixed point locus $X^{\mathbb{G}_m}$ admits a geometric torification such that the restrictions of the fibrations ϕ_i to the individual tori of the torification of Z_i are trivializable, then X also admits a geometric torification.*

Proof. The Bialynicki-Birula decomposition, [9], [10], [11], see also [42], shows that a smooth projective k -variety X endowed with a \mathbb{G}_m action has smooth closed fixed point locus $X^{\mathbb{G}_m}$ which decomposes into a finite union of components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, of dimensions $\dim Z_i$ the dimension of $TX_z^{\mathbb{G}_m}$ at $z \in Z_i$. The variety X has a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$ of relative dimension d_i equal to the dimension of the positive eigenspace of the \mathbb{G}_m -action on the tangent space of X at points of Z_i . The scheme $X_i \setminus X_{i-1}$ is identified with $\{x \in X : \lim_{t \rightarrow 0} tx \in Z_i\}$ under the \mathbb{G}_m -action $t : x \mapsto tx$, with $\phi_i(x) = \lim_{t \rightarrow 0} tx$. As shown in [18], the object $M(X)$ in the category of correspondences Corr_k with integral coefficients (and in the category of Chow motives) decomposes as

$$(5.3) \quad M(X) = \bigoplus_i M(Z_i)(d_i),$$

where $M(Z_i)$ are the motives of the components of the fixed point set and $M(Z_i)(d_i)$ are Tate twists. The class in the Grothendieck ring $K_0(\mathcal{V}_k)$ decomposes then as

$$(5.4) \quad [X] = \sum_i [Z_i] \mathbb{L}^{d_i}.$$

It is then immediate that, if the components Z_i admit a geometric torification (respectively, a torification of the Grothendieck class) then the variety X also does. If $Z_i = \cup_{j=1}^{n_i} T_{ij}$ with $T_{ij} = \mathbb{G}_m^{a_{ij}}$ or, respectively $[Z_i] = \sum_{j=1}^{n_i} (\mathbb{L} - 1)^{a_{ij}}$, then $X = \cup_{i=0}^n (X_i \setminus X_{i-1}) = \cup_{i=0}^n \mathcal{F}^{d_i}(Z_i)$, where $\mathcal{F}^{d_i}(Z_i)$ denotes the total space of the affine fibration $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$ with fibers \mathbb{A}^{d_i} . The Grothendieck class is then torified by

$$[X] = \sum_{i=1}^n \sum_{j=1}^{n_i} \mathbb{T}^{a_{ij}} \left(1 + \sum_{k=1}^{d_i} \binom{d_i}{k} \mathbb{T}^k \right),$$

with $\mathbb{T} = \mathbb{L} - 1$ the class of the multiplicative group $\mathbb{T} = [\mathbb{G}_m]$, and where the affine spaces are torified by

$$\mathbb{L}^n - 1 = \sum_{k=1}^n \binom{n}{k} \mathbb{T}^k.$$

If the restriction of the fibration $\mathcal{F}^{d_i}(Z_i)$ to the individual tori T_{ij} of the torification of Z_i is trivial, then it can be torified by a products $T_{ij} \times T_k$ of the torus T_{ij} and the tori T_k of a torification of the fiber affine space \mathbb{A}^{d_i} . This determines a a geometric torification of the affine fibrations $\mathcal{F}^{d_i}(Z_i)$, hence of X . \square

5.3. An example of torified varieties. A physically significant example of torified varieties of the type described in Lemma 5.2 arises in the context of BPS state counting of [22]. Refined BPS state counting computes the multiplicities of BPS particles with charges in a lattice (K -theory changes of even D -branes) for assigned spin quantum numbers of a $\text{Spin}(4) = SU(2) \times SU(2)$ representation, see [22], [23], [33].

We mention here the following explicit example from [23], namely the case of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4, 1)$ of Gieseker semi-stable shaved on \mathbb{P}^2 with Hilbert polynomial equal to $4m + 1$. In this case, it is proved in [23] that $\mathcal{M}_{\mathbb{P}^2}(4, 1)$ has a torus action of \mathbb{G}_M^2 for which the fixed point locus consists of 180 isolated points and 6 components isomorphic to \mathbb{P}^1 . The Grothendieck class, obtained through the Bialynicki-Birula decomposition [23] is given by

$$\begin{aligned} [\mathcal{M}_{\mathbb{P}^2}(4, 1)] &= 1 + 2\mathbb{L} + 6\mathbb{L}^2 + 10\mathbb{L}^3 + 14\mathbb{L}^4 + 15\mathbb{L}^5 \\ &\quad + 16\mathbb{L}^6 + 16\mathbb{L}^7 + 16\mathbb{L}^8 + 16\mathbb{L}^9 + 16\mathbb{L}^{10} + 16\mathbb{L}^{11} \\ &\quad + 15\mathbb{L}^{12} + 14\mathbb{L}^{13} + 10\mathbb{L}^{14} + 6\mathbb{L}^{15} + 2\mathbb{L}^{16} + \mathbb{L}^{17}. \end{aligned}$$

Note that, for a smooth projective variety with Grothendieck class that is a polynomial in the Lefschetz motive \mathbb{L} , the Poincaré polynomial and the Grothendieck class are related by replacing x^2 with \mathbb{L} , since the variety is Hodge–Tate. In torified form the above gives

$$\begin{aligned} [\mathcal{M}_{\mathbb{P}^2}(4, 1)] &= \mathbb{T}^{17} + 19 \mathbb{T}^{16} + 174 \mathbb{T}^{15} + 1020 \mathbb{T}^{14} + 4284 \mathbb{T}^{13} + 13665 \mathbb{T}^{12} + 34230 \mathbb{T}^{11} \\ &\quad + 68678 \mathbb{T}^{10} + 111606 \mathbb{T}^9 + 147653 \mathbb{T}^8 + 159082 \mathbb{T}^7 + 139008 \mathbb{T}^6 \\ &\quad + 97643 \mathbb{T}^5 + 54320 \mathbb{T}^4 + 23370 \mathbb{T}^3 + 7468 \mathbb{T}^2 + 1632 \mathbb{T} + 192, \end{aligned}$$

where $192 = \chi(\mathcal{M}_{\mathbb{P}^2}(4, 1))$ is the Euler characteristics, which is also the number of points over \mathbb{F}_1 . The number of points over \mathbb{F}_{1^m} gives 864045 for $m = 1$ (the number of tori in the torification), 383699680 for $m = 2$ (roots of unity of order two), 36177267945 for $m = 3$ (roots of unity of order three), etc.

In this example, the Euler characteristic $\chi(\mathcal{M}_{\mathbb{P}^2}(4, 1))$, which can also be seen as the number of \mathbb{F}_1 -points, is interpreted physically as determining the BPS counting. It is natural to ask whether the counting of \mathbb{F}_{1^m} -points, which corresponds to the counting of roots of unity in the tori of the torification, can also carry physically significant information.

Other examples of torified varieties relevant to physics can be found in the context of quantum field theory, see [6] and [60].

5.3.1. *BPS counting and the virtual motive.* The formulation of the refined BPS counting given in [22] can be summarized as follows. The virtual motive $[X]_{\text{vir}} = \mathbb{L}^{-n/2}[X]$, with $n = \dim(X)$, is a class in the ring of motivic weights $K_0(\mathcal{V})[\mathbb{L}^{-1/2}]$, see [12]. When X admits a \mathbb{G}_m action and a Bialynicki-Birula decomposition as discussed in the previous section, where all the components Z_i of the fixed point locus of the \mathbb{G}_m -action have Tate classes $[Z_i] = \sum_j c_{ij} \mathbb{L}^{b_{ij}} \in K_0(\mathcal{V})$, with $c_{ij} \in \mathbb{Z}$ and $b_{ij} \in \mathbb{Z}_+$, the virtual motive $[X]_{\text{vir}}$ is a Laurent polynomial in the square root $\mathbb{L}^{1/2}$ of the Lefschetz motive,

$$(5.5) \quad [X]_{\text{vir}} = \sum_{i,j} c_{ij} \mathbb{L}^{b_{ij} + d_i - 1/2},$$

where, as before, d_i is the dimension of the positive eigenspace of the \mathbb{G}_m -action on the tangent space of X at points of Z_i . In applications to BPS counting, one considers the virtual motive of a moduli space M that admits a perfect obstruction theory, so that it has virtual dimension zero and an associated invariant $\#_{\text{vir}} M$ which is computed by a virtual index

$$\#_{\text{vir}} M = \chi_{\text{vir}}(M, K_{M,\text{vir}}^{1/2}) = \chi(M, K_{M,\text{vir}}^{1/2} \otimes \mathcal{O}_{M,\text{vir}}),$$

where $\mathcal{O}_{M,\text{vir}}$ is the virtual structure sheaf and $K_{M,\text{vir}}^{1/2}$ is a square root of the virtual canonical bundle, see [35].

5.3.2. *The formal square root of the Lefschetz motive.* The formal square root $\mathbb{L}^{1/2}$ of the Lefschetz motive that occurs in (5.5) as Grothendieck class can be introduced, at the level of the category of motives, as shown in §3.4 of [48], using the Tannakian formalism, [30]. Let $\mathcal{C} = \text{Num}_{\mathbb{Q}}^{\dagger}$ be the Tannakian category of pure motives with the numerical equivalence relation and the Koszul sign rule twist \dagger in the tensor structure, with motivic Galois group $G = \text{Gal}(\mathcal{C})$. The inclusion of the Tate motives (with motivic Galois group \mathbb{G}_m) determines a group homomorphism $t : G \rightarrow \mathbb{G}_m$, which satisfies $t \circ w = 2$ with the weight homomorphism $w : \mathbb{G}_m \rightarrow G$ (see §5 of [31]). The category $\mathcal{C}(\mathbb{Q}(\frac{1}{2}))$ obtained by adjoining a square root of the Tate motive to \mathcal{C} is then obtained as the Tannakian category whose Galois group is the fibered product

$$G^{(2)} = \{(g, \lambda) \in G \times \mathbb{G}_m : t(g) = \lambda^2\}.$$

The construction of square roots of Tate motives described in [48] was generalized in [49] to arbitrary n -th roots of Tate motives, obtained via the same Tannakian construction, with the category $\mathcal{C}(\mathbb{Q}(\frac{1}{n}))$ obtained by adjoining an n -th root of the Tate motive determined by its Tannakian Galois group

$$G^{(n)} = \{(g, \lambda) \in G \times \mathbb{G}_m : t(g) = \sigma_n(\lambda)\},$$

with $\sigma_n : \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_n(\lambda) = \lambda^n$. The category $\hat{\mathcal{C}}$ obtained by adjoining to $\mathcal{C} = \text{Num}_{\mathbb{Q}}^{\dagger}$ arbitrary roots of the Tate motives is the Tannakian category with Galois group

$\hat{G} = \varprojlim_n G^{(n)}$. The category $\hat{\mathcal{C}}$ has an action of \mathbb{Q}_+^* by automorphisms induced by the endomorphisms σ_n of \mathbb{G}_m . These roots of Tate motives give rise to a good formalism of \mathbb{F}_ζ -geometry, with ζ a root of unity, lying “below” \mathbb{F}_1 -geometry and expressed at the motivic level in terms of a Habiro ring type object associated to the Grothendieck ring of orbit categories of $\hat{\mathcal{C}}$, see [49].

5.4. Counting \mathbb{F}_1 -points and zeta function. For a variety X over \mathbb{Z} that is polynomially countable (that is, the counting functions $N_X(q) = \#X_p(\mathbb{F}_q)$ with X_p the mod p reduction is a polynomial in q with \mathbb{Z} coefficients) the counting of points over the “extensions” \mathbb{F}_{1^m} (in the sense of [45]) can be obtained as the values $N_X(m+1)$ (see Theorem 4.10 of [24] and Theorem 1 of [29]). As we discussed earlier, in the case of a torified variety, with Grothendieck class $[X] = \sum_{i \geq 0} a_i \mathbb{T}^i$ with $a_i \in \mathbb{Z}_+$, this corresponds to the counting given in (5.2). This is the counting of the number of m -th roots of unity in each torus $\mathbb{T}^i = [\mathbb{G}_m^i]$ of the torification.

For a variety X over a finite field \mathbb{F}_q the Hasse–Weil zeta function is given, in logarithmic form by

$$(5.6) \quad \log Z_{\mathbb{F}_q}(X, t) = \sum_{m \geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m.$$

In the case of torified varieties, there is an analogous zeta function over \mathbb{F}_1 . We think of this \mathbb{F}_1 -zeta function as defined on torified Grothendieck classes, $Z_{\mathbb{F}_1}([X], t)$. In the case of geometric torifications, we can regard it as a function of the variety and the torification, $Z_{\mathbb{F}_1}((X, T), t)$. For simplicity of notation, we will simply write $Z_{\mathbb{F}_1}(X, t)$ by analogy to the Hasse–Weil zeta function, with

$$(5.7) \quad \log Z_{\mathbb{F}_1}(X, t) := \sum_{m \geq 1} \frac{\#X(\mathbb{F}_{1^m})}{m} t^m.$$

Lemma 5.3. *Let X be a variety over \mathbb{Z} with a torified Grothendieck class $[X] = \sum_{k \geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$. Then the \mathbb{F}_1 -zeta function is given by*

$$(5.8) \quad \log Z_{\mathbb{F}_1}(X, t) = \sum_{k=0}^N a_k \operatorname{Li}_{1-k}(t),$$

where $\operatorname{Li}_s(t)$ is the polylogarithm function with $\operatorname{Li}_1(t) = -\log(1-t)$ and for $k \geq 1$

$$\operatorname{Li}_{1-k}(t) = \left(t \frac{d}{dt}\right)^{k-1} \frac{t}{1-t}.$$

Proof. For $[X] = \sum_{k \geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$ as above, we can consider a similar zeta function based on the counting of \mathbb{F}_{1^m} -points described above. Using (5.2), we obtain an expression of the form

$$\log Z_{\mathbb{F}_1}(X, t) = \sum_{m \geq 1} \frac{\#X(\mathbb{F}_{1^m})}{m} t^m = \sum_{k=0}^N a_k \sum_{m \geq 1} m^{k-1} t^m = \sum_{k=0}^N a_k \operatorname{Li}_{1-k}(t),$$

given by a linear combination of polylogarithm functions $\text{Li}_s(t)$ at integer values $s \leq 1$. \square

Such polylogarithm functions can be expressed explicitly in the form $\text{Li}_1(t) = -\log(1-t)$ and for $k \geq 1$

$$\text{Li}_{1-k}(t) = \left(t \frac{d}{dt}\right)^{k-1} \frac{t}{1-t} = \sum_{\ell=0}^{k-1} \ell! S(k, \ell+1) \left(\frac{t}{1-t}\right)^{\ell+1},$$

with $S(k, r)$ the Stirling numbers of the second kind

$$S(k, r) = \frac{1}{r!} \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} j^k.$$

As in the case of the Hasse–Weil zeta function over \mathbb{F}_q (see [62]), the \mathbb{F}_1 -zeta function gives an exponentiable motivic measure.

Proposition 5.4. *The \mathbb{F}_1 -zeta function is an exponentiable motivic measure, that is, a ring homomorphism $Z_{\mathbb{F}_1} : K_0(\mathcal{T})^a \rightarrow W(\mathbb{Z})$ from the Grothendieck ring of torified varieties (with either $a = w, o, s$) to the Witt ring.*

Proof. Clearly with respect to addition in the Grothendieck ring of torified varieties we have $[X] + [X'] = \sum_{i \geq 0} a_i \mathbb{T}^i + \sum_{j \geq 0} a'_j \mathbb{T}^j = \sum_{k \geq 0} b_k \mathbb{T}^k$ with $b_k = a_k + a'_k$, hence

$$\log Z_{\mathbb{F}_1}([X] + [X'], t) = \sum_{k=0}^N b_k \text{Li}_{1-k}(t) = \log Z_{\mathbb{F}_1}([X], t) + \log Z_{\mathbb{F}_1}([X'], t).$$

The behavior with respect to products $[X] \cdot [Y]$ in the Grothendieck ring of torified varieties can be analyzed as in [62] for the Hasse–Weil zeta function. We view the \mathbb{F}_1 -zeta function

$$Z_{\mathbb{F}_1}(X, t) = \exp\left(\sum_{k=0}^N a_k \text{Li}_{1-k}(t)\right)$$

as the element in the Witt ring $W(\mathbb{Z})$ with ghost components $\#X(\mathbb{F}_{1^m}) = \sum_{k=0}^N m^k$, by writing the ghost map $\text{gh} : W(\mathbb{Z}) \rightarrow \mathbb{Z}^{\mathbb{N}}$ as

$$\text{gh} : Z(t) = \exp\left(\sum_{m \geq 1} \frac{N_m}{m} t^m\right) \mapsto t \frac{d}{dt} \log Z(t) = \sum_{m \geq 1} N_m t^m \mapsto (N_m)_{m \geq 1}.$$

The ghost map is an injective ring homomorphism. Thus, it suffices to see that on the ghost components $N_m(X \times Y) = N_m(X) \cdot N_m(Y)$. If $[X] = \sum_{k \geq 0} a_k \mathbb{T}^k$ and $[Y] = \sum_{\ell \geq 0} b_\ell \mathbb{T}^\ell$ then $[X \times Y] = \sum_{n \geq 0} \sum_{k+\ell=n} a_k b_\ell \mathbb{T}^n$ and $N_m(X \times Y) = \sum_{n \geq 0} \sum_{k+\ell=n} a_k b_\ell m^n = N_m(X) \cdot N_m(Y)$. \square

5.5. Relation to the Hasse–Weil zeta function. We discuss here the relation between the \mathbb{F}_1 -zeta function $Z_{\mathbb{F}_1}(X, t)$ introduced in (5.7) above, for a variety X over \mathbb{Z} with torified Grothendieck class $[X] = \sum_{k \geq 0} a_k \mathbb{T}^k$, and the Hesse–Weil zeta function $Z_{\mathbb{F}_q}(X, t)$, defined as in (5.6).

Definition 5.5. Consider the following elements in the Witt ring $W(\mathbb{Z})$, for $k \geq 0$:

$$(5.9) \quad Z_{0,k,q}(t) := \exp\left(\sum_{m \geq 1} (q-1)^k \frac{t^m}{m}\right) = \frac{1}{(1-t)^{(q-1)^k}}$$

$$(5.10) \quad Z_{1,k,q}(t) := \exp\left(\sum_{m \geq 1} (\#\mathbb{F}^{m-1}(\mathbb{F}_q))^k \frac{t^m}{m}\right) = \exp\left(\sum_{m \geq 1} \left(\sum_{\ell=0}^{m-1} q^\ell\right)^k \frac{t^m}{m}\right).$$

Lemma 5.6. *Let $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ be the Hasse–Weil zeta function of a torus \mathbb{T}^k . The function $Z_{0,k,q}(t)$ of (5.9) divides $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ in the Witt ring with quotient the function $Z_{1,k,q}(t)$ of (5.10).*

Proof. Given elements $Q = Q(t)$ and $P = P(t)$ in the Witt ring $W(\mathbb{Z})$, we have that Q divides P iff the ghost components q_m of Q in $\mathbb{Z}^{\mathbb{N}}$ divide the corresponding ghost components p_m of P . There is then an element $S = S(t)$ in $W(\mathbb{Z})$, with ghost components $s_m = p_m/q_m$, such that the Witt product gives $S \star_W Q = P$. The m -th ghost components of $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ is $(q^m - 1)^k = \#\mathbb{T}^k(\mathbb{F}_{q^m})$, and we have $(q^m - 1)^k / (q - 1)^k = (1 + q + \cdots + q^{m-1})^k$. \square

Given elements $Q, P \in W(\mathbb{Z})$ such that $Q|P$ as above, we write $S = P/_W Q$ for the resulting element $S \in W(\mathbb{Z})$ with $S \star_W Q = P$.

The \mathbb{F}_1 -zeta function of (5.7) is obtained from the Hasse–Weil zeta function of (5.6) in the following way.

Proposition 5.7. *Let X be a variety X over \mathbb{Z} with torified Grothendieck class $[X] = \sum_{k \geq 0} a_k \mathbb{T}^k$. The \mathbb{F}_1 -zeta function is given by*

$$(5.11) \quad Z_{\mathbb{F}_1}(X, t) = \lim_{q \rightarrow 1} {}^W \sum_{k \geq 0} (Z_{\mathbb{F}_q}(\mathbb{T}^k, t) /_W Z_{0,k,q}(t))^{a_k} = \lim_{q \rightarrow 1} {}^W \sum_{k \geq 0} Z_{1,k,q}(t)^{a_k},$$

while the Hasse–Weil zeta function is given by

$$(5.12) \quad Z_{\mathbb{F}_q}(X, t) = {}^W \sum_{k \geq 0} Z_{\mathbb{F}_q}(\mathbb{T}^k, t)^{a_k},$$

where ${}^W \sum$ denotes the sum in the Witt ring.

Proof. For the Hasse–Weil zeta function we have

$$\begin{aligned} Z_{\mathbb{F}_q}(X, t) &= \exp\left(\sum_{m \geq 1} \#X(\mathbb{F}_{q^m}) \frac{t^m}{m}\right) = \exp\left(\sum_{k \geq 0} a_k \sum_{m \geq 1} (q^m - 1)^k \frac{t^m}{m}\right) \\ &= \prod_{k \geq 0} \exp(a_k \log Z_{\mathbb{F}_q}(\mathbb{T}^k, t)), \end{aligned}$$

hence we get (5.12). To obtain the \mathbb{F}_1 -zeta function we then use Lemma 5.6 and the fact that $(q^m - 1)^k / (q - 1)^k = (1 + q + \cdots + q^{m-1})^k$, with $\lim_{q \rightarrow 1} (1 + q + \cdots + q^{m-1})^k = m^k$. \square

5.6. Dynamical zeta functions. The dynamical approach to \mathbb{F}_1 -structures proposed in [54] is based on the existence of an endomorphism $f : X \rightarrow X$ that induces a quasi-unipotent morphism f_* on the homology $H_*(X, \mathbb{Z})$. In particular, this means that the map f_* has eigenvalues that are roots of unity.

In the case of a variety X endowed with a torification $X = \sqcup_i T^{d_i}$, one can consider in particular endomorphisms $f : X \rightarrow X$ that preserve the torification and that restrict to endomorphisms of each torus T^{d_i} .

We recall the definition and main properties of the relevant dynamical zeta functions, which we will consider in Proposition 5.8.

5.6.1. Properties of dynamical zeta functions. In general to a self-map $f : X \rightarrow X$, one can associate the dynamical Artin–Mazur zeta function and the homological Lefschetz zeta function. A particular class of maps with the property that they induce quasi-unipotent morphisms in homology is given by the Morse–Smale diffeomorphisms of smooth manifolds, see [66]. These are diffeomorphisms characterized by the properties that the set of nonwandering points is finite and hyperbolic, consisting of a finite number of periodic points, and for any pair of these points x, y the stable and unstable manifolds $W^s(x)$ and $W^u(y)$ intersect transversely. Morse–Smale diffeomorphisms are structurally stable among all diffeomorphisms, [36], [66].

The Lefschetz zeta function is given by

$$(5.13) \quad \zeta_{\mathcal{L},f}(t) = \exp \left(\sum_{m \geq 1} \frac{L(f^m)}{m} t^m \right),$$

with $L(f^m)$ the Lefschetz number of the m -th iterate f^m ,

$$L(f^m) = \sum_{k \geq 0} (-1)^k \text{Tr}((f^m)_* | H_k(X, \mathbb{Q})).$$

For a function with finite sets of fixed points $\text{Fix}(f^m)$ this is also equal to

$$L(f^m) = \sum_{x \in \text{Fix}(f^m)} \mathcal{I}(f^m, x),$$

with $\mathcal{I}(f^m, x)$ the index of the fixed point. The zeta function can then be written as a rational function of the form

$$\zeta_{\mathcal{L},f}(t) = \prod_k \det(1 - t f_* | H_k(X, \mathbb{Q}))^{(-1)^{k+1}}.$$

In the case of a map f with finitely many periodic points, all hyperbolic, the Lefschetz zeta function can be equivalently written (see [36]) as the rational function

$$\zeta_{\mathcal{L},f}(t) = \prod_{\gamma} (1 - \Delta_{\gamma} t^{p(\gamma)})^{(-1)^{u(\gamma)+1}}.$$

Here the product is over periodic orbits γ with least period $p(\gamma)$ and $u(\gamma) = \dim E_x^u$, for $x \in \gamma$, is the dimension of the span of eigenvectors of $Df_x^{p(\gamma)} : T_x M \rightarrow T_x M$ with eigenvalues λ with $|\lambda| > 1$. One has $\Delta_{\gamma} = \pm 1$ according to whether $Df_x^{p(\gamma)}$ is orientation preserving or reversing. The relation comes from the identity $\mathcal{I}(f^m, x) = (-1)^{u(\gamma)} \Delta_{\gamma}$. The Artin–Mazur zeta function is given by

$$(5.14) \quad \zeta_{AM,f}(t) = \exp \left(\sum_{m \geq 1} \frac{\#\text{Fix}(f^m)}{m} t^m \right).$$

The case of Morse–Smale diffeomorphisms can be treated as in [37] to obtain rationality and a description in terms of the homological zeta functions.

In the setting of real tori $\mathbb{R}^d / \mathbb{Z}^d$, one can consider the case of a toral endomorphism specified by a matrix $M \in M_d(\mathbb{Z})$. In the hyperbolic case, the counting of isolated fixed points of M^m is given by $|\det(1 - M^m)|$ and the dynamical Artin–Mazur zeta function is expressible in terms of the Lefschetz zeta function, associated to the signed counting of fixed points, through the fact that the Lefschetz zeta function agrees with the zeta function

$$(5.15) \quad \zeta_M(t) = \exp \left(\sum_{n \geq 1} \frac{t^n}{n} a_n \right), \quad \text{with} \quad a_n = \det(1 - M^n),$$

where $a_n = \det(1 - M^n)$ is a signed fixed point counting. The general relation between the zeta functions for the signed $\det(1 - M^n)$ and for $|\det(1 - M^n)|$ is shown in [4] for arbitrary toral endomorphisms, with $M \in M_d(\mathbb{Z})$.

In the case of complex algebraic tori $T^d = \mathbb{G}_m^d(\mathbb{C})$, one can similarly consider the endomorphisms action of the semigroup of matrices $M \in M_d(\mathbb{Z})^+$ by the linear action on \mathbb{C}^d preserving \mathbb{Z}^d and the exponential map $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow 1$ so that, for $M = (m_{ab})$ and $\lambda_a = \exp(2\pi i u_a)$, with the action given by

$$\lambda = (\lambda_a) \mapsto M(\lambda) = \exp(2\pi i \sum_b m_{ab} u_b).$$

The subgroup $\text{SL}_n(\mathbb{Z}) \subset M_n(\mathbb{Z})^+$ acts by automorphisms. These generalize the Bost–Connes endomorphisms $\sigma_n : \mathbb{G}_m \rightarrow \mathbb{G}_m$, which correspond to the ring homomorphisms of $\mathbb{Z}[t, t^{-1}]$ given by $\sigma_n : P(t) \mapsto P(t^n)$ and determine multivariable versions of the Bost–Connes algebra, see [55]. We can consider in this way maps of complex algebraic tori $T_{\mathbb{C}}^d = \mathbb{G}_m^d(\mathbb{C})$ that induce maps of the real tori obtained as the subgroup $T_{\mathbb{R}}^d = U(1)^d \subset \mathbb{G}_m^d(\mathbb{C})$, and associate to these maps the Lefschetz and Artin–Mazur zeta functions of the induced map of real tori.

5.6.2. *Torifications and dynamical zeta functions.* In the case of a variety with a torification, we consider endomorphisms $f : X \rightarrow X$ that preserves the tori of the torification and restricts to each torus T^{d_i} to a diffeomorphism $f_i : T_{\mathbb{R}}^{d_i} \rightarrow T_{\mathbb{R}}^{d_i}$. In particular, we consider toral endomorphism with a matrix $M_i \in M_{d_i}(\mathbb{Z})$, we can associate to the pair (X, f) a zeta function of the form

$$(5.16) \quad \zeta_{\mathcal{L},f}(X, t) = \prod_i \zeta_{\mathcal{L},f_i}(t), \quad \zeta_{AM,f}(X, t) = \prod_i \zeta_{AM,f_i}(t).$$

Proposition 5.8. *The zeta functions (5.13) and (5.14) define exponentiable motivic measures on the Grothendieck ring $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ of §6 of [54] with values in the Witt ring $W(\mathbb{Z})$. The zeta functions (5.16) define exponentiable motivic measures on the Grothendieck ring $K_0(\mathcal{T})^a$ of torified varieties with values in $W(\mathbb{Z})$.*

Proof. The Grothendieck ring $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ considered in §6 of [54] consists of pairs (X, f) of a complex quasi-projective variety and an automorphism $f : X \rightarrow X$ that induces a quasi-unipotent map f_* in homology. The addition is simply given by the disjoint union, and both the counting of periodic points $\#\text{Fix}(f^m)$ and the Lefschetz numbers $L(f^m)$ behave additively under disjoint unions. Thus, the zeta functions $\zeta_{\mathcal{L},f}(t)$ and $\zeta_{AM,f}(t)$, seen as elements in the Witt ring $W(\mathbb{Z})$ add

$$\begin{aligned} \zeta_{\mathcal{L},f_1 \sqcup f_2}(t) &= \exp \left(\sum_{m \geq 1} \frac{L((f_1 \sqcup f_2)^m)}{m} t^m \right) = \\ &= \exp \left(\sum_{m \geq 1} \frac{L(f_1^m)}{m} t^m \right) \cdot \exp \left(\sum_{m \geq 1} \frac{L(f_2^m)}{m} t^m \right) = \zeta_{\mathcal{L},f_1}(t) +_W \zeta_{\mathcal{L},f_2}(t) \end{aligned}$$

and similarly for $\zeta_{AM,f_1 \sqcup f_2}(t) = \zeta_{AM,f_1}(t) +_W \zeta_{AM,f_2}(t)$. The product is given by the Cartesian product $(X_1, f_1) \times (X_2, f_2)$. Since $\text{Fix}((f_1 \times f_2)^m) = \text{Fix}(f_1^m) \times \text{Fix}(f_2^m)$ and the same holds for Lefschetz numbers since

$$L((f_1 \times f_2)^m) = \sum_{k \geq 0} \sum_{\ell+r=k} (-1)^{\ell+r} \text{Tr}((f_1^m)_* \otimes (f_2^m)_* | H_{\ell}(X_1, \mathbb{Q}) \otimes H_r(X_2, \mathbb{Q}))$$

which gives $L(f_1^m) \cdot L(f_2^m)$. Thus, we can use as in Proposition 5.4 the fact that the ghost map $\text{gh} : W(\mathbb{Z}) \rightarrow \mathbb{Z}^{\mathbb{N}}$

$$\text{gh} : \exp \left(\sum_{m \geq 1} \frac{N_m}{m} t^m \right) \mapsto \sum_{m \geq 1} N_m t^m \mapsto (N_m)_{m \geq 1}$$

is an injective ring homomorphism to obtain the multiplicative property. The case of the torified varieties and the zeta functions (5.16) is analogous, combining the additive and multiplicative behavior of the fixed point counting and the Lefschetz numbers on each torus and of the decomposition into tori as in Proposition 5.4. \square

In the case of quasi-unipotent maps of tori the Lefschetz zeta function can be computed completely explicitly. Indeed, it is shown in [7], [8] that, for a quasi-unipotent self map $f : T_{\mathbb{R}}^n \rightarrow T_{\mathbb{R}}^n$, the Lefschetz zeta function has an explicit form

that is completely determined by the map on the first homology. Under the quasi-unipotent assumption all the eigenvalues of the induced map on H_1 are roots of unity, hence the characteristic polynomial $\det(1 - t f_* | H_1(X))$ is a product of cyclotomic polynomials $\Phi_{m_1}(t) \cdots \Phi_{m_N}(t)$ where

$$\Phi_m(t) = \prod_{d|m} (1 - t^d)^{\mu(m/d)},$$

with $\mu(n)$ the Möbius function. It is shown in [8] that the Lefschetz zeta function has the form

$$(5.17) \quad \zeta_{\mathcal{L},f}(t) = \prod_{d|m} (1 - t^d)^{-s_d},$$

where $m = \text{lcm}\{m_1, \dots, m_N\}$ and

$$s_d = \frac{1}{d} \sum_{k|d} F_k \mu(d/k)$$

$$F_k = \prod_{i=1}^N (\Phi_{m_i/(k, m_i)}(1))^{\varphi(m_i)/\varphi(m_i/(k, m_i))}$$

where the Euler function

$$\varphi(m) = m \prod_{p|m, p \text{ prime}} (1 - p^{-1})$$

is the degree of $\Phi_m(t)$.

Remark 5.9. The properties of dynamical Artin–Mazur zeta functions change significantly when, instead of considering varieties over \mathbb{C} one considers varieties in positive characteristic, [16], [19]. The prototype model of this phenomenon is illustrated by considering the Bost–Connes endomorphisms $\sigma_n : \lambda \mapsto \lambda^n$ of $\mathbb{G}_m(\overline{\mathbb{F}}_p)$. In this case, the dynamical zeta function of σ_n is rational or transcendental depending on whether p divides n (Theorem 1.2 and 1.3 and §3 of [16] and Theorem 1 of [17]). Similar phenomena in the more general case of endomorphisms of Abelian varieties in positive characteristic have been investigated in [19]. In the positive characteristic setting, where one is considering the characteristic p version of the Bost–Connes system of [27], one should then replace the dynamical zeta function by the tame zeta function considered in [19].

6. SPECTRA AND ZETA FUNCTIONS

We have already discussed in §5.4 and §5.6 zeta functions arising from certain counting functions that define ring homomorphisms from suitable Grothendieck rings to the Witt ring $W(\mathbb{Z})$. We consider here a more general setting of exponentiable motivic measures.

A motivic measure is a ring homomorphism $\mu : K_0(\mathcal{V}) \rightarrow R$, from the Grothendieck ring of varieties $K_0(\mathcal{V})$ to a commutative ring R . Examples include the counting

measure, for varieties defined over finite fields, which counts the number of algebraic points over \mathbb{F}_q , the topological Euler characteristic or the Hodge–Deligne polynomials for complex algebraic varieties.

The Kapranov motivic zeta function [46] is defined as $\zeta(X, t) = \sum_{n=0}^{\infty} [S^n(X)] t^n$, where $S^n(X) = X^n/S_n$ are the symmetric products of X and $[S^n(X)]$ are the classes in $K_0(\mathcal{V})$. Similarly, the zeta function of a motivic measure is defined as

$$(6.1) \quad \zeta_{\mu}(X, t) = \sum_{n=0}^{\infty} \mu(S^n(X)) t^n.$$

It is viewed as an element in the Witt ring $W(R)$. The addition in $K_0(\mathcal{V})$ is mapped by the zeta function to the addition in $W(R)$, which is the usual product of the power series,

$$(6.2) \quad \zeta_{\mu}(X \sqcup Y, t) = \zeta_{\mu}(X, t) \cdot \zeta_{\mu}(Y, t) = \zeta_{\mu}(X, t) +_{W(R)} \zeta_{\mu}(Y, t).$$

The motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$ is said to be exponentiable (see [62], [63]) if the zeta function (6.1) defines a ring homomorphism

$$\zeta_{\mu} : K_0(\mathcal{V}) \rightarrow W(R),$$

that is, in addition to (6.2) one also has

$$(6.3) \quad \zeta_{\mu}(X \times Y, t) = \zeta_{\mu}(X, t) \star_{W(R)} \zeta_{\mu}(Y, t).$$

We investigate here how to lift the zeta functions of exponentiable motivic measures to the level of spectra. To this purpose, we first investigate how to construct a spectrum whose π_0 is a dense subring $W_0(R)$ of the Witt ring $W(R)$ and then we consider how to lift the ring homomorphisms given by zeta functions ζ_{μ} of exponentiable measures with a rationality and a factorization condition.

6.1. The Endomorphism Category. Let R be a commutative ring. We denote by \mathcal{E}_R the endomorphism category of R , which is defined as follows (see [1], [2], [32]).

Definition 6.1. The category \mathcal{E}_R has objects given by the pairs (E, f) of a finite projective module E over R and an endomorphism $f \in \text{End}_R(E)$, and morphisms given by morphisms $\phi : E \rightarrow E'$ of finite projective modules that commute with the endomorphisms, $f' \circ \phi = \phi \circ f$. The endomorphism category has direct sum $(E, f) \oplus (E', f') = (E \oplus E', f \oplus f')$ and tensor product $(E, f) \otimes (E', f') = (E \otimes E', f \otimes f')$.

The category of finite projective modules over R is identified with the subcategory corresponding to the objects $(E, 0)$ with trivial endomorphism.

An exact sequence in \mathcal{E}_R is a sequence of objects and morphisms in \mathcal{E}_R which is exact as a sequence of finite projective modules over R (forgetting the endomorphisms). This determines a collection of admissible short exact sequence (and of admissible monomorphisms and epimorphisms). The endomorphism category \mathcal{E}_R is then an exact category, hence it has an associated K -theory defined via the Quillen Q -construction,

[61]. This assigns to the exact category \mathcal{E}_R the category $\mathcal{Q}\mathcal{E}_R$ with the same objects and morphisms $\text{Hom}_{\mathcal{Q}\mathcal{E}_R}((E, f), (E', f'))$ given by diagrams

$$\begin{array}{ccc} & (E'', f'') & \\ & \swarrow \quad \searrow & \\ (E, f) & & (E', f'), \end{array}$$

where the first arrow is an admissible epimorphism and the second an admissible monomorphism, with composition given by pullback. By the Quillen construction K -theory of \mathcal{E}_R is then $K_{n-1}(\mathcal{E}_R) = \pi_n(\mathcal{N}(\mathcal{Q}\mathcal{E}_R))$, with $\mathcal{N}(\mathcal{Q}\mathcal{E}_R)$ the nerve of $\mathcal{Q}\mathcal{E}_R$.

The forgetful functor $(E, f) \mapsto E$ induces a map on K -theory

$$K_n(\mathcal{E}_R) \rightarrow K_n(\mathcal{P}_R) = K_n(R),$$

which is a split surjection. Let

$$\mathcal{E}_n(R) := \text{Ker}(K_n(\mathcal{E}_R) \rightarrow K_n(R)).$$

In the case of K_0 an explicit description is given by the following, [1], [2]. Let $K_0(\mathcal{E}_R)$ denote the K_0 of the endomorphism category \mathcal{E}_R . It is a ring with the product structure induced by the tensor product. It is proved in [1], [2] that the quotient

$$(6.4) \quad W_0(R) = K_0(\mathcal{E}_R)/K_0(R)$$

embeds as a dense subring of the big Witt ring $W(R)$ via the map

$$(6.5) \quad L : (E, f) \mapsto \det(1 - tM(f))^{-1},$$

with $M(f)$ the matrix associated to $f \in \text{End}_R(E)$, where $\det(1 - tM(f))^{-1}$ is viewed as an element in $\Lambda(R) = 1 + tR[[t]]$. As a subring $W_0(R) \hookrightarrow W(R)$ of the big Witt ring, $W_0(R)$ consists of the rational Witt vectors

$$W_0(R) = \left\{ \frac{1 + a_1t + \cdots + a_nt^n}{1 + b_1t + \cdots + b_mt^m} \mid a_i, b_i \in R, n, m \geq 0 \right\}.$$

Equivalently, one can consider the ring $\mathcal{R} = (1 + tR[t])^{-1}R[t]$ and identify the above with $1 + t\mathcal{R}$, where the multiplication in $1 + t\mathcal{R}$ corresponds to the addition in the Witt ring, and the Witt product is determined by the identity $(1 - at) \star (1 - bt) = (1 - abt)$.

This description of Witt rings in terms of endomorphism categories was applied to investigate the arithmetic structures of the Bost–Connes quantum statistical mechanical system, see [24], [56], [57].

This relation between the Grothendieck ring and Witt vectors was extended to the higher K -theory in [38], where an explicit description for the kernels $\mathcal{E}_n(R)$ is obtained, by showing that

$$\mathcal{E}_{n-1}(R) = \text{Coker}(K_n(R) \rightarrow K_n(\mathcal{R})),$$

where $\mathcal{R} = (1 + tR[t])^{-1}R[t]$ and $K_n(R) \rightarrow K_n(\mathcal{R})$ is a split injection. The identification above is obtained in [38] by showing that there is an exact sequence

$$(6.6) \quad 0 \rightarrow K_n(R) \rightarrow K_n(\mathcal{R}) \rightarrow K_{n-1}(\mathcal{E}_R) \rightarrow K_{n-1}(R) \rightarrow 0.$$

The identification (6.4) for K_0 is then recovered as the case with $n = 0$ that gives an identification $\mathcal{E}_0(R) \simeq 1 + t\mathcal{R}$.

6.2. Spectrum of the Endomorphism Category and Witt vectors. Let \mathcal{P}_R denote the category of finite projective modules over a commutative ring R with unit. Also let \mathcal{E}_R be the endomorphism category recalled in §3.2. By the Segal construction described in §3.2, we obtain associated Γ -spaces $F_{\mathcal{P}_R}$ and $F_{\mathcal{E}_R}$ and spectra $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$, the K -theory spectrum of R , and $F_{\mathcal{E}_R}(\mathbb{S})$, the spectrum of the endomorphism category.

We obtain in the following way a functorial “spectrification” of the Witt ring $W_0(R)$, namely a spectrum $\mathbb{W}(R)$ with $\pi_0\mathbb{W}(R) = W_0(R)$.

Definition 6.2. For a commutative ring R , with \mathcal{P}_R the category of finite projective modules and \mathcal{E}_R the category of endomorphisms, the spectrum $\mathbb{W}(R)$ is defined as the cofiber $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ obtained from the Γ -spaces $F_{\mathcal{P}_R} : \Gamma^0 \rightarrow \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \rightarrow \Delta_*$ associated to the categories \mathcal{P}_R and \mathcal{E}_R .

Lemma 6.3. *For a commutative ring R , the inclusion of the category \mathcal{P}_R of finite projective modules as the subcategory of the endomorphism category \mathcal{E}_R determines a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \cdots \\ \cdots \rightarrow \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \end{aligned}$$

of the homotopy groups of the spectra $F_{\mathcal{P}_R}(\mathbb{S})$, $F_{\mathcal{E}_R}(\mathbb{S})$ with cofiber $\mathbb{W}(R)$ as in Definition 6.2. The spectrum $\mathbb{W}(R)$ satisfies $\pi_0\mathbb{W}(R) = W_0(R)$.

Proof. The functoriality of the Segal construction implies that the inclusion of \mathcal{P}_R as the subcategory of \mathcal{E}_R given by objects $(E, 0)$ with trivial endomorphism determines a map of Γ -spaces $F_{\mathcal{P}_R} \rightarrow F_{\mathcal{E}_R}$, which is a natural transformation of the functors $F_{\mathcal{P}_R} : \Gamma^0 \rightarrow \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \rightarrow \Delta_*$. After passing to endofunctors $F_{\mathcal{P}_R} : \Delta_* \rightarrow \Delta_*$ and $F_{\mathcal{E}_R} : \Delta_* \rightarrow \Delta_*$ we obtain a map of spectra $K(R) \rightarrow F_{\mathcal{E}_R}(\mathbb{S})$, induced by the inclusion of \mathcal{P}_R as subcategory of \mathcal{E}_R . The category Δ_* of simplicial sets has products and equalizers, hence pullbacks. Thus, given two functors $F, F' : \Gamma^0 \rightarrow \Delta_*$, a natural transformation $\alpha : F \rightarrow F'$ is mono if and only if for all objects $X \in \Gamma^0$ the morphism $\alpha_X : F(X) \rightarrow F'(X)$ is a monomorphism in Δ_* . An embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$ determines by composition an embedding $\Sigma_{\mathcal{C}}(X) \hookrightarrow \Sigma_{\mathcal{C}'}(X)$ of the categories of summing functors, for each object $X \in \Gamma^0$. This gives a monomorphism $F_{\mathcal{C}}(X) = \mathcal{N}\Sigma_{\mathcal{C}}(X) \rightarrow F_{\mathcal{C}'}(X) = \mathcal{N}\Sigma_{\mathcal{C}'}(X)$, hence a monomorphism $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}'}$ of Γ -spaces. Arguing as in Lemma 1.3 of [64] we then obtain from such a map $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}'}$ of Γ -spaces a long exact sequence of homotopy groups of the associated spectra

$$\begin{aligned} \cdots \rightarrow \pi_n(F_{\mathcal{C}}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{C}'}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{C}'}(\mathbb{S})/F_{\mathcal{C}}(\mathbb{S})) \rightarrow \pi_{n-1}(F_{\mathcal{C}}(\mathbb{S})) \rightarrow \cdots \\ \cdots \rightarrow \pi_0(F_{\mathcal{C}}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{C}'}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{C}'}(\mathbb{S})/F_{\mathcal{C}}(\mathbb{S})), \end{aligned}$$

where $F_{C'}(\mathbb{S})/F_C(\mathbb{S})$ is the cofiber. When applied to the subcategory $\mathcal{P}_R \hookrightarrow \mathcal{E}_R$ this gives the long exact sequence

$$\begin{aligned} \cdots \rightarrow \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \cdots \\ \cdots \rightarrow \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})). \end{aligned}$$

Here we have $\pi_n(F_{\mathcal{P}_R}(\mathbb{S})) = K_n(R)$. Moreover, by construction we have $\pi_0(F_{\mathcal{E}_R}(\mathbb{S})) = K_0(\mathcal{E}_R)$ so that we identify

$$\pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) = W_0(R) = K_0(\mathcal{E}_R)/K_0(R).$$

Thus, the spectrum $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ given by the cofiber of $F_{\mathcal{P}_R}(\mathbb{S}) \rightarrow F_{\mathcal{E}_R}(\mathbb{S})$ provides a spectrum whose zeroth homotopy group is the Witt ring $W_0(R)$. \square

The forgetful functor $\mathcal{E}_R \rightarrow \mathcal{P}_R$ also induces a corresponding map of Γ -spaces $F_{\mathcal{E}_R} \rightarrow F_{\mathcal{P}_R}$. Moreover, one can also construct a spectrum with π_0 equal to $W_0(R)$ using the characterization given in [38], that we recalled above, in terms of the map on K -theory (and on K -theory spectra) $K(R) \rightarrow K(\mathcal{R})$ with $\mathcal{R} = (1 + rR[t])^{-1}R[t]$. One can obtain in this way a reformulation in terms of spectra of the result of [38]. However, for our purposes here, it is preferable to work with the spectrum constructed in Lemma 6.3.

We give a variant of Lemma 6.3 that will be useful in the following. We denote by \mathcal{P}_R^\pm and \mathcal{E}_R^\pm , respectively, the categories of $\mathbb{Z}/2\mathbb{Z}$ -graded finite projective R -modules and the $\mathbb{Z}/2\mathbb{Z}$ -graded endomorphism category with objects given by pairs $\{(E_+, f_+), (E_-, f_-)\}$, which we write simply as (E_\pm, f_\pm) and with morphisms $\phi : E_\pm \rightarrow E'_\pm$ of $\mathbb{Z}/2\mathbb{Z}$ -graded finite projective modules that commute with f_\pm . The sum in \mathcal{E}_R^\pm is given by

$$(E_\pm, f_\pm) \oplus (E'_\pm, f'_\pm) = ((E_+ \oplus E'_+, E_- \oplus E'_-), (f_+ \oplus f'_+, f_- \oplus f'_-))$$

while the tensor product $(E_\pm, f_\pm) \otimes (E'_\pm, f'_\pm)$ is given by

$$((E_+ \otimes E'_+ \oplus E_- \otimes E'_-, f_+ \otimes f'_+ \oplus f_- \otimes f'_-), (E_+ \otimes E'_- \oplus E_- \otimes E'_+, f_+ \otimes f'_- \oplus f_- \otimes f'_+)).$$

Again we consider \mathcal{P}_R^\pm as a subcategory of \mathcal{E}_R^\pm with trivial endomorphisms.

Lemma 6.4. *The map $\delta : K_0(\mathcal{E}_R^\pm) \rightarrow K_0(\mathcal{E}_R)$ given by $[E_\pm, f_\pm] \mapsto [E_+, f_+] - [E_-, f_-]$ is a ring homomorphism and it descends to a ring homomorphism*

$$K_0(\mathcal{E}_R^\pm)/K_0(\mathcal{P}_R^\pm) \rightarrow K_0(\mathcal{E}_R)/K_0(R) \simeq W_0(R).$$

Proof. The map is clearly compatible with sums. Compatibility with product also holds since $[E_\pm, f_\pm] \cdot [E'_\pm, f'_\pm] \mapsto ([E_+, f_+] - [E_-, f_-]) \cdot ([E'_+, f'_+] - [E'_-, f'_-])$. Moreover, it maps $K_0(\mathcal{P}_R^\pm)$ to $K_0(\mathcal{P}_R)$. \square

As before, the categories \mathcal{P}_R^\pm and \mathcal{E}_R^\pm have associated Γ -spaces $F_{\mathcal{P}_R^\pm} : \Gamma^0 \rightarrow \Delta_*$ and $F_{\mathcal{E}_R^\pm} : \Gamma^0 \rightarrow \Delta_*$ and spectra $F_{\mathcal{P}_R^\pm}(\mathbb{S})$ and $F_{\mathcal{E}_R^\pm}(\mathbb{S})$. The following result follows as in Lemma 6.3.

Lemma 6.5. *The inclusion of \mathcal{P}_R^\pm as a subcategory of \mathcal{E}_R^\pm induces a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \pi_n(F_{\mathcal{P}_R^\pm}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R^\pm}(\mathbb{S})) \rightarrow \pi_n(F_{\mathcal{E}_R^\pm}(\mathbb{S})/F_{\mathcal{P}_R^\pm}(\mathbb{S})) \rightarrow \pi_{n-1}(F_{\mathcal{P}_R^\pm}(\mathbb{S})) \rightarrow \cdots \\ \cdots \rightarrow \pi_0(F_{\mathcal{P}_R^\pm}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R^\pm}(\mathbb{S})) \rightarrow \pi_0(F_{\mathcal{E}_R^\pm}(\mathbb{S})/F_{\mathcal{P}_R^\pm}(\mathbb{S})) \end{aligned}$$

of the homotopy groups of the spectra $F_{\mathcal{P}_R^\pm}(\mathbb{S})$ and $F_{\mathcal{E}_R^\pm}(\mathbb{S})$, which at the level of π_0 gives $K_0(\mathcal{P}_R^\pm) \rightarrow K_0(\mathcal{E}_R^\pm) \rightarrow K_0(\mathcal{E}_R^\pm)/K_0(\mathcal{P}_R^\pm)$.

We denote by $\mathbb{W}^\pm(R) = F_{\mathcal{E}_R^\pm}(\mathbb{S})/F_{\mathcal{P}_R^\pm}(\mathbb{S})$ the cofiber of $F_{\mathcal{P}_R^\pm}(\mathbb{S}) \rightarrow F_{\mathcal{E}_R^\pm}(\mathbb{S})$.

Remark 6.6. It is important to point out that our treatment of Witt vectors and their spectrification, as presented in this section, differs from the one in [40] (see especially Theorem 2.2.9 and equation (2.2.11) in that paper), and in [20]. Nonetheless, the circle action on THH that is used to obtain the spectrum TR is closely related to the Bost–Connes structure investigated in the present paper. A more direct relation between Bost–Connes structures and topological Hochschild and cyclic homology will also relate naturally to the point of view on \mathbb{F}_1 -geometry developed in [26]. We will leave this topic for future work.

6.3. Exponentiable measures and maps of Γ -spaces. The problem of lifting to the level of spectra the Hasse–Weil zeta function associated to the counting motivic measure for varieties over finite fields was discussed in [21]. We consider here a very similar setting and procedure, where we want to lift a zeta function $\zeta_\mu : K_0(\mathcal{V}) \rightarrow W(R)$ associated to an exponentiable motivic measure to the level of spectra. To this purpose, we make some assumptions of rationality and the existence of a factorization for our zeta functions of exponentiable motivic measures. We then consider the spectrum $K(\mathcal{V})$ of [70], [72] with $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V})$ and a spectrum, obtained from a Γ -space, associated to the subring $W_0(R)$ of the big Witt ring $W(R)$.

Definition 6.7. A motivic measure, that is, a ring homomorphism $\mu : K_0(\mathcal{V}) \rightarrow R$ of the Grothendieck ring of varieties to a commutative ring R , is called factorizable if it satisfies the following three properties:

- (1) **exponentiability:** the associated zeta function $\zeta_\mu(X, t)$ is a ring homomorphism $\zeta_\mu : K_0(\mathcal{V}) \rightarrow W(R)$ to the Witt ring of R ;
- (2) **rationality:** the homomorphism ζ_μ factors through the inclusion of the subring $W_0(R)$ of the Witt ring, $\zeta_\mu : K_0(\mathcal{V}) \rightarrow W_0(R) \hookrightarrow W(R)$.
- (3) **factorization:** the rational functions $\zeta_\mu(X, t)$ admit a factorization into linear factors

$$\zeta_\mu(X, t) = \frac{\prod_i (1 - \alpha_i t)}{\prod_j (1 - \beta_j t)} = \zeta_{\mu,+}(X, t) \underset{W}{-} \zeta_{\mu,-}(X, t)$$

where $\zeta_{\mu,+}(X, t) = \prod_j (1 - \beta_j t)^{-1}$ and $\zeta_{\mu,-}(X, t) = \prod_i (1 - \alpha_i t)^{-1}$ and $-_W$ is the difference in the Witt ring, that is the ratio of the two polynomials.

Lemma 6.8. *A factorizable motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$, as in Definition 6.7, determines a functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \rightarrow \mathcal{E}_R^\pm$ where $\mathcal{C}_\mathcal{V}$ is the assembler category encoding the scissor-congruence relations of the Grothendieck ring $K_0(\mathcal{V})$ and \mathcal{E}_R^\pm is the $\mathbb{Z}/2\mathbb{Z}$ -graded endomorphism category.*

Proof. The objects of $\mathcal{C}_\mathcal{V}$ are varieties X and the morphisms are locally closed embeddings, [70], [72]. To an object X we assign an object of \mathcal{E}_R obtained in the following way. Consider a factorization

$$\zeta_\mu(X, t) = \frac{\prod_{i=1}^n (1 - \alpha_i t)}{\prod_{j=1}^m (1 - \beta_j t)} = \zeta_{\mu,+}(X, t) -_W \zeta_{\mu,-}(X, t)$$

as above of the zeta function of X . Let $E_+^{X,\mu} = R^{\oplus m}$ and $E_-^{X,\mu} = R^{\oplus n}$ with endomorphisms $f_\pm^{X,\mu}$ respectively given in matrix form by $M(f_+^{X,\mu}) = \text{diag}(\beta_j)_{j=1}^m$ and $M(f_-^{X,\mu}) = \text{diag}(\alpha_i)_{i=1}^n$. The pair $(E_\pm^{X,\mu}, f_\pm^{X,\mu})$ is an object of the endomorphism category \mathcal{E}_R^\pm . Given an embedding $Y \hookrightarrow X$, the zeta function satisfies

$$\zeta_\mu(X, t) = \zeta_\mu(Y, t) \cdot \zeta_\mu(X \setminus Y, t) = \zeta_\mu(Y, t) +_W \zeta_\mu(X \setminus Y, t).$$

Using the factorizations of each term, this gives

$$(E_\pm^{X,\mu}, f_\pm^{X,\mu}) = (E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \oplus (E_\pm^{X \setminus Y,\mu}, f_\pm^{X \setminus Y,\mu}),$$

hence a morphism in \mathcal{E}_R^\pm given by the canonical morphism to the direct sum

$$(E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \rightarrow (E_\pm^{X,\mu}, f_\pm^{X,\mu}).$$

□

Proposition 6.9. *The functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \rightarrow \mathcal{E}_R^\pm$ of Lemma 6.8 induces a map of Γ -spaces and of the associated spectra $\Phi_\mu : K(\mathcal{V}) \rightarrow F_{\mathcal{E}_R^\pm}(\mathbb{S})$. The induced maps on the homotopy groups has the property that the composition*

$$(6.7) \quad K_0(\mathcal{V}) \xrightarrow{\Phi_\mu} K_0(\mathcal{E}_R^\pm) \xrightarrow{\delta} K_0(\mathcal{E}_R) \rightarrow K_0(\mathcal{E}_R)/K_0(R) = W_0(R)$$

with δ as in Lemma 6.4, is given by the zeta function $\zeta_\mu : K_0(\mathcal{V}) \rightarrow W_0(R)$.

Proof. The Γ -space associated to the assembler category $\mathcal{C}_\mathcal{V}$ is obtained in the following way, [70], [72]. One first associates to the assembler category $\mathcal{C}_\mathcal{V}$ another category $\mathcal{W}(\mathcal{C}_\mathcal{V})$ whose objects are finite collections $\{X_i\}_{i \in I}$ of non-initial objects of $\mathcal{C}_\mathcal{V}$ with morphisms $\varphi = (f, f_i) : \{X_i\}_{i \in I} \rightarrow \{X'_j\}_{j \in J}$ given by a map of the indexing sets $f : I \rightarrow J$ and morphisms $f_i : X_i \rightarrow X'_{f(i)}$ in $\mathcal{C}_\mathcal{V}$, such that, for every fixed $j \in J$ the collection $\{f_i : X_i \rightarrow X'_j : i \in f^{-1}(j)\}$ is a disjoint covering family of the assembler $\mathcal{C}_\mathcal{V}$. This means, in the case of the assembler $\mathcal{C}_\mathcal{V}$ underlying the Grothendieck ring of varieties, that the f_i are closed embeddings of the varieties X_i in the given X'_j with disjoint images. We first show that the functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \rightarrow \mathcal{E}_R^\pm$ of Lemma 6.8 extends to a functor (for which we still use the same notation) $\Phi_\mu : \mathcal{W}(\mathcal{C}_\mathcal{V}) \rightarrow \mathcal{E}_R^\pm$. We define $\Phi_\mu(\{X_i\}_{i \in I}) = \bigoplus_{i \in I} \Phi_\mu(X_i) = \bigoplus_{i \in I} (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu})$. Given a covering family $\{f_i : X_i \rightarrow X'_j : i \in f^{-1}(j)\}$ as above, each morphism $f_i : X_i \rightarrow X'_j$ determines a morphism $\Phi_\mu(f_i) : (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \rightarrow (E_\pm^{X'_j,\mu}, f_\pm^{X'_j,\mu})$ given by the canonical morphism

to the direct sum $(E_{\pm}^{X_i, \mu}, f_{\pm}^{X_i, \mu}) \rightarrow (E_{\pm}^{X_i, \mu}, f_{\pm}^{X_i, \mu}) \oplus (E_{\pm}^{X'_j \setminus X_i, \mu}, f_{\pm}^{X'_j \setminus X_i, \mu})$. This determines a morphism $\Phi_{\mu}(\varphi) : \bigoplus_{i \in I} (E_{\pm}^{X_i, \mu}, f_{\pm}^{X_i, \mu}) \rightarrow \bigoplus_{j \in J} (E_{\pm}^{X'_j, \mu}, f_{\pm}^{X'_j, \mu})$. We then show that the functor $\Phi_{\mu} : \mathcal{W}(\mathcal{C}_{\mathcal{V}}) \rightarrow \mathcal{E}_R^{\pm}$ constructed in this way determines a map of the associated Γ -spaces. The Γ -space associated to $\mathcal{W}(\mathcal{C}_{\mathcal{V}})$ is constructed in [70], [72] as the functor that assigns to a finite pointed set $S \in \Gamma^0$ the simplicial set given by the nerve $\mathcal{NW}(S \wedge \mathcal{C}_{\mathcal{V}})$, where the coproduct of assemblers $S \wedge \mathcal{C}_{\mathcal{V}} = \bigvee_{s \in S \setminus \{s_0\}} \mathcal{C}_{\mathcal{V}}$ has an initial object and a copy of the non-initial objects of $\mathcal{C}_{\mathcal{V}}$ for each point $s \in S \setminus \{s_0\}$ and morphisms induced by those of $\mathcal{C}_{\mathcal{V}}$. This means that we can regard objects of $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ as collections $\{X_{s,i}\}_{i \in I}$, for some $s \in S \setminus \{s_0\}$ and morphisms $\varphi_s = (f_s, f_{s,i}) : \{X_{s,i}\}_{i \in I} \rightarrow \{X'_{s,j}\}_{j \in J}$ as above. In order to obtain a map of Γ -spaces between $F_{\mathcal{V}} : S \mapsto \mathcal{NW}(S \wedge \mathcal{C}_{\mathcal{V}})$ and $F_{\mathcal{E}_R^{\pm}} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^{\pm}}(S)$, we construct a functor $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}}) \rightarrow \Sigma_{\mathcal{E}_R^{\pm}}(S)$ from the category $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ described above to the category of summing functors $\Sigma_{\mathcal{E}_R^{\pm}}(S)$. To an object $X_{S,I} := \{X_{s,i}\}_{i \in I}$ in $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ we associate a functor $\Phi_{X_{S,I}} : \mathcal{P}(S) \rightarrow \mathcal{E}_R^{\pm}$ that maps a subset $A_+ = \{s_0\} \sqcup A \in \mathcal{P}(X)$ to $\Phi_{X_{S,I}}(A_+) = \bigoplus_{a \in A} \Phi_{\mu}(\{X_{a,i}\}_{i \in I})$ where $\Phi_{\mu} : \mathcal{W}(\mathcal{C}_{\mathcal{V}}) \rightarrow \mathcal{E}_R^{\pm}$ is the functor constructed above. It is a summing functor since $\Phi_{X_{S,I}}(A_+ \cup B_+) = \Phi_{X_{S,I}}(A_+) \oplus \Phi_{X_{S,I}}(B_+)$ for $A_+ \cap B_+ = \{s_0\}$. This induces a map of simplicial sets $\mathcal{NW}(S \wedge \mathcal{C}_{\mathcal{V}}) \rightarrow \mathcal{N}\Sigma_{\mathcal{E}_R^{\pm}}(S)$ which determines a natural transformation of the functors $F_{\mathcal{V}} : S \mapsto \mathcal{NW}(S \wedge \mathcal{C}_{\mathcal{V}})$ and $F_{\mathcal{E}_R^{\pm}} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^{\pm}}(S)$. This map of Γ -spaces in turn determines a map of the associated spectra and an induced map of their homotopy groups. It remains to check that the induced map at the level of π_0 agrees with the expected map of Grothendieck rings $K_0(\mathcal{V}) \rightarrow K_0(\mathcal{E}_R^{\pm})$, hence with the zeta function when further mapped to $K_0(\mathcal{E}_R)$ and to the quotient $K_0(\mathcal{E}_R)/K_0(R)$. This is the case since by construction the induced map $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V}) \rightarrow K_0(\mathcal{E}_R^{\pm}) = \pi_0 F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$ is given by the assignment $[X] \mapsto [E_{\pm}^{X, \mu}, f_{\pm}^{X, \mu}]$. \square

Corollary 6.10. *The map of Grothendieck rings given by the composition (6.7) also lifts to a map of spectra.*

Proof. It is possible to realize the map $\delta : K_0(\mathcal{E}_R^{\pm}) \rightarrow K_0(\mathcal{E}_R)$ of Lemma 6.4 at the level of spectra. The K -theory spectrum of an abelian category \mathcal{A} is weakly equivalent to the K -theory spectrum of the category of bounded chain complexes over \mathcal{A} . In fact, this holds more generally for \mathcal{A} an exact category closed under kernels. Thus, in the case of the category \mathcal{E}_R , there is a weak equivalence $K(\text{Ch}^b(\mathcal{E}_R)) \xrightarrow{\sim} K(\mathcal{E}_R)$ which descends on the level π_0 to the map $K_0(\text{Ch}^b(\mathcal{E}_R)) \xrightarrow{\sim} K_0(\mathcal{E}_R)$ given by $[E^{\cdot}, f^{\cdot}] \mapsto \sum_k (-1)^k [E^k, f^k]$. To an object (E^{\pm}, f^{\pm}) of \mathcal{E}_R^{\pm} we can assign a chain complex in $\text{Ch}^b(\mathcal{E}_R)$ of the form $0 \rightarrow (E^-, f^-) \xrightarrow{0} (E^+, f^+) \rightarrow 0$, where (E^+, f^+) sits in degree 0. This descends on the level of K -theory to a map $K(\mathcal{E}_R^{\pm}) \rightarrow K(\text{Ch}^b(\mathcal{E}_R))$, which at the level of π_0 gives the map $[E^{\pm}, f^{\pm}] \mapsto [E^+, f^+] - [E^-, f^-]$. The functor $\mathcal{E}_R^{\pm} \rightarrow \text{Ch}^b(\mathcal{E}_R)$ used here does not respect tensor products, although the induced map $\delta : K_0(\mathcal{E}_R^{\pm}) \rightarrow K_0(\mathcal{E}_R)$ at the level of K_0 is compatible with products. Thus, the composition (6.7) can also be lifted at the level of spectra. \square

It should be noted that the construction of a derived motivic zeta function outlined above is not the first to appear in the literature. In [21], the authors describe a derived motivic measure $\zeta : K(\mathcal{V}_k) \rightarrow K(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell))$ from the Grothendieck spectrum of varieties to the K -theory spectrum of the category of continuous ℓ -adic Galois representations. This map corresponds to the assignment $X \mapsto H_{\text{et},c}^*(X \times_k k^s, \mathbb{Z}_\ell)$. In particular, they show that when $k = \mathbb{F}_q$ for ℓ coprime to q , on the level of π_0 , ζ corresponds to the Hasse-Weil zeta function. They then use ζ to prove that $K_1(\mathcal{V}_{\mathbb{F}_q})$ is not only nontrivial, but contains interesting algebro-geometric data.

Essentially, the approach in [21] was to start with a Weil Cohomology theory (in this case, ℓ -adic cohomology) and then to construct a derived motivic measure realizing on the level of K -theory the assignment to a variety X of its corresponding cohomology groups. The methods used in the case of ℓ -adic cohomology may not immediately generalize to other Weil cohomology theories. This method has yielded deep insight into the world of algebraic geometry. Our approach here, in contrast, is to take an interesting class of motivic measures, namely Kapranov motivic zeta functions (exponentiable motivic measures, [46], [62], [63]), and to determine reasonable conditions under which such a motivic measure can be derived directly. This method still needs to be studied further to yield additional insights into what it captures about the geometry of varieties.

6.4. Bost–Connes type systems via motivic measures. The lifting of the integral Bost–Connes algebra to various Grothendieck rings, their assembler categories, and the associated spectra, that we discussed in [54] and in the earlier sections of this paper, can be viewed as an instance of a more general kind of operation. As discussed in [25], there is a close relation between the endomorphisms σ_n and the maps $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the operation of Frobenius and Verschiebung in the Witt ring. Thus, we can formulate a more general form of the question investigated above, of lifting of the integral Bost–Connes algebra to a Grothendieck ring through an Euler characteristic map, in terms of lifting the Frobenius and Verschiebung operations of a Witt ring to a Grothendieck ring through the zeta function ζ_μ of an exponentiable motivic measure. A prototype example of this more general setting is provided by the Hasse–Weil zeta function $Z : K_0(\mathcal{V}_{\mathbb{F}_q}) \rightarrow W(\mathbb{Z})$, which has the properties that the action of the Frobenius F_n on the Witt ring $W(\mathbb{Z})$ corresponds to passing to a field extension, $F_n Z(X_{\mathbb{F}_q}, t) = Z(X_{\mathbb{F}_{q^n}}, t)$ and the action of the Verschiebung V_n on the Witt ring $W(\mathbb{Z})$ is related to the Weil restriction of scalars from \mathbb{F}_{q^n} to \mathbb{F}_q (see [62] for a precise statement).

Recall that, if one denotes by $[a]$ the elements $[a] = (1 - at)^{-1}$ in the Witt ring $W(R)$, for $a \in R$, then the Frobenius ring homomorphisms $F_n : W(R) \rightarrow W(R)$ of the Witt ring are determined by $F_n([a]) = [a^n]$ and the Verschiebung group homomorphisms $V_n : W(R) \rightarrow W(R)$ are defined on an arbitrary $P(t) \in W(R)$ as $F_n : P(t) \mapsto P(t^n)$. These operations satisfy an analog of the Bost–Connes relations

$$(6.8) \quad F_n \circ F_m = F_{nm}, \quad V_n \circ V_m = V_{nm}, \quad F_n \circ V_n = n \cdot \text{id}, \quad F_n \circ V_m = V_m F_n \text{ if } (n, m) = 1.$$

These correspond, respectively, to the semigroup structure of the σ_n and $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the relations $\sigma_n \circ \tilde{\rho}_n = n \cdot \text{id}$, while the last relation is determined in the Bost–Connes case by the commutation of the generators $\tilde{\mu}_n$ and μ_m^* for $(n, m) = 1$.

Definition 6.11. A factorizable motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$, in the sense of Definition 6.7, is of *Bost–Connes type* if there is a lift to $K_0(\mathcal{V})$ of the Frobenius F_n and Verschiebung V_n of the Witt ring $W(R)$ to $K_0(\mathcal{V})$ such that the diagrams commute

$$\begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \\ \downarrow \sigma_n & & \downarrow F_n \\ K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \end{array} \quad \begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \\ \downarrow \tilde{\rho}_n & & \downarrow V_n \\ K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \end{array}$$

Such a motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$ is of *homotopic Bost–Connes type* if the maps σ_n and $\tilde{\rho}_n$ in the diagrams above also lift to endofunctors of the assembler category $\mathcal{C}_{\mathcal{V}}$ of the Grothendieck ring $K_0(\mathcal{V})$ with the endofunctors σ_n compatible with the monoidal structure.

Definition 6.12. The Frobenius and Verschiebung on the category \mathcal{E}_R^\pm are defined as the endofunctors $F_n(E, f) = (E, f^n)$ and $V_n(E_\pm, f_\pm) = (E_\pm^{\oplus n}, V_n(f_\pm))$ with $V_n(f)$ defined by

$$(6.9) \quad V_n : (E, f) \mapsto (E^{\oplus n}, V_n(f)), \quad V_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & f \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

It is worth noting that the endofunctors of Definition 6.12 are akin to those used in the definitions of topological cyclic and topological restriction homology, [41].

Lemma 6.13. *The Frobenius and Verschiebung F_n and V_n of Definition 6.12 are endofunctors of the category \mathcal{E}_R^\pm with the property that the maps they induce on $W_0(R) = K_0(\mathcal{E}_R)/K_0(R)$ agree with the restrictions to $W_0(R) \subset W(R)$ of the Frobenius and Verschiebung maps. These endofunctors determine natural transformations (still denoted F_n and V_n) of the Γ -space $F_{\mathcal{E}_R^\pm} : \Gamma^0 \rightarrow \Delta_*$.*

Proof. The homomorphism $K_0(\mathcal{E}_R) \rightarrow W_0(R)$ given by

$$(E, f) \mapsto L(E, f) = \det(1 - tM(f))^{-1}$$

sends the pair (R, f_a) with f_a acting on R as multiplication by $a \in R$ to the element $[a] = (1 - at)^{-1}$ in the Witt ring. The action of the Frobenius $F_n([a]) = [a^n]$ is induced from the Frobenius $F_n(E, f) = (E, f^n)$ which is an endofunctor of \mathcal{E}_R . This extends to a compatible endofunctor of \mathcal{E}_R^\pm by $F_n(E_\pm, f_\pm) = (E_\pm, f_\pm^n)$. Similarly, the Verschiebung map that sends $\det(1 - tM(f))^{-1} \mapsto \det(1 - t^n M(f))^{-1}$ is induced

from the Verschiebung on \mathcal{E}_R given by (6.9), since we have $L(E^{\oplus n}, V_n(f)) = \det(1 - t^n M(f))^{-1}$, with compatible endofunctors $V_n(E_{\pm}, f_{\pm}) = (E_{\pm}^{\oplus n}, V_n(f_{\pm}))$ on \mathcal{E}_R^{\pm} . The Frobenius and Verschiebung on \mathcal{E}_R^{\pm} induce natural transformations of the Γ -space $F_{\mathcal{E}_R^{\pm}} : \Gamma^0 \rightarrow \Delta_*$ by composition of the summing functors $\Phi : \mathcal{P}(X) \rightarrow \mathcal{E}_R^{\pm}$ in $\Sigma_{\mathcal{E}_R^{\pm}}(X)$ with the endofunctors F_n and V_n of \mathcal{E}_R^{\pm} . \square

Proposition 6.14. *Let $\mu : K_0(\mathcal{V}) \rightarrow R$ be a factorizable motivic measure, as in Definition 6.7, that is of homotopical Bost–Connes type. Then the endofunctors σ_n and $\tilde{\rho}_n$ of the assembler category $\mathcal{C}_{\mathcal{V}}$ determine natural transformations (still denoted by σ_n and $\tilde{\rho}_n$) of the associated Γ -space $F_{\mathcal{V}} : \Gamma^0 \rightarrow \Delta_*$ that fit in the commutative diagrams*

$$\begin{array}{ccc} F_{\mathcal{V}} & \xrightarrow{\Phi_{\mu}} & F_{\mathcal{E}_R^{\pm}} \\ \downarrow \sigma_n & & \downarrow F_n \\ F_{\mathcal{V}} & \xrightarrow{\Phi_{\mu}} & F_{\mathcal{E}_R^{\pm}} \end{array} \quad \begin{array}{ccc} F_{\mathcal{V}} & \xrightarrow{\Phi_{\mu}} & F_{\mathcal{E}_R^{\pm}} \\ \downarrow \tilde{\rho}_n & & \downarrow V_n \\ F_{\mathcal{V}} & \xrightarrow{\Phi_{\mu}} & F_{\mathcal{E}_R^{\pm}} \end{array}$$

where $\Phi_{\mu} : F_{\mathcal{V}} \rightarrow F_{\mathcal{E}_R^{\pm}}$ is the natural transformation of Γ -spaces of (6.9) and F_n and V_n are the natural transformations of Lemma 6.13.

Proof. The natural transformation Φ_{μ} is determined as in Proposition 6.9 by the functor $\Phi_{\mu} : \mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{E}_R^{\pm}$ that assigns $\Phi_{\mu} : X \mapsto (E_{\pm}^X, f_{\pm}^X)$ constructed as in Lemma 6.8. Suppose we have endofunctors σ_n and $\tilde{\rho}_n$ of the assembler category $\mathcal{C}_{\mathcal{V}}$ that induce maps σ_n and $\tilde{\rho}_n$ on $K_0(\mathcal{V})$ that lift the Frobenius and Verschiebung maps of $W(R)$ through the zeta function $\zeta_{\mu} : K_0(\mathcal{V}) \rightarrow W(R)$. This means that $\zeta_{\mu}(\sigma_n(X), t) = F_n \zeta_{\mu}(X, t)$ and $\zeta_{\mu}(\tilde{\rho}_n(X), t) = V_n \zeta_{\mu}(X, t) = \zeta_{\mu}(X, t^n)$. By Lemma 6.13, we have $F_n \zeta_{\mu}(X, t) = L(F_n(E_{\pm}^X, f_{\pm}^X)) = L(E_{\pm}^X, (f_{\pm}^X)^n)$ and $V_n \zeta_{\mu}(X, t) = L(V_n(E_{\pm}^X, f_{\pm}^X)) = L((E_{\pm}^X)^{\oplus n}, V_n(f_{\pm}^X))$. This shows the compatibilities of the natural transformations in the diagrams above. \square

6.5. Spectra and spectra. We apply a construction similar to the one discussed in the previous subsections to the case of the map $(X, f) \mapsto \sum_{\lambda \in \text{Spec}(f_*)} m_{\lambda} \lambda$ that assigns to a variety over \mathbb{C} with a quasi-unipotent map the spectrum of the induced map f_* in homology, seen as an element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, as in §6 of [54].

In this section the term spectrum will appear both in its homotopy theoretic sense and in its operator sense. Indeed, we consider here a lift to the level of spectra (in the homotopy theoretic sense) of the construction described in §6 of [54], based on the spectrum (in the operator sense) Euler characteristic.

We consider here a setting as in [34], [39], where (X, f) is a pair of a variety over \mathbb{C} and an endomorphism $f : X \rightarrow X$ such that the induced map f_* on $H_*(X, \mathbb{Z})$ has spectrum consisting of roots of unity. As discussed in [54] and in a related form in [34] the spectrum determines a ring homomorphism (an Euler characteristic)

$$(6.10) \quad \sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

where $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ denotes the Grothendieck ring of pairs (X, f) with the operations defined by the disjoint union and the Cartesian product. It is shown in [54] that one can lift the operations σ_n and $\tilde{\rho}_n$ of the integral Bost–Connes algebra from $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the “spectral Euler characteristic” (6.10), and that the operations can further be lifted from $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to a (homotopy theoretic) spectrum with π_0 equal to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the assembler category construction of [70].

In the next sub section we discuss how to lift the right hand side of (6.10), namely the original Bost–Connes algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the operations σ_n and $\tilde{\rho}_n$ to the level of a homotopy theoretic spectrum, so that the spectral Euler characteristic (6.10) becomes induced by a map of spectra.

6.5.1. *Bost–Connes Tannakian categorification and lifting of the spectral Euler characteristic.* To construct a categorification of the map (6.10) compatible with the Bost–Connes structure, we use the lift of the left-hand-side of (6.10) to an assembler category, as in Proposition 6.6 of [54], while for the right-hand-side of (6.10) we use the categorification of Bost–Connes system constructed in [57].

We begin by recalling the categorification of the Bost–Connes algebra of [57]. Let $\text{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ be the category of pairs $(W, \bigoplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{W}_r)$ with W a finite dimensional \mathbb{Q} -vector space and $\bigoplus_r \bar{W}_r$ a \mathbb{Q}/\mathbb{Z} -graded vector space with $\bar{W} = W \otimes \bar{\mathbb{Q}}$. This is a neutral Tannakian category with fiber functor the forgetful functor $\omega : \text{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q}) \rightarrow \text{Vect}(\mathbb{Q})$ and with $\text{Aut}^{\otimes}(\omega) = \text{Spec}(\bar{\mathbb{Q}}[\mathbb{Q}/\mathbb{Z}]^G)$ and $G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, see Theorem 3.2 of [57]. The category $\text{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ is endowed with additive symmetric monoidal functors $\sigma_n(W) = W$ and $\overline{\sigma_n(W)}_r = \bigoplus_{r' : \sigma_n(r')=r} \bar{W}_{r'}$ if r is in the range of σ_n and zero otherwise and additive functors $\tilde{\rho}_n(W) = W^{\oplus n}$ and $\overline{\tilde{\rho}_n(W)}_r = \bar{W}_{\sigma_n(r)}$ satisfying $\sigma_n \circ \tilde{\rho}_n = n \cdot \text{id}$ that induce the Bost–Connes maps on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.

As shown in Theorem 3.18 of [57], this category can be equivalently described as a category of automorphisms $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ with objects pairs (W, ϕ) of a \mathbb{Q} -vector space V and a G -equivariant diagonalizable automorphism of \bar{W} with eigenvalues that are roots of unity (seen as elements in \mathbb{Q}/\mathbb{Z}). There is an equivalence of categories between $\text{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ and $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ under which the functors σ_n and $\tilde{\rho}_n$ correspond, respectively, to the Frobenius and Verschiebung on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$, given by

$$(6.11) \quad F_n : (W, \phi) \mapsto (W, \phi^n), \quad V_n : (W, \phi) \mapsto (W^{\oplus n}, V_n(\phi)),$$

with

$$(6.12) \quad V_n(\phi) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The equivalence is realized by mapping $(W, \phi) \mapsto (W, \bigoplus_r \bar{W}_r)$ where \bar{W}_r are the eigenspaces of ϕ with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$.

Remark 6.15. Conceptually, the first description of the categorification in terms of the Tannakian category $\text{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ is closer to the integral Bost–Connes algebra as introduced in [27], while its equivalent description in terms of $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ is closer to the reinterpretation of the Bost–Connes algebra in terms of Frobenius and Verschiebung operators, as in [25]. Since we have introduced here the Bost–Connes algebra in the form of [27], we are recalling both of these descriptions of the categorification, even though in the following we will be using only the one in terms of $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$.

Proposition 6.16. *Let $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ be the assembler category underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$, as in Proposition 6.6 of [54]. The assignment $\Phi(X, f) = (H_*(X, \mathbb{Q}), \bigoplus_r E_r(f_*))$, where $E_r(f_*)$ is the eigenspace with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$, determines a functor $\Phi : \mathcal{C}_{\mathbb{C}}^{\mathbb{Z}} \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ that lifts the Frobenius and Verschiebung functors on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ to the endofunctors σ_n and $\tilde{\rho}_n$ of $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ implementing the Bost–Connes structure.*

Proof. We can construct the functor from the assembler category $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ of §6 of [54], underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ by following along the lines of Lemma 6.8 and Proposition 6.9, where we assign $\Phi(X, f) = (H_*(X, \mathbb{Q}), \bigoplus_r E_r(f_*))$ where $E_r(f_*)$ is the eigenspace with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$. The Bost–Connes algebra then lifts to the Frobenius and Verschiebung functors on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ and the latter lift to geometric Frobenius and Verschiebung operations on the pairs (X, f) mapping to (X, f^n) and to $(X \times Z_n, \Phi_n(f))$. \square

This point of view, that replaces the Bost–Connes algebra with its categorification in terms of the Tannakian category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ as in [57] will also be useful in Section 7, where we reformulate our categorical setting, by passing from Grothendieck rings, assemblers and spectra, to Tannakian categories of Nori motives, and we compare in Lemma 7.4 and Theorem 7.7 the categorification of the Bost–Connes algebra obtained via Nori motives with the one of [57] recalled here.

7. BOST–CONNES SYSTEMS IN CATEGORIES OF NORI MOTIVES

We introduce in this section a motivic framework, with Bost–Connes type systems that on Tannakian categories of motives. The main result in this part of the paper will be Theorem 7.7, showing the existence of a fiber functor from the Tannakian category of Nori motives with good effectively finite $\hat{\mathbb{Z}}$ -action to the Tannakian category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ that lifts the Bost–Connes system given by Frobenius and Verschiebung on the target category to a Bost–Connes system on Nori motives. Proposition 7.9 then extends this Bost–Connes structure to the relative case of motivic sheaves.

This is a natural generalization of the approach to Grothendieck rings via assemblers, which can be extended in an interesting way to the domain of motives, namely, Nori motives.

Roughly speaking, the theory of Nori motives starts with lifting the relations

$$(7.1) \quad [f : X \rightarrow S] = [f|_Y : Y \rightarrow S] + [f|_{X \setminus Y} : X \setminus Y \rightarrow S]$$

of (relative) Grothendieck rings $K_0(\mathcal{V}_S)$ to the level of “diagrams”, which intuitively can be imagined as “categories without multiplication of morphisms.”

7.1. Nori diagrams. More precisely, (cf. Definition 7.1.1 of [43], p. 137), we have the following definitions.

Definition 7.1. A diagram (also called a quiver) D is a family consisting of a set of vertices $V(D)$ and a set of oriented edges, $E(D)$. Each edge e either connects two different vertices, going, say, from a vertex $\partial_{\text{out}}e = v_1$ to a vertex $\partial_{\text{in}}e = v_2$, or else is “an identity”, starting and ending with one and the same vertex v . We will consider only diagrams with one identity for each vertex.

Diagrams can be considered as objects of a category, with obvious morphisms.

Definition 7.2. Each small category \mathcal{C} can be considered as a diagram $D(\mathcal{C})$, with $V(D(\mathcal{C})) = \text{Ob } \mathcal{C}$, $E(D(\mathcal{C})) = \text{Mor } \mathcal{C}$, so that each morphism $X \rightarrow Y$ “is” an oriented edge from X to Y . More generally, a *representation* T of a diagram D in a (small) category \mathcal{C} is a morphism of directed graphs $T : D \rightarrow D(\mathcal{C})$.

Notice that a considerably more general treatment of graphs with markings, including diagrams etc. in the operadic environment, can be found in [52]. We do not use it here, although it might be highly relevant.

7.2. From geometric diagrams to Nori motives. We recall the main idea in the construction of Nori motives from geometric diagrams. For more details, see [43], pp. 140–144.

- (1) *Start* with the following data:
 - a) a diagram D ;
 - b) a noetherian commutative ring with unit R and the category of finitely generated R -modules $R\text{-Mod}$;
 - c) a representation T of D in $R\text{-Mod}$, in the sense of Definition 7.2.
- (2) *Produce* from them the category $C(D, T)$ defined in the following way:
 - d1) If D is finite, then $C(D, T)$ is the category of finitely generated R -modules equipped with an R -linear action of $\text{End}(T)$.
 - d2) If D is infinite, first consider its all finite subdiagrams F .

- d3) For each F construct $C(F, T|_F)$ as in d1). Then apply the following limiting procedure:

$$C(D, T) := \operatorname{colim}_{F \subseteq D \text{ finite}} C(F, T|_F)$$

Thus, the category $C(D, T)$ has the following structure:

- Objects of $C(D, T)$ will be all objects of the categories $C(F, T|_F)$. If $F \subset F'$, then each object X_F of $C(F, T|_F)$ can be canonically extended to an object of $C(F', T|_{F'})$.
 - Morphisms from X to Y in $C(D, T)$ will be defined as colimits over F of morphisms from X_F to Y_F with respect to these extensions.
- d4) The fact that $C(D, T)$ has a functor to $R\text{-Mod}$ follows directly from the definition and the finite case.

- (3) *The result is called the diagram category $C(D, T)$.*

It is an R -linear abelian category which is endowed with R -linear faithful exact forgetful functor

$$f_T : C(D, T) \rightarrow R\text{-Mod}.$$

7.2.1. *Universal diagram category.* The following results explain why abstract diagram categories play a central role in the formalism of Nori motives: they formalise the Grothendieck intuition of motives as objects of the universal cohomology theory.

Theorem 7.3. [43]

- (i) *Any representation $T : D \rightarrow R\text{-Mod}$ can be presented as post-composition of the forgetful functor f_T with an appropriate representation $\tilde{T} : D \rightarrow C(D, T)$:*

$$T = f_T \circ \tilde{T}.$$

with the following universal property:

Given any R -linear abelian category A with a representation $F : D \rightarrow A$ and R -linear faithful exact functor $f : A \rightarrow R\text{-Mod}$ with $T = f \circ F$, it factorizes through a faithful exact functor $L(F) : C(D, T) \rightarrow A$ compatibly with the decomposition

$$T = f_T \circ \tilde{T}.$$

- (ii) *The functor $L(F)$ is unique up to unique isomorphism of exact additive functors.*

For proofs, cf. [43], pp. 140–141 and p. 167.

7.2.2. *Nori geometric diagrams.* If we start not with an abstract category but with a “geometric” category \mathcal{C} (in the sense that its objects are spaces/varieties/schemes, possibly endowed with additional structures), in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \setminus Y \rightarrow X$, we can define the Nori diagram of *effective pairs* $D(\mathcal{C})$ in the following way (see [43], pp. 207–208).

- a) One vertex of $D(\mathcal{C})$ is a triple (X, Y, i) where $Y \hookrightarrow X$ is a closed embedding, and i is an integer.
- b) Besides obvious identities, there are edges of two types.
 - b1) Let (X, Y) and (X', Y') be two pairs of closed embeddings. Every morphism $f : X \rightarrow X'$ such that $f(Y) \subset Y'$ produces functoriality edges f^* (or rather (f^*, i)) going from (X', Y', i) to (X, Y, i) .
 - b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings. Then it defines coboundary edges ∂ from (Y, Z, i) to $(X, Y, i + 1)$.

7.2.3. *(Co)homological representatons of Nori geometric diagrams.* If we start not just from the initial category of spaces \mathcal{C} , but rather from a pair (\mathcal{C}, H) where H is a cohomology theory, then assuming reasonable properties of this pair, we can define the respective representation T_H of $D(\mathcal{C})$ that we will call a *(co)homological representation of $D(\mathcal{C})$* .

For a survey of such pairs (\mathcal{C}, H) that were studied in the context of Grothendieck’s motives, see [43], pp. 31–133. The relevant cohomology theories include, in particular, singular cohomology, and algebraic and holomorphic de Rham cohomologies.

Below we will consider the basic example of cohomological representations of Nori diagrams that leads to Nori motives.

7.2.4. *Effective Nori motives.* We follow [43], pp. 207–208. Take as a category \mathcal{C} , the starting object in the definition of Nori geometric diagrams above, the category \mathcal{V}_k of varieties X defined over a subfield $k \subset \mathbb{C}$.

We can then define the Nori diagram $D(\mathcal{C})$ as above. This diagram will be denoted Pairs^{eff} from now on,

$$\text{Pairs}^{eff} = D(\mathcal{V}_k).$$

The category of effective mixed Nori motives is the diagram category $\mathcal{C}(\text{Pairs}^{eff}, H^*)$ where $H^i(X, \mathbb{Z})$ is the respective singular cohomology of the analytic space X^{an} (cf. [43], pp. 31–34 and further on).

It turns out (see [43], Proposition 9.1.2. p. 208) that the map

$$H^* : \text{Pairs}^{eff} \rightarrow \mathbb{Z}\text{-Mod}$$

sending (X, Y, i) to the relative singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, naturally extends to a representation of the respective Nori diagram in the category of finitely generated abelian groups $\mathbb{Z}\text{-Mod}$.

7.3. Category of equivariant Nori motives. We now introduce the specific category of Nori motives that we will be using for the construction of the associated Bost–Connes system.

Let $D(\mathcal{V})$ the Nori geometric diagrams associated to the category \mathcal{V} of varieties over \mathbb{Q} , constructed as described in §7.2.

As in [54] and in § 2 of this paper, we consider here the category $\mathcal{V}^{\hat{\mathbb{Z}}}$ of varieties X with a good effectively finite action of $\hat{\mathbb{Z}}$. We can view $\mathcal{V}^{\hat{\mathbb{Z}}}$ as an enhancement $\hat{\mathcal{V}}$ of the category \mathcal{V} , in the sense described in §3.3.

Define the Nori diagram of *effective pairs* $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ as we recalled earlier in §7.2:

- a) One vertex of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ is a triple $((X, \alpha_X), (Y, \alpha_Y), i)$, of varieties X and Y with good effectively finite $\hat{\mathbb{Z}}$ actions, $\alpha_X : \hat{\mathbb{Z}} \times X \rightarrow X$ and $\alpha_Y : \hat{\mathbb{Z}} \times Y \rightarrow Y$, and an integer i , together with a closed embedding $j : Y \hookrightarrow X$ that is equivariant with respect to the $\hat{\mathbb{Z}}$ actions. For brevity, we will denote such a triple (\hat{X}, \hat{Y}, i) and call it a closed embedding in the enhancement $\hat{\mathcal{V}}$.
- b) Identity edges, functoriality edges, and coboundary edges are obvious enhancements of the respective edges defined in §7.2, with the requirement that all these maps are $\hat{\mathbb{Z}}$ -equivariant.
- b1) Let (\hat{X}, \hat{Y}) and (\hat{X}', \hat{Y}') be two pairs of closed embeddings in $\hat{\mathcal{V}}$. Every morphism $f : X \rightarrow X'$ such that $f(Y) \subset Y'$ and $f \circ \alpha_X = \alpha_{X'} \circ f$ produces functoriality edges f^* (or rather (f^*, i)) going from $((X', \alpha_{X'}), (Y', \alpha_{Y'}), i)$ to (X, Y, i) .
- b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings compatible with enhancements (equivariant with respect to the $\hat{\mathbb{Z}}$ -actions). Then it defines coboundary edges ∂

$$((Y, \alpha_Y), (Z, \alpha_Z), i) \rightarrow ((X, \alpha_X), (Y, \alpha_Y), i + 1).$$

We have thus defined the Nori geometric diagram of enhanced effective pairs, which we denote equivalently by $D(\hat{\mathcal{V}})$ or $D(\mathcal{V}^{\hat{\mathbb{Z}}})$.

Notice that forgetting in this diagram all enhancements, we obtain the map $D(\hat{\mathcal{V}}) \rightarrow D(\mathcal{V})$ which is *injective* both on vertices and edges.

7.4. Bost–Connes system on Nori motives. We now construct a Bost–Connes system on a category of Nori motives obtained from the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ described above, which lifts to the level of motives the categorification of the Bost–Connes algebra constructed in [57].

As we recalled in §6.5.1, we can describe the categorification of the Bost–Connes algebra of [57] in terms of the Tannakian category $\text{Vec}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ with suitable functors σ_n and $\tilde{\rho}_n$ constructed as in Theorem 3.7 of [57] or in terms of an equivalent Tannakian category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ endowed with Frobenius and Verschiebung functors. We are going to use here the second description.

Lemma 7.4. *The assignment $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines a representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ constructed above.*

Proof. As discussed in the previous subsection, we view elements $((X, \alpha_X), (Y, \alpha_Y), i)$ of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ in terms of an enhancement $\hat{\mathcal{V}}$ of the category \mathcal{V} defined as in §3.3, by choosing a primitive root of unity that generates the cyclic group $\mathbb{Z}/N\mathbb{Z}$, so that the actions α_X and α_Y are determined by self maps v_X and v_Y as in §3.3. We identify the element above with $((X, v_X), (Y, v_Y), i)$, which we also denoted by (\hat{X}, \hat{Y}, i) in the previous subsection. Since the embedding $Y \hookrightarrow X$ is $\hat{\mathbb{Z}}$ -equivariant, the map v_Y is the restriction to Y of the map v_X under this embedding. We denote by ϕ^i the induced map on the cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$. The eigenspaces of ϕ^i are the subspaces of the decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ according to characters of $\hat{\mathbb{Z}}$, that is, elements in $\text{Hom}(\hat{\mathbb{Z}}, \mathbb{C}^*) = \nu^* \simeq \mathbb{Q}/\mathbb{Z}$. Thus, we obtain an object $(H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}), \phi^i)$ in the category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$. Edges in the diagram are $\hat{\mathbb{Z}}$ -equivariant maps so they induce morphisms between the corresponding objects in the category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$. \square

One can also see in a similar way that the fiber functor $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines an object in the category $\text{Vec}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$. Indeed, the pair (X, Y) with $Y \subset X$ is endowed with compatible good effectively finite $\hat{\mathbb{Z}}$ -actions α_X and α_Y , hence the singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ carries a resulting $\hat{\mathbb{Z}}$ -representation. Thus, the vector space $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ can be decomposed into eigenspaces of this representations according to characters $\chi \in \text{Hom}(\hat{\mathbb{Z}}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$. Thus, we obtain a decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}) = \bigoplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{V}_r$ as a \mathbb{Q}/\mathbb{Z} -graded vector space. We choose to work with the category $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ because the Bost–Connes structure is more directly expressed in terms of Frobenius and Verschiebung, which will make the lifting of this structure to the resulting category of Nori motives more immediately transparent, as we discuss below.

The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ replaces, at this motivic level, our previous use in [54] of the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ (see [50]) as a way to lift the Bost–Connes algebra. We proceed in the following way to obtain the Bost–Connes structure in this setting.

Definition 7.5. Let D be a diagram, endowed with a representation $T : D \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$, and let $\mathcal{C}(D, T)$ be the associated diagram category, obtained as in §7.2, with the induced functor $\tilde{T} : \mathcal{C}(D, T) \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$. We say that the functor \tilde{T} intertwines the Bost–Connes structure, if there are endofunctors σ_n and $\tilde{\rho}_n$ of $\mathcal{C}(D, T)$ (where the σ_n but not the $\tilde{\rho}_n$ are compatible with the tensor product structure) such

that the following diagrams commute,

$$\begin{array}{ccc}
\mathcal{C}(D, T) & \xrightarrow{\tilde{T}} & \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q}) \\
\downarrow \sigma_n & & \downarrow F_n \\
\mathcal{C}(D, T) & \xrightarrow{\tilde{T}} & \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})
\end{array}
\quad
\begin{array}{ccc}
\mathcal{C}(D, T) & \xrightarrow{\tilde{T}} & \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q}) \\
\tilde{\rho}_n \uparrow & & \uparrow V_n \\
\mathcal{C}(D, T) & \xrightarrow{\tilde{T}} & \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})
\end{array}$$

where on the right-hand-side of the diagrams, the F_n and V_n are the Frobenius and Verschiebung on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$, defined as in (6.11) and (6.12).

Definition 7.6. For $((X, \alpha_X), (Y, \alpha_Y), i)$ in the category $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ of Nori motives associated to the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ define

$$(7.2) \quad \sigma_n : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto ((X, \alpha_X \circ \sigma_n), (Y, \alpha_Y \circ \sigma_n), i)$$

$$(7.3) \quad \tilde{\rho}_n : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto (X \times Z_n, \Phi_n(\alpha_X), (Y \times Z_n, \Phi_n(\alpha_Y)), i),$$

where $Z_n = \text{Spec}(\mathbb{Q}^n)$ and $\Phi_n(\alpha)$ is the geometric Verschiebung defined as in §2.5.

Theorem 7.7. *The σ_n and $\tilde{\rho}_n$ of (7.2) and (7.3) determine a Bost–Connes system on the category $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ of Nori motives associated to the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ constructed above has the property that the induced functor*

$$\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T) \rightarrow \text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$$

intertwines the endofunctors σ_n and $\tilde{\rho}_n$ of the Bost–Connes system on $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ and the Frobenius F_n and Verschiebung V_n of the Bost–Connes structure on $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$.

Proof. Consider the mappings σ_n and $\tilde{\rho}_n$ defined in (7.2) and (7.3). The effect of the transformation σ_n , when written in terms of the data $((X, v_X), (Y, v_Y), i)$ is to send $v_X \mapsto v_X^n$ and $v_Y \mapsto v_Y^n$, hence it induces the Frobenius map F_n acting on $(H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}), \phi^i)$ in $\text{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$. Similarly, we have $T(X \times Z_n, \Phi_n(\alpha_X), (Y \times Z_n, \Phi_n(\alpha_Y)), i) = H^i(X \times Z_n, Y \times Z_n, \mathbb{Q})$ where by the relative version of the Künneth formula we have $(H^i(X(\mathbb{C}) \times Z_n(\mathbb{C}), Y(\mathbb{C}) \times Z_n(\mathbb{C}), \mathbb{Q}) \simeq H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})^{\oplus n}$ with the induced map $V_n(\phi^i)$. The maps σ_n and $\tilde{\rho}_n$ defined as above determine self maps of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. By Lemma 7.2.6 of [43] given a map $F : D_1 \rightarrow D_2$ of diagrams and a representation $T : D_2 \rightarrow R\text{-Mod}$, there is an R -linear exact functor

$\mathcal{F} : C(D_1, T \circ F) \rightarrow C(D_2, T)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 D_1 & \xrightarrow{F} & D_2 \\
 \downarrow & & \downarrow \\
 C(D_1, T \circ F) & \xrightarrow{\mathcal{F}} & C(D_2, T) \\
 & \searrow & \swarrow \\
 & R\text{-Mod} &
 \end{array}$$

We still denote by σ_n and $\tilde{\rho}_n$ the endofunctors induced in this way on $C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$. To check the compatibility of the σ_n functors with the monoidal structure, we use the fact that for Nori motives the product structure is constructed using “good pairs” (see §9.2.1 of [43]), that is, elements (X, Y, i) with the property that $H^j(X, Y, \mathbb{Z}) = 0$ for $j \neq i$. For such elements the product is given by $(X, Y, i) \times (X', Y', j) = (X \times X', X \times Y' \cup Y \times X', i + j)$. The diagram category $C(\text{Good}^{eff}, T)$ obtained by replacing effective pairs Pairs^{eff} with good effective pairs Good^{eff} is equivalent to $C(\text{Pairs}^{eff}, T)$ (Theorem 9.2.22 of [43]), hence the tensor structure defined in this way on $C(\text{Good}^{eff}, T)$ determines the tensor structure of $C(\text{Pairs}^{eff}, T)$ and on the resulting category of Nori motives, see §9.3 of [43]. Thus, to check the compatibility of the functors σ_n with the tensor structure it suffices to see that on a product of good pairs, where indeed we have

$$\begin{aligned}
 & \sigma_n((X, \alpha_X), (Y, \alpha_Y), i) \times \sigma_n((X', \alpha'_X), (Y', \alpha'_Y), j) = \\
 & ((X \times X', (\alpha_X \times \alpha'_X) \circ \Delta \circ \sigma_n), ((X \times Y', (\alpha_X \times \alpha'_Y) \circ \Delta \circ \sigma_n) \cup (Y \times X', (\alpha_Y \times \alpha'_X) \circ \Delta \circ \sigma_n)), i + j) \\
 & = \sigma_n(((X, \alpha_X), (Y, \alpha_Y), i) \times ((X', \alpha'_X), (Y', \alpha'_Y), j)).
 \end{aligned}$$

The functors $\tilde{\rho}_n$ are not compatible with the tensor product structure, as expected. \square

Remark 7.8. In [57] a motivic interpretation of the categorification of the Bost–Connes algebra is given by identifying the Tannakian category $\text{Vec}_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{Q}}(\mathbb{Q})$ with a limit of orbit categories of Tate motives. Here we presented a different motivic categorification of the Bost–Connes algebra by lifting the Bost–Connes structure to the level of the category of Nori motives. In [57] a motivic Bost–Connes structure was also constructed using the category of motives over finite fields and the larger class of Weil numbers replacing the roots of unity of the Bost–Connes system.

7.5. Motivic sheaves and the relative case. The argument presented in Theorem 7.7 lifting the Bost–Connes structure to the category of Nori motives, which provides a Tannakian category version of the list to Grothendieck rings via the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, can also be generalized to the relative setting, where we considered the Euler characteristic

$$\chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \rightarrow K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$$

with values in the Grothendieck ring of constructible sheaves, discussed in §2 of this paper. The categorical setting of Nori motives that is appropriate for this relative case is the Nori category of motivic sheaves introduced in [3].

We recall here briefly the construction of the category of motivic sheaves of [3] and we show that the Bost–Connes structure on the category of Nori motives described in Theorem 7.7 extends to this relative setting.

Consider pairs $(X \rightarrow S, Y)$ of varieties over a base S with $Y \subset X$ endowed with the restriction $f_Y : Y \rightarrow S$. Morphisms $f : (X \rightarrow S, Y) \rightarrow (X' \rightarrow S, Y')$ are morphisms of varieties $h : X \rightarrow X'$ satisfying the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ & \searrow f & \swarrow f' \\ & S & \end{array}$$

and such that $h(Y) \subset Y'$. As before, we consider varieties endowed with good effectively finite $\hat{\mathbb{Z}}$ -action. We denote by (S, α) the base with its good effectively finite $\hat{\mathbb{Z}}$ -action and by $((X, \alpha_X) \rightarrow (S, \alpha), (Y, \alpha_Y))$ the pairs as above where we assume that the map $f : X \rightarrow S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$ -equivariant.

Following [3], a diagram $D(\mathcal{V}_S)$ is obtained by considering as vertices elements of the form $(X \rightarrow S, Y, i, w)$ with $(X \rightarrow S, Y)$ a pair as above, $i \in \mathbb{N}$ and $w \in \mathbb{Z}$. The edges are given by the three types of edges

- (1) geometric morphisms $h : (X \rightarrow S, Y) \rightarrow (X' \rightarrow S, Y')$ as above determine edges $h^* : (X' \rightarrow S, Y', i, w) \rightarrow (X \rightarrow S, Y, i, w)$;
- (2) connecting morphisms $\partial : (Y \rightarrow S, Z, i, w) \rightarrow (X \rightarrow S, Y, i + 1, w)$ for a chain of inclusions $Z \subset Y \subset X$;
- (3) twisted projections: $(X, Y, i, w) \rightarrow (X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

For consistency with our previous notation we have here written the morphisms in the contravariant (cohomological) way rather than in the covariant (homological) way used in §3.3 of [3].

Note that in the previous section, following [43] we described the effective Nori motives as $\mathcal{MN}^{eff} = C(\text{Pairs}^{eff}, T)$, with the category of Nori motives \mathcal{MN} being then obtained as the localization of \mathcal{MN}^{eff} at $(\mathbb{G}_m, \{1\}, 1)$ (inverting the Lefschetz motive). Here in the setting of [3] the Tate motives are accounted for in the diagram construction by the presence of the twist w and the last class of edges.

Given $f : X \rightarrow S$ and a sheaf \mathcal{F} on X one has $H_S^i(X; \mathcal{F}) = R^i f_* \mathcal{F}$. In the case of a pair $(f : X \rightarrow S, Y)$, let $j : X \setminus Y \hookrightarrow X$ be the inclusion and consider $H_S^i(X, Y; \mathcal{F}) = R^i f_* j_* \mathcal{F}|_{X \setminus Y}$. The diagram representation T in this case maps $T(X \rightarrow S, Y, i, w) = H_S^i(X, Y, \mathcal{F})(w)$ to the (Tate twisted) constructible sheaf $H_S^i(X, Y; \mathcal{F})$. It is shown in [3] that the Nori formalism of geometric diagrams applies to this setting and gives rise to a Tannakian category of motivic sheaves \mathcal{MN}_S . In particular one considers the case where \mathcal{F} is constant with $\mathcal{F} = \mathbb{Q}$, so that the diagram representation $T : D(\mathcal{V}_S) \rightarrow \mathbb{Q}_S$

and the induced functor on \mathcal{MN}_S replace at the motivic level the Euler characteristic map on the relative Grothendieck ring $K_0(\mathcal{V}_S) \rightarrow K_0(\mathbb{Q}_S)$.

As in the previous cases, we consider an enhancement of this category of motivic sheaves, in the sense of §3.3, by introducing good effectively finite $\hat{\mathbb{Z}}$ -actions. We modify the construction of [3] in the following way.

We consider a diagram $D(\mathcal{V}_{(S,\alpha)}^{\hat{\mathbb{Z}}})$ where the vertices are elements

$$((X, \alpha_X) \rightarrow (S, \alpha), (Y, \alpha_Y), i, w)$$

so that the maps $f : X \rightarrow S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$ -equivariant, and with morphisms as above, where all the maps are required to be compatible with the $\hat{\mathbb{Z}}$ -actions. One obtains by the same procedure as in [3] a category of equivariant motivic sheaves $\mathcal{MN}_S^{\hat{\mathbb{Z}}}$. The representation above maps $D(\mathcal{V}_{(S,\alpha)}^{\hat{\mathbb{Z}}})$ to $\hat{\mathbb{Z}}$ -equivariant constructible sheaves over (S, α) . Then the same argument we used §2 at the level of Grothendieck rings, assemblers and spectra applies to this setting and gives the following result.

Proposition 7.9. *The maps of diagrams*

$$\sigma_n : D(\mathcal{V}_{(S,\alpha)}^{\hat{\mathbb{Z}}}) \rightarrow D(\mathcal{V}_{(S,\alpha \circ \sigma_n)}^{\hat{\mathbb{Z}}})$$

$$\tilde{\rho}_n : D(\mathcal{V}_{(S,\alpha)}^{\hat{\mathbb{Z}}}) \rightarrow D(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha))}^{\hat{\mathbb{Z}}})$$

given by

$$\sigma_n((X, \alpha_X) \rightarrow (S, \alpha), (Y, \alpha_Y), i, w) = ((X, \alpha_X \circ \sigma_n) \rightarrow (S, \alpha \circ \sigma_n), (Y, \alpha_Y \circ \sigma_n), i, w)$$

$$\tilde{\rho}_n((X, \alpha_X) \rightarrow (S, \alpha), (Y, \alpha_Y), i, w) =$$

$$((X \times Z_n, \Phi_n(\alpha_X)) \rightarrow (S \times Z_n, \Phi_n(\alpha)), (Y \times Z_n, \Phi_n(\alpha_Y)), i, w)$$

determine functors of the resulting category of motivic sheaves $\mathcal{MN}_S^{\hat{\mathbb{Z}}}$ such that $\sigma_n \circ \tilde{\rho}_n = n \text{id}$ and $\tilde{\rho}_n \circ \sigma_n$ is a product with (Z_n, α_n) . Thus, one obtains on the category $\mathcal{MN}_S^{\hat{\mathbb{Z}}}$ a Bost–Connes system as in Definition 3.11.

Proof. The argument is as in Proposition 2.6, using again, as in Theorem 7.7 the fact that maps of diagrams induce functors of the resulting categories of Nori motives. \square

7.6. Nori geometric diagrams for assemblers, and a challenge. We conclude this section on Bost–Connes systems and Nori motives by formulating a question about Nori diagrams and assembler categories.

According to the Nori formalism as it is presented in [43], we must start with a “geometric” category C of spaces/varieties/schemes, possibly endowed with additional structures, in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \setminus Y \rightarrow X$. Then the Nori diagram of *effective pairs* $D(C)$ is defined as in [43], pp. 207–208, see §7.2.2.

In the current context, *objects* of our category \mathcal{C} will be *assemblers* \mathcal{C} (of course, described in terms of a category of lower level). In particular, each such \mathcal{C} is endowed with a Grothendieck topology.

A *vertex* of the Nori diagram $D(\mathcal{C})$ will be a triple $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$ where its first two terms are taken from an abstract scissors congruence in \mathcal{C} , and i is an integer. Intuitively, this means that we are considering the canonical embedding $\mathcal{C} \setminus \mathcal{D} \hookrightarrow \mathcal{C}$ as an analog of closed embedding. This intuition makes translation of the remaining components of Nori’s diagrams obvious, except for one: *what is the geometric meaning of the integer i in $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$?*

The answer in the general context of assemblers, seemingly, was not yet suggested, and already in the algebraic–geometric contexts is non–obvious and non–trivial. Briefly, i translates to the level of Nori geometric diagrams the *weight filtration* of various cohomology theories (cf. [43], 10.2.2, pp. 238–241), and the existence of such translation and its structure are encoded in several versions of *Nori’s Basic Lemma* independently and earlier discovered by A. Beilinson and K. Vilonen (cf. [43], 2.5, pp. 45–59).

The most transparent and least technical version of the Basic Lemma ([43], Theorem 2.5.2, p. 46) shows that in algebraic geometry the existence of weight filtration is based upon special properties of *affine schemes*. As we will see in the last section, lifts of Bost–Connes algebras to the level of cohomology based upon the techniques of *enhancement* also require a definition of *affine* assemblers. Since we do not know its combinatorial version, the enhancements that we can study now, force us to return to algebraic geometry.

This challenge suggests to think about other possible geometric contexts in which dimensions/weights of the relevant objects may take, say, p –adic values (as in the theory of p –adic weights of automorphic forms inaugurated by J. P. Serre), or rational values (as it happens in some corners of “geometries below $\text{Spec } \mathbb{Z}$ ”), or even real values (as in various fractal geometries).

Can one transfer the scissors congruences imagery there?

See, for example, the formalism of Farey semi–intervals as base of ∞ –adic topology.

Acknowledgment. We thank the referee for many very detailed and useful comments and suggestions on how to improve the structure and presentation of the paper. The first and third authors were supported in part by the Perimeter Institute for Theoretical Physics. The third author is also partially supported by NSF grant DMS-1707882, and by NSERC Discovery Grant RGPIN-2018-04937 and Accelerator Supplement grant RGPAS-2018-522593.

REFERENCES

- [1] G. Almkvist, *Endomorphisms of finitely generated projective modules over a commutative ring*, Ark. Mat. 11 (1973) 263–301.
- [2] G. Almkvist, *The Grothendieck ring of the category of endomorphisms*, J. Algebra 28 (1974) 375–388.
- [3] D. Arapura, *An abelian category of motivic sheaves*, Adv. Math. 233 (2013), 135–195. [arXiv:0801.0261]
- [4] M. Baake, E. Lau, V. Paskunas, *A note on the dynamical zeta function of general toral endomorphisms*, Monatsh. Math., Vol.161 (2010) 33–42. [arXiv:0810.1855]
- [5] S. del Baño, *On the Chow motive of some moduli spaces*, J. Reine Angew. Math. 532 (2001) 105–132
- [6] D. Bejleri, M. Marcolli, *Quantum field theory over \mathbb{F}_1* , J. Geom. Phys. 69 (2013) 40–59. [arXiv:1209.4837]
- [7] P. Berrizbeitia, V.F. Sirvent, *On the Lefschetz zeta function for quasi-unipotent maps on the n -dimensional torus*, J. Difference Equ. Appl. 20 (2014), no. 7, 961–972.
- [8] P. Berrizbeitia, M.J. González, A. Mendoza, V.F. Sirvent, *On the Lefschetz zeta function for quasi-unipotent maps on the n -dimensional torus. II: The general case*, Topology Appl. 210 (2016), 246–262.
- [9] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. Math. (2) 98 (1973) 480–497.
- [10] A. Bialynicki-Birula, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976) 667–674
- [11] A. Bialynicki-Birula, J.B. Carrell, W.M. McGovern, *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Vol.131 of Encyclopaedia of Mathematical Sciences. Invariant Theory and Algebraic Transformation Groups, II, Springer 2002.
- [12] Kai Behrend, Jim Bryan, Balázs Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, Invent. Math. 192 (2013) 111–160. [arXiv:0909.5088]
- [13] J. Borger, B. de Smit, *Galois theory and integral models of Λ -rings*, Bull. Lond. Math. Soc. 40 (2008), no. 3, 439–446. [arXiv:0801.2352]
- [14] L. Borisov, *The class of the affine line is a zero divisor in the Grothendieck ring*, arXiv:1412.6194.
- [15] J.B. Bost, A. Connes. *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*. Selecta Math. (N.S.) 1 (1995) no. 3, pp. 411–457.
- [16] A. Bridy, *The Artin–Mazur zeta function of a dynamically affine rational map in positive characteristic*, Journal de Théorie des Nombres de Bordeaux, Vol.28 (2016) 301–324. [arXiv:1306.5267]
- [17] A. Bridy, *Transcendence of the Artin–Mazur zeta function for polynomial maps of $\mathbb{A}^1(\mathbb{F}_p)$* , Acta Arith. 156 (2012), no. 3, 293–300. [arXiv:1202.0362]
- [18] P. Brosnan, *On motivic decompositions arising from the method of Bialynicki-Birula*, Invent. Math. 161 (2005) 91–111. [arXiv:math/0407305]
- [19] J. Byszewski, G. Cornelissen, *Dynamics on Abelian varieties in positive characteristic*, arXiv:1802.07662
- [20] J.A. Campbell, *Facets of the Witt Vectors*, arXiv:1910.10206.
- [21] J.A. Campbell, J. Wolfson, I. Zakharevich, *Derived ℓ -adic zeta functions*, arXiv:1703.09855.
- [22] J. Choi, S. Katz, A. Klemm, *The refined BPS index from stable pair invariants*, Commun. Math. Phys. 328 (2014) 903–954. [arXiv:1210.4403]
- [23] J. Choi, M. Maican, *Torus action on the moduli spaces of torsion plane sheaves of multiplicity four*, Journal of Geometry and Physics 83 (2014) 18–35. [arXiv:1304.4871]
- [24] A. Connes, C. Consani, *Schemes over \mathbb{F}_1 and zeta functions*, Compos. Math. 146 (2010) 1383–1415. [arXiv:0903.2024]

- [25] A. Connes, C. Consani, *On the arithmetic of the BC-system*, J. Noncommut. Geom. 8 (2014) no. 3, 873–945. [arXiv:1103.4672]
- [26] A. Connes, C. Consani, *Absolute algebra and Segal’s Γ -rings: au dessous de $\text{Spec}(\mathbb{Z})$* , J. Number Theory 162 (2016), 518–551 [arXiv:1502.05585]
- [27] A. Connes, C. Consani, M. Marcolli, *Fun with \mathbb{F}_1* , J. Number Theory 129 (2009) 1532–1561. [arXiv:0806.2401]
- [28] A. Connes, M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, Vol.55, American Mathematical Society, 2008.
- [29] A. Deitmar, *Remarks on zeta functions and K-theory over \mathbb{F}_1* , Proc. Japan Acad. Ser. A Math. Sci. 82 (2006) 141–146. [arXiv:math/0605429]
- [30] P. Deligne, *Catégories tensorielles*, Mosc. Math. J. 2 (2002), no. 2, 227–248.
- [31] P. Deligne, J.S. Milne, *Tannakian categories, in Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics, Vol. 900, Springer 1982, pp. 101–228
- [32] A.W.M. Dress, C. Siebeneicher, *The Burnside ring of profinite groups and the Witt vector construction*, Advances in Mathematics Vol.70 (1988) N.1, 87–132.
- [33] J.M. Drézet, M Maican, *On the geometry of the moduli spaces of semi-stable sheaves supported on plane quartics*, Geom. Dedicata 152 (2011) 17–49. [arXiv:0910.5327]
- [34] W. Ebeling, S.M. Gusein-Zade, *Higher-order spectra, equivariant Hodge–Deligne polynomials, and Macdonald-type equations*, in “Singularities and computer algebra”, pp. 97–108, Springer, 2017. [arXiv:1507.08088]
- [35] B. Fantechi, L. Göttsche, *Riemann-Roch theorems and elliptic genus for virtually smooth schemes*, Geom. Topol. 14 (2010) no. 1, 83–115. [arXiv:0706.0988]
- [36] J.M. Franks, *Some smooth maps with infinitely many hyperbolic periodic points*, Trans. Amer. Math. Soc. 226 (1977), 175–179.
- [37] J.M. Franks, *Homology and the zeta function for diffeomorphisms*, International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975), pp. 79–88. Astérisque, No. 40, Soc. Math. France, 1976.
- [38] D.R. Grayson, *The K-theory of endomorphisms*, J. Algebra 48 (1977) no. 2, 439–446.
- [39] S.M. Gusein-Zade, *Equivariant analogues of the Euler characteristic and Macdonald type equations*, Russian Math. Surveys 72 (2017) 1, 1–32.
- [40] L. Hesselholt, *Witt vectors of non-commutative rings and topological cyclic homology*, Acta Math., Vol. 178 (1997) N.1, 109–141.
- [41] L. Hesselholt, I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology, 36 (1997) N.1, 29–101.
- [42] W.H. Hesselink, *Concentration under actions of algebraic groups*, Lect. Notes Math., vol. 867 (1981) 55–89.
- [43] A. Huber, St. Müller–Stach. *Periods and Nori motives*. With contributions by Benjamin Friedrich and Jonas von Wangenheim. Erg der Math und ihrer Grenzgebiete, vol. 65, Springer 2017.
- [44] Z. Jin, M. Marcolli, *Endomotives of toric varieties*, J. Geom. Phys. 77 (2014), 48–71. [arXiv:1309.4101]
- [45] M. Kapranov, A. Smirnov, *Cohomology determinants and reciprocity laws: number field case*, Unpublished manuscript.
- [46] M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, arXiv:math/0001005.
- [47] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Springer, 2005.
- [48] M. Kontsevich, Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. Vol. 5 (2011) N.2, 231–352. [arXiv:1006.2706]
- [49] C. Lo, M. Marcolli, *\mathbb{F}_ζ -geometry, Tate motives, and the Habiro ring*, International Journal of Number Theory, 11 (2015), no. 2, 311–339. [arXiv:1310.2261]

- [50] E. Looijenga, *Motivic measures*, Séminaire N. Bourbaki, 1999-2000, exp. no 874, 267–297.
- [51] J. López Peña, O. Lorscheid, *Torified varieties and their geometries over \mathbb{F}_1* , Math. Z. 267 (2011) 605–643. [arXiv:0903.2173]
- [52] Yu. Manin, D. Borisov. *Generalized operads and their inner cohomomorphisms*. In: Geometry and Dynamics of Groups and Spaces (In memory of Aleksander Reznikov). Ed. by M. Kapranov et al. Progress in Math., vol. 265, Birkhäuser, Boston, pp. 247–308. Preprint math.CT/0609748.
- [53] Yuri I. Manin, Matilde Marcolli, *Moduli operad over \mathbb{F}_1* , in “Absolute Arithmetic and \mathbb{F}_1 -Geometry”, 331–361, Eur. Math. Soc., 2016. [arXiv:1302.6526]
- [54] Yuri I. Manin, Matilde Marcolli, *Homotopy types and geometries below $\text{Spec}(\mathbb{Z})$* , in “Dynamics: topology and numbers”, pp. 27–56, Contemp. Math., 744, Amer. Math. Soc., 2020. [arXiv:1806.10801]
- [55] M. Marcolli, *Cyclotomy and endomotives*, p-Adic Numbers Ultrametric Anal. Appl. 1 (2009), no. 3, 217–263. [arXiv:0901.3167]
- [56] M. Marcolli, Z. Ren, *q-Deformations of statistical mechanical systems and motives over finite fields*, p-Adic Numbers Ultrametric Anal. Appl. 9 (2017) no. 3, 204–227. [arXiv:1704.06367]
- [57] M. Marcolli, G. Tabuada, *Bost-Connes systems, categorification, quantum statistical mechanics, and Weil numbers*, J. Noncommut. Geom. 11 (2017) no. 1, 1–49. [arXiv:1411.3223]
- [58] N. Martin, *The class of the affine line is a zero divisor in the Grothendieck ring: an improvement*, C. R. Math. Acad. Sci. Paris 354 (2016), no. 9, pp. 936–939. [arXiv:1604.06703]
- [59] L. Maxim, J. Schürmann, *Equivariant characteristic classes of external and symmetric products of varieties*, Geom. Topol. 22 (2018) no. 1, 471–515. [arXiv:1508.04356]
- [60] S. Müller-Stach, B. Westrich, *Motives of graph hypersurfaces with torus operations*, Transform. Groups 20 (2015), no. 1, 167–182. [arXiv:1301.5221]
- [61] D. Quillen, *Higher algebraic K-theory. I*, in “Algebraic K-theory, I: Higher K-theories”, Lecture Notes in Math., Vol. 341 (1973) 85–147.
- [62] N. Ramachandran, *Zeta functions, Grothendieck groups, and the Witt ring*, Bull. Sci. Math. Soc. Math. Fr. 139 (2015) N.6, 599–627 [arXiv:1407.1813]
- [63] N. Ramachandran, G. Tabuada, *Exponentiable motivic measures*, J. Ramanujan Math. Soc. 30 (2015), no. 4, 349–360. [arXiv:1412.1795]
- [64] S. Schwede, *Stable homotopical algebra and Γ -spaces*, Math. Proc. Phil. Soc. Vol.126 (1999) 329–356.
- [65] G. Segal, *Categories and cohomology theories*, Topology, Vol.13 (1974) 293–312.
- [66] M. Shub, D. Sullivan, *Homology theory and dynamical systems*, Topology 14 (1975) 109–132.
- [67] C. Soulé, *Les variétés sur le corps à un élément*, Mosc. Math. J. 4 (2004), 217–244
- [68] R.W. Thomason, *Symmetric monoidal categories model all connective spectra*, Theory and Applications of Categories, Vol.1 (1995) N.5, 78–118.
- [69] J.L. Verdier, *Caractéristique d’Euler-Poincaré*, Bull. Soc. Math. France 101 (1973) 441–445.
- [70] I. Zakharevich, *The K-theory of assemblers*, Adv. Math. 304 (2017), 1176–1218. [arXiv:1401.3712]
- [71] I. Zakharevich, *On K_1 of an assembler*, J. Pure Appl. Algebra 221 (2017), no. 7, 1867–1898. [arXiv:1506.06197]
- [72] I. Zakharevich, *The annihilator of the Lefschetz motive*, Duke Math. J. 166 (2017), no. 11, 1989–2022. [arXiv:1506.06200]

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, USA
 Email address: jlieber@caltech.edu

MAX PLANCK INSTITUTE FOR MATHEMATICS, BONN, GERMANY
 Email address: manin@mpim-bonn.mpg.de

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, USA

Email address: `matilde@caltech.edu`