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A Langevin equation for stochastic climate models with periodic feedback and forcing variance

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ABSTRACT

The concept of stochastic climate models developed by Hasselmann is generalized to include periodic feedback coefficients and random forcing with periodic variance, in order to take into account the seasonal variability of the mean atmosphere–ice–ocean interaction and of the atmospheric noise. Our results show marked departures from the original model, seen as seasonal modulations of the amplitudes of the covariances both with respect to seasonal and to lag times.

1. Introduction

In a recent publication, Hasselmann (1976) has considered a stochastic model of climatic variability in which the slow changes of climate were interpreted as the continuous response of climatic variables to random excitations by "weather" disturbances. Later, this model was applied to a very simple climatic system (Frankignoul and Hasselmann, 1977) consisting of the upper layer of the ocean, driven by random fluxes of heat and momentum across the air–sea interface. The interest was centered in the features and orders of magnitude of the sea-surface temperature (SST) variability in the time-scale range of months to a few years. Lemke (1977) used a similar approach to estimate climate variability on longer time-scales (10^1–10^4 years) with a stochastically forced energy balanced climate model.

Stochastic models were also fitted (in the least squares sense) to observed SST anomalies (Reynolds, 1978; Herterich and Hasselmann, 1982) and to sea-ice anomalies (Lemke et al., 1980), thereby achieving a statistical verification as well as yielding information on the statistics of the atmospheric forcing and on the deterministic feedback governing parts of the climate system. Essentially, in these studies it is assumed that for small excursions about an equilibrium state the evolution of a climate variable $y(t)$ is governed by a Langevin-type equation

$$\dot{y}(t) = -v(t)\,y(t) + w(t), \tag{1.1}$$

where $w(t)$ is a stationary Gaussian, delta correlated, stochastic process representing the "weather" disturbances. The function $v(t)$ describes the resistance of $y(t)$ to changes produced by the forcing. The variable $y(t)$ may be associated with climate variables such as SST anomalies, ice coverage, etc. Typical time-scales for $y(t)$ are of the order of several months to several years or longer, while for $w(t)$ they are of the order of a few days.

In all models hitherto discussed, the feedback $v(t)$ and the variance of the stochastic forcing $w(t)$ were approximated by constants determined by physical considerations or from data fitting. However, the mean atmosphere–ice–ocean interaction represented by $v(t)$ and also the variance of the atmospheric noise will in many cases be seasonally variable. Therefore, we present here a generalization of the initial stationary stochastic

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model by including periodic modulations of the feedback coefficient and stochastic forcing.

This paper will be mainly concerned with the statistical properties of the one-dimensional process (1.1) since they can be readily compared with those estimated from observed climatic time series. In order to determine the relative importance of the seasonal dependence in the feedback and forcing terms, the covariance function and the variance spectrum are derived from (1.1) for three different models. The general model (denoted by the superscript III) includes a seasonal varying feedback coefficient as well as a seasonal varying forcing. Model II includes a periodicity only in the feedback coefficient and model I only in the forcing variance. For comparison reasons we will also use model 0. In this model both the feedback coefficient and the forcing variance are constant (Hasselmann, 1976).

In this theoretical study we investigate only the mathematical models. A comparison with climatic data will be the subject of a subsequent publication.

2. The model covariance

The governing equation is given by

$$\dot{y}(t) = -\{\beta + \beta_1 \cos(\omega_a t + \phi)\}\, y(t) + w(t), \qquad (2.1)$$

where ω_a denotes the (usually annual) angular frequency and ϕ the phase of the feedback modulation.

Eq. (2.1) is a first-order ordinary inhomogeneous linear differential equation with time dependent coefficients. For $t > -\infty$ its solution is given as the response to the forcing $w(t)$ by

$$y(t) = \exp\{-\beta t - \alpha \sin(\omega_a t + \phi)\} \times \int_{-\infty}^{t} \mathrm{d}s\, w(s) \exp\{\beta s + \alpha \sin(\omega_a s + \phi)\} \qquad (2.2)$$

with $\alpha = \beta_1/\omega_a$.

The covariance function $C(t, \tau)$ of the climate variable $y(t)$ is defined as

$$C(t, \tau) = \langle y(t + \tau) y(t) \rangle - \langle y(t + \tau) \rangle \langle y(t) \rangle, \qquad (2.3)$$

where the angular brackets $\langle \dots \rangle$ denote ensemble averages over a set of realizations of the "weather" variables represented by the forcing $w(t)$.

Inserting the expression (2.2) into (2.3) we find for the covariance function

$$C(t, \tau) = \exp\{-\beta(t + \tau) - \alpha \sin(\omega_a(t + \tau) + \phi) - \beta t - \alpha \sin(\omega_a t + \phi)\} \int_{-\infty}^{t+\tau} \mathrm{d}r \int_{-\infty}^{t} \mathrm{d}s \times \langle w(r) w(s) \rangle \exp\{\beta r + \alpha \sin(\omega_a r + \phi) + \beta s + \alpha \sin(\omega_a s + \phi)\}. \qquad (2.4)$$

The assumption of cyclo-stationary delta correlated random forcing allows us to write

$$\langle w(r) w(r) \rangle = D(s)\, \delta(r - s) \qquad (2.5)$$

with

$$D(s) = D_0 + D_1 \cos(\omega_a s + \gamma), \qquad (2.6)$$

and consequently

$$C(t, \tau) = \exp\{-\beta(t + \tau) - \alpha \sin(\omega_a(t + \tau) + \phi) - \beta t - \alpha \sin(\omega_a t + \phi)\} \int_{-\infty}^{z} \mathrm{d}s \times \exp\{2\beta s + 2\alpha \sin(\omega_a s + \phi)\}\, D(s), \qquad (2.7)$$

where $z = \min(t + \tau, t)$.

The integration of eq. (2.7) can be performed expanding $\exp\{2\alpha \sin(\omega_a s + \phi)\}$ into a Bessel series:

$$\exp\{2\alpha \sin(\omega_a s + \phi)\} = \sum_{n=-\infty}^{\infty} J_n(2i\alpha) \times \exp\{-in(\omega_a s + \phi)\}. \qquad (2.8)$$

Introducing the variable transformation $s \to s + z$, we obtain

$$\int_{-\infty}^{0} \mathrm{d}s \exp\{2\beta(s + z) + 2\alpha \sin(\omega_a(s + z) + \phi)\} \times D(s + z) = \exp\{2\beta z\} \sum_{p=-\infty}^{\infty} J_p(2i\alpha) \times \exp\{-ip((\omega_a z + \phi)\} \int_{-\infty}^{0} \times D(s + z) \exp\{2\beta s - ip\omega_a s\} = \exp\{2\beta z\} \times \sum_{p=-\infty}^{\infty} J_p(2i\alpha) \exp\{-ip(\omega_a z + \phi)\} \times (a + b \cos(\omega_a z + \gamma) + c \sin(\omega_a z + \gamma)), \qquad (2.9)$$

where

$$a = D_0/(2\beta - ip\omega_a)$$
$$b = D_1(2\beta - ip\omega_a)/((2\beta - ip\omega_a)^2 + \omega_a^2) \qquad (2.10)$$
$$c = D_1\, \omega_a/((2\beta - ip\omega_a)^2 + \omega_a^2).$$

Therefore the covariance function for the complete model III can be represented by the series

$$C^{\mathrm{III}}(t,\tau)=\exp\{-\beta\tau-\alpha\sin(\omega_a(t+\tau)+\phi)-\alpha\sin(\omega_a t+\phi)\}\times\sum_{p=-\infty}^{\infty}J_p(2i\alpha)\exp\{-ip(\omega_a t+\phi)\}(a+b\times\cos(\omega_a t+\gamma)+c\sin(\omega_a t+\gamma));\qquad \tau\geqslant 0$$
$$=\exp\{\beta\tau-\alpha\sin(\omega_a(t+\tau)+\phi)-\alpha\sin(\omega_a t+\phi)\}\times\sum_{p=-\infty}^{\infty}J_p(2i\alpha)\exp\{-ip(\omega_a(t+\tau)+\phi)\}\times(a+b\cos(\omega_a(t+\tau)+\gamma)+c\sin(\omega_a(t+\tau)+\gamma));\qquad \tau\leqslant 0. \tag{2.11}$$

The covariance function for model II (only periodic feedback) can be obtained from eq. (2.11) by setting D_1, and consequently b and c equal to zero

$$C^{\mathrm{II}}(t,\tau)=D_0\exp\{-\beta\tau-\alpha\sin(\omega_a(t+\tau)+\phi)\times\alpha\sin(\omega_a t+\phi)\}\sum_{p=-\infty}^{\infty}J_p(2i\alpha)\times\exp\{-ip(\omega_a t+\phi)\}/(2\beta-ip\omega_a);\qquad \tau\geqslant 0$$
$$=D_0\exp\{\beta\tau-\alpha\sin(\omega_a(t+\tau)+\phi)-\alpha\sin(\omega_a t+\phi)\}\sum_{p=-\infty}^{\infty}J_p(2i\alpha)\times\exp\{-ip(\omega_a(t+\tau)+\phi)\}/(2\beta-ip\omega_a);\qquad \tau\leqslant 0. \tag{2.12}$$

The covariance function for model I follows from eq. (2.11) by setting $\alpha=0$ and noting that

$$J_p(0)=1\quad \text{for } p=0$$
$$=0\quad \text{otherwise;}$$

$$C^{\mathrm{I}}(t,\tau)=\exp\{-\beta\tau\}\left[\frac{D_0}{2\beta}+\frac{2\beta D_1}{4\beta^2+\omega_a^2}\cos(\omega_a t+\gamma)+\frac{\omega_a D_1}{4\beta^2+\omega_a^2}\sin(\omega_a t+\gamma)\right];\qquad \tau\geqslant 0$$
$$=\exp\{\beta\tau\}\left[\frac{D_0}{2\beta}+\frac{2\beta D_1}{4\beta^2+\omega_a^2}\cos(\omega_a(t+\tau)+\gamma)+\frac{\omega_a D_1}{4\beta^2+\omega_a^2}\sin(\omega_a(t+\tau)+\gamma)\right]\quad \tau\leqslant 0. \tag{2.13}$$

To exhibit the features of the three models we must introduce numerical values into the above formulae. We may think of $y(t)$ as representing a dimensionless zonal averaged SST anomaly. For this case we have chosen $\beta=0.222$ month^{-1}, (see Frankignoul and Hasselmann, 1977), and $\omega_a=(2\pi/12)$ month^{-1}, corresponding to the seasonal cycle. We have also chosen $\beta_1=0.8$ month^{-1} to represent the case of a strong periodic modulation of the feedback coefficient. In all three models the form of the covariance function depends on the phases ϕ and γ. We have chosen to present the covariance functions with the same phase for all three models, and we have arbitrarily taken $\phi=\gamma=0$. As the forcing has been supposed to be normalized, we have taken $D_0=1$. D_1 should be smaller than D_0 for any physical model. We have chosen it to be 0.5.

With this set of values we present in Figs. 1 and 2 a plot of the covariance functions $C(t,\tau)$ for two different seasonal times and for the four models. We observe first a marked variation in the amplitude with the seasonal time t, which goes in opposite directions for models I and II. We also observe a marked variation of the form of the covariances with respect to model 0, with new peaks for time-lags around 6 and 12 months. We observe in the figures that the response of the model climatic variable $y(t)$ to the periodic forcing is out-of-phase, with respect to seasonal time t, with the response to a non-periodic forcing when the system has a periodic feedback. In this case it is evident that model III must show a behaviour intermediate between that of models I and II.

In Figs. 3–5 we have plotted the time-lag (τ) dependence of the first three coefficients of a Fourier series expansion of the covariance functions $C(t,\tau)$ for the models I, II and III

$$C(t,\tau)=a_0(\tau)+a_1(\tau)\cos(\omega_a t)+b_1(\tau)\sin(\omega_a t)+\dots \tag{2.14}$$

For comparison, a plot of $a_0(\tau)$ for model 0 is also presented.

We observe in Fig. 3 that only for model II is an appreciable deviation from model 0 obtained, and that this deviation has a small hump for a time-lag of about 12 months. In Figs. 4 and 5 we observe that the most pronounced seasonal dependence is obtained for time-lags around, but not equal to zero, while again for $\tau=12$ months

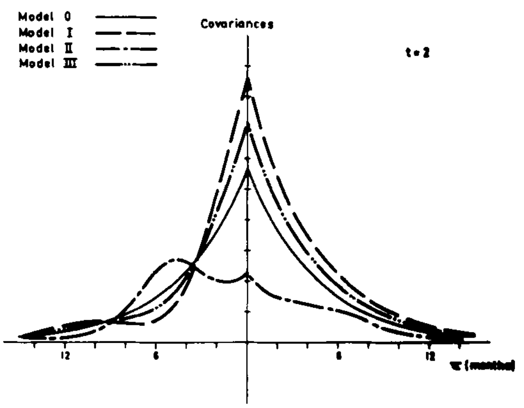

Fig. 1. Covariance functions $C(t, \tau)$ of the response of the stochastically forced climate models 0, I, II, III. Seasonal time is $t = 2$ months. The y-axis units are arbitrary.

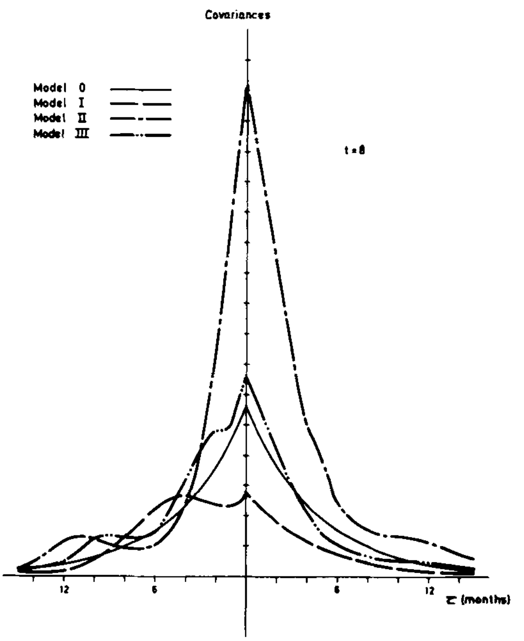

Fig. 2. Covariance functions $C(t, \tau)$ of the response of the stochastically forced models 0, I, II, III. Seasonal time is $t = 8$ months. The y-axis units are the same as in Fig. 1.

we obtain a small variation. We also observe here that models I and II show effects in opposite directions.

3. The model spectrum

In analogy with the stationary case, the spectrum $P(\omega', \omega)$ of the cyclo-stationary process

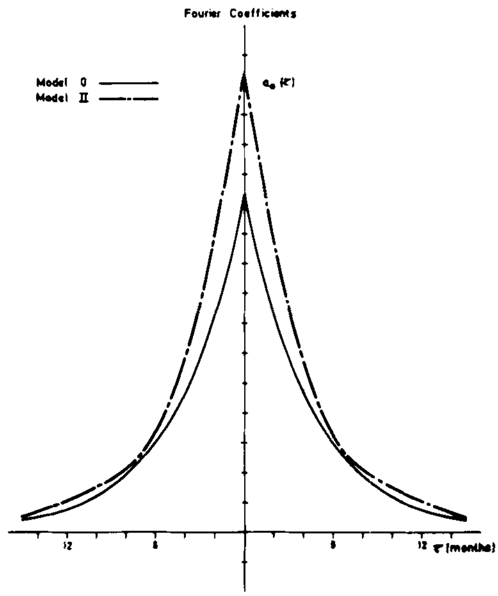

Fig. 3. First Fourier coefficient $a_0(\tau)$ of the Fourier series expansion of the covariance functions for model 0 and II. Models I and III Fourier coefficients are indistinguishable from that of Model 0. The y-axis units are arbitrary.

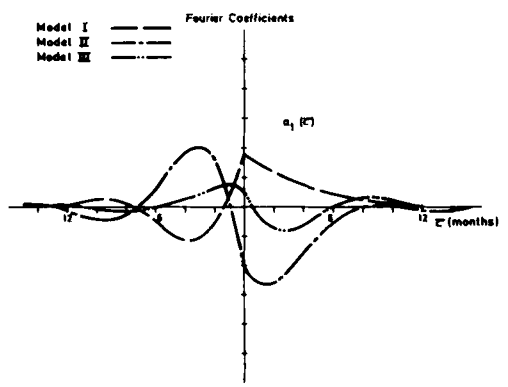

Fig. 4. Second Fourier coefficient $a_1(\tau)$ of the Fourier series expansion of the covariance functions for the three models I, II, III. The coefficient for Model 0 is identically zero. The y-axis units are the same as in Fig. 3.

$y(t)$ may be defined by the covariance matrix of the Fourier components $y(\omega)$, where

$$y(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, y(\omega) \exp(i\omega t), \tag{3.1}$$

or as the Fourier transform of the covariance function $C(t, \tau)$.

Since the process $y(t)$ is non-stationary, the

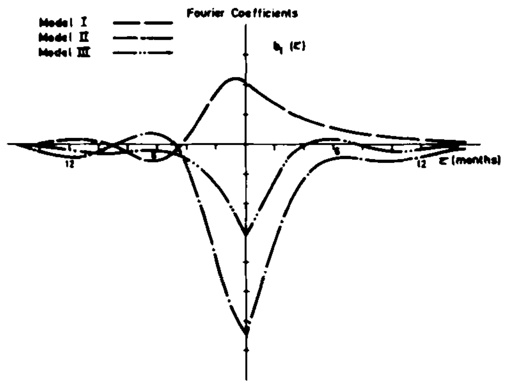

Fig. 5. Third Fourier coefficient $b_1(\tau)$ of the Fourier series expansion of the covariance functions for the three models I, II, III. The coefficient for Model 0 is identically zero. The y-axis units are the same as in Fig. 3.

covariance matrix of the Fourier components is non-diagonal, and the spectrum $P(\omega', \omega)$ depends on two frequency coordinates, a continuous frequency ω, corresponding to the time-lag variable τ of $C(t, \tau)$, and a discrete frequency $\omega' = q\omega_a$, corresponding to the periodic dependence of $C(t, \tau)$ on the seasonal time t. This covariance matrix is given by

$$\langle y^*(\omega) y(\omega + \omega') \rangle = \sum_{q=-\infty}^{\infty} P(\omega', \omega)\, \delta(\omega' - q\omega_a), \tag{3.2}$$

where

$$P(\omega', \omega) = (2/T_a) \int_0^{T_a} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau \times C(t, \tau) \exp\{-i(\omega' t + \omega\tau)\}; \qquad T_a = (2\pi/\omega_a). \tag{3.3}$$

Using the expression (2.7) we find

$$\begin{aligned} P(\omega', \omega) = {} & (2/T_a) \int_0^{T_a} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau \\ & \times \exp\{-i(\omega' t + \omega\tau)\} \exp\{-\beta(t + \tau) \\ & - \alpha \sin(\omega_a(t + \tau) + \phi)\} \exp\{-\beta t - \alpha \\ & \times \sin(\omega_a t + \phi)\} \int_{-\infty}^{z} \mathrm{d}s \exp\{2\beta s + 2\alpha \\ & \times \sin(\omega_a s + \phi)\} D(s). \end{aligned} \tag{3.4}$$

The last integral in (3.4) is the same as in eq. (2.7) and is therefore given by (2.9).

After expanding $\exp\{-\alpha \sin(\omega_a(t + \tau) + \phi)\}$ and $\exp\{-\alpha \sin(\omega_a t + \phi)\}$ into a Bessel series, the spectrum for model III can be written as

$$\begin{aligned} P^{\mathrm{III}}(\omega', \omega) = {} & (2/T_a) \sum_{m,n,p=-\infty}^{\infty} A_{mnp}(\alpha) \\ & \times \exp\{i\phi(m + n - p)\} \int_0^{T_a} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau \\ & \times \exp\{i\omega_a(mt + n(t + \tau) - pz)\} \\ & \times \exp\{-i\omega' t - i\omega\} \exp\{\beta(2z - 2t - \tau)\} \\ & \times (a + b\cos(\omega_a z + \gamma) + c \sin(\omega_a z + \gamma)), \end{aligned} \tag{3.5}$$

$$A_{mnp}(\alpha) = J_m(i\alpha) J_n(i\alpha) J_p(2i\alpha), \tag{3.6}$$

and a, b and c are given again by eq. (2.10).

Since $z = \min(t + \tau, t)$, the double integration in (3.5) (denoted by H) can be written as

$$H = \int_0^{T_a} \mathrm{d}t \left[\int_{-\infty}^{0} \mathrm{d}\tau_{(z=t+\tau)} + \int_0^{\infty} \mathrm{d}\tau_{(z=t)}\right]. \tag{3.7}$$

Using the Euler formula for the sine and cosine terms in (3.5) we can determine H. We obtain for the spectral matrix the final expression

$$\begin{aligned} P^{\mathrm{III}}(\omega', \omega) = {} & \pi \sum_{m,n,p=-\infty}^{\infty} A_{mnp}(\alpha) \\ & \times \exp\{i\phi(m + n - p)\} \\ & \times \left[\frac{2D_0\, \delta_{q,m+n-p}}{\{\beta - i\omega + i\omega_a(n - p)\}\{\beta + i\omega - in\omega_a\}}\right. \\ & + \frac{D_1 \exp\{i\gamma\}\, \delta_{q,m+n-p+1}}{\{\beta - i\omega + i\omega_a(n - (p - 1))\}\{\beta + i\omega - in\omega_a\}} \\ & \left. + \frac{D_1 \exp\{-i\gamma\}\, \delta_{q,m+n-p-1}}{\{\beta - i\omega + i\omega_a(n - (p + 1))\}\{\beta + i\omega - in\omega_a\}}\right] \end{aligned} \tag{3.8}$$

where $\delta_{q,k}$ are Kronecker deltas.

The spectrum for model II is calculated from eq. (3.8) by setting $D_1 = 0$

$$\begin{aligned} P^{\mathrm{II}}(\omega', \omega) = {} & 2D_0 \sum_{m,n,p=-\infty}^{\infty} A_{mnp}(\alpha) \\ & \times \frac{\exp\{i\phi(m + n - p)\}\, \delta_{q,m+n-p}}{\{\beta - i\omega + i\omega_a(n - p)\}\{\beta + i\omega - in\omega_a\}}. \end{aligned} \tag{3.9}$$

Similarly, the spectral matrix for model I is derived from eq. (3.8) with $\alpha = 0$ and noting that

$$A_{mnp}(\alpha) = 1 \quad \text{for } m = n = p = 0, \qquad = 0 \quad \text{otherwise;}$$

$$P^{\text{I}}(\omega', \omega) = \frac{2D_0}{\beta^2 + \omega^2}\,\delta_{q,0} + \frac{\pi D_1 \exp\{i\gamma\}}{\{\beta - i(\omega - \omega_a)\}\{\beta + i\omega\}}\,\delta_{q,-1} + \frac{\pi D_1 \exp\{-i\gamma\}}{\{\beta - i(\omega + \omega_a)\}\{\beta + i\omega\}}\,\delta_{q,1}. \tag{3.10}$$

We note that for $\alpha = 0$ and $D_1 = 0$ the spectrum (3.10) is reduced to the well-known red-shaped form given by Hasselmann (1976).

The effect of a periodic forcing is most easily seen if we substitute $P^{\text{I}}(\omega', \omega)$ into eq. (3.2)

$$\langle y^*(\omega)\, y(\omega + \omega')\rangle = \sum_{q=-\infty}^{\infty} P^{\text{I}}(\omega', \omega)\, \delta(\omega' - q\omega_a)$$
$$= \frac{2\pi D_0}{\beta^2 + \omega^2} + \frac{\pi D_1 \exp\{i\gamma\}}{\{\beta - i(\omega - \omega_a)\}\{\beta + i\omega\}} \times \delta(\omega' + \omega_a) + \frac{\pi D_1 \exp\{-i\gamma\}}{\{\beta - i(\omega + \omega_a)\}\{\beta + i\omega\}} \times \delta(\omega' - \omega_a). \tag{3.11}$$

In this case the response is split into side bands at frequencies $\omega + \omega_a$ and $\omega - \omega_a$.

In order to interpret the effect of a periodic feedback coefficient, another perhaps more evident way of considering the spectrum $P^{\text{II}}(\omega', \omega)$ can be obtained from eq. (2.2) if we first calculate the Fourier transform

$$y(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, y(t) \exp\{-i\omega t\}. \tag{3.12}$$

As is shown in the Appendix, we can write

$$y(\omega) = \sum_{p=-\infty}^{\infty} C_p\, w(\omega - p\omega_a), \tag{3.13}$$

where

$$w(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, w(t) \exp\{-i\omega t\} \tag{3.14}$$

and

$$C_p = \sum_{n=-\infty}^{\infty} J_{p+n}(i\alpha)\, J_n(i\alpha)\, \frac{1}{\beta + i(\omega + (p + n)\,\omega_a)}.$$

Thus we observe that each line input at frequency ω is split into two-sided infinite bands with separation ω_a (Fig. 6).

With the expression (3.13) for $y(\omega)$ we can write for the spectrum of model II

$$\langle y^*(\omega)\, y(\omega + \omega')\rangle = \sum_{p,q=-\infty}^{\infty} D_{pq}\, P_w(\omega + p\omega_a, \omega' + q\omega_a), \tag{3.16}$$

where $D_{pq} = C_p^*\, C_q$ and

$$P_w(\omega', \omega) = \langle w^*(\omega)\, w(\omega + \omega')\rangle. \tag{3.17}$$

In Figs. 7–11 we present the ω-dependence of the first three coefficients of the expansion

$$P(\omega', \omega) = a_0(\omega) + a_1(\omega) \cos(\omega_a t) + b_1(\omega) \times \cos(\omega_a t) + \ldots \tag{3.18}$$

for all three models.

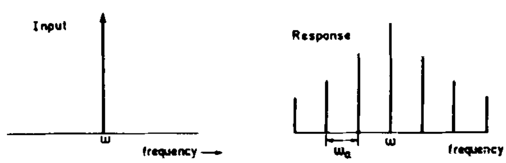

Fig. 6. Frequency response to a delta-shaped input in the frequency domain (Model II).

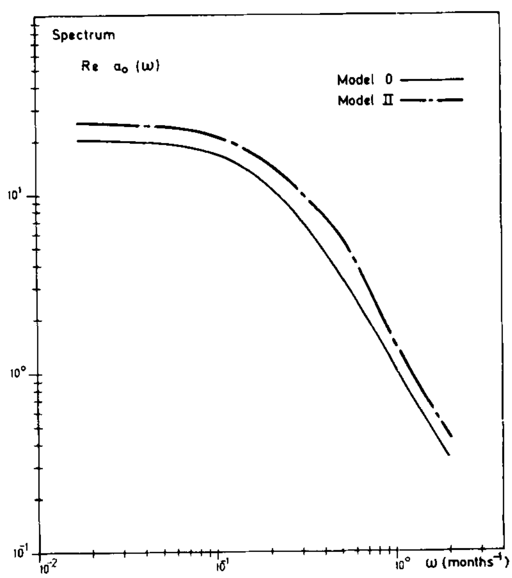

Fig. 7. Spectrum of the response of the stochastically forced climate models. Fourier transform of the first Fourier coefficient, $a_0(\omega)$. The transforms for models I and III are indistinguishable from that of model 0. The y-axis units are arbitrary.

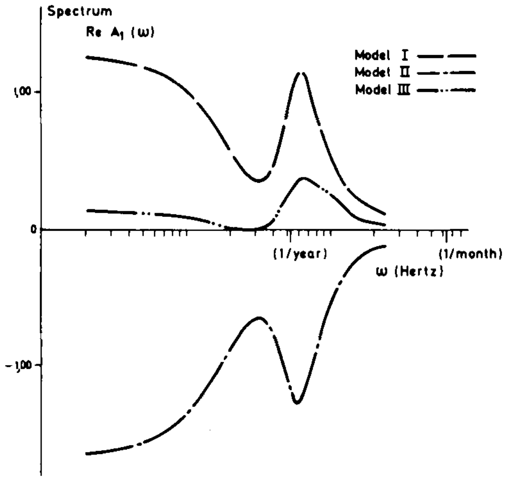

Fig. 8. Spectrum of the response of the stochastically forced climate models. Real part of the Fourier transform of the second Fourier coefficient, $a_1(\omega)$. The y-axis units are the same as in Fig. 7.

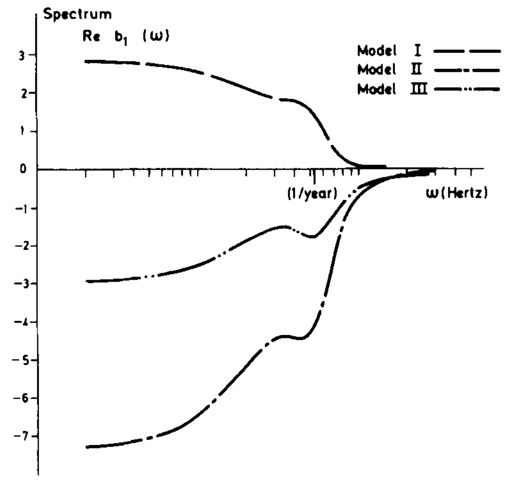

Fig. 10. Spectrum of the response of the stochastically forced climate models. Real part of the Fourier transform of the third Fourier coefficient, $b_1(\omega)$. The y-axis units are the same as in Fig. 7.

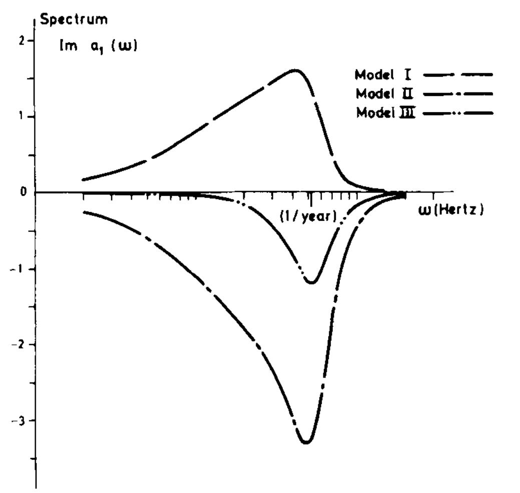

Fig. 9. Spectrum of the response of the stochastically forced climate models. Imaginary part of the Fourier transform of the second Fourier coefficient, $a_1(\omega)$. The y-axis units are the same as in Fig. 7.

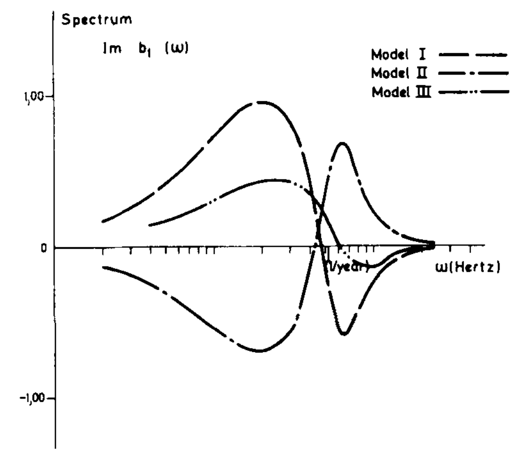

Fig. 11. Spectrum of the response of the stochastically forced climate models. Imaginary part of the Fourier transform of the third Fourier coefficient, $b_1(\omega)$. The y-axis units are the same as in Fig. 7.

The marked peaks around frequencies corresponding to periods of 12 months to be seen in these figures again show the periodic features of the three models, the effects of model I and model II being again reversed in sign.

4. Conclusions

We have analyzed here three complementary generalizations of the stochastic climate model proposed by Hasselmann (1976). Model I included a forcing term with periodic variance, while retaining a constant feedback coefficient. Model II included only a periodically varying feedback coefficient, while in model III both periodically varying terms were included.

We have analyzed the covariances between the responses at different times, and the spectrum for all three models. Our results show marked departures from those of the original model, seen as strong seasonal modulations in the amplitudes of the covariances both with respect to seasonal and lag-times.

We have observed that the effects of periodic feedback and of periodic modulated noise can be considered to be out-of-phase with respect to seasonal time. Consequently, model III, which includes both modulations presents results intermediate between those of model I and model II.

To determine which of these models should be favoured, all three models can be fitted to climatic data by the well-known least-squares method (Linnik, 1961; Müller et al, 1978; Lemke et al., 1980). This determination will be the subject of a subsequent publication.

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6. Appendix

We want to show here that it is possible to write the response $y(\omega)$ in terms of a series of the Fourier transforms of the input noise $w(t)$.

Writing

$$y(t) = \exp\{-\beta t - \alpha \sin(\omega_a t + \phi)\} \int_{-\infty}^{t} ds\, w(s) \times \exp\{\beta s + \alpha \sin(\omega_a s + \phi)\}, \tag{A.1}$$

$$w(s) = \int_{-\infty}^{\infty} d\omega_1\, w(\omega_1) \exp\{i\omega_1 s\}, \tag{A.2}$$

and expanding again $\exp\{-\alpha \sin(\omega_a t + \phi)\}$ and $\exp\{\alpha \sin(\omega_a s + \phi)\}$ into two series of Bessel functions, we obtain

$$\begin{aligned} y(t) = {} & \sum_{m,n=-\infty}^{\infty} J_m(i\alpha) J_n(i\alpha) \exp\{i\phi(m-n)\} \\ & \times \int_{-\infty}^{\infty} d\omega_1 w(\omega_1) \exp\{-\beta t + im\omega_a t\} \int_{-\infty}^{t} ds \\ & \times \exp\{\beta s + i(n\omega_a + \omega_1)s\} = \sum_{m,n=-\infty}^{\infty} \\ & \times J_m(i\alpha) J_n(i\alpha) \exp\{i\phi(m-n)\} \int_{-\infty}^{\infty} d\omega_1 \\ & \times \frac{w(\omega_1) \exp\{it(\omega_1 - (m-n)\,\omega_a)\}}{\beta + i(\omega_1 - n\omega_a)}. \end{aligned} \tag{A.3}$$

If we now substitute

$$\omega_1 - (m-n)\,\omega_a = \omega, \qquad (m-n) = p, \tag{A.4}$$

we obtain

$$\begin{aligned} y(t) = {} & \sum_{p,n=-\infty}^{\infty} J_{n+p}(i\alpha) J_n(i\alpha) \exp\{i\phi p\} \int_{-\infty}^{\infty} d\omega \\ & \times \frac{w(\omega - p\omega_a) \exp\{i\omega t\}}{\beta + i(\omega + (p+n)\,\omega_a)} \end{aligned} \tag{A.5}$$

or

$$\begin{aligned} y(\omega) = {} & \sum_{p,n=-\infty}^{\infty} \\ & \times \frac{J_{n+p}(i\alpha) J_n(i\alpha) \exp\{i\phi p\}}{\beta + i(\omega + (p+n)\,\omega_a)} w(\omega - p\omega_a). \end{aligned} \tag{A.6}$$

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