CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXIII-2

A 2CAT-INSPIRED MODEL STRUCTURE FOR DOUBLE CATEGORIES

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Résumé. On construit une stucture de modèles sur la catégorie DblCat des doubles catégories et doubles foncteurs. Contrairement aux structures de modèles existantes sur les doubles catégories, ces nouvelles structures de modèles recouvrent la structure de modèles de Lack sur les 2-catégories via le plongement horizontal $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Ce dernier est à la fois un adjoint de Quillen à gauche et à droite, et est homotopiquement plein et fidèle. De plus, on obtient un enrichissement sur $2\mathrm{Cat}$ de notre structure de modèles sur DblCat, en utilisant une variante du produit tensoriel de Gray.

Sous certaines conditions, on prouve un théorème de Whitehead qui caractérise nos équivalences faibles comme étant les doubles foncteurs qui admettent un pseudo-inverse à équivalence horizontale pseudo-naturelle près.

Abstract. We construct a model structure on the category DblCat of double categories and double functors. Unlike previous model structures for double categories, it recovers the homotopy theory of 2-categories through the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which is both left and right Quillen, and homotopically fully faithful. Furthermore, we show that Lack's model structure on $2\mathrm{Cat}$ is both left- and right-induced along $\mathbb{H}$ from our model structure on DblCat. In addition, we obtain a $2\mathrm{Cat}$-enrichment of our model structure on DblCat, by using a variant of the Gray tensor product.

Under certain conditions, we prove a Whitehead theorem, characterizing our weak equivalences as the double functors which admit an inverse pseudo double functor up to horizontal pseudo natural equivalence.

Keywords. Double categories, 2-categories, homotopy theory, enriched

model categories, Whitehead theorem.
Mathematics Subject Classification (2020). 18N10; 18N40; 18D20.

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1. Introduction

In category theory as well as homotopy theory, we strive to find the correct notion of "sameness", often with a specific context or perspective in mind. When working with categories themselves, it is commonly agreed that having an isomorphism between categories is much too strong a requirement, and we instead concur that the right condition to demand is the existence of an *equivalence* of categories.

There are many ways one can justify this in practice, but, at heart, it is due to the fact that the category Cat of categories and functors actually forms a 2-category, with 2-cells given by the natural transformations. Therefore, instead of asking that a functor $F\colon \mathcal{A} \to \mathcal{B}$ has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ such that their composites are *equal* to the identities, it is more natural to ask for the existence of *natural isomorphisms* $\mathrm{id}_{\mathcal{A}} \cong GF$ and $FG \cong \mathrm{id}_{\mathcal{B}}$. In particular, this characterizes F as a functor that is surjective on objects up to isomorphism, and fully faithful on morphisms.

Ever since Quillen's seminal work [21], and even more so in the last two decades, we have come to expect that any reasonable notion of equivalence in a category should lend itself to defining the class of weak equivalences of a model structure. This is in fact the case of the categorical equivalences: the category Cat can be endowed with a model structure, called the *canonical model structure*, in which the weak equivalences are precisely the equivalences of categories.

Going one dimension up and focusing on 2-categories, the 2-functors themselves now form a 2-category, with higher cells given by the pseudo natural transformations, and the so-called modifications between them. We can then define a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to be a *biequivalence* if it has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{B}}$, i.e., equivalences in the corresponding 2-categories of 2-dimensional functors. Note that this inverse G is in general a pseudo functor rather than a 2-functor. Furthermore, a Whitehead theorem for 2-categories [14, Theorem 7.4.1] is available, and characterizes the biequivalences as the 2-functors that are surjective on objects up to equivalence, full on morphisms up to invertible 2-cell, and fully faithful on 2-cells.

As in the case of the equivalences of categories, the biequivalences of 2-categories are part of the data of a model structure. Indeed, in [15, 16], Lack defines a model structure on the category $2\mathrm{Cat}$ of 2-categories and 2-functors in which the weak equivalences are precisely the biequivalences; we henceforth refer to it as the *Lack model structure*. In particular, the canonical homotopy theory of categories embeds reflectively in this homotopy theory of 2-categories.

In this paper, we consider another type of 2-dimensional objects, called *double categories*, which have both horizontal and vertical morphisms between pairs of objects, related by 2-dimensional cells called *squares*. These are more structured than 2-categories, in the sense that a 2-category $\mathcal{A}$ can be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ with only trivial vertical morphisms. As a consequence, the study of various notions of 2-category theory benefits from a passage to double categories. For example, a 2-limit of a 2-functor F does not coincide with a 2-terminal object in the slice 2-category of cones, as shown in [2, Counter-example 2.12]. However, by considering the 2-functor F as a horizontal double functor $\mathbb{H}F$, Grandis and Paré prove that a 2-limit of F is precisely a double terminal object in the slice double category of cones over $\mathbb{H}F$; see [9, 11, §4.2] and [8, Theorem 5.6.5].

This horizontal embedding of 2-categories into double categories is fully faithful, and we expect to have a homotopy theory of double categories that contains that of 2-categories; constructing such a homotopy theory is the aim of this paper.

The idea of defining a model structure on the category of double categories is scarcely a new one. In [5], Fiore and Paoli construct a Thomason model structure on the category DblCat of double categories and double functors (more precisely, on the category of n-fold categories), and in [6], Fiore, Paoli, and Pronk construct several categorical model structures on DblCat. However, the horizontal embedding of 2-categories does not induce a Quillen pair between the Lack model structure on $2\mathrm{Cat}$ and any of these model structures on DblCat; this follows from Lemma 8.8. Some intuition is provided by the fact that their categorical model structures on DblCat are constructed from the canonical model structure on Cat. As a result, the weak equivalences in each of these model structures induce two equivalences of categories: one between the categories of objects and horizontal morphisms, and one between the categories of vertical morphisms and squares. However, a biequivalence between 2-categories does not generally induce an equivalence between the underlying categories. Therefore, the horizontal embedding of $2\mathrm{Cat}$ into DblCat will not preserve weak equivalences.

In order to remedy this loss of higher data, we aim to extract from a double category $\mathbb{A}$ two 2-categories whose underlying categories are precisely the ones mentioned above. First, we can promote the underlying category of objects and horizontal morphisms of $\mathbb{A}$ to a 2-category by using the right adjoint to the horizontal embedding $\mathbb{H}$: this is a well-known construction given by the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$, whose 2-cells are given by those squares of $\mathbb{A}$ with trivial vertical boundaries. As shown by Ehresmann and Ehresmann in [4], the category DblCat is cartesian closed, and we denote by $[-,-]$ its internal hom double categories. We can then alternatively describe the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ as the 2-category $\mathbf{H}[\mathbb{1},\mathbb{A}]$, where $\mathbb{1}$ denotes the terminal category.

From this perspective, the category of vertical morphisms and squares can be seen as the underlying horizontal category of the double category $[\mathbb{V}2, \mathbb{A}]$, where $\mathbb{V}2$ is the free double category on a vertical morphism. To promote this to a 2-category we can simply consider instead the underlying horizontal 2-category $\mathbf{H}[\mathbb{V}2, \mathbb{A}]$; this defines a new functor $\mathcal{V}$ that sends a double category $\mathbb{A}$ to a 2-category $\mathcal{V}\mathbb{A}$ of vertical morphisms, squares, and 2-cells as described in Definition 2.11.

Using these constructions, we introduce a new notion of weak equivalences between double categories, that we suggestively call *double biequivalences*; these are given by the double functors F such that the induced 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. This provides a 2-categorical analogue of notions of equivalences between double categories already present in the literature. Notably, double biequivalences are the natural 2-categorical version of equivalences described by Grandis in [8, Theorem 4.4.5 (iv)], which are precisely the double functors inducing equivalences between the categories of objects and horizontal morphisms, and the categories of vertical morphisms and squares.

Since biequivalences can be characterized as the 2-functors which are surjective on objects up to equivalence, full on morphisms up to invertible 2-cell, and fully faithful on 2-cells, our double biequivalences admit a similar description. To give such a description, we introduce new notions of weak invertibility for horizontal morphisms and squares in a double category $\mathbb{A}$; namely, those of *horizontal equivalences* and *weakly horizontally invertible squares*, which correspond to the equivalences in the 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$, respectively. These notions were independently developed by Grandis and Paré in [10, §2], where the weakly horizontally invertible squares are called *equivalence cells*. Now the double biequivalences can be described as the double functors which are surjective on objects up to horizontal equivalence, full on horizontal morphisms up to vertically invertible square, surjective on vertical morphisms up to weakly horizontally invertible square, and fully faithful on squares.

The double biequivalences are designed in such a way that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if its associated horizontal double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence. This can be seen as a first step towards showing that the homotopy theory of 2-categories sits inside that of double categories. Note that "surjectivity" rather than "fullness" on vertical morphisms is necessary to achieve our goal of defining a model structure on DblCat compatible with the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Indeed, as we want $\mathbb{H}$ to preserve weak equivalences, and as the 2-category E_{adj} given by the free-living adjoint equivalence is biequivalent to the terminal category $\mathbb{1}$, the double functor $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ should be a weak equivalence in DblCat. It is then straightforward to check that such a double functor cannot be full on vertical morphisms, as there is no vertical morphism between the two distinct objects of the horizontal double category $\mathbb{H}E_{\mathrm{adj}}$.

Our first main result, Theorem 3.18, provides the desired model structure on the category of double categories.

Theorem A. *Consider the adjunction*

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,,$$

where each copy of $2\mathrm{Cat}$ *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence in this model structure if and only if it is a double biequivalence.*

Since the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated, so is the model structure on DblCat constructed above. Moreover, every double category is fibrant, since all objects are fibrant in $2\mathrm{Cat}$.

By taking a closer look at the homotopy equivalences in our model structure on DblCat, we identify them as the double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ and two horizontal pseudo natural equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$. In particular, the usual Whitehead theorem for model structures (see [3, Lemma 4.24]) allows us to identify the double biequivalences between cofibrant double categories as the homotopy equivalences described above.

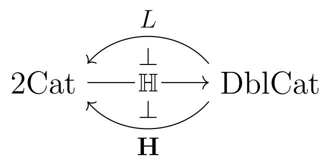
In fact, we show in Theorem 5.13 that a more lax version of this result, involving a horizontally pseudo double functor G, holds for an even larger class of double categories containing the cofibrant objects; this mirrors the definition of biequivalences in $2\mathrm{Cat}$, which further supports the fact that our double biequivalences provide a good notion of weak equivalences between double categories. As a corollary, we retrieve the Whitehead theorem for 2-categories mentioned above.

Theorem B. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories such that the underlying vertical category $U\mathbf{V}\mathbb{B}$ is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence if and only if there is a normal horizontally pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$, and horizontal pseudo natural equivalences $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ and $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$.*

This Whitehead Theorem is reminiscent of a result by Grandis in [8, Theorem 4.4.5] which characterizes the 1-categorical version of our double biequivalences under a different assumption on the double categories involved; namely, that of horizontal invariance. In [20, Definition 2.10], the authors introduce a notion of *weakly horizontally invariant* double categories, and use them to prove yet another Whitehead Theorem for double biequivalences; see [20, Theorem 8.1]. Moreover, the weakly horizontally invariant double categories are identified as the fibrant objects in a different model structure on DblCat, whose study is the purpose of [20].

We now address our original motivation of constructing a homotopy theory for double categories that contains that of 2-categories through the horizontal embedding. Our model structure on DblCat successfully achieves this goal, and moreover, exhibits the greatest possible compatibility with respect to the horizontal embedding $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ that one could hope for, as studied in Section 6.

Theorem C. *The adjunctions*



are both Quillen pairs between the Lack model structure on $\mathrm{2Cat}$ and the model structure on DblCat of Theorem A. Moreover, the functor $\mathbb{H}$ is homotopically fully faithful, and the Lack model structure on $\mathrm{2Cat}$ is both left- and right-induced from our model structure on DblCat along $\mathbb{H}$.

As a consequence, a 2-functor F is a cofibration, fibration or weak equivalence in $2\mathrm{Cat}$ if and only if the double functor $\mathbb{H}F$ is a cofibration, fibration or weak equivalence in DblCat, respectively.

Having established the exceptional behavior of our model structure with the horizontal embedding, we want to further investigate its relation with the Lack model structure on $2\mathrm{Cat}$. Lack shows in [15] that the model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product. In the double categorical setting, there is an analogous monoidal structure on DblCat given by the Gray tensor product constructed by Böhm in [1]. However, this monoidal structure is not compatible with our model structure on DblCat (see Remark 7.3), since it treats the vertical and horizontal directions symmetrically, while our model structure does not. Nevertheless, restricting this Gray tensor product for double categories in one of the variables to $2\mathrm{Cat}$ via $\mathbb{H}$ removes this symmetry and provides an enrichment of DblCat over $2\mathrm{Cat}$ that is compatible with our model structure. More precisely, this enrichment is given by the hom 2-categories of double functors, horizontal pseudo natural transformations, and modifications between them, denoted by $\mathbf{H}[-,-]_{\mathrm{ps}}$.

Theorem D. *The model structure on* DblCat *of Theorem A is a* $2\mathrm{Cat}$*-enriched model structure, where the enrichment is given by* $\mathbf{H}[-,-]_{\mathrm{ps}}$.

The fact that horizontal pseudo natural transformations play a key role was to be expected, since they are the type of transformations that detect our weak equivalences, as established in our version of the Whitehead theorem above.

Acknowledgements

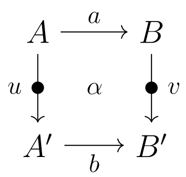
The authors would like to thank Martina Rovelli for reading an early version of this paper and providing many helpful comments; especially, for suggesting at the beginning of this project that we could induce a model structure on DblCat from two copies of $2\mathrm{Cat}$. Philip Hackney and Martina Rovelli pointed out that the Lack model structure was further left-induced along the horizontal embedding from our model structure, and Jérôme Scherer pointed us towards a better construction of path objects. The authors are also grateful to tslil clingman, for suggesting a construction that became our functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, and to Viktoriya Ozornova and Jérôme Scherer, for spotting many typos in previous versions. The authors would also like to thank the anonymous referee for helpful comments that improved the readability of the paper.

This work started when the first- and third-named authors were at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. During the realization of this work, the first-named author was supported by the Swiss National Science Foundation under the project P1ELP2_188039 and the Max Planck Institute of Mathematics. The third-named author was supported by an international Macquarie University Research Excellence Scholarship.

2. Double categorical preliminaries

In this section, we recall the basic notions about double categories, and also introduce non-standard definitions and terminology that will be used throughout the paper. The reader familiar with double categories may wish to jump directly to Definition 2.11.

Definition 2.1. A **double category** $\mathbb{A}$ consists of objects, horizontal morphisms, vertical morphisms, and squares, which we denote by



with horizontal compositions for horizontal morphisms and squares and vertical compositions for vertical morphisms and squares, which are associative and unital, and such that the horizontal and vertical compositions of squares satisfy the interchange law.

We write id_A and e_A for the horizontal and vertical identity at an object A, e_a for the vertical identity square at a horizontal morphism a, and id_u for the horizontal identity square at a vertical morphism u.

Definition 2.2. Let $\mathbb{A}$, $\mathbb{B}$ be double categories. A **double functor** $F\colon \mathbb{A} \to \mathbb{B}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with domains and codomains and preserve all double categorical compositions and identities strictly.

Notation 2.3. We write DblCat for the category of double categories and double functors.

Proposition 2.4 ([6, Proposition 2.11])**.** *The category* DblCat *is cartesian closed. We denote by* $[\mathbb{A},\mathbb{B}]$ *the* ***hom double category*** *for* $\mathbb{A},\mathbb{B}\in\mathrm{DblCat}$. *In particular, for every double category* $\mathbb{A}$*, there is an adjunction*

$$\mathrm{DblCat} \underset{[\mathbb{A},-]}{\overset{-\times\mathbb{A}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{DblCat}\,.$$

There is another monoidal structure on the category of double categories introduced by Böhm in [1], similar to the Gray tensor product for 2-categories.

Proposition 2.5 ([1, §3])**.** *There is a symmetric monoidal structure on the category* DblCat *given by the Gray tensor product*

$$\otimes_{\mathrm{Gr}}\colon \mathrm{DblCat}\times\mathrm{DblCat}\to\mathrm{DblCat}.$$

Moreover, this monoidal structure is closed and we denote by $[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$ *the* ***pseudo hom double category*** *for* $\mathbb{A},\mathbb{B}\in\mathrm{DblCat}$. *In particular, for every double category* $\mathbb{A}$*, there is an adjunction*

$$\mathrm{DblCat} \underset{[\mathbb{A},-]_{\mathrm{ps}}}{\overset{-\otimes_{\mathrm{Gr}}\mathbb{A}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{DblCat}\,.$$

Remark 2.6. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a horizontal morphism in the pseudo hom $[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$ is a **horizontal pseudo natural transformation** $h\colon F\Rightarrow G\colon\mathbb{A}\to\mathbb{B}$. It consists of

(i) a horizontal morphism $h_A\colon FA\to GA$ in $\mathbb{B}$, for each object $A\in\mathbb{A}$,

(ii) a square $h_u\colon (Fu\,{}^{h_A}_{h_{A'}}\,Gu)$ in $\mathbb{B}$, for each vertical morphism $u\colon A\twoheadrightarrow A'$ in $\mathbb{A}$, and

(iii) a vertically invertible square $h_a\colon \left(e_{FA}\,{}^{(Ga)h_A}_{h_B(Fa)}\,e_{GB}\right)$ in $\mathbb{B}$, for each horizontal morphism $a\colon A \to B$ in $\mathbb{A}$, expressing a pseudo naturality condition for horizontal morphisms.

These assignments of squares are functorial with respect to compositions of horizontal and vertical morphisms, and these data satisfy a naturality condition with respect to squares.

In comparison, the horizontal morphisms in the (strict) hom $[\mathbb{A}, \mathbb{B}]$ are horizontal pseudo natural transformations h such that the vertically invertible squares h_a are identity squares for all a. See [8, §3.2.7] for an explicit description of the data of the hom double category $[\mathbb{A}, \mathbb{B}]$ and [8, §3.8] or [1, §2.2] for the pseudo hom $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$.

As mentioned in the introduction, there is a full horizontal embedding of the category $2\mathrm{Cat}$ of 2-categories and 2-functors into DblCat.

Definition 2.7. The **horizontal embedding functor** $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is defined as follows. It takes a 2-category $\mathcal{A}$ to the double category $\mathbb{H}\mathcal{A}$ having the same objects as $\mathcal{A}$, the morphisms of $\mathcal{A}$ as horizontal morphisms, only identities as vertical morphisms, and squares

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & B \\ \| & \alpha & \| \\ A & \xrightarrow[\;b\;]{} & B \end{array}$$

given by the 2-cells $\alpha\colon a \Rightarrow b$ in $\mathcal{A}$. It sends a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ that acts as F does on the corresponding data.

The functor $\mathbb{H}$ admits a right adjoint given by the following.

Definition 2.8. We define the functor $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$. It takes a double category $\mathbb{A}$ to its **underlying horizontal 2-category** $\mathbf{H}\mathbb{A}$, i.e., the 2-category whose objects are the objects of $\mathbb{A}$, whose morphisms are the horizontal morphisms of $\mathbb{A}$, and whose 2-cells $\alpha\colon a \Rightarrow b$ are given by the squares in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \Vert & \alpha & \Vert \\ A & \xrightarrow[b]{} & B \end{array}.$$

It sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ that acts as F does on the corresponding data.

Proposition 2.9 ([6, Proposition 2.5]). *The functors* $\mathbf{H}$ *and* $\mathbb{H}$ *form an adjunction*

$$2\mathrm{Cat} \underset{\mathbf{H}}{\overset{\mathbb{H}}{\underset{\perp}{\rightleftarrows}}} \mathrm{DblCat}\,.$$

Moreover, the unit $\eta\colon \mathrm{id} \Rightarrow \mathbf{H}\mathbb{H}$ *is the identity.*

Remark 2.10. We can also define a functor $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, sending a 2-category to its associated vertical double category with only trivial horizontal morphisms, and a functor $\mathbf{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, sending a double category to its underlying vertical 2-category. These form an adjunction $\mathbb{V} \dashv \mathbf{V}$.

We now introduce a new functor between DblCat and $2\mathrm{Cat}$ that extracts, from a double category, a 2-category whose objects and morphisms are the vertical morphisms and squares; this is the functor $\mathcal{V}$ mentioned in the introduction. In order to do this, we use the category $\mathbb{V}\mathbb{2}$, where $\mathbb{2}$ is the (2-)category $\{0 \to 1\}$ free on a morphism. This double category $\mathbb{V}\mathbb{2}$ is therefore the double category free on a vertical morphism.

Definition 2.11. We define the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ as the composite

$$\mathrm{DblCat} \xrightarrow{[\mathbb{V}\mathbb{2},-]} \mathrm{DblCat} \xrightarrow{\mathbf{H}} 2\mathrm{Cat}.$$

Explicitly, it sends a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A} = \mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{A}]$ given by the following data.

(i) An object in $\mathcal{V}\mathbb{A}$ is a vertical morphism $u\colon A \to\!\!\!\!\bullet\;\, A'$ in $\mathbb{A}$.

(ii) A morphism $(a, b, \alpha)\colon u \to v$ is a square in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\ A' & \xrightarrow[b]{} & B' . \end{array}$$

(iii) A 2-cell $(\sigma_0, \sigma_1)\colon (a, b, \alpha) \Rightarrow (c, d, \beta)$ consists of two squares σ_0 and σ_1 in $\mathbb{A}$ such that the following pasting equality holds.

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \Vert\!\bullet & \sigma_0 & \bullet\!\Vert \\ A & \xrightarrow{c} & B \\ {\scriptstyle u}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle v} \\ A' & \xrightarrow[d]{} & B' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\ A' & \xrightarrow{b} & B' \\ \Vert\!\bullet & \sigma_1 & \bullet\!\Vert \\ A' & \xrightarrow[d]{} & B' \end{array}$$

By Propositions 2.4 and 2.9, we obtain the following.

Proposition 2.12. *The functor $\mathcal{V}$ has a left adjoint $\mathbb{L}$*

$$2\mathrm{Cat} \underset{\mathcal{V}}{\overset{\mathbb{L}}{\rightleftarrows}} \mathrm{DblCat} \quad (\mathbb{L} \dashv \mathcal{V})$$

given by $\mathbb{L} = \mathbb{H}(-) \times \mathbb{V}\mathbb{2}$.

Notation 2.13. We denote by $\otimes_2\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$ the Gray tensor product for 2-categories. It makes $2\mathrm{Cat}$ into a closed symmetric monoidal category with internal homs given by $\mathrm{Ps}[\mathcal{A}, \mathcal{B}]$: the 2-category of 2-functors from $\mathcal{A}$ to $\mathcal{B}$, pseudo natural transformations, and modifications.

The following technical result, which exhibits the behavior of the functors $\mathbb{H}$, $\mathbf{H}$, and $\mathcal{V}$ with respect to pseudo homs, will be of use when we prove the existence of the desired model structure.

Lemma 2.14. *Let $\mathcal{B}$ be a 2-category and $\mathbb{A}$ be a double category. Then there are isomorphisms of 2-categories*

$$\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{A}] \ \textit{ and } \ \mathcal{V}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathcal{V}\mathbb{A}]$$

natural in $\mathcal{B}$ and $\mathbb{A}$.

Proof. We first consider the isomorphism $\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{A}]$. On objects, this follows from the adjunction $\mathbb{H} \dashv \mathbf{H}$ given in Proposition 2.9. On morphisms, as there are no non-trivial vertical morphisms in $\mathbb{H}\mathcal{B}$, horizontal pseudo natural transformations out of $\mathbb{H}\mathcal{B}$ are canonically the same as pseudo natural transformations out of $\mathcal{B}$. The argument for 2-morphisms is similar.

For the second isomorphism, first note that $[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}} = [\mathbb{V}2,\mathbb{A}]$, since there are no non-trivial horizontal morphisms in $\mathbb{V}2$, and therefore horizontal pseudo natural transformations out of $\mathbb{V}2$ correspond to horizontal (strict) natural transformations out of $\mathbb{V}2$. Therefore, we have that

$$\begin{aligned}\mathcal{V}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} &= \mathbf{H}[\mathbb{V}2,[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}} \cong \mathbf{H}[\mathbb{H}\mathcal{B},[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}} \\ &\cong \mathrm{Ps}[\mathcal{B},\mathbf{H}[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}}] = \mathrm{Ps}[\mathcal{B},\mathcal{V}\mathbb{A}],\end{aligned}$$

where the first isomorphism follows from the symmetry of the Gray tensor product on DblCat; see Proposition 2.5 below. □

We conclude this section by introducing new notions of weak invertibility for horizontal morphisms and squares in a double category, together with some technical results that will be of use later in the paper. We do not prove these results here, but instead refer the reader to work by the first author [18, Appendix A]. These notions and results were independently developed by Grandis and Paré in [10, §2].

Definition 2.15. A horizontal morphism $a\colon A \to B$ in a double category $\mathbb{A}$ is a **horizontal equivalence** if it is an equivalence in the 2-category $\mathbf{H}\mathbb{A}$.

Definition 2.16. A square $\alpha\colon (u \ {}^{a}_{b} \ v)$ in a double category $\mathbb{A}$ is **weakly horizontally invertible** if it is an equivalence in the 2-category $\mathcal{V}\mathbb{A}$. See [20, Definition 2.5] for a more detailed description.

Remark 2.17. In particular, the horizontal boundaries a and b of a weakly horizontally invertible square α are horizontal equivalences, which we refer to as the *horizontal equivalence data* of α.

Since any equivalence in a 2-category can be promoted to an adjoint equivalence (see, for example, [22, Lemma 2.1.11]), we get the following result.

Lemma 2.18. *Every horizontal equivalence can be promoted to a horizontal* adjoint *equivalence. Similarly, every weakly horizontally invertible square can be promoted to one with horizontal* adjoint *equivalence data.*

Finally, we conclude with a result concerning weakly horizontally invertible squares.

Lemma 2.19 ([18, Lemma A.2.1])**.** *A square whose horizontal boundaries are horizontal equivalences, and whose vertical boundaries are identities, is weakly horizontally invertible if and only if it is vertically invertible.*

Remark 2.20. It follows that, for a 2-category $\mathcal{A}$, a weakly horizontally invertible square in the double category $\mathbb{H}\mathcal{A}$ corresponds to an invertible 2-cell in $\mathcal{A}$.

3. Model structure for double categories

This section contains our first main result, which proves the existence of a model structure on DblCat constructed as a right-induced model structure along the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$, where both copies of $2\mathrm{Cat}$ are endowed with the Lack model structure.

An analogue construction could be done for weak double categories and strict double functors, by considering Lack's model structure on bicategories and strict functors. These enjoy the same relations as the ones studied in this paper; we exclude them for expositional purposes.

3.1 Lack model structure on $2\mathrm{Cat}$

We start by recalling the main features of Lack's model structure on $2\mathrm{Cat}$; see [15, 16]. Its class of weak equivalences is given by the *biequivalences*, and we refer to the fibrations in this model structure as *Lack fibrations*.

Definition 3.1. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **biequivalence** if (b1) for every object $B \in \mathcal{B}$, there is an object $A \in \mathcal{A}$ and an equivalence $B \xrightarrow{\simeq} FA$ in $\mathcal{B}$,

(b2) for every pair of objects $A, C \in \mathcal{A}$ and every morphism $b\colon FA \to FC$ in $\mathcal{B}$, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $b \cong Fa$ in $\mathcal{B}$, and

(b3) for every pair of morphisms $a, c\colon A \to C$ in $\mathcal{A}$ and every 2-cell $\beta\colon Fa \Rightarrow Fc$ in $\mathcal{B}$, there is a unique 2-cell $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

Definition 3.2. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **Lack fibration** if

(f1) for every object $C \in \mathcal{A}$ and every equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathcal{B}$, there is an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $Fa = b$, and

(f2) for every morphism $c\colon A \to C$ in $\mathcal{A}$ and every invertible 2-cell $\beta\colon b \cong Fc$ in $\mathcal{B}$, there is an invertible 2-cell $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

Theorem 3.3 ([16, Theorem 4])**.** *There is a cofibrantly generated model structure on* $2\mathrm{Cat}$*, called the* Lack model structure*, in which the weak equivalences are the biequivalences and the fibrations are the Lack fibrations.*

Remark 3.4. Note that every 2-category is fibrant in the Lack model structure.

Recall that a *monoidal model category* is a closed monoidal category which admits a model structure compatible with the monoidal structure; see [19, Definition 5.1]. The Lack model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product.

Theorem 3.5 ([15, Theorem 7.5])**.** *The category* $2\mathrm{Cat}$ *endowed with the Lack model structure is a monoidal model category with respect to the closed symmetric monoidal structure given by the Gray tensor product.*

3.2 Constructing the model structure for DblCat

We introduce double biequivalences in DblCat inspired by the definition of biequivalences in $2\mathrm{Cat}$. Our convention of regarding 2-categories as horizontal double categories justifies the choice of directions when emulating the definition of biequivalences in the context of double categories.

Definition 3.6. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double biequivalence** if

(db1) for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ and a horizontal equivalence $B \xrightarrow{\simeq} FA$ in $\mathbb{B}$,

(db2) for every pair of objects $A, C \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \cong & \| \\ FA & \xrightarrow[Fa]{} & FC \end{array},$$

(db3) for every vertical morphism $v\colon B \to B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ and a weakly horizontally invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & FA \\ v\downarrow & \simeq & \downarrow Fu \\ B' & \xrightarrow[\simeq]{} & FA' \end{array},$$

(db4) for every data in $\mathbb{A}$ as below left, and every square in $\mathbb{B}$ as below right,

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ u\downarrow & & \downarrow u' \\ A' & \xrightarrow[c]{} & C' \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ Fu\downarrow & \beta & \downarrow Fu' \\ FA' & \xrightarrow[Fc]{} & FC' \end{array}$$

there is a unique square $\alpha\colon (u \, {}^{a}_{c} \, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 3.7. In $2\mathrm{Cat}$, one can prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there is a pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{B}}$. Under certain hypotheses, we can show a similar characterization of double biequivalences using *horizontal* pseudo natural equivalences. This is done in Section 5.2.

Similarly to the definition of double biequivalence, we take inspiration from the Lack fibrations to define a notion of *double fibrations*.

Definition 3.8. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double fibration** if

(df1) for every object $C \in \mathbb{A}$ and every horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$, there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $Fa = b$,

(df2) for every horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ and for every vertically invertible square $\beta\colon (e_{FA} \;{}^{b}_{Fc}\; e_{FC})$ in $\mathbb{B}$ as depicted below left, there is a vertically invertible square $\alpha\colon (e_A \;{}^{a}_{c}\; e_C)$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$,

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \beta \cong & \| \\ FA & \xrightarrow[Fc]{} & FC \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \alpha \cong & \| \\ A & \xrightarrow[c]{} & C \end{array}$$

(df3) for every vertical morphism $u'\colon C \to\!\!\!\bullet\; C'$ in $\mathbb{A}$ and every weakly horizontally invertible square $\beta\colon (v \;{}^{\simeq}_{\simeq}\; Fu')$ in $\mathbb{B}$ as depicted below left, there is a weakly horizontally invertible square $\alpha\colon (u \;{}^{\simeq}_{\simeq}\; u')$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$.

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & FC \\ {\scriptstyle v}\downarrow & \beta \simeq & \downarrow{\scriptstyle Fu'} \\ B' & \xrightarrow[\simeq]{} & FC' \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\simeq} & C \\ {\scriptstyle u}\downarrow & \alpha \simeq & \downarrow{\scriptstyle u'} \\ A' & \xrightarrow[\simeq]{} & C' \end{array}$$

By requiring that a double functor is both a double biequivalence and a double fibration, we get a notion of *double trivial fibration*, which can be described as follows.

Definition 3.9. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double trivial fibration** if it satisfies (db4) of Definition 3.6, and the following conditions:

(dt1) for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ such that $B = FA$, (dt2) for every pair of objects $A, C \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ such that $b = Fa$, and

(dt3) for every vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ such that $v = Fu$.

Remark 3.10. Note that (dt2) says that a double trivial fibration is *full* on horizontal morphisms, while (dt3) says that a double trivial fibration is only *surjective* on vertical morphisms.

We can use the functors $\mathbf{H}, \mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ to characterize double biequivalences and double fibrations through biequivalences and Lack fibrations in $2\mathrm{Cat}$. We state these characterizations here, and defer their proofs to Section 5.1.

Proposition 3.11. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence in* DblCat *if and only if the* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are biequivalences in* $2\mathrm{Cat}$.

Proposition 3.12. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration in* DblCat *if and only if the* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are Lack fibrations in* $2\mathrm{Cat}$.

This is intuitively sound, since horizontal equivalences and weakly horizontally invertible squares were defined to be the equivalences in the 2-categories induced by $\mathbf{H}$ and $\mathcal{V}$, respectively.

As a corollary, we get a similar characterization for double trivial fibrations.

Corollary 3.13. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double trivial fibration in* DblCat *if and only if the induced* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are trivial fibrations in the Lack model structure on* $2\mathrm{Cat}$.

To build a model structure on DblCat with these classes of morphisms as its weak equivalences and (trivial) fibrations, we use the notion of *right-induced model structure*. Given a model category $\mathcal{M}$ and an adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N}, \quad L \dashv R \tag{3.14}$$

we can, under certain conditions, induce a model structure on $\mathcal{N}$ along the right adjoint R, in which a weak equivalence (resp. fibration) is a morphism F in $\mathcal{N}$ such that RF is a weak equivalence (resp. fibration) in $\mathcal{M}$.

Propositions 3.11 and 3.12 suggest that the model structure on DblCat we desire, with double biequivalences as the weak equivalences and double fibrations as the fibrations, corresponds to the right-induced model structure, if it exists, along the adjunction

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,, \quad \bot$$

where each copy of $2\mathrm{Cat}$ is endowed with the Lack model structure. To prove the existence of this model structure, we use results by Garner, Hess, Kędziorek, Riehl, and Shipley in [7, 12]. In particular, we use the following theorem, inspired by the original Quillen Path Object Argument [21].

Theorem 3.15. *Let $\mathcal{M}$ be an accessible model category, and let $\mathcal{N}$ be a locally presentable category. Suppose we have an adjunction $L \dashv R$ between them as in (3.14). Suppose moreover that every object in $\mathcal{M}$ is fibrant and that, for every object $X \in \mathcal{N}$, there is a factorization*

$$X \xrightarrow{W} \mathrm{Path}(X) \xrightarrow{P} X \times X$$

of the diagonal morphism in $\mathcal{N}$ such that RP is a fibration in $\mathcal{M}$ and RW is a weak equivalence in $\mathcal{M}$. Then the right-induced model structure on $\mathcal{N}$ exists.

Proof. This follows directly from [19, Theorem 6.2], which is the dual of [12, Theorem 2.2.1]. Indeed, if every object in $\mathcal{M}$ is fibrant, then the underlying fibrant replacement of conditions (i) and (ii) of [19, Theorem 6.2] are trivially given by the identity. □

Our strategy is then to construct a path object $\mathrm{Path}(\mathbb{A})$ for a double category $\mathbb{A}$ together with double functors W and P factorizing the diagonal morphism $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$, such that their images under $(\mathbf{H}, \mathcal{V})$ give a weak equivalence and a fibration in $2\mathrm{Cat} \times 2\mathrm{Cat}$ respectively.

Definition 3.16. Let $\mathbb{A}$ be a double category. We define a **path object** for $\mathbb{A}$ as the double category $\mathrm{Path}(\mathbb{A}) \coloneqq [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$, where the 2-category E_{adj} is the free-living adjoint equivalence. It comes with a factorization of the diagonal double functor

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A},$$

where W is the double functor $\mathbb{A} \cong [\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} \to [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} = \mathrm{Path}(\mathbb{A})$ induced by the unique map $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ and P is the double functor $\mathrm{Path}(\mathbb{A}) = [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \to [\mathbb{1} \sqcup \mathbb{1}, \mathbb{A}]_{\mathrm{ps}} \cong \mathbb{A} \times \mathbb{A}$ induced by the inclusion $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ at the two endpoints. Note that, since the composite $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ is the unique map, the composite PW is the diagonal double functor $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$.

Proposition 3.17. *For every double category $\mathbb{A}$, the path object of Definition 3.16*

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A},$$

is such that $(\mathbf{H}, \mathcal{V})W$ is a weak equivalence and $(\mathbf{H}, \mathcal{V})P$ is a fibration in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Proof. We first prove that $\mathbf{H}W$ and $\mathcal{V}W$ are biequivalences in $2\mathrm{Cat}$. By Lemma 2.14, we have commutative squares

$$\begin{array}{ccc} \mathbf{H}[\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} & \xrightarrow{\mathbf{H}W} & \mathbf{H}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Ps}[\mathbb{1}, \mathbf{H}\mathbb{A}] & \xrightarrow[(\mathbf{H}W)^\sharp]{} & \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \end{array} \qquad \begin{array}{ccc} \mathcal{V}[\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} & \xrightarrow{\mathcal{V}W} & \mathcal{V}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Ps}[\mathbb{1}, \mathcal{V}\mathbb{A}] & \xrightarrow[(\mathcal{V}W)^\sharp]{} & \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}] \end{array}$$

where the 2-functors $(\mathbf{H}W)^\sharp$ and $(\mathcal{V}W)^\sharp$ are induced by the unique map $E_{\mathrm{adj}} \to \mathbb{1}$. As the inclusion $\mathbb{1} \to E_{\mathrm{adj}}$ is a trivial cofibration in $2\mathrm{Cat}$ and $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$ are fibrant 2-categories, by monoidality of the Lack model structure, we get that the induced 2-functors

$$R\colon \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \to \mathrm{Ps}[\mathbb{1}, \mathbf{H}\mathbb{A}] \ \text{ and } \ S\colon \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}] \to \mathrm{Ps}[\mathbb{1}, \mathcal{V}\mathbb{A}]$$

are trivial fibrations in $2\mathrm{Cat}$. As $R(\mathbf{H}W)^\sharp$ and $S(\mathcal{V}W)^\sharp$ compose to the identity, by 2-out-of-3, we get that $(\mathbf{H}W)^\sharp$ and $(\mathcal{V}W)^\sharp$ are biequivalences.

Again, by 2-out-of-3 applied to the commutative squares above, we conclude that $\mathbf{H}W$ and $\mathcal{V}W$ are biequivalences.

Similarly, one can show that $\mathbf{H}P$ and $\mathcal{V}P$ are Lack fibrations in $2\mathrm{Cat}$, since the 2-functor $\mathbb{1} \sqcup \mathbb{1} \to E_{\mathrm{adj}}$ is a cofibration in $2\mathrm{Cat}$. Therefore, the induced 2-functors

$$\mathrm{Ps}[\mathbb{1} \sqcup \mathbb{1}, \mathbf{H}\mathbb{A}] \to \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \text{ and } \mathrm{Ps}[\mathbb{1} \sqcup \mathbb{1}, \mathcal{V}\mathbb{A}] \to \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}]$$

are fibrations in $2\mathrm{Cat}$, by monoidality of the Lack model structure. □

We are finally ready to prove the existence of the right-induced model structure on DblCat along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$.

Theorem 3.18. *Consider the adjunction*

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,, \quad \bot$$

where each copy of $2\mathrm{Cat}$ *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence (resp. fibration) in this model structure if and only if it is a double biequivalence (resp. double fibration).*

Proof. We first describe the weak equivalences and fibrations in the right-induced model structure on DblCat. These are given by the double functors F such that $(\mathbf{H}, \mathcal{V})F$ is a weak equivalence (resp. fibration) in $2\mathrm{Cat} \times 2\mathrm{Cat}$, or equivalently, such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences (resp. Lack fibrations) in $2\mathrm{Cat}$. Then it follows from Propositions 3.11 and 3.12 that the weak equivalences (resp. fibrations) in DblCat are precisely the double biequivalences (resp. double fibrations).

We now prove the existence of the model structure. For this purpose, we want to apply Theorem 3.15 to our setting. First note that $2\mathrm{Cat}$ and DblCat are locally presentable, and that the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated. In particular, this implies that the product $2\mathrm{Cat} \times 2\mathrm{Cat}$ endowed with two copies of the Lack model structure is combinatorial, hence accessible. Moreover, every pair of 2-categories is fibrant in $2\mathrm{Cat} \times 2\mathrm{Cat}$, since every object is fibrant in the Lack model structure. Finally, for every double category $\mathbb{A}$, Proposition 3.17 gives a factorization

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

such that W is a double biequivalence and P is a double fibration. By Theorem 3.15, this proves that the right-induced model structure along $(\mathbf{H}, \mathcal{V})$ on DblCat exists. □

Remark 3.19. Note that every double category is fibrant in this model structure. Indeed, this follows directly from the fact that it is right-induced from a model structure in which every object is fibrant.

4. Generating (trivial) cofibrations and cofibrant objects

In this section, we take a closer look at the (trivial) cofibrations and cofibrant objects in our model structure on DblCat, and we show that the latter is cofibrantly generated.

4.1 Generating sets of (trivial) cofibrations

Recall from Theorem 3.3 that the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated. As a consequence, our model structure on DblCat is also cofibrantly generated.

Proposition 4.1. *Let $\mathcal{I}_2$ and $\mathcal{J}_2$ denote sets of generating cofibrations and generating trivial cofibrations, respectively, for the Lack model structure on* $2\mathrm{Cat}$. *Then, the sets of morphisms in* DblCat

$$\mathcal{I} = \{\mathbb{H}i,\ \mathbb{H}i \times \mathbb{V}\mathbb{2} \mid i \in \mathcal{I}_2\}, \ \text{ and } \ \mathcal{J} = \{\mathbb{H}j,\ \mathbb{H}j \times \mathbb{V}\mathbb{2} \mid j \in \mathcal{J}_2\}$$

give sets of generating cofibrations and generating trivial cofibrations, respectively, for the model structure on DblCat *of Theorem 3.18.*

Proof. Since the model structure on DblCat is right-induced from two copies of the Lack model structure on $2\mathrm{Cat}$ along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, sets of generating cofibrations and of generating trivial cofibrations can be given by the images under the left adjoint $\mathbb{H}\sqcup\mathbb{L}$ of the fixed sets of generating cofibrations and generating trivial cofibrations in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Let i and i' be generating cofibrations in $\mathcal{I}_2$ in $2\mathrm{Cat}$. Then $\mathbb{H}i$ and $\mathbb{L}i = \mathbb{H}i \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. To see this apply $\mathbb{H}\sqcup\mathbb{L}$ to the cofibrations $(i, \mathrm{id}_\emptyset)$ and $(\mathrm{id}_\emptyset, i)$, respectively. Similarly, $\mathbb{H}i'$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. Since coproducts of cofibrations are cofibrations, then $(\mathbb{H} \sqcup \mathbb{L})(i, i') = \mathbb{H}i \sqcup \mathbb{L}i'$ can be obtained from $\mathbb{H}i$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$. This shows that $\mathcal{I}$ is a set of generating cofibrations of DblCat.

Similarly, we can show that $\mathcal{J}$ is a set of generating trivial cofibrations of DblCat. □

However, we can find sets of generating (trivial) cofibrations, which are both smaller and more descriptive than the ones given above, by using the characterization of fibrations and trivial fibrations in our model structure given in Proposition 3.12 and Corollary 3.13.

Notation 4.2. Let $\mathbb{S}$ be the double category free on a square, $\delta\mathbb{S}$ be its boundary, and $\mathbb{S}_2$ be the double category free on two squares with the same boundary.

$$\mathbb{S} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow\!\bullet & \alpha & \downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array} \;; \quad \delta\mathbb{S} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow\!\bullet & & \downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array} \;; \quad \mathbb{S}_2 = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow\!\bullet & \alpha_0 \;\; \alpha_1 & \downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array}$$

We fix notation for the following double functors, which form a set of generating cofibrations for our model structure on DblCat:

- the unique map $I_1 \colon \emptyset \to \mathbb{1}$,
- the inclusion $I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$,
- the unique map $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$,
- the inclusion $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$, and
- the double functor $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$ sending both squares in $\mathbb{S}_2$ to the non-trivial square of $\mathbb{S}$.

We also fix notation for the following double functors, which form a set of generating trivial cofibrations for our model structure on DblCat:

- the inclusion $J_1\colon \mathbb{1} \to \mathbb{H}E_{\text{adj}}$, where the 2-category E_{adj} is the free-living adjoint equivalence,
- the inclusion $J_2\colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\text{inv}}$, where the 2-category C_{inv} is the free-living invertible 2-cell, and
- the inclusion $J_3\colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\text{adj}} \times \mathbb{V}\mathbb{2}$; note that the double category $\mathbb{H}E_{\text{adj}} \times \mathbb{V}\mathbb{2}$ is the free-living weakly horizontally invertible square (with horizontal adjoint equivalence data).

Proposition 4.3. *In the model structure on* DblCat *of Theorem 3.18, a set* $\mathcal{I}'$ *of generating cofibrations is given by*

$$\{I_1\colon \emptyset \to \mathbb{1},\ I_2\colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2},\ I_3\colon \emptyset \to \mathbb{V}\mathbb{2},\ I_4\colon \delta\mathbb{S} \to \mathbb{S},\ I_5\colon \mathbb{S}_2 \to \mathbb{S}\}$$

and a set $\mathcal{J}'$ *of generating trivial cofibrations is given by*

$$\{J_1\colon \mathbb{1} \to \mathbb{H}E_{\text{adj}},\ J_2\colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\text{inv}},\ J_3\colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\text{adj}} \times \mathbb{V}\mathbb{2}\}.$$

Proof. It is a routine exercise to check that a double functor is a double trivial fibration as defined in Definition 3.9 if and only if it has the right-lifting property with respect to the cofibrations in $\mathcal{I}'$, and that a double functor is a double fibration as defined in Definition 3.8 if and only if it has the right-lifting property with respect to the trivial cofibrations of $\mathcal{J}'$. This shows that $\mathcal{I}'$ and $\mathcal{J}'$ are sets of generating cofibrations and generating trivial cofibration for DblCat, respectively. □

4.2 Cofibrations and cofibrant double categories

Our next goal is to provide a characterization of the cofibrations in DblCat. In [15, Lemma 4.1], Lack shows that a 2-functor is a cofibration in $2\mathrm{Cat}$ if and only if its underlying functor has the left lifting property with respect to all surjective on objects and full functors. A similar result applies to our model structure.

First, we state a characterization of the functors in Cat which have the left lifting property with respect to all surjective on objects and full (resp. surjective on morphisms) functors, that will be useful to understand the characterization of cofibrations in Proposition 4.7.

Lemma 4.4. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ has the left lifting property with respect to surjective on objects and full (resp. surjective) on morphisms functors if and only if*

(i) the functor F is injective on objects and faithful, and

(ii) there are functors $I\colon \mathcal{B} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{B}$ such that $RI = \mathrm{id}_{\mathcal{B}}$, where the category $\mathcal{C}$ is obtained from the image of F by freely adjoining objects and then freely adjoining morphisms between specified objects (resp. by freely adjoining objects and morphisms).

Moreover, a functor $\emptyset \to \mathcal{A}$ has the left lifting property with respect to surjective on objects and full (resp. surjective) on morphisms functors if and only if the category $\mathcal{A}$ is free (resp. a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$).

Proof. The statement about "full on morphisms" is proven in [15, Corollary 4.12]. For the "surjective on morphisms" case, the proof is analogous, replacing $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{2}$ by $\emptyset \to \mathbb{2}$.

The second statement about $\emptyset \to \mathcal{A}$ follows from the fact that a retract of a free category is itself free, and similarly for disjoint unions of copies of $\mathbb{1}$ and $\mathbb{2}$. □

Notation 4.5. We write $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$ for the functor that sends a 2-category to its underlying category.

Remark 4.6. The functor $U\mathbf{H}\colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying category of objects and horizontal morphisms, has a right adjoint. It is given by the functor $R_h\colon \mathrm{Cat} \to \mathrm{DblCat}$ that sends a category $\mathcal{C}$ to the double category with the same objects as $\mathcal{C}$, horizontal morphisms given by the morphisms of $\mathcal{C}$, a unique vertical morphism between every pair of objects, and a unique square $!\colon (!\ {}^{f}_{g}\ !)$ for every pair of morphisms f, g in $\mathcal{C}$.

Similarly, the functor $U\mathbf{V}\colon \mathrm{DblCat} \to \mathrm{Cat}$ admits a right adjoint R_v.

Proposition 4.7. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in* DblCat *if and only if*

(i) the underlying horizontal functor $U\mathbf{H}F\colon U\mathbf{H}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and full functors, and

(ii) the underlying vertical functor $U\mathbf{V}F\colon U\mathbf{V}\mathbb{A} \to U\mathbf{V}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and surjective on morphisms functors.

Proof. Suppose first that $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in DblCat, i.e., it has the left lifting property with respect to all double trivial fibrations. In order to show (i), let $P\colon \mathcal{X} \to \mathcal{Y}$ be a surjective on objects and full functor. By the adjunction $U\mathbf{H} \dashv R_h$, saying that $U\mathbf{H}F$ has the left lifting property with respect to P is equivalent to saying that F has the left lifting property with respect to R_hP. We now prove this latter statement.

Note that the double functor $R_hP\colon R_h\mathcal{X} \to R_h\mathcal{Y}$ is surjective on objects and full on horizontal morphisms, since P is so. Moreover, by construction of R_h, there is exactly one vertical morphism and one square for each boundary in both its source and target; therefore R_hP is surjective on vertical morphisms and fully faithful on squares. Hence R_hP is a double trivial fibration, and F has the left lifting property with respect to R_hP since it is a cofibration in DblCat.

Similarly, one can show that (ii) holds, by considering the adjunction $U\mathbf{V} \dashv R_v$ and replacing fullness by surjectivity on morphisms.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ satisfies (i) and (ii). Let $P\colon \mathbb{X} \to \mathbb{Y}$ be a double trivial fibration and consider a commutative square as below left. We want to find a lift $L\colon \mathbb{B} \to \mathbb{X}$ in this square as depicted below. Using (ii), since $U\mathbf{V}P$ is surjective on objects and surjective on morphisms, we have a lift L_v in the below middle diagram. We now wish to find a lift L_h in the diagram below right, that agrees with L_v on objects. Using the characterization of $U\mathbf{H}F$ given in Lemma 4.4, and the fact that $U\mathbf{H}P$ is full, we can extend the given assignment on objects to a functor $L_h\colon U\mathbf{H}\mathbb{B} \to U\mathbf{H}\mathbb{X}$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\ {\scriptstyle F}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle P} \\ \mathbb{B} & \xrightarrow[Q]{} & \mathbb{Y} \end{array} \qquad \begin{array}{ccc} U\mathbf{V}\mathbb{A} & \xrightarrow{U\mathbf{V}G} & U\mathbf{V}\mathbb{X} \\ {\scriptstyle U\mathbf{V}F}\downarrow & \overset{L_v}{\nearrow} & \downarrow{\scriptstyle U\mathbf{V}P} \\ U\mathbf{V}\mathbb{B} & \xrightarrow[U\mathbf{V}Q]{} & U\mathbf{V}\mathbb{Y} \end{array} \qquad \begin{array}{ccc} U\mathbf{H}\mathbb{A} & \xrightarrow{U\mathbf{H}G} & U\mathbf{H}\mathbb{X} \\ {\scriptstyle U\mathbf{H}F}\downarrow & \overset{L_h}{\nearrow} & \downarrow{\scriptstyle U\mathbf{H}P} \\ U\mathbf{H}\mathbb{B} & \xrightarrow[U\mathbf{H}Q]{} & U\mathbf{H}\mathbb{Y} \end{array}$$

Then, since $P\colon \mathbb{X} \to \mathbb{Y}$ is fully faithful on squares, the assignment on objects, horizontal morphisms, and vertical morphisms given by L_h and L_v

uniquely extend to a double functor $L\colon \mathbb{B} \to \mathbb{Y}$, which gives the desired lift. □

Remark 4.8. From Lemma 4.4 and Proposition 4.7, it is straightforward to see that a cofibration in DblCat is in particular injective on objects, and faithful on horizontal morphisms and vertical morphisms.

Finally, as a corollary of Lemma 4.4 and Proposition 4.7, we obtain a characterization of the cofibrant double categories.

Corollary 4.9. *A double category $\mathbb{A}$ is cofibrant if and only if its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free and its underlying vertical category $U\mathbf{V}\mathbb{A}$ is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$.*

5. Fibrations, weak equivalences, and Whitehead theorems

The purpose of this section is to describe the weak equivalences and fibrations of our model structure. Section 5.1 provides proofs of Propositions 3.11 and 3.12, which claim that the weak equivalences and fibrations of the right-induced model structure on DblCat of Theorem 3.18 are precisely the double biequivalences and the double fibrations.

In Section 5.2 we turn our attention to another characterization of the weak equivalences, known as the Whitehead theorem. Recall that, in the 2-categorical case, a 2-functor is a biequivalence if and only if it has a pseudo inverse up to pseudo natural equivalence (see [14, Theorem 7.4.1]). A similar statement does not hold in general for double biequivalences, but it does if we assume cofibrancy on the target double category. In particular, we retrieve the usual Whitehead theorem for model categories applied to our setting, and also the characterization of biequivalences stated above. Another version of the Whitehead theorem for double biequivalences is given in [20, Theorem 8.1], which in turn holds for the fibrant objects of the model structure on DblCat defined therein.

5.1 Characterizations of weak equivalences and fibrations

We first prove Proposition 3.11, dealing with weak equivalences. In order to characterize the double functors F such that $(\mathbf{H}, \mathcal{V})F$ is a weak equivalence in $2\mathrm{Cat} \times 2\mathrm{Cat}$, we express what it means for $\mathbf{H}F$ and $\mathcal{V}F$ to be biequivalences in $2\mathrm{Cat}$; this is done by translating (b1-3) of Definition 3.1 for these 2-functors.

Remark 5.1. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a biequivalence in $2\mathrm{Cat}$ if and only if F satisfies (db1-2) of Definition 3.6, and the following condition:

(hb3) for every pair of horizontal morphisms $a, c\colon A \to C$ in $\mathbb{A}$ and every square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ \Vert\!\bullet & \beta & \Vert\!\bullet \\ FA & \xrightarrow[Fc]{} & FC \end{array},$$

there is a unique square $\alpha\colon (e_A \,{}^{a}_{c}\, e_C)$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 5.2. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a biequivalence in $2\mathrm{Cat}$ if and only if F satisfies (db3) of Definition 3.6, and the following conditions:

(vb2) for every pair of vertical morphisms $u\colon A \nrightarrow A'$ and $u'\colon C \nrightarrow C'$ in $\mathbb{A}$ and every square $\beta\colon (Fu \,{}^{b}_{d}\, Fu')$ in $\mathbb{B}$, there is a square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ and two vertically invertible squares in $\mathbb{B}$ such that the following pasting equality holds,

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \Vert\!\bullet & \cong & \Vert\!\bullet \\ FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow\!\bullet & F\alpha & \bullet\!\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[Fc]{} & FC' \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\downarrow\!\bullet & \beta & \bullet\!\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \\ \Vert\!\bullet & \cong & \Vert\!\bullet \\ FA' & \xrightarrow[Fc]{} & FC' \end{array}$$

(vb3) for every tuple of squares $\alpha\colon (u \,{}^{a}_{c}\, u')$ and $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ in $\mathbb{A}$, and τ_0 and τ_1 in $\mathbb{B}$ as in the pasting equality below left, there are unique squares $\sigma_0\colon (e_A \,{}^{a}_{a'}\, e_C)$ and $\sigma_1\colon (e_{A'} \,{}^{c}_{c'}\, e_{C'})$ in $\mathbb{A}$ satisfying the pasting equality below right, and with the property that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$.

$$
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
\Vert & \tau_0 & \Vert \\
FA & \xrightarrow{Fa'} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha' & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc]{} & FC' \\
\Vert & \tau_1 & \Vert \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\Vert & \sigma_0 & \Vert \\
A & \xrightarrow{a'} & C \\
{\scriptstyle u}\downarrow & \alpha' & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c']{} & C'
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C' \\
\Vert & \sigma_1 & \Vert \\
A' & \xrightarrow[c']{} & C'
\end{array}
$$

The reader may have noticed that condition (db4) in Definition 3.6 regarding fully faithfulness on squares has not been mentioned so far, but it is recovered by the conditions (hb3) and (vb2-3) above.

Lemma 5.3. *Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double functor satisfying (hb3) of Remark 5.1, and (vb2-3) of Remark 5.2. Then F satisfies (db4) of Definition 3.6.*

Proof. Suppose $\beta\colon (Fu \,{}^{Fa}_{Fc}\, Fu')$ is a square in $\mathbb{B}$ as in (db4) of Definition 3.6. By (vb2) of Remark 5.2, there is a square $\overline{\alpha}\colon (u \,{}^{\overline{a}}_{\overline{c}}\, u')$ in $\mathbb{A}$ and two vertically invertible squares ψ_0, ψ_1 in $\mathbb{B}$ such that the following pasting equality holds.

$$
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
\Vert & \psi_0 \,\Vert\!\wr & \Vert \\
FA & \xrightarrow{F\overline{a}} & FC \\
{\scriptstyle Fu}\downarrow & F\overline{\alpha} & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[F\overline{c}]{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
{\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc]{} & FC' \\
\Vert & \psi_1 \,\Vert\!\wr & \Vert \\
FA' & \xrightarrow[F\overline{c}]{} & FC'
\end{array}
$$

By (hb3) of Remark 5.1 applied to ψ_0 and ψ_1, there are unique squares $\varphi_0\colon (e_A \,{}^{a}_{\overline{a}}\, e_C)$ and $\varphi_1\colon (e_{A'} \,{}^{c}_{\overline{c}}\, e_{C'})$ in $\mathbb{A}$ such that $F\varphi_0 = \psi_0$ and $F\varphi_1 = \psi_1$.

Moreover, the squares φ_0 and φ_1 are vertically invertible by the unicity condition in (hb3). Therefore, the square α given by the following vertical pasting

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ u\big\downarrow & \alpha & \big\downarrow u' \\ A' & \xrightarrow[c]{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \varphi_0 \,\cong & \| \\ A & \xrightarrow{\overline{a}} & C \\ u\big\downarrow & \overline{\alpha} & \big\downarrow u' \\ A' & \xrightarrow[\overline{c}]{} & C' \\ \| & \varphi_1^{-1} \cong & \| \\ A' & \xrightarrow[c]{} & C' \end{array}$$

is such that $F\alpha = \beta$. This settles the matter of the existence of the square α. Now suppose there are two squares $\alpha\colon (u \,{}^{a}_{c}\, u')$ and $\alpha'\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta = F\alpha'$. Take $\tau_0 = e_{Fa}$ and $\tau_1 = e_{Fc}$ in (vb3) of Remark 5.2. This gives unique squares σ_0 and σ_1 in $\mathbb{A}$ such that the following pasting equality holds

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \sigma_0 & \| \\ A & \xrightarrow{a} & C \\ u\big\downarrow & \alpha' & \big\downarrow u' \\ A' & \xrightarrow[c]{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & C \\ u\big\downarrow & \alpha & \big\downarrow u' \\ A' & \xrightarrow[c]{} & C' \\ \| & \sigma_1 & \| \\ A' & \xrightarrow[c]{} & C' \end{array}$$

and $F\sigma_0 = e_{Fa}$ and $F\sigma_1 = e_{Fc}$. By unicity in (hb3), we must have $\sigma_0 = e_a$ and $\sigma_1 = e_c$. Replacing σ_0 and σ_1 by e_a and e_c in the pasting diagram above implies that $\alpha = \alpha'$. This proves unicity. □

We can now use the above results to obtain the desired characterization of the weak equivalences in our model structure on DblCat.

Proof of Proposition 3.11. Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double functor such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. By Remarks 5.1 and 5.2, we directly have that F satisfies (db1-3) of Definition 3.6. Moreover, by Lemma 5.3, we also have that F satisfies (db4) of Definition 3.6. This shows that F is a double biequivalence.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence. We want to show that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. To show that $\mathbf{H}F$ is a biequivalence, it suffices to show that (hb3) of Remark 5.1 is satisfied; this follows directly from taking u and u' to be vertical identities in (db4) of Definition 3.6.

It remains to show that $\mathcal{V}F$ is a biequivalence; we do so by proving (vb2-3) of Remark 5.2. To prove (vb2), let $u\colon A \twoheadrightarrow A'$ and $u'\colon C \twoheadrightarrow C'$ be vertical morphisms in $\mathbb{A}$ and β be a square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\big\downarrow & \beta & \big\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \end{array}.$$

By (db2) of Definition 3.6, there are horizontal morphisms $a\colon A \to C$ and $c\colon A' \to C'$ in $\mathbb{A}$ and vertically invertible squares $\varphi_0\colon (e_{FA}\ {}^{b}_{Fa}\ e_{FC})$ and $\varphi_1\colon (e_{FA'}\ {}^{d}_{Fc}\ e_{FC'})$ in $\mathbb{B}$. By (db4) of Definition 3.6, there is a unique square $\alpha\colon (u\ {}^{a}_{c}\ u')$ in $\mathbb{A}$ such that

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\big\downarrow & F\alpha & \big\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[Fc]{} & FC' \end{array} = \begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ \big\| & \varphi_0^{-1}\ \cong & \big\| \\ FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\big\downarrow & \beta & \big\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \\ \big\| & \varphi_1\ \cong & \big\| \\ FA' & \xrightarrow[Fc]{} & FC' \end{array},$$

which gives (vb2). Finally, we prove (vb3). Suppose we have a pasting equality in $\mathbb{B}$ as below left.

$$
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
\Vert & \tau_0 & \Vert \\
FA & \xrightarrow{Fa'} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha' & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc]{} & FC' \\
\Vert & \tau_1 & \Vert \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\Vert & \sigma_0 & \Vert \\
A & \xrightarrow{a'} & C \\
{\scriptstyle u}\downarrow & \alpha' & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c']{} & C'
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C' \\
\Vert & \sigma_1 & \Vert \\
A' & \xrightarrow[c']{} & C'
\end{array}
$$

By applying (db4) of Definition 3.6 to τ_0 and τ_1, we obtain unique squares $\sigma_0\colon (e_A \,{}^{a}_{a'}\, e_C)$ and $\sigma_1\colon (e_{A'} \,{}^{c}_{c'}\, e_{C'})$ in $\mathbb{A}$ such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Moreover, by unicity in (db4) of Definition 3.6, we have the pasting equality above right, since applying F to each vertical composite yields the same squares in $\mathbb{B}$. This proves (vb3), and thus concludes the proof. □

Now we turn our attention to Proposition 3.12, dealing with fibrations. For this, we first translate (f1-2) of Definition 3.2 for $\mathbf{H}F$ and $\mathcal{V}F$.

Remark 5.4. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a fibration in $2\mathrm{Cat}$ if and only if F satisfies (df1-2) of Definition 3.8.

Remark 5.5. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a fibration in $2\mathrm{Cat}$ if and only if F satisfies (df3) of Definition 3.8, and the following condition:

(vf2) for every square $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ in $\mathbb{A}$ and every square $\beta\colon (Fu \,{}^{b}_{d}\, Fu')$ in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the pasting equality below left, there is a square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$, together with vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$ as in the pasting equality below right, such that $F\alpha = \beta$, $F\sigma_0 = \tau_0$, $F\sigma_1 = \tau_1$.

$$
\begin{array}{ccc}
FA & \xrightarrow{b} & FC \\
\Vert & \tau_0\,\cong & \Vert \\
FA & \xrightarrow{Fa'} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha' & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{b} & FC \\
{\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[d]{} & FC' \\
\Vert & \tau_1\,\cong & \Vert \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\Vert & \sigma_0\,\cong & \Vert \\
A & \xrightarrow{a'} & C \\
{\scriptstyle u}\downarrow & \alpha' & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c']{} & C'
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C' \\
\Vert & \sigma_1\,\cong & \Vert \\
A' & \xrightarrow[c']{} & C'
\end{array}
$$

We can now use the above remarks to show the desired characterization of the fibrations in our model structure.

Proof of Proposition 3.12. It is clear that if a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is such that both $\mathbf{H}F$ and $\mathcal{V}F$ are Lack fibrations in $2\mathrm{Cat}$, then it is a double fibration, by Remarks 5.4 and 5.5.

Suppose now that $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration. By Remark 5.4, we directly get that $\mathbf{H}F$ is a Lack fibration in $2\mathrm{Cat}$. To show that $\mathcal{V}F$ is also a Lack fibration, it suffices to show that (vf2) of Remark 5.5 is satisfied. Let $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ be a square in $\mathbb{A}$ and $\beta\colon (Fu \,{}^{b}_{d}\, Fu')$ be a square in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the leftmost pasting equality diagram in (vf2). By (df2) of Definition 3.8, there are vertically invertible squares $\sigma_0\colon (e_A \,{}^{a}_{a'}\, e_C)$ and $\sigma_1\colon (e_{A'} \,{}^{c}_{c'}\, e_{C'})$ in $\mathbb{A}$ such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Then the square α given by the vertical composite

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ u \big\downarrow & \alpha & \big\downarrow u' \\ A' & \xrightarrow[c]{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \sigma_0 \,\cong & \| \\ A & \xrightarrow{a'} & C \\ u \big\downarrow & \alpha' & \big\downarrow u' \\ A' & \xrightarrow[c']{} & C' \\ \| & \sigma_1^{-1} \,\cong & \| \\ A' & \xrightarrow[c]{} & C' \end{array}$$

is such that $F\alpha = \beta$, which proves (vf2). □

5.2 Homotopy equivalences and the Whitehead theorem

Any model category satisfies a Whitehead theorem, stating that the weak equivalences between cofibrant-fibrant objects are precisely the homotopy equivalences; i.e., the morphisms $f\colon X \to Y$ such that there is a morphism $g\colon Y \to X$ with the property that fg and gf are homotopic to the identity. We begin by studying what the notion of homotopy entails in our setting; for this, let us first introduce the notion of horizontal pseudo natural equivalences.

Definition 5.6. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. A horizontal pseudo natural transformation $h\colon F \Rightarrow G$ is a **horizontal pseudo natural equivalence** if

(i) the horizontal morphism $h_A\colon FA \to GA$ is a horizontal equivalence in $\mathbb{B}$, for each object $A \in \mathbb{A}$, and

(ii) the square $h_u\colon (Fu \,{}^{h_A}_{h_{A'}}\, Gu)$ is weakly horizontally invertible in $\mathbb{B}$, for each vertical morphism $u\colon A \to A'$ in $\mathbb{A}$.

If the horizontal morphisms $h_A\colon FA \to GA$ are in addition horizontal adjoint equivalences in $\mathbb{B}$, we say that h is a **horizontal pseudo natural adjoint equivalence**.

We write $h\colon F \simeq G$ for such a horizontal pseudo natural transformation.

Remark 5.7. By [18, Lemma A.3.3], a horizontal pseudo natural (adjoint) equivalence as above is precisely an (adjoint) equivalence in the 2-category $\mathbf{H}[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, or equivalently, a horizontal (adjoint) equivalence in the double category $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$.

With this definition in hand, we get the following characterization of homotopic double functors.

Proposition 5.8. *Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. Then F and G are homotopic via the path object $\mathrm{Path}(\mathbb{B})$ of Definition 3.16 if and only if there is a horizontal pseudo natural adjoint equivalence $F \simeq G$.*

Proof. Recall that the path object $\mathrm{Path}(\mathbb{B})$ of Definition 3.16 is given by the pseudo hom double category $[\mathbb{H}E_{\mathrm{adj}}, \mathbb{B}]_{\mathrm{ps}}$, where the 2-category E_{adj} is the free-living adjoint equivalence $\{0 \xrightarrow{\simeq} 1\}$. Therefore, a homotopy between double functors $F, G\colon \mathbb{A} \to \mathbb{B}$ via the path object $\mathrm{Path}(\mathbb{B})$ is a double functor $h\colon \mathbb{A} \to [\mathbb{H}E_{\mathrm{adj}}, \mathbb{B}]_{\mathrm{ps}}$ such that $Ph = (F, G)$ or, equivalently, a double functor

$$\widehat{h}\colon \mathbb{H}E_{\mathrm{adj}} \to [\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$$

such that $\widehat{h}(0) = F$ and $\widehat{h}(1) = G$. This corresponds to a horizontal pseudo natural adjoint equivalence $F \simeq G$ by Remark 5.7. $\square$

Remark 5.9. By the usual Whitehead theorem (see, for example, [3, Lemma 4.24]), a morphism between cofibrant-fibrant objects in a model category is a weak equivalence if and only if it is a homotopy equivalence. Hence, since all double categories are fibrant in the model structure of Theorem 3.18, we can use Proposition 5.8 to characterize double biequivalences between cofibrant objects in DblCat as those double functors which admit an inverse up to horizontal pseudo natural adjoint equivalence, i.e., double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal pseudo natural adjoint equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$.

In our double categorical setting, we can prove a version of the Whitehead theorem for a wider class of weak equivalences, by only imposing a condition on their target double categories. However, in some cases, the homotopy inverse is not a strict double functor anymore, but it is rather pseudo in the horizontal direction.

Definition 5.10. A **horizontally pseudo double functor** $F\colon \mathbb{A} \to \mathbb{B}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with domains and codomains. These maps preserve identities and compositions of vertical morphisms and of squares strictly, but they preserve identities and compositions of horizontal morphisms only up to vertically invertible squares. These are submitted to associativity, unitality, and naturality conditions. See [8, Definition 3.5.1] for details (note, however, that our definition has reversed the roles of the horizontal and vertical directions).

If F strictly preserves horizontal identities, we say that F is **normal**.

Remark 5.11. Analogously to Remark 2.6 and Definition 5.6, we have notions of horizontal pseudo natural transformations and horizontal pseudo natural equivalences between horizontally pseudo double functors. See [8, §3.8] for precise definitions; note that our definition has reversed the roles of the horizontal and vertical directions.

Our class of double biequivalences contains in particular the double functors that have a horizontally pseudo inverse up to horizontal pseudo natural equivalence.

Proposition 5.12. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. If there is a normal horizontally pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal* *pseudo natural equivalences* $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$*, then* F *is a double biequivalence.*

Proof. Under these assumptions, the double functor F is in particular a horizontal biequivalence as introduced in [20, Definition 8.8]. Therefore F is a double biequivalence by [20, Proposition 8.11]. □

By only requiring that the target of a double biequivalence F does not contain any non-trivial composites of vertical morphisms, we can construct a horizontally pseudo double functor which gives a homotopy inverse of F. As the construction of this homotopy inverse is practically identical to the one in [20, Proposition 8.12], we only specify here the data of the pseudo inverse and of one of the horizontal pseudo natural equivalences, and refer the reader to the proof of [20, Proposition 8.12] for details.

Theorem 5.13. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be double categories such that the underlying vertical category* $U\mathbf{V}\mathbb{B}$ *is a disjoint union of copies of* $\mathbb{1}$ *and* $\mathbb{2}$*. Then a double functor* $F\colon \mathbb{A} \to \mathbb{B}$ *is a double biequivalence if and only if there is a normal horizontally pseudo double functor* $G\colon \mathbb{B} \to \mathbb{A}$*, and horizontal pseudo natural equivalences* $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$*.*

Proof. By Proposition 5.12, we directly get the converse implication.

Now suppose that F is a double biequivalence. We highlight the definition of the horizontally pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ and the horizontal pseudo natural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$ on objects and vertical morphisms as it is the only part of the construction that differs from [20, Proposition 8.12]. One can easily check that the rest of the proof of [20, Proposition 8.12] does not depend on the weakly horizontally invariant condition that is not required in this statement, and thus can be applied verbatim.

To define G and ϵ on objects and vertical morphisms, we give the values of G and ϵ on each copy of $\mathbb{1}$ and $\mathbb{2}$ in $U\mathbf{V}\mathbb{B}$.

- Given a copy of the form $B\colon \mathbb{1} \to U\mathbf{V}\mathbb{B}$, by (db1) applied to the object $B \in \mathbb{B}$, we get an object $A \in \mathbb{A}$ and a horizontal equivalence $f\colon FA \xrightarrow{\simeq} B$ in $\mathbb{B}$. We set $GB := A$ and $\epsilon_B := f\colon FGB \xrightarrow{\simeq} B$.

- Given a copy of the form $v\colon \mathbb{2} \to U\mathbf{V}\mathbb{B}$, by (db3) applied to the vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{B}$, we get a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square β in $\mathbb{B}$ as follows.

$$\begin{array}{ccc} FA & \xrightarrow[\simeq]{f} & B \\ {\scriptstyle Fu}\big\downarrow\!\bullet & \beta \ \simeq & \bullet\!\big\downarrow{\scriptstyle v} \\ FA' & \xrightarrow[g]{\simeq} & B' \end{array}$$

We set $GB := A$, $GB' := A'$, and $Gv := u$, and we set $\epsilon_B := f\colon FGB \xrightarrow{\simeq} B$, $\epsilon_{B'} := g\colon FGB' \xrightarrow{\simeq} B'$, and $\epsilon_v := \beta\colon (FGv \, {}^{\epsilon_B}_{\epsilon_{B'}} \, v)$.

As there are no composites of vertical morphisms in $\mathbb{B}$, G and ϵ are trivially compatible with vertical morphisms. □

Remark 5.14. If we further require that the double category $\mathbb{B}$ in Theorem 5.13 is cofibrant, we can construct the weak inverse $G\colon \mathbb{B} \to \mathbb{A}$ of F in such a way that it is a strict double functor, since the underlying horizontal category of $\mathbb{B}$ is free. This subsumes the usual Whitehead theorem mentioned in Remark 5.9.

Finally, as a horizontal double category has a discrete underlying vertical category, the result applies in particular to the case where $\mathbb{B}$ is horizontal. We then retrieve the Whitehead theorem for 2-categories, which can be found in [14, Theorem 7.4.1].

Corollary 5.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. Then a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there is a normal pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\eta\colon \mathrm{id}_{\mathcal{A}} \simeq GF$ and $\epsilon\colon FG \simeq \mathrm{id}_{\mathcal{B}}$.*

Proof. Since F is a biequivalence if and only if $\mathbb{H}F$ is a double biequivalence, as we will see in Theorem 6.5, and $\mathbb{H}\mathcal{B}$ is horizontal, we can apply Theorem 5.13 to $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$. Then $\mathbb{H}F$ is a double biequivalence if and only if there is a normal horizontally pseudo double functor $G'\colon \mathbb{H}\mathcal{B} \to \mathbb{H}\mathcal{A}$ together with horizontal pseudo natural equivalences $\eta'\colon \mathrm{id}_{\mathbb{H}\mathcal{A}} \simeq G'(\mathbb{H}F)$ and $\epsilon'\colon (\mathbb{H}F)G' \simeq \mathrm{id}_{\mathbb{H}\mathcal{B}}$. As normal horizontally pseudo double functors and horizontal pseudo natural equivalence between double categories in the image of $\mathbb{H}$ are equivalently normal pseudo functors and pseudo natural equivalences between their preimages, the data (G', η', ϵ') for $\mathbb{H}F$ uniquely correspond to a data (G, η, ϵ) for F as required. □

6. Quillen pairs between DblCat and 2Cat

In this paper, the model structure on DblCat was constructed in such a way as to be compatible with the Lack model structure on 2Cat through the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. We now study the precise relation between these model structures.

We present here the two Quillen pairs involving the horizontal embedding functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ and its right and left adjoints.

Proposition 6.1. *The adjunction*

$$2\mathrm{Cat} \underset{\mathbf{H}}{\overset{\mathbb{H}}{\rightleftarrows}} \mathrm{DblCat} \quad (\mathbb{H} \dashv \mathbf{H})$$

is a Quillen pair, where 2Cat *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.18. Moreover, its derived unit is levelwise a biequivalence; in particular, this says that the functor* $\mathbb{H}$ *is homotopically fully faithful.*

Proof. Since the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat}\times 2\mathrm{Cat}$ and the projection $\mathrm{pr}_1\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$ are right Quillen, then so is their composite $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, which proves that $\mathbb{H} \dashv \mathbf{H}$ is a Quillen pair. Moreover, since every object in DblCat is fibrant, the derived unit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ is given by the components of the unit at cofibrant objects, and is therefore levelwise an identity, by Proposition 2.9. □

The functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is also right Quillen. The existence of its left adjoint is given by the Adjoint Functor Theorem, since $\mathbb{H}$ preserves all limits and colimits between locally presentable categories.

Theorem 6.2. *The adjunction*

$$\mathrm{DblCat} \underset{\mathbb{H}}{\overset{L}{\rightleftarrows}} 2\mathrm{Cat} \quad (L \dashv \mathbb{H})$$

is a Quillen pair, where 2Cat *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.18.*

Proof. We show that $\mathbb{H}$ is right Quillen, i.e., it preserves fibrations and trivial fibrations.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a fibration in $2\mathrm{Cat}$; we prove that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a fibration, (df1-2) of Definition 3.8 are satisfied. It remains to show (df3) of Definition 3.8. Let us consider a weakly horizontally invertible square in $\mathbb{H}\mathcal{B}$

$$\begin{array}{ccc} B & \xrightarrow[b]{\simeq} & FC \\ \| & \beta \cong & \| \\ B & \xrightarrow[d]{\simeq} & FC \,. \end{array}$$

Note that its vertical boundaries must be trivial, since all vertical morphisms in $\mathbb{H}\mathcal{B}$ are identities. Then the square β is, in particular, vertically invertible by Lemma 2.19. Since F is a fibration in $2\mathrm{Cat}$, there is an equivalence $c\colon A \xrightarrow{\simeq} C$ such that $Fc = d$, by (f1) of Definition 3.2. Now β can be rewritten as

$$\begin{array}{ccc} FA & \xrightarrow[b]{\simeq} & FC \\ \| & \beta \cong & \| \\ FA & \xrightarrow[Fc]{\simeq} & FC \,. \end{array}$$

Then β is equivalently an invertible 2-cell $\beta\colon b \cong Fc$ in $\mathcal{B}$. Since F is a fibration in $2\mathrm{Cat}$, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $F\alpha = \beta$, by (f2) of Definition 3.2. In particular, since c is an equivalence in $\mathcal{A}$, then so is a. This gives a vertically invertible square in $\mathbb{H}\mathcal{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow[a]{\simeq} & C \\ \| & \alpha \cong & \| \\ A & \xrightarrow[c]{\simeq} & C \end{array}$$

such that $F\alpha = \beta$; furthermore, by Lemma 2.19, the square α is weakly horizontally invertible. This shows that $\mathbb{H}F$ is a double fibration.

Now let the 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ be a trivial fibration. We show that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double trivial fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a trivial fibration, it satisfies (dt1-2) of Definition 3.9. Then (dt3) of Definition 3.9 follows from the fact that F is surjective on objects, since all vertical morphisms are identities. Finally, (dt4) of Definition 3.9 is a direct consequence of F being fully faithful on 2-cells, since all squares in $\mathbb{H}\mathcal{A}$ and $\mathbb{H}\mathcal{B}$ are equivalently 2-cells in $\mathcal{A}$ and $\mathcal{B}$, respectively. This shows that $\mathbb{H}F$ is a double trivial fibration, and concludes the proof. □

Remark 6.3. As we have seen in Proposition 6.1, the functor $\mathbb{H}$ is homotopically fully faithful, and therefore the derived counit of the adjunction $L \dashv \mathbb{H}$ is levelwise a biequivalence.

Remark 6.4. As a consequence of Proposition 6.1 and Theorem 6.2, we see that the functor $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ preserves all cofibrations, fibrations, and weak equivalences. Indeed, the fact that it preserves cofibrations and fibrations follows from the fact that $\mathbb{H}$ is both left and right Quillen, while the fact that it preserves weak equivalences is a consequence of Ken Brown's Lemma (see [13, Lemma 1.1.12]), since all objects in $\mathrm{2Cat}$ are fibrant.

In fact, more is true: the horizontal embedding $\mathbb{H}$ also reflects cofibrations, fibrations, and weak equivalences, as we deduce from the following.

Theorem 6.5. *The Lack model structure on* $\mathrm{2Cat}$ *is both left- and right-induced along the adjunctions*

$$\mathrm{2Cat} \underset{\mathbf{H}}{\overset{L}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}}\!\!\xrightarrow{\;\mathbb{H}\;} \mathrm{DblCat}\,,$$

where DblCat *is endowed with the model structure of Theorem 3.18.*

Proof. To show this result, it is enough to prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence (resp. Lack fibration, cofibration) in $\mathrm{2Cat}$ if and only if the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence (resp. double fibration, cofibration) in DblCat, as a model structure is uniquely determined by its classes of weak equivalences and fibrations, or alternatively by its classes of weak equivalences and cofibrations.

By Remark 6.4, we have that if F is a biequivalence (resp. Lack fibration, cofibration) in $2\mathrm{Cat}$, then $\mathbb{H}F$ is a double biequivalence (resp. double fibration, cofibration) in DblCat, as $\mathbb{H}$ preserves all of these classes of morphisms.

Conversely, if $\mathbb{H}F$ is a double biequivalence (resp. double fibration), then $\mathbf{H}\mathbb{H}F = F$ is a biequivalence (resp. Lack fibration) by definition of the model structure on DblCat.

It remains to show that if $\mathbb{H}F$ is a cofibration, then so is F. For this, suppose that $\mathbb{H}F$ is a cofibration in DblCat; we show that F has the left lifting property with respect to all trivial fibrations in $2\mathrm{Cat}$. Let $P\colon \mathcal{X} \to \mathcal{Y}$ be a trivial fibration in $2\mathrm{Cat}$ and consider a commutative square as below.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{B} & \xrightarrow[H]{} & \mathcal{Y} \end{array}$$

Since $\mathbb{H}$ preserves trivial fibrations, we have that $\mathbb{H}P$ is a double trivial fibration. Then, as $\mathbb{H}F$ is a cofibration, there is a lift in the diagram below left. By the adjunction $\mathbb{H} \dashv \mathbf{H}$, this corresponds to a lift in the diagram below right, which concludes the proof.

$$\begin{array}{ccc} \mathbb{H}\mathcal{A} & \xrightarrow{\mathbb{H}G} & \mathbb{H}\mathcal{X} \\ {\scriptstyle \mathbb{H}F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbb{H}P} \\ \mathbb{H}\mathcal{B} & \xrightarrow[\mathbb{H}H]{} & \mathbb{H}\mathcal{Y} \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathbf{H}\mathbb{H}\mathcal{X} = \mathcal{X} \\ {\scriptstyle F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbf{H}\mathbb{H}P = P} \\ \mathcal{B} & \xrightarrow[H]{} & \mathbf{H}\mathbb{H}\mathcal{Y} = \mathcal{Y} \end{array}$$

□

We saw that the derived unit (resp. counit) of the adjunction $\mathbb{H} \dashv \mathbf{H}$ (resp. $L \dashv \mathbb{H}$) is levelwise a biequivalence. However, these adjunctions are not expected to be Quillen equivalences, since the homotopy theory of double categories should be richer than that of 2-categories. This is indeed the case, as shown in the following remarks.

Remark 6.6. The components of the derived counit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ are not double biequivalences. To see this, consider the double category $\mathbb{V}\mathbb{2}$ free on a vertical morphism. Since $\mathbf{H}\mathbb{V}\mathbb{2} \cong \mathbb{1} \sqcup \mathbb{1}$ is cofibrant in $2\mathrm{Cat}$, the component of the derived counit at $\mathbb{V}\mathbb{2}$ is given by the component of the counit

$$\epsilon_{\mathbb{V}\mathbb{2}} \colon \mathbb{H}\mathbf{H}(\mathbb{V}\mathbb{2}) \cong \mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}\mathbb{2},$$

which is not a double biequivalence, as it does not satisfy (db3) of Definition 3.6.

Remark 6.7. The components of the derived unit of the adjunction $L \dashv \mathbb{H}$ are not double biequivalences. By Proposition 4.3, the unique map $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat, so that $\mathbb{V}\mathbb{2}$ is cofibrant. Since all objects in $2\mathrm{Cat}$ are fibrant, the component of the derived unit at $\mathbb{V}\mathbb{2}$ is given by the component of the unit

$$\eta_{\mathbb{V}\mathbb{2}} \colon \mathbb{V}\mathbb{2} \to \mathbb{H}L(\mathbb{V}\mathbb{2}) \cong \mathbb{1},$$

which is not a double biequivalence, as it does not satisfy (db2) of Definition 3.6. Note that the isomorphism above comes from the fact that the left adjoint L collapses the vertical structure and thus $L\mathbb{V}\mathbb{2} \cong \mathbb{1}$.

Remark 6.8. Since we induced the model structure on DblCat along $\mathbb{H}\sqcup\mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, we also get that the adjunction $\mathbb{L} \dashv \mathcal{V}$ forms a Quillen pair between $2\mathrm{Cat}$ and DblCat. However, note that neither the derived unit nor counit of $\mathbb{L} \dashv \mathcal{V}$ are levelwise weak equivalences.

7. $2\mathrm{Cat}$-enrichment of the model structure on DblCat

The aim of this section is to provide a $2\mathrm{Cat}$-enrichment on DblCat which is compatible with the model structure introduced in Theorem 3.18. Recall that a model category $\mathcal{M}$ is said to be *enriched* over a closed monoidal category $\mathcal{N}$ that is also a model category, if it is a tensored and cotensored $\mathcal{N}$-enriched category and it satisfies the pushout-product axiom (see for example [19, §5] for more details). In particular, the category $\mathcal{N}$ is said to be a *monoidal model category* if its model structure is enriched over itself.

7.1 The model structure on DblCat is not monoidal

In [15, Example 7.2], it is shown that the Lack model structure is not monoidal with respect to the cartesian product. As shown in the remark below, a similar argument also applies in the case of DblCat.

Remark 7.1. By Proposition 4.3, the inclusion $I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$ is a generating cofibration in DblCat. However, the pushout product $I_2 \,\square\, I_2$ with respect to the cartesian product is the double functor from the non-commutative square of horizontal morphisms to the commutative square of horizontal morphisms, as in [15, Example 7.2]. Since cofibrations in DblCat are in particular faithful on horizontal morphisms by Remark 4.8, the pushout-product $I_2 \,\square\, I_2$ cannot be a cofibration in DblCat.

As stated in Theorem 3.5, Lack's model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product. However, since the cofibrations in DblCat are not as well behaved in the vertical direction as in the horizontal direction; e.g., the underlying vertical category of a cofibrant double category is only a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$ rather than a free category, our model structure is not compatible with the Gray tensor product on DblCat (see Proposition 2.5), as we show below.

Notation 7.2. Let $I \colon \mathbb{A} \to \mathbb{B}$ and $J \colon \mathbb{A}' \to \mathbb{B}'$ be double functors in DblCat. We write $I \,\square_{\mathrm{Gr}}\, J$ for their pushout-product

$$I \,\square_{\mathrm{Gr}}\, J \colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B}' \coprod_{\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{A}'} \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{A}' \to \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{B}'$$

with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$ on DblCat.

Remark 7.3. The model structure defined in Theorem 3.18 is not compatible with the Gray tensor product $\otimes_{\mathrm{Gr}}$. To see this, recall that $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat by Proposition 4.3. However the pushout-product

$$I_3 \,\square_{\mathrm{Gr}}\, I_3 \colon \emptyset \to \mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$$

is not a cofibration, where $\mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$ is the double category generated by the following data

$$\begin{array}{ccc} 0 & = & 0 \\ \big\downarrow\!\bullet & & \big\downarrow\!\bullet \\ 0' & \cong & 1 \\ \big\downarrow\!\bullet & & \big\downarrow\!\bullet \\ 1' & = & 1' \,. \end{array}$$

Indeed, since the underlying vertical category of $\mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$ has non-trivial composites of vertical morphisms, this is not a cofibrant double category by Corollary 4.9.

7.2 $2\mathrm{Cat}$-enrichment of the model structure on DblCat

By restricting the Gray tensor product on DblCat along $\mathbb{H}$ in one of the variables, we get rid of the issue concerning the vertical structure that obstructs the compatibility with the model structure of Theorem 3.18. With this variation, we show that DblCat is a tensored and cotensored $2\mathrm{Cat}$-enriched category, and that the corresponding enrichment is now compatible with our model structure.

Definition 7.4. The tensoring functor $\otimes\colon 2\mathrm{Cat} \times \mathrm{DblCat} \to \mathrm{DblCat}$ is defined to be the composite

$$2\mathrm{Cat} \times \mathrm{DblCat} \xrightarrow{\mathbb{H} \times \mathrm{id}} \mathrm{DblCat} \times \mathrm{DblCat} \xrightarrow{\otimes_{\mathrm{Gr}}} \mathrm{DblCat}.$$

Proposition 7.5. *The category* DblCat *is enriched, tensored, and cotensored over* $2\mathrm{Cat}$*, with*

(i) hom 2*-categories given by* $\mathbf{H}[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{A}, \mathbb{B} \in \mathrm{DblCat}$*,*

(ii) tensors given by $\mathcal{C} \otimes \mathbb{A}$*, for all* $\mathbb{A} \in \mathrm{DblCat}$ *and* $\mathcal{C} \in 2\mathrm{Cat}$*, where* $\otimes$ *is the tensoring functor of Definition 7.4, and*

(iii) cotensors given by $[\mathbb{H}\mathcal{C}, \mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{B} \in \mathrm{DblCat}$ *and* $\mathcal{C} \in 2\mathrm{Cat}$*,*

where $[-,-]_{\mathrm{ps}}$ *is the pseudo hom double category of Proposition 2.5.*

Proof. This follows directly from the definition of $\otimes$, and the universal properties of the tensor $\otimes_{\mathrm{Gr}}$ and of the adjunction $\mathbb{H} \dashv \mathbf{H}$. □

We now present the main result of this section.

Theorem 7.6. *The model structure on* DblCat *of Theorem 3.18 is a* $2\mathrm{Cat}$*-enriched model structure, where the enrichment is given by* $\mathbf{H}[-,-]_{\mathrm{ps}}$.

The rest of this section is devoted to the proof of this theorem. With that goal, we first prove several auxiliary lemmas.

Notation 7.7. Let $i\colon \mathcal{A}\to\mathcal{B}$ and $j\colon \mathcal{A}'\to\mathcal{B}'$ be 2-functors in $2\mathrm{Cat}$, and let $I\colon \mathbb{A}\to\mathbb{B}$ be a double functor in DblCat. We denote by $i\square_2 j$ the pushout-product

$$i\,\square_2\, j\colon \mathcal{A}\otimes_2\mathcal{B}'\coprod_{\mathcal{A}\otimes_2\mathcal{A}'}\mathcal{B}\otimes_2\mathcal{A}'\to\mathcal{B}\otimes_2\mathcal{B}'$$

with respect to the Gray tensor product $\otimes_2$ on $2\mathrm{Cat}$ (see Notation 2.13), and we denote by $i\,\square\, I$ the pushout-product

$$i\,\square\, I\colon \mathcal{A}\otimes\mathbb{B}\coprod_{\mathcal{A}\otimes\mathbb{A}}\mathcal{B}\otimes\mathbb{A}\to\mathcal{B}\otimes\mathbb{B}$$

with respect to the tensoring functor $\otimes\colon 2\mathrm{Cat}\times\mathrm{DblCat}\to\mathrm{DblCat}$. In particular, we have that $i\,\square\, I=\mathbb{H}i\,\square_{\mathrm{Gr}}\, I$.

Lemma 7.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. There is an isomorphism of double categories*

$$\mathcal{A}\otimes\mathbb{H}\mathcal{B}\cong\mathbb{H}(\mathcal{A}\otimes_2\mathcal{B}),$$

natural in $\mathcal{A}$ and $\mathcal{B}$.

Proof. By the universal properties of $\otimes$ and $\otimes_2$, the adjunction $\mathbb{H}\dashv\mathbf{H}$, and Lemma 2.14, we have an isomorphism

$$\begin{aligned}\mathrm{DblCat}(\mathcal{A}\otimes\mathbb{H}\mathcal{B},\mathbb{C})&\cong 2\mathrm{Cat}(\mathcal{A},\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{C}]_{\mathrm{ps}})\cong 2\mathrm{Cat}(\mathcal{A},\mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{C}])\\&\cong 2\mathrm{Cat}(\mathcal{A}\otimes_2\mathcal{B},\mathbf{H}\mathbb{C})\cong\mathrm{DblCat}(\mathbb{H}(\mathcal{A}\otimes_2\mathcal{B}),\mathbb{C}),\end{aligned}$$

for every double category $\mathbb{C}$, which is natural in $\mathcal{A}$, $\mathcal{B}$, and $\mathbb{C}$. The result then follows from the Yoneda lemma. □

Remark 7.9. The natural isomorphism $\mathbf{H}[\mathbb{H}(-),-]_{\mathrm{ps}}\cong\mathrm{Ps}[-,\mathbf{H}(-)]$ implies that the adjunction $\mathbb{H}\dashv\mathbf{H}$ is enriched with respect to the $2\mathrm{Cat}$-enrichments $\mathbf{H}[-,-]_{\mathrm{ps}}$ and $\mathrm{Ps}[-,-]$ of DblCat and $2\mathrm{Cat}$, respectively.

Lemma 7.10. *Let $\mathcal{A}$ be a 2-category. There is an isomorphism of double categories*

$$\mathcal{A}\otimes\mathbb{V}\mathbb{2}\cong\mathbb{H}\mathcal{A}\times\mathbb{V}\mathbb{2},$$

natural in $\mathcal{A}$.

Proof. By the universal properties of $\otimes$ and $\times$, and the fact that by the proof of Lemma 2.14 $\mathbf{H}[\mathbb{V}\mathbb{2},\mathbb{B}]_{\mathrm{ps}} = \mathbf{H}[\mathbb{V}\mathbb{2},\mathbb{B}]$ for all $\mathbb{B} \in \mathrm{DblCat}$, we have an isomorphism

$$\begin{aligned}\mathrm{DblCat}(\mathcal{A}\otimes\mathbb{V}\mathbb{2},\mathbb{B}) &\cong 2\mathrm{Cat}(\mathcal{A},\mathbf{H}[\mathbb{V}\mathbb{2},\mathbb{B}]_{\mathrm{ps}}) = 2\mathrm{Cat}(\mathcal{A},\mathbf{H}[\mathbb{V}\mathbb{2},\mathbb{B}]) \\ &\cong \mathrm{DblCat}(\mathbb{H}\mathcal{A},[\mathbb{V}\mathbb{2},\mathbb{B}]) \cong \mathrm{DblCat}(\mathbb{H}\mathcal{A}\times\mathbb{V}\mathbb{2},\mathbb{B}),\end{aligned}$$

for every double category $\mathbb{B}$, which is natural in $\mathcal{A}$ and $\mathbb{B}$. The result then follows from the Yoneda lemma. □

Lemma 7.11. *Let $i\colon \mathcal{A}\to\mathcal{B}$ and $j\colon \mathcal{A}'\to\mathcal{B}'$ be 2-functors in* $2\mathrm{Cat}$. *There are isomorphisms*

$$i\,\square\,\mathbb{H}j \cong \mathbb{H}(i\,\square_2\,j) \;\text{ and }\; i\,\square\,(\mathbb{H}j\times\mathbb{V}\mathbb{2}) \cong \mathbb{H}(i\,\square_2\,j)\times\mathbb{V}\mathbb{2}$$

in the arrow category $\mathrm{DblCat}^{\mathbb{2}}$.

Proof. Since $\mathbb{H}$ is a left adjoint, it preserves pushouts and, by Lemma 7.8, we have that it is compatible with the tensors $\otimes$ and $\otimes_2$. Therefore, we have $i\,\square\,\mathbb{H}j \cong \mathbb{H}(i\,\square_2\,j)$. By Lemma 7.10, by associativity of $\otimes_{\mathrm{Gr}}$, and by the first isomorphism, we then get that

$$\begin{aligned}i\,\square\,(\mathbb{H}j\times\mathbb{V}\mathbb{2}) &\cong i\,\square\,(j\otimes\mathbb{V}\mathbb{2}) \cong (i\,\square\,\mathbb{H}j)\otimes_{\mathrm{Gr}}\mathbb{V}\mathbb{2} \\ &\cong (i\,\square_2\,j)\otimes\mathbb{V}\mathbb{2} \cong \mathbb{H}(i\,\square_2\,j)\times\mathbb{V}\mathbb{2}.\end{aligned}$$ □

We are now ready to prove Theorem 7.6.

Proof of Theorem 7.6. Recall from Proposition 4.1 that a set $\mathcal{I}$ of generating cofibrations and a set $\mathcal{J}$ of generating trivial cofibrations for the model structure on DblCat are given by morphisms of the form $\mathbb{H}j$ and $\mathbb{L}j = \mathbb{H}j\times\mathbb{V}\mathbb{2}$, where j is a generating cofibration or a generating trivial cofibration in $2\mathrm{Cat}$, respectively.

We show that the pushout-product of a generating cofibration in $\mathcal{I}$ with any (trivial) cofibration in $2\mathrm{Cat}$ is a (trivial) cofibration in DblCat, and that the pushout-product of a generating trivial cofibration in $\mathcal{J}$ with any cofibration in $2\mathrm{Cat}$ is a trivial cofibration in DblCat.

Given cofibrations i and j in $2\mathrm{Cat}$, we know by Lemma 7.11 that

$$i\,\square\,\mathbb{H}j \cong \mathbb{H}(i\,\square_2\,j) \;\text{ and }\; i\,\square\,(\mathbb{H}j\times\mathbb{V}\mathbb{2}) \cong \mathbb{H}(i\,\square_2\,j)\times\mathbb{V}\mathbb{2} = \mathbb{L}(i\,\square_2\,j),$$

and by Theorem 3.5 that $i \square_2 j$ is also a cofibration in $2\mathrm{Cat}$, which is trivial when either i or j is. Since $\mathbb{H}$ and $\mathbb{L}$ preserve (trivial) cofibrations by Proposition 6.1 and Remark 6.8, then $\mathbb{H}(i \square_2 j)$ and $\mathbb{L}(i \square_2 j)$ are cofibrations in DblCat, which are trivial if either i or j is. Taking j to be a generating cofibration or generating trivial cofibration in $2\mathrm{Cat}$, we get the desired results. □

8. Comparison with other model structures on DblCat

In [6], Fiore, Paoli, and Pronk construct several model structures on the category DblCat of double categories. We show in this section that our model structure on DblCat is not related to their model structures in the following sense: the identity adjunction on DblCat is not a Quillen pair between the model structure of Theorem 3.18 and any of the model structures of [6]. This is not surprising, since our model structure was constructed in such a way that the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ embeds the homotopy theory of $2\mathrm{Cat}$ into that of DblCat, while there seems to be no such relation between their model structures on DblCat and the Lack model structure on $2\mathrm{Cat}$, e.g. see end of Section 9 in [6]. Further evidence is given by the fact that our double biequivalences are 2-categorically induced, while the weak equivalences in the model structures of [6] are rather 1-categorically induced.

We start by recalling the categorical model structures on DblCat constructed in [6]. Since our primary interest is to compare them to our model structure, we only describe the weak equivalences; the curious reader is encouraged to visit their paper for further details.

The first model structure we recall is induced from the canonical model structure on Cat by means of the *vertical nerve*.

Definition 8.1 ([6, Definition 5.1]). The **vertical nerve** of double categories is the functor

$$N_v\colon \mathrm{DblCat} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$$

sending a double category $\mathbb{A}$ to the simplicial category $N_v\mathbb{A}$ such that $(N_v\mathbb{A})_0$ is the category of objects and horizontal morphisms of $\mathbb{A}$, $(N_v\mathbb{A})_1$ is the category of vertical morphisms and squares of $\mathbb{A}$ and, for $n \geq 2$,

$$(N_v\mathbb{A})_n = (N_v\mathbb{A})_1 \times_{(N_v\mathbb{A})_0} \cdots \times_{(N_v\mathbb{A})_0} (N_v\mathbb{A})_1.$$

Proposition 8.2 ([6, Theorem 7.17])**.** *There is a model structure on* DblCat *in which a double functor F is a weak equivalence if and only if N_vF is levelwise an equivalence of categories.*

The next model structure on DblCat requires a different perspective. For a 2-category $\mathcal{A}$ that admits limits and colimits, there is a model structure on the underlying category $U\mathcal{A}$ in which the weak equivalences are precisely the equivalences of the 2-category $\mathcal{A}$; see [17]. When applying this construction to the 2-category DblCat_h of double categories, double functors, and horizontal natural transformations, one obtains the following model structure on DblCat; see [6, §8.4].

Proposition 8.3. *There is a model structure on* DblCat*, called the* trivial model structure*, in which a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence if and only if it is an equivalence in the 2-category* DblCat_h*, i.e., there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ and two horizontal natural isomorphisms* $\mathrm{id}_{\mathbb{A}} \cong GF$ *and* $FG \cong \mathrm{id}_{\mathbb{B}}$.

Remark 8.4. By comparing this to our Whitehead theorems (see Section 5.2), we see that the weak equivalences in the model structure of Proposition 8.3 require stricter conditions than double biequivalences. Indeed, the units and counits in the statement above are horizontal *strict* natural *isomorphisms*, while in our Whitehead theorems they are horizontal *pseudo* natural *equivalences*. This further supports our claim that the weak equivalences in our model structure are a 2-categorical analogue, and therefore carry more information, than the weak equivalences already present in the literature.

The last model structure is of a more algebraic flavor. Let T be a 2-monad on a 2-category $\mathcal{A}$. In [17], Lack gives a construction of a model structure on the category of T-algebras, in which the weak equivalences are the morphisms of T-algebras whose underlying morphism in $\mathcal{A}$ is an equivalence. In particular, double categories can be seen as the algebras over a 2-monad on the 2-category $\mathrm{Cat}(\mathrm{Graph})$ whose objects are the category objects in graphs; see [6, §9]. This gives the following model structure.

Proposition 8.5. *There is a model structure on* DblCat*, called the* algebra model structure*, in which a double functor F is a weak equivalence if and only if its underlying morphism in the 2-category* $\mathrm{Cat}(\mathrm{Graph})$ *is an equivalence.*

Remark 8.6. In [6, Corollary 8.29 and Theorems 8.52 and 9.1], Fiore, Paoli, and Pronk show that the model structures on DblCat of Propositions 8.2, 8.3 and 8.5 coincide with model structures given by Grothendieck topologies, when double categories are seen as internal categories to Cat. Then, it follows from [6, Propositions 8.24 and 8.38] that a weak equivalence in the algebra model structure is in particular a weak equivalence in the model structure induced by the vertical nerve N_v.

Remark 8.7. At this point, we must mention that [5, 6] define other model structures on DblCat, which are not equivalent to any of the above. However, these are Thomason-like model structures, and are therefore not expected to have any relation to our model structure, which is categorical.

We now proceed to compare these three model structures on DblCat to the one defined in Theorem 3.18. Our strategy will be to find a trivial cofibration in our model structure that is not a weak equivalence in any of the other model structures. Let E_{adj} be the free-living adjoint equivalence 2-category $\{0 \xrightarrow{\simeq} 1\}$. By Proposition 4.3, the inclusion double functor $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ at 0 is a generating trivial cofibration in our model structure on DblCat.

Lemma 8.8. *The double functor $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is not a weak equivalence in any of the model structures on* DblCat *of Propositions 8.2, 8.3 and 8.5.*

Proof. We first prove that J_1 is not a weak equivalence in the model structure on DblCat of Proposition 8.2 induced by the vertical nerve. For this, we need to show that

$$N_v(J_1) \colon N_v(\mathbb{1}) = \Delta\mathbb{1} \to N_v(\mathbb{H}E_{\mathrm{adj}})$$

is not a levelwise equivalence of categories. Indeed, the category $N_v(\mathbb{H}E_{\mathrm{adj}})_0$ is the free category generated by $\{0 \rightleftarrows 1\}$ which is not equivalent to $\mathbb{1}$.

By Remark 8.6, a weak equivalence in the algebra model structure on DblCat of Proposition 8.5 is in particular a weak equivalence in the model structure induced by the vertical nerve. Therefore J_1 is not a weak equivalence in the algebra model structure either.

Finally, we show that J_1 is not a weak equivalence in the trivial model structure on DblCat of Proposition 8.3. If J_1 was an equivalence in the 2-category DblCat_h, then its weak inverse would be given by the unique double functor $!\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ and we would have a horizontal natural isomorphism $\mathrm{id}_{\mathbb{H}E_{\mathrm{adj}}} \cong J_1\,!$, where $J_1\,!$ is constant at 0. But such a horizontal natural isomorphism does not exist since 1 is not isomorphic to 0 in $\mathbb{H}E_{\mathrm{adj}}$. Therefore J_1 is not an equivalence. □

Proposition 8.9. *The identity adjunction on* DblCat *is not a Quillen pair between the model structure of Theorem 3.18 and any of the model structures of Propositions 8.2, 8.3 and 8.5.*

Proof. We consider the identity functor $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}$ from the model structure of Theorem 3.18 to any of the other model structures of Propositions 8.2, 8.3 and 8.5, and show that it is neither left nor right Quillen.

Since J_1 is a trivial cofibration in the model structure of Theorem 3.18, but is not a weak equivalence in any of the other model structures as shown in Lemma 8.8, we see that id does not preserve trivial cofibrations; therefore, it is not left Quillen. Moreover, every object is fibrant in the model structure of Theorem 3.18, so that if id was right Quillen, it would preserve all weak equivalences by Ken Brown's Lemma (see [13, Lemma 1.1.12]). However, it does not preserve the weak equivalence J_1, and thus it is not right Quillen. □

References

[1] Gabriella Böhm. The Gray monoidal product of double categories. *Appl. Categ. Structures*, 2019.

[2] tslil clingman and Lyne Moser. 2-limits and 2-terminal objects are too different. Preprint on arXiv:2004.01313, 2020.

[3] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[4] Andrée Ehresmann and Charles Ehresmann. Multiple functors. II. The monoidal closed category of multiple categories. *Cah. Topol. Géom. Différ.*, 19(3):295–333, 1978.

[5] Thomas M. Fiore and Simona Paoli. A Thomason model structure on the category of small n-fold categories. *Algebr. Geom. Topol.*, 10(4):1933–2008, 2010.

[6] Thomas M. Fiore, Simona Paoli, and Dorette Pronk. Model structures on the category of small double categories. *Algebr. Geom. Topol.*, 8(4):1855–1959, 2008.

[7] Richard Garner, Magdalena Kędziorek, and Emily Riehl. Lifting accessible model structures. *J. Topol.*, 13(1):59–76, 2020.

[8] Marco Grandis. *Higher dimensional categories*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[9] Marco Grandis and Robert Pare. Limits in double categories. *Cah. Topol. Géom. Différ. Catég.*, 40(3):162–220, 1999.

[10] Marco Grandis and Robert Paré. Persistent double limits. *Cah. Topol. Géom. Différ. Catég.*, 60(3):255–297, 2019.

[11] Marco Grandis and Robert Paré. Persistent double limits and flexible weighted limits. 2019.

[12] Kathryn Hess, Magdalena Kędziorek, Emily Riehl, and Brooke Shipley. A necessary and sufficient condition for induced model structures. *J. Topol.*, 10(2):324–369, 2017.

[13] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[14] Niles Johnson and Donald Yau. 2-dimensional categories. Preprint on arXiv:2002.06055, 2020.

[15] Stephen Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.

[16] Stephen Lack. A Quillen model structure for bicategories. *K-Theory*, 33(3):185–197, 2004.

[17] Stephen Lack. Homotopy-theoretic aspects of 2-monads. *J. Homotopy Relat. Struct.*, 2(2):229–260, 2007.

[18] Lyne Moser. A double $(\infty, 1)$-categorical nerve for double categories. Preprint on arXiv:2007.01848.

[19] Lyne Moser. Injective and projective model structures on enriched diagram categories. *Homology Homotopy Appl.*, 21(2):279–300, 2019.

[20] Lyne Moser, Maru Sarazola, and Paula Verdugo. A model structure for weakly horizontally invariant double categories. Preprint on arXiv:2007.00588, 2020.

[21] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[22] Emily Riehl and Dominic Verity. Elements of ∞-category theory. 2018.

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