

# FOLDING OF HITCHIN SYSTEMS AND CREPANT RESOLUTIONS

FLORIAN BECK, RON DONAGI, AND KATRIN WENDLAND

ABSTRACT. Folding of ADE-Dynkin diagrams according to graph automorphisms yields irreducible Dynkin diagrams of ABCDEFG-types. This folding procedure allows to trace back the properties of the corresponding simple Lie algebras or groups to those of ADE-type. In this article, we implement the techniques of folding by graph automorphisms for Hitchin integrable systems. We show that the fixed point loci of these automorphisms are isomorphic as algebraic integrable systems to the Hitchin systems of the folded groups away from singular fibers. The latter Hitchin systems are isomorphic to the intermediate Jacobian fibrations of Calabi–Yau orbifold stacks constructed by the first author. We construct simultaneous crepant resolutions of the associated singular quasi-projective Calabi–Yau threefolds and compare the resulting intermediate Jacobian fibrations to the corresponding Hitchin systems.

## CONTENTS

1. Introduction	2
1.1. Relation to other works	4
1.2. Structure	5
1.3. Notation	5
1.4. Acknowledgments	5
2. Folding in Lie and singularity theory	6
2.1. Folding of root systems	6
2.2. Folding and Cartan subalgebras	7
2.3. Folding of simple complex Lie algebras	10
2.4. Folding of simple complex Lie groups	12
2.5. Folding of ADE singularities and Slodowy slices	13
3. Folding of Hitchin systems	18
3.1. Hitchin systems	19
3.2. Generic Hitchin fibers	20
3.3. Folding of cameral curves	21
3.4. Folding of Hitchin systems	23
4. Crepant resolutions and Hitchin systems	26
4.1. Quasi-projective Calabi–Yau threefolds over Hitchin bases	27
4.2. Crepant resolutions and their intermediate Jacobians	28
Appendix A. An isomorphism $S \cong S_{h (t_h/W_h)\mathbb{C}}$ in our running example	33
References	36

## 1. INTRODUCTION

Any non-simply-laced Dynkin diagram  $\Delta$  is obtained from a simply-laced one  $\Delta_h$  (i.e. of type ADE<sup>1</sup>) by folding. Folding is the process of identifying nodes of  $\Delta_h$  according to a cyclic subgroup  $\mathbf{C} \subset \text{Aut}(\Delta_h)$  of the graph automorphism of  $\Delta_h$ , for example:

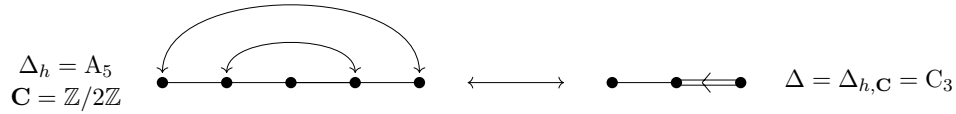


FIGURE 1. Folding of  $\Delta_h = A_5$  to  $\Delta = \Delta_{h,\mathbf{C}} = C_3$ .

In Lie theory, folding effectively reduces the study of simple complex Lie algebras  $\mathfrak{g} = \mathfrak{g}(\Delta)$  with folded Dynkin diagram  $\Delta = \Delta_{h,\mathbf{C}}$  to simple complex Lie algebras  $\mathfrak{g}_h = \mathfrak{g}(\Delta_h)$  with an ADE-Dynkin diagram  $\Delta_h$  and a lift of  $\mathbf{C} \subset \text{Aut}(\Delta_h)$  to outer automorphisms of  $\mathfrak{g}_h$ . The same applies to simple complex Lie groups.

In singularity theory, Slodowy [Slo80] used folding to *define* a  $\Delta$ -singularity of a surface for any irreducible Dynkin diagram  $\Delta$ , thereby generalizing ADE-surface singularities. The idea is as before: if  $\Delta = \Delta_{h,\mathbf{C}}$ , then a  $\Delta$ -singularity is a  $\Delta_h$ -singularity  $Y$  together with an appropriate lift of  $\mathbf{C}$  to  $\text{Aut}(Y)$ . The dual graph of the exceptional divisor of the minimal resolution of  $Y$  coincides with  $\Delta_h$ . Then the lift of the  $\mathbf{C}$ -action to  $\text{Aut}(Y)$  induces a  $\mathbf{C}$ -action on  $\Delta_h$  which is required to agree with the original  $\mathbf{C}$ -action on  $\Delta_h$  by graph automorphisms.

Since the ADE-classification through simply-laced Dynkin diagrams is ubiquitous in mathematics, it is natural to expect applications of folding in other situations as well. Indeed, the ADE-surface singularities have been directly linked to ADE-Hitchin integrable systems by the second author with Diaconescu and Pantev in [DDP07]. Building on results of Szendrői's [Sze04], to each ADE-surface singularity one associates a family of non-compact Calabi–Yau threefolds obtained from the semi-universal  $\mathbb{C}^*$ -deformation of the singularity, whose Griffiths' intermediate Jacobians are compact and together form an algebraic integrable system. The latter is called a *Calabi–Yau integrable system*, and it generalizes integrable systems constructed in [DM96] by the second author with Markman from compact Calabi–Yau threefolds. According to [DDP07], the Calabi–Yau integrable system obtained from an ADE-surface singularity is isomorphic to an ADE-Hitchin integrable system.

Motivated by these results, the first aim of this article is to develop folding of Hitchin systems. More specifically, let  $H$  denote a reductive complex Lie group and  $\mathcal{M}(\Sigma, H)$  the neutral component of the moduli space of semistable  $H$ -Higgs bundles over a compact Riemann surface  $\Sigma$  of genus  $g_\Sigma \geq 2$ . The smooth locus  $\mathcal{M}(\Sigma, H)^{sm} \subset \mathcal{M}(\Sigma, H)$  possesses

<sup>1</sup>Some authors, for example Slodowy [Slo80], call the ADE- or simply-laced Dynkin diagrams ‘homogeneous’. This explains the subscript  $h$ .

a holomorphic symplectic structure  $\omega_H$  and carries the structure of an algebraic integrable system

$$\chi: (\mathcal{M}(\Sigma, H)^{sm}, \omega_H) \longrightarrow \mathbf{B}(\Sigma, H),$$

the  $H$ -Hitchin system. The Hitchin map  $\chi$  is a global analogue of the adjoint quotient

$$(1.1) \quad \chi: \mathfrak{h} \longrightarrow \mathfrak{t}/W$$

where  $\mathfrak{h}$  is the Lie algebra of  $H$  and  $\mathfrak{t} \subset \mathfrak{h}$  is a Cartan subalgebra with Weyl group  $W$ . Then the Hitchin base  $\mathbf{B}(\Sigma, H)$  is given by  $H^0(\Sigma, K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W)$ , where the canonical bundle  $K_\Sigma$  of  $\Sigma$  is considered as a  $\mathbb{C}^*$ -bundle and  $\mathfrak{t}/W$  is equipped with its natural  $\mathbb{C}^*$ -action.

By folding of Hitchin systems, we mean the following: let  $\Delta = \Delta_{h, \mathbf{C}}$  be an irreducible folded Dynkin diagram and let  $G$  and  $G_h$  be the simple complex Lie groups of adjoint type with Dynkin diagrams  $\Delta$  and  $\Delta_h$  respectively. Then  $\mathbf{C}$  lifts to a group of outer automorphisms of  $G_h$  (see e.g. [Spr09, §10.3]) which in turn induces a  $\mathbf{C}$ -action on  $\mathcal{M}(\Sigma, G_h)$ . Our first main result is

**Theorem A** (Theorem 3.4.6). *Over a Zariski-open and dense subset  $\mathbf{B}^\circ \subset \mathbf{B}(\Sigma, G)$ , the restriction  $\mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ}$  of  $\mathcal{M}(\Sigma, G)$  is isomorphic to  $\mathcal{M}(\Sigma, G_h)|_{\mathbf{B}^\circ}^{\mathbf{C}}$  as an algebraic integrable system.*

The part of the Theorem that isomorphically identifies  $\mathbf{B}^\circ$  with an open subset of  $\mathbf{B}(\Sigma, G_h)^{\mathbf{C}}$  is more or less covered by the known commutative diagram

$$(1.2) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\cong} & \mathfrak{g}_h^{\mathbf{C}} \\ \downarrow \chi & & \downarrow \chi_h \\ \mathfrak{t}/W & \xrightarrow{\cong} & (\mathfrak{t}_h/W_h)^{\mathbf{C}} \end{array}$$

where all  $\mathbf{C}$ -actions are induced from the  $\mathbf{C}$ -action on  $\Delta_h$  (see for example [Slo80, §8.8] and our Section 2.3). Theorem A is then a global version of (1.1). The key new ingredient is the relation between cameral covers of  $G$  and  $G_h$ , which we work out in Proposition 3.3.1 in Section 3.3.

In Section 4, we turn to the above-mentioned Calabi–Yau integrable systems associated to Hitchin systems [DDP07, Bec17, Bec19]. More specifically, the first author in [Bec19] showed that the outer automorphism group  $\mathbf{C}$  acts on the family  $\mathcal{X} \longrightarrow \mathbf{B}(\Sigma, G)$  of quasi-projective Gorenstein threefolds which is isomorphic to the restriction of the family  $\mathcal{X}_h \longrightarrow \mathbf{B}(\Sigma, G_h)$  from [DDP07] to  $\mathbf{B}(\Sigma, G_h)^{\mathbf{C}} \cong \mathbf{B}(\Sigma, G)$ . Moreover, the canonical class of these threefolds is  $\mathbf{C}$ -trivializable, thereby inducing the family  $[\mathcal{X}/\mathbf{C}] \longrightarrow \mathbf{B}(\Sigma, G)$  of Calabi–Yau orbifold stacks. In [Bec19], the first author further showed that over an open and Zariski-dense subset  $\mathbf{B}^\circ \subset \mathbf{B}(\Sigma, G)$  with  $\mathcal{X}^\circ := \mathcal{X}|_{\mathbf{B}^\circ}$  the associated intermediate Jacobian fibration

$$(1.3) \quad \begin{aligned} & J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \longrightarrow \mathbf{B}^\circ, \\ & J_{\mathbf{C}}^2(X_b) = H^3(X_b, \mathbb{C})^{\mathbf{C}} / (F^2 H^3(X_b, \mathbb{C})^{\mathbf{C}} + H^3(X_b, \mathbb{Z})^{\mathbf{C}}), \quad b \in \mathbf{B}^\circ, \end{aligned}$$

is a Calabi–Yau integrable system. The group  $H^3(X_b, \mathbb{Z})^{\mathbf{C}}$  of  $\mathbf{C}$ -invariants is isomorphic to the third integral equivariant cohomology group  $H_{\mathbf{C}}^3(X_b, \mathbb{Z})$  [Bec19, Proposition 4], which is the third singular cohomology group  $H^3([X_b/\mathbf{C}], \mathbb{Z})$  of the orbifold stack  $[X_b/\mathbf{C}]$ , see [Beh04]. The main result of [Bec17] is the construction of an isomorphism

$$(1.4) \quad J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \cong \mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ}$$

of algebraic integrable systems over  $\mathbf{B}^\circ$ , see [Bec17, Theorem 5.2.1]. This generalizes [DDP07, Theorem 3] where  $\mathbf{C} = \{\text{id}\}$ , i.e.  $G$  has an ADE-Dynkin diagram.

Our second main result concerns crepant resolutions of the quotient varieties  $X_b/\mathbf{C}$  associated to  $[X_b/\mathbf{C}]$ ,  $b \in \mathbf{B}^\circ$ . One might expect that, if they exist, their intermediate Jacobian fibrations are isomorphic to  $J_{\mathbf{C}}^2(X_b)$  and hence to Hitchin fibers. However, we show:

**Theorem B** (Proposition 4.2.3 and Corollary 4.2.8). *Let  $\mathbf{C} \subset \text{Aut}(\Delta_h)$  be a cyclic subgroup generated by an automorphism  $a$  of order  $|a| > 0$  such that  $\Delta = \Delta_{h, \mathbf{C}}$ . Then the family  $\mathcal{X}^\circ/\mathbf{C} \rightarrow \mathbf{B}^\circ$  admits a simultaneous crepant resolution  $\mathcal{Z} \rightarrow \mathbf{B}^\circ$ . Moreover, there is an isogeny*

$$J^2(\mathcal{Z}/\mathbf{B}^\circ) \simeq J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \times J(\mathcal{X}|_{\mathbf{B}^\circ}^{\mathbf{C}})^{|a|-1}$$

over  $\mathbf{B}^\circ$ . Here  $J(\mathcal{X}|_{\mathbf{B}^\circ}^{\mathbf{C}}) \rightarrow \mathbf{B}^\circ$  is the family of Jacobians of the fixed point loci  $X_b^{\mathbf{C}} \subset X_b$ ,  $b \in \mathbf{B}^\circ$ .

We point out that the fixed point loci  $X_b^{\mathbf{C}}$ ,  $b \in \mathbf{B}^\circ$ , are branched coverings of  $\Sigma$  and therefore have genus strictly larger than  $g_\Sigma$ . Because of (1.4),  $J^2(\mathcal{Z}/\mathbf{B}^\circ)$  contains the  $G$ -Hitchin system  $\mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ}$  over  $\mathbf{B}^\circ$  (up to isogeny), but this inclusion is proper. Therefore  $J^2(\mathcal{Z}/\mathbf{B}^\circ)$  cannot be an algebraic integrable system over  $\mathbf{B}^\circ$  for dimensional reasons. It would be interesting to extend  $J^2(\mathcal{Z}/\mathbf{B}^\circ) \rightarrow \mathbf{B}^\circ$  to an algebraic integrable system over a larger base, a problem which Krichever solved for a special case in a different context [Kri05].

The proofs of Theorems A and B rely on a precise understanding of folding in Lie theory and in singularity theory. Since we were not able to locate a coherent account thereof in the literature, we provide one for the convenience of the reader. As they may have noticed, there are in fact two ways of folding an ADE-Dynkin diagram  $\Delta_h$  according to a cyclic subgroup  $\mathbf{C} \subset \text{Aut}(\Delta_h)$ : one corresponds to taking  $\mathbf{C}$ -coinvariants in the root system corresponding to  $\Delta_h$ , resulting in  $\Delta = \Delta_{h, \mathbf{C}}$  (as depicted in Figure 1), the other one to taking  $\mathbf{C}$ -invariants, resulting in the dual Dynkin diagram  $\Delta^\vee = \Delta_h^{\mathbf{C}}$ . We explain the interplay between these two procedures in Section 2. In particular, Figure 1 is extended to Figure 2 below.

**1.1. Relation to other works.** Folding in Lie theory is well-known (see for example [Spr09, §10.3]). However, we could not locate a concise account in the literature which considers the adjoint quotients (1.2) as well.

Our construction of the folding of Hitchin systems (Theorem A) is closely related to the ideas of [GR16], where the action of outer automorphisms of  $G$  on the entire total space

$\mathcal{M}(\Sigma, G)$  is considered. However, our result makes a statement about the fixed point locus relative to the Hitchin base  $\mathbf{B}(\Sigma, G)$  which is not considered in [GR16], cf. Remark 3.4.4. In particular, the folding of cameral curves (Section 3.3) has not been considered before.

The families  $\mathcal{X} \rightarrow \mathbf{B}(\Sigma, G)$  have been constructed in [Bec19]. As pointed out before, [Bec19] works directly with the global orbifold stacks  $[X_b/\mathbf{C}]$ ,  $b \in \mathbf{B}^\circ$ , instead of the crepant resolutions of the quotient varieties  $X_b/\mathbf{C}$ .

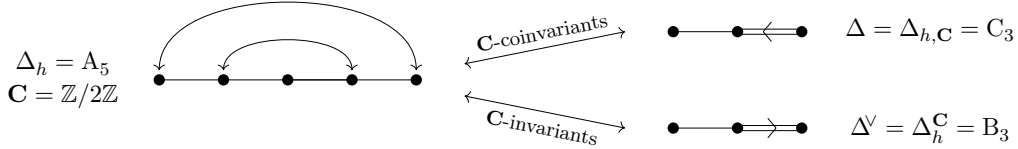


FIGURE 2. Folding of  $\Delta_h = A_5$  to  $\Delta = \Delta_{h,\mathbf{C}} = C_3$  or dually to  $\Delta^V = \Delta_h^{\mathbf{C}} = B_3$ .

**1.2. Structure.** In Section 2, we give a concise account of folding in Lie and singularity theory. In Section 3 we first review Hitchin systems and generic Hitchin fibers by introducing cameral curves. Then we consider the folding of cameral curves and prove Theorem A. In Section 4 we briefly recall the construction of the families  $\mathcal{X} \rightarrow \mathbf{B}(\Sigma, G)$ , determine their crepant resolutions, and prove Theorem B. Most of our findings are illustrated by a running example to guide the reader.

### 1.3. Notation.

- $\Delta$ : irreducible Dynkin diagram, denoted  $\Delta_h$  if required to be simply-laced, that is, ‘homogeneous’, according to [Slo80, §6.1].
- $R = R(\Delta)$ : root system  $R \subset (V, (-, -))$  associated with  $\Delta$ , denoted  $R_h \subset (V_h, (-, -))$  if  $\Delta = \Delta_h$ ; we identify  $\Delta$  with a choice of simple roots in  $R$
- $\Delta^V$ : irreducible Dynkin diagram of the coroot system  $R^V$  associated to  $R = R(\Delta)$ ; we view  $R^V$  as a root system in  $V^*$ , throughout.
- $(\Delta_h, \mathbf{C})$ : pair consisting of the irreducible simply-laced Dynkin diagram  $\Delta_h$  and a cyclic subgroup  $\mathbf{C} := \langle a \rangle$  of the group  $\text{Aut}_D(\Delta_h)$  of Dynkin graph automorphisms<sup>2</sup> of  $\Delta_h$  such that  $\Delta^V = \Delta_h^{\mathbf{C}}$ , cf. Section 2.1.
- $\mathfrak{g} = \mathfrak{g}(\Delta) = \mathfrak{g}(R)$ : simple complex Lie algebra associated with the irreducible Dynkin diagram  $\Delta$  and root system  $R$ , denoted  $\mathfrak{g}_h$  if  $\Delta = \Delta_h$ .

**1.4. Acknowledgments.** Florian Beck thanks Lara Anderson and Laura Schaposnik for an invitation to the Simons Center for Geometry and Physics, Stony Brook, where part of this research was conducted, and kindly acknowledges the funding by the DFG Emmy Noether grant AL 1407/2-1. During the preparation of this work, Ron Donagi was supported in part by NSF grant DMS 2001673 and by Simons HMS Collaboration grant #

<sup>2</sup>These are graph automorphisms  $a$  such that  $(a(\alpha), \alpha) \in \{0, 2\}$  for each root  $\alpha \in R_h$ . Note that with this terminology, Dynkin diagrams of type  $\Delta_h = A_{2n}$  have no non-trivial Dynkin graph automorphisms,  $\text{Aut}_D(\Delta_h) = \{\text{id}\}$ , while  $\text{Aut}(\Delta_h) \cong \mathbb{Z}_2$ . Moreover,  $\mathbf{C} \cong \{\text{id}\}$  or  $\mathbf{C} \cong \mathbb{Z}_2$  or  $\mathbf{C} \cong \mathbb{Z}_3$ .

390287. Katrin Wendland thanks Igor Krichever, Motohico Mulase and Kenji Ueno for very helpful discussions. The authors thank the Simons Center for Geometry and Physics, Stony Brook, as well as the Max-Planck-Institute for Mathematics, Bonn, for their hospitality, as some of the work presented here was carried out during our visits to these institutions.

## 2. FOLDING IN LIE AND SINGULARITY THEORY

In this section, we provide the relevant background for folding from a Lie theoretical perspective, and we introduce a running example for folding.

**2.1. Folding of root systems.** Let  $\Delta_h$  be any irreducible Dynkin diagram of type ADE and  $R_h \subset V_h$  the corresponding root system in the Euclidean vector space  $(V_h, (-, -))$ . We identify the nodes of  $\Delta_h$  with a choice of simple roots in  $R_h$ . Let  $\mathbf{C} = \langle a \rangle \subset \text{Aut}(\Delta_h)$  be a subgroup generated by a Dynkin graph automorphism  $a \in \text{Aut}_D(\Delta_h)$  of finite order  $|a| = \text{ord}(a)$ . The definition of Dynkin graph automorphisms (cf. footnote 2) implies, for example, that  $\mathbf{C}$  is trivial if  $\Delta_h$  is of type  $A_n$  with even  $n$ . The  $\mathbf{C}$ -coinvariants in  $R_h$  are defined by

$$(2.1) \quad R_{h,\mathbf{C}} := R_h / (1 - a)R_h, \quad R_{h,\mathbf{C}} \subset V_{h,\mathbf{C}} \text{ with } V_{h,\mathbf{C}} := V_h / (1 - a)V_h.$$

We denote the class of  $\alpha \in R_h$  in  $R_{h,\mathbf{C}}$  by  $\alpha_{\mathbf{O}}$ . The following is a standard result (see, for example, [Spr09, §10.3], [Ste]):

**Lemma 2.1.1.** *The coinvariants  $R_{h,\mathbf{C}}$  define an irreducible root system in  $V_{h,\mathbf{C}} = V_h / (1 - a)V_h$ . Its irreducible Dynkin diagram  $\Delta = \Delta_{h,\mathbf{C}}$  is determined by the following table*

$\Delta_h$	$\text{ord}(a)$	$\Delta = \Delta_{h,\mathbf{C}}$
$A_{2n-1}$	2	$C_n$ ( $n \geq 2$ )
$D_{n+1}$	2	$B_n$ ( $n \geq 3$ )
$D_4$	3	$G_2$
$E_6$	2	$F_4$ .

*In particular, this establishes a one-to-one correspondence between irreducible Dynkin diagrams  $\Delta$  and pairs  $(\Delta_h, \mathbf{C})$  of irreducible ADE-Dynkin diagrams and subgroups  $\mathbf{C} = \langle a \rangle \subset \text{Aut}(\Delta_h)$  generated by a Dynkin graph automorphism (see footnote 2 for the definition). We say that  $\Delta$  is obtained from  $(\Delta_h, \mathbf{C})$  through folding.*

The dual version of folding replaces  $\mathbf{C}$ -coinvariants by  $\mathbf{C}$ -invariants and thereby interchanges short and long roots. More precisely, let  $O(\alpha)$  be the orbit of  $\alpha \in R_h$  under the action of  $\mathbf{C} = \langle a \rangle$ . Then we define

$$(2.2) \quad \alpha^{\mathbf{O}} := \sum_{\alpha' \in O(\alpha)} \alpha'.$$

Note that the sum does not involve multiplicities, so in particular,  $a \cdot \alpha = \alpha$  implies  $\alpha^{\mathbf{O}} = \alpha$ . By construction, we have

$$(2.3) \quad R_h^{\mathbf{C}} = \{\alpha^{\mathbf{O}} \mid \alpha \in R_h\} \subset V_h^{\mathbf{C}}.$$

**Lemma 2.1.2.** *If  $\Delta = \Delta_{h,\mathbf{C}}$  as in Lemma 2.1.1, then  $R_h^{\mathbf{C}} \subset V_h^{\mathbf{C}}$  is an irreducible root system with Dynkin diagram  $\Delta^\vee$ , that is,  $\Delta^\vee = \Delta_h^{\mathbf{C}}$ . Moreover, if for each  $\alpha_h \in R_h$  we define  $\alpha^\vee \in V_h^*$  by*

$$(2.4) \quad \forall v \in V_h: \quad \alpha^\vee(v) := 2 \frac{(\alpha, v)}{(\alpha, \alpha)},$$

then the map

$$(R_{h,\mathbf{C}})^\vee \longrightarrow (R_h^\vee)^{\mathbf{C}}, \quad (\alpha_O)^\vee \mapsto (\alpha^\vee)^O$$

is an isomorphism of root systems in  $(V_{h,\mathbf{C}})^* \cong (V_h^*)^{\mathbf{C}}$ . In other words,  $(\Delta_{h,\mathbf{C}})^\vee \cong (\Delta_h^\vee)^{\mathbf{C}}$ .

Since in our conventions,  $\Delta_h$  is simply-laced, we have  $\Delta_h^\vee \cong \Delta_h$ , so Lemma 2.1.2 also implies  $\Delta \cong (\Delta_h^{\mathbf{C}})^\vee$ .

**Example 2.1.3.** As a running example throughout this work, we present the folding of  $\Delta_h = A_3$  to  $\Delta = C_2$ .

On the level of root systems, for  $\Delta_h = A_3$  we choose a basis  $(\alpha_1, \alpha_2, \alpha_3)$  of simple roots with

$$(\alpha_j, \alpha_j) = 2 \text{ for } j \in \{1, 2, 3\}, \quad (\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1, \quad (\alpha_1, \alpha_3) = 0.$$

To fold  $\Delta_h = A_3$  to  $\Delta = C_2$ , we must implement the automorphism  $a$  that is induced by  $\alpha_1 \leftrightarrow \alpha_3$ . Since  $\Delta \cong (\Delta_h^{\mathbf{C}})^\vee$ , we may work with the basis  $(\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee)$  of  $V_h^*$  which is dual to  $(\alpha_1, \alpha_2, \alpha_3)$  under (2.4), such that  $\Delta$  corresponds to a coroot system with simple basis  $(\frac{1}{2}(\alpha_1^\vee + \alpha_3^\vee), \alpha_2^\vee)$ .

**2.2. Folding and Cartan subalgebras.** As before, let  $R = R_{h,\mathbf{C}}$  be a folded root system, i.e.  $R \subset V$ , and  $R^\vee \subset V^*$  its coroot system with  $V = V_{h,\mathbf{C}}$  and  $V^* = (V_h^*)^{\mathbf{C}}$ . They have Dynkin diagrams  $\Delta = \Delta_{h,\mathbf{C}}$  and  $\Delta^\vee = \Delta_h^{\mathbf{C}}$ , respectively.

We next work out the relation between the action of the Weyl group  $W_h = W(R_h^\vee)$  of  $R_h^\vee \subset V_h^*$  and the Weyl group  $W = W(R^\vee)$  of  $R^\vee \subset V^*$  on the Cartan tori  $t_h = V_h^* \otimes_{\mathbb{R}} \mathbb{C}$  and  $t = t_h^{\mathbf{C}}$ , respectively: we may identify  $W$  as a subgroup of  $W_h$  and thereby  $t/W$  with the  $\mathbf{C}$ -invariants in  $t_h/W_h$ . This statement is known, see e.g. [Slo80, §8, Remarks, p. 144], however we find it useful to present a self-contained proof. Indeed, first we note

**Proposition 2.2.1.** *Let  $W_h^{\mathbf{C}} := \{w \in W_h \mid aw = wa\} \subset \text{Aut}(V_h^*)$ . Then the restriction to  $V^* = (V_h^*)^{\mathbf{C}} \subset V_h^*$ ,*

$$(2.5) \quad W_h^{\mathbf{C}} \longrightarrow \text{Hom}(V^*, V_h^*), \quad w \mapsto w|_{V^*}$$

induces an isomorphism of groups  $W_h^{\mathbf{C}} \cong W$ .

*Proof.* First observe that by construction,  $w \in W_h^{\mathbf{C}}$  implies  $w|_{V^*} \in \text{Aut}(V^*)$ , and  $w(R^\vee) = R^\vee$  is immediate by (2.2) (also see (2.6) below). Since  $R^\vee$  is a folded root system,  $\text{Aut}(R^\vee) = W \rtimes \text{Aut}(\Delta^\vee) = W$  and therefore  $w|_{V^*} \in W$ . It remains to prove that  $W_h^{\mathbf{C}} \longrightarrow W$ ,  $w \mapsto w|_{V^*}$  is bijective.

Injectivity: Let  $\rho_h = \frac{1}{2} \sum_{\alpha \in R_h^+} \alpha$  be the Weyl vector associated to the positive coroots  $R_h^+ \subset R_h^\vee$  and  $\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta$  be the Weyl vector associated to the positive coroots  $R^+ \subset R^\vee$ . Since  $R^\vee = (R_h^\vee)^\mathbb{C}$ , by (2.3) (replacing  $R_h$  by  $R_h^\vee$ ) we have  $\rho = \rho_h \in V_h$ . If  $w|_{V^*} = \text{id}$  for some  $w \in W_h^\mathbb{C}$ , then  $w(\rho_h) = w(\rho) = \rho = \rho_h$ . Hence  $w$  fixes a vector in the fundamental Weyl chamber associated to  $R_h^+$  and thus a fundamental domain for  $W_h$ , and therefore  $w = \text{id}$ .

Surjectivity onto  $W$ : It suffices to prove that every simple reflection  $s_\beta$ ,  $\beta \in \Delta^\vee$ , is in the image of the homomorphism (2.5). Let  $\beta = \alpha^O$  for  $\alpha \in \Delta_h$  as in (2.2). Then

$$(2.6) \quad s_\beta = \tilde{s}_{\beta|_{V^*}} \quad \text{with} \quad \tilde{s}_\beta := \prod_{\alpha' \in O(\alpha)} s_{\alpha'},$$

where we have used the very definition of Dynkin graph automorphisms (footnote 2). Since  $a$  permutes  $O(\alpha)$  and  $as_{\alpha'}a^{-1} = s_{a\alpha'}$  for every  $\alpha' \in O(\alpha)$ , we have  $\tilde{s}_\beta \in W_h^\mathbb{C}$ , and surjectivity onto  $W$  of our map follows.  $\square$

We may thus indeed realize  $W$  naturally as a subgroup of  $W_h$ . For its action on the regular elements of  $\mathfrak{t} = \mathfrak{t}_h^\mathbb{C}$ , we note

**Lemma 2.2.2.** *Assume that  $t \in \mathfrak{t}$  and  $w \in W_h$  with  $wt \in \mathfrak{t}$ . Then the orbits of  $t$  and  $wt$  under  $W$  agree,  $W(wt) = W(t)$ . In particular, if  $t$  belongs to the set  $\mathfrak{t}^\circ$  of regular elements of  $\mathfrak{t}$ , then  $w \in W$ .*

*Proof.* First note that  $\mathfrak{t}^\circ \subset \mathfrak{t}_h^\circ$ . Indeed, if  $t \in \mathfrak{t}$  but  $t \notin \mathfrak{t}_h^\circ$ , then  $\alpha(t) = 0$  for some  $\alpha \in \Delta_h$ . Now  $a(t) = t$  because  $\mathfrak{t} = \mathfrak{t}_h^\mathbb{C}$ , and since  $a \in \text{Aut}(\Delta_h)$ , for  $\beta := \alpha_O$ , we conclude  $\beta(t) = 0$  with  $\beta \in \Delta_{h,\mathbb{C}}$ . In other words,  $t \notin \mathfrak{t}^\circ$ .

Now let  $t \in \mathfrak{t}^\circ$ , so by the above,  $t \in \mathfrak{t}_h^\circ$ . Then  $w \in W$  if and only if  $W(wt) = W(t)$ . To show that indeed  $w \in W$ , choose  $w_1, w_2 \in W$  such that  $t' := w_1t$  and  $w_2wt$  both belong to the fundamental Weyl chamber in  $\mathfrak{t}$  corresponding to our choice of simple roots of  $R$ . By construction and assumption, with  $w' := w_2w(w_1)^{-1}$ , both  $t'$  and  $w't'$  belong to the same fundamental Weyl chamber in  $\mathfrak{t}_h$  with  $w' \in W_h$ , hence  $w' = \text{id}$  and  $w = (w_2)^{-1}w_1 \in W$  follows.

If  $t \notin \mathfrak{t}^\circ$  is non-zero, then we work with the root subsystem  $R(t)$  of  $R$  defined by all roots  $\alpha \in R$  such that  $\alpha(t) \neq 0$ . Then  $t$  is regular with respect to  $R(t)$  and the above argument applies. If  $t = 0$ , there is nothing to show.  $\square$

According to classical results by Chevalley<sup>3</sup> [Che55], explained, for example, in [Hum78, Section 23], the coordinate ring  $\mathbb{C}[\mathfrak{t}]^W$  of  $\mathfrak{t}/W$  is freely generated by polynomials  $\chi_1, \dots, \chi_r$  of degrees  $d_1, \dots, d_r$ , respectively, where  $r$  denotes the rank of  $\mathfrak{g}$  and  $\epsilon_j = d_j - 1$ ,  $j \in \{1, \dots, r\}$ , are the exponents of this Lie algebra, and similarly for  $\mathfrak{g}_h$ . This induces natural  $\mathbb{C}^*$ -actions on  $\mathfrak{t}/W$  and  $\mathfrak{t}_h/W_h$ , as well as non-canonical isomorphisms  $\mathfrak{t}/W \cong \mathbb{C}^r$ ,  $\mathfrak{t}_h/W_h \cong \mathbb{C}^{r_h}$  as  $\mathbb{C}^*$ -spaces with weights  $d_1, \dots, d_r$ .

**Corollary 2.2.3.**

<sup>3</sup>These hold for any complex Lie algebra that corresponds to a complex reductive algebraic group  $H$ .



i) The reflection hyperplanes  $\mathfrak{t}^{\alpha^O} \subset \mathfrak{t}$ ,  $\alpha^O \in R_h^{\mathbf{C}} = R^{\vee}$ , are given by

$$(2.7) \quad \mathfrak{t}^{\alpha^O} = \bigcap_{\alpha' \in O(\alpha)} \mathfrak{t}_h^{\alpha'} \cap \mathfrak{t} \subset \mathfrak{t}_h.$$

ii) The inclusion  $\mathfrak{t} = \mathfrak{t}_h^{\mathbf{C}} \hookrightarrow \mathfrak{t}_h$  induces the isomorphism

$$\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbf{C}}$$

of  $\mathbf{C}^*$ -spaces.

*Proof.*

i) This is a direct consequence of (2.6).

ii) As stated before, the claim follows from Slodowy's work, see [Slo80, §8, Corollary and Remarks, p. 144]. But it also follows from what was shown in this section, so far. Indeed,  $\mathfrak{t} \hookrightarrow \mathfrak{t}_h \rightarrow \mathfrak{t}_h/W_h$  factorizes over  $\mathfrak{t}/W$  since  $W \subset W_h$  by Proposition 2.2.1. Moreover, since  $W_h$  is a normal subgroup of  $\text{Aut}(\Delta_h)$ ,  $\mathbf{C}$  acts naturally on  $\mathfrak{t}_h/W_h$  such that  $\mathfrak{t}/W$  maps to  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$ . The  $\mathbf{C}$ -action commutes with the  $\mathbf{C}^*$ -action by construction, so this is a homomorphism of  $\mathbf{C}^*$ -spaces. We need to show that it is injective and surjective.

Injectivity: If  $t, t' \in \mathfrak{t}$  with  $t' = wt$  for some  $w \in W_h$ , we need to show that  $t' = w't$  for some  $w' \in W$ . But this is immediate from Lemma 2.2.2.

Surjectivity: Assume that  $t \in \mathfrak{t}_h$  and  $at = wt$  for some  $w \in W_h$ , that is,  $t$  represents a class in  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$ . Since  $W_h$  is a normal subgroup of  $\text{Aut}(\Delta_h)$ ,  $\mathbf{C}$  acts on the orbit  $W_h(t)$  of  $t$  under  $W_h$ . On the other hand,  $W_h(t)$  intersects the closure of the fundamental Weyl chamber in a unique element  $w't \in W_h(t)$ ,  $w' \in W_h$ . Hence  $aw't = w't$ , and  $w't \in \mathfrak{t}$  represents a class in  $\mathfrak{t}/W$  which maps to the class of  $t$  in  $\mathfrak{t}_h/W_h$ . □

It is important to keep in mind that the simple roots in the root system  $\Delta_h$  form a basis of  $\mathfrak{t}_h^*$ ; descending to  $\mathbf{C}$ -invariants in  $\mathfrak{t}_h/W_h$  as in Corollary 2.2.3 above thus corresponds to descending to coinvariants on the level of root systems,  $\Delta = \Delta_{h, \mathbf{C}}$ . This justifies our choice of notations, where  $\Delta^{\vee} = \Delta_h^{\mathbf{C}}$  in contrast to the conventions in Slodowy's work [Slo80], cf. footnote 5 on page 13 below.

**Example 2.2.4.** The Lie algebras encoded by the Dynkin data  $\Delta_h = A_3$  and  $\Delta = C_2$  are  $\mathfrak{g}_h = \mathfrak{sl}_4(\mathbf{C})$  and  $\mathfrak{g} = \mathfrak{sp}_4(\mathbf{C})$ , respectively. We may thus use

$$\begin{aligned} \mathfrak{g}_h &= \{A \in \text{Mat}_{4 \times 4}(\mathbf{C}) \mid \text{tr}(A) = 0\}, \\ \mathfrak{g} &= \left\{ \begin{pmatrix} -d^T & b \\ c & d \end{pmatrix} \mid b, c, d \in \text{Mat}_{2 \times 2}(\mathbf{C}), b = b^T, c = c^T \right\}. \end{aligned}$$

As our Cartan subalgebras, we choose the subalgebras of diagonal matrices in each case,

$$\mathfrak{t}_h = \{\text{diag}(u, v, w, -u - v - w) \mid u, v, w \in \mathbf{C}\}, \quad \mathfrak{t} = \{\text{diag}(u, v, -u, -v) \mid u, v \in \mathbf{C}\}.$$

For later convenience we also choose simple coroots for  $\mathfrak{g}_h = \mathfrak{sl}_4(\mathbb{C})$ , namely

$$\alpha_1^\vee := \text{diag}(1, -1, 0, 0), \quad \alpha_2^\vee := \text{diag}(0, 1, -1, 0), \quad \alpha_3^\vee := \text{diag}(0, 0, 1, -1),$$

while for  $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$  we use

$$\beta_1^\vee := \frac{1}{2}\text{diag}(-1, -1, 1, 1), \quad \beta_2^\vee := \text{diag}(0, 1, 0, -1).$$

The generators  $s_j$  of the Weyl group  $W_h$  corresponding to our choices of simple roots  $\alpha_j^\vee$  act by transpositions  $(j, j+1)$ , exchanging the  $j^{\text{th}}$  and the  $(j+1)^{\text{th}}$  diagonal entry of any matrix  $t \in \mathfrak{t}_h$ , while  $\beta_1^\vee$  corresponds to  $s_1 s_2 s_3 s_2 s_1 s_2 = (1, 4)(2, 3)$  and  $\beta_2^\vee$  corresponds to  $s_2 s_3 s_2 = (2, 4)$ . As remarked in Example 2.1.3, the generator  $a$  of the  $\mathbf{C}$ -action for  $\Delta_h$  is induced by  $\alpha_1^\vee \leftrightarrow \alpha_3^\vee$ . Let us explicitly verify  $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbf{C}}$  as  $\mathbf{C}^*$ -spaces in this example. To describe  $\mathfrak{t}/W$ , we work with the fundamental Weyl chamber in  $\mathfrak{t}$  corresponding to the simple root basis  $(\beta_1^\vee, \beta_2^\vee)$ , and analogously for  $\mathfrak{t}_h/W_h$ . Then the isomorphism  $\mathfrak{t} \cong \mathfrak{t}_h^{\mathbf{C}}$  given by

$$(2.8) \quad \beta_1^\vee \mapsto \frac{1}{2}(\alpha_1^\vee + \alpha_3^\vee), \quad \beta_2^\vee \mapsto \alpha_2^\vee$$

induces an isomorphism  $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbf{C}}$  which is  $\mathbf{C}^*$ -equivariant. Indeed, the coordinate ring  $\mathbb{C}[\mathfrak{t}/W] = \mathbb{C}[\mathfrak{t}]^W$  is generated by the even elementary symmetric polynomials  $\sigma_j$ ,  $j \in \{2, 4\}$ , of degree  $j$  in the diagonal entries of each element of  $\mathfrak{t}$ . This yields natural coordinates  $\xi = (\sigma_2, \sigma_4)$  on  $\mathfrak{t}/W$ . On  $\mathfrak{t}_h/W_h$  we similarly have coordinates  $\xi_h = (\sigma_2, \sigma_3, \sigma_4)$  where  $\sigma_3$  vanishes on  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$ . We thus may use  $\check{\xi}_h = (\sigma_2, \sigma_4)$  as coordinates on  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$  and check

$$\check{\xi}_h(u(\alpha_1^\vee + \alpha_3^\vee) + (u+v)\alpha_2^\vee) = (-u^2 - v^2, u^2v^2) = \xi(2u\beta_1^\vee + (u+v)\beta_2^\vee).$$

Hence the isomorphism  $\check{\xi}_h \mapsto \xi$  of coordinate rings yields the isomorphism  $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbf{C}}$  induced by (2.8), which is therefore  $\mathbf{C}^*$ -equivariant.

**2.3. Folding of simple complex Lie algebras.** Let  $\mathfrak{g}_h = \mathfrak{g}(\Delta_h)$  be the simple complex Lie algebra associated with  $\Delta_h$ . If  $G_h$  denotes the simple adjoint complex Lie group of  $\mathfrak{g}_h$ , then we have the short exact sequence

$$(2.9) \quad 1 \longrightarrow G_h \longrightarrow \text{Aut}(\mathfrak{g}_h) \longrightarrow \text{Aut}(\Delta_h) \longrightarrow 1,$$

see [Slo80, §8.8]. This short exact sequence splits: let  $\alpha_i \in \Delta_h$ ,  $i \in \{1, \dots, r\}$ , denote a choice of simple roots, and choose a Chevalley basis  $(e_\alpha, \alpha_i^\vee \mid \alpha \in R_h, i \in \{1, \dots, r\})$  of  $\mathfrak{g}_h$ . This means that for  $\alpha \in R_h$ ,  $e_\alpha$  forms a basis of the root space  $\mathfrak{g}_{h,\alpha}$ , and  $(\alpha_1^\vee, \dots, \alpha_r^\vee)$  is a simple basis of  $\mathfrak{t}_h$ , with normalizations as follows. For every  $\alpha \in R_h$ ,  $[e_\alpha, e_{-\alpha}] = h_\alpha$  with  $\alpha(h_\alpha) = 2$ . If  $\alpha, \beta, \alpha + \beta \in R_h$ , then

$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta} \quad \text{with} \quad c_{-\alpha,-\beta} = -c_{\alpha,\beta} \in \mathbb{Z}.$$

These normalizations ensure that for any  $a \in \text{Aut}(\Delta_h)$  we may define, by abuse of notation, the automorphism  $a: \mathfrak{g}_h \longrightarrow \mathfrak{g}_h$  by

$$(2.10) \quad a(e_\alpha) := e_{a \cdot \alpha}, \quad a(\alpha_i^\vee) := (a \cdot \alpha_i)^\vee.$$

Note that this splitting is compatible with the  $\mathbf{C}$ -action on  $\mathfrak{t}_h$  already used in Section 2.2.

**Proposition 2.3.1.** *Let  $\mathfrak{g}_h = \mathfrak{g}(\Delta_h)$  be the simple complex Lie algebra associated to the irreducible Dynkin diagram  $\Delta_h$  and  $\mathbf{C} = \langle a \rangle$  the subgroup of  $\text{Aut}(\Delta_h)$  generated by the Dynkin graph automorphism  $a$ . Further let  $\mathbf{C} \hookrightarrow \text{Aut}(\mathfrak{g}_h)$  be determined by the above splitting of (2.9). Then*

$$\mathfrak{g}_h^{\mathbf{C}} \cong \mathfrak{g}(\Delta_{h,\mathbf{C}}).$$

*Proof.* Set  $\mathfrak{g} := \mathfrak{g}_h^{\mathbf{C}}$ ,  $\mathfrak{t} := \mathfrak{t}_h^{\mathbf{C}}$  and let

$$(2.11) \quad p: \mathfrak{g}_h \longrightarrow \mathfrak{g}, \quad \xi \mapsto \frac{1}{|a|} \sum_{k=1}^{|a|} a^k(\xi)$$

be the (vector space) projection. Firstly note that  $a$  is a Lie algebra homomorphism, which implies that the Killing form  $\kappa_h$  on  $\mathfrak{g}_h$  restricts to a non-degenerate form  $\kappa$ , the Killing form of  $\mathfrak{g}$ . Thus by [Hum78, §5.1],  $\mathfrak{g}$  is semisimple.

We next show that  $\mathfrak{t}$  is a Cartan subalgebra. Since  $\mathbb{C}$  has characteristic zero and  $\mathfrak{g}$  is semisimple, by [Hum78, §15.3] it suffices to show that  $\mathfrak{t} \subset \mathfrak{g}$  is a maximal toral subalgebra. That  $\mathfrak{t}$  is toral follows immediately from the fact that  $a$  is a Lie algebra homomorphism. Since

$$\dim(\mathfrak{t}) = \text{rk}(R^\vee) = \dim(\mathfrak{t}_h^{\mathbf{C}}), \quad R^\vee = R_h^{\mathbf{C}},$$

$\mathfrak{t}$  is also maximal for dimensional reasons.

By the duality between invariants and coinvariants, we have

$$(2.12) \quad (\mathfrak{t}_h^{\mathbf{C}})^* \cong (\mathfrak{t}_h^*)_{\mathbf{C}}.$$

In particular,  $R_{h,\mathbf{C}} \hookrightarrow \mathfrak{t}^*$ .

Next we claim that<sup>4</sup>

$$(2.13) \quad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha_O \in R_{h,\mathbf{C}}} \mathfrak{g}_{\alpha_O}$$

is a root space decomposition of  $\mathfrak{g}$ . Indeed, let  $\eta = p(\xi)$  with  $\xi \in \mathfrak{g}_{h,\alpha}$ ,  $s = p(t)$  with  $t \in \mathfrak{t}$  and compute

$$\begin{aligned} [s, \eta] &= \frac{1}{|a|^2} \sum_{k,l} [a^k(t), a^l(\xi)] \\ &= \frac{1}{|a|^2} \sum_{k,l} a^l([a^{k-l}(t), \xi]) \\ &= \frac{1}{|a|^2} \sum_{k,l} \alpha(a^{k-l}(t)) a^l(\xi) \\ &= \frac{1}{|a|} \sum_l \alpha(s) a^l(\xi) = \alpha_O(s) \eta. \end{aligned}$$

It follows that  $p$  surjects  $\mathfrak{g}_{h,\alpha}$  onto  $\mathfrak{g}_{\alpha_O}$ . In particular, this shows (2.13), thus concluding the proof.  $\square$

<sup>4</sup>Note that  $\alpha_O = (a \cdot \alpha)_O$  for all  $\alpha \in R_h$ .

The previous result combined with Corollary 2.2.3 yields the commutative diagram

$$(2.14) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & \mathfrak{g}_h \\ \chi \downarrow & & \downarrow \chi_h \\ \mathfrak{t}/W & \xrightarrow{\quad} & \mathfrak{t}_h/W_h. \end{array}$$

Here  $\chi$  and  $\chi_h$  are the respective adjoint quotients.

**Example 2.3.2.** For  $\mathfrak{g}_h = \mathfrak{sl}_4(\mathbb{C})$  and  $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$  as in Example 2.2.4, we lift the action of  $a$  to  $\mathfrak{g}_h$  by

$$(2.15) \quad A \mapsto \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} A^{\tilde{T}} \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where  $\mathbf{1}$  denotes the identity matrix in  $\text{Mat}_{2 \times 2}(\mathbb{C})$  and  $A^{\tilde{T}}$  is obtained from  $A$  by reflecting in the northeast-southwest diagonal. One immediately checks that this induces the action  $\alpha_1^\vee \leftrightarrow \alpha_3^\vee$  on  $\mathfrak{t}_h$ . Moreover,

$$\text{with } \mathbf{N} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{\alpha_1} := \begin{pmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad e_{\alpha_2} := \begin{pmatrix} \mathbf{0} & \mathbf{N}^T \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad e_{\alpha_3} := \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{N} \end{pmatrix}$$

generate a Chevalley basis, where  $e_{-\alpha} = e_\alpha^T$  for all  $\alpha \in R_h$ . On this basis, (2.15) implements the action (2.10). We then have

$$\mathfrak{g}_h^{\mathbf{C}} = \left\{ \left( \begin{array}{cc} -d^{\tilde{T}} & b \\ c & d \end{array} \right) \mid b, c, d \in \text{Mat}_{2 \times 2}(\mathbb{C}), b = b^{\tilde{T}}, c = c^{\tilde{T}} \right\}.$$

Using the description of  $\mathfrak{g}$  from Example 2.2.4, we thus have  $\mathfrak{g}_h^{\mathbf{C}} \cong \mathfrak{g}$  under the automorphism which permutes rows and columns by the transposition  $(1, 2)$ . We continue to use the notations introduced in Example 2.2.4. Then (2.14) is obtained from  $\chi: \mathfrak{g} \rightarrow \mathfrak{t}/W$ ,  $\chi_h: \mathfrak{g}_h \rightarrow \mathfrak{t}_h/W_h$ , where

$$(2.16) \quad \xi \circ \chi(A) = (\text{tr}(\Lambda^2 A), \text{tr}(\Lambda^4 A)), \quad \xi_h \circ \chi_h(A) = (\text{tr}(\Lambda^2 A), \text{tr}(\Lambda^3 A), \text{tr}(\Lambda^4 A)),$$

with  $\text{tr}(\Lambda^3 A) = 0$  for all  $A \in \mathfrak{sp}_4(\mathbb{C})$ .

**2.4. Folding of simple complex Lie groups.** Let  $G_h$  be the simple *adjoint* complex Lie group with irreducible ADE-Dynkin diagram  $\Delta_h$  and  $\mathbf{C} = \langle a \rangle$  as before. In the following we fix a maximal torus  $T_h \subset G_h$ . Its character and its cocharacter lattice are given by

$$(2.17) \quad \Lambda_h^\vee = \text{Hom}(T_h, \mathbb{C}^*) \cong \langle R_h \rangle_{\mathbb{Z}},$$

$$(2.18) \quad \Lambda_h = \text{Hom}(\mathbb{C}^*, T_h).$$

Here,  $\Lambda_h^\vee \cong \langle R_h \rangle_{\mathbb{Z}}$  follows since  $G_h$  is of adjoint type. This further implies that  $\text{Aut}(G_h) \cong \text{Aut}(\mathfrak{g}_h)$ , and thus our choice of splitting of the sequence (2.9) yields  $\mathbf{C} \hookrightarrow \text{Aut}(G_h)$ .

**Proposition 2.4.1.** [Spr09, Proposition 10.3.5] *The identity component  $(G_h^{\mathbf{C}})^\circ$  of the subgroup  $G_h^{\mathbf{C}} \subset G_h$  fixed by the automorphism group  $\mathbf{C} \subset \text{Aut}(G_h)$  is the simple adjoint*

complex Lie group  $G$  with Dynkin diagram  $\Delta_{h,\mathbf{C}}$ . In particular, its character and cocharacter lattices satisfy

$$(2.19) \quad \Lambda^\vee = (\Lambda_h^\vee)_{\mathbf{C}}, \quad \Lambda = \Lambda_h^{\mathbf{C}}.$$

**Example 2.4.2.** For Dynkin type  $\Delta_h = A_3$ , the adjoint form is  $G_h = PSL_4(\mathbb{C})$ , while  $SL_4(\mathbb{C})$  is the simply connected form. On the other hand, the adjoint form for  $\Delta = C_2$  is  $G = Sp_4(\mathbb{C})/\{\pm 1\} = PSp_4(\mathbb{C})$ , while  $Sp_4(\mathbb{C})$  is the simply connected form. We lift the action of  $\mathbf{C}$  used in Example 2.3.2 to  $G_h$ , and we obtain

$$\forall A \in SL_4(\mathbb{C}): \quad A \mapsto -\tilde{J}(A^T)^{-1}\tilde{J}, \quad \text{where } \tilde{J} := \begin{pmatrix} \mathbf{0} & \mathbf{Q} \\ -\mathbf{Q} & \mathbf{0} \end{pmatrix} = -\tilde{J}^{-1}, \quad \mathbf{Q} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here,

$$\tilde{J} = PJP \quad \text{with} \quad P = \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix},$$

so  $J$  is the standard symplectic matrix. It follows that  $(G_h)^{\mathbf{C}}$  arises from  $G$  by conjugation with  $P$ . Hence  $(G_h)^{\mathbf{C}} \cong G$  holds.

**2.5. Folding of ADE singularities and Slodowy slices.** The construction of quasi-projective Calabi–Yau threefolds of [DDP07] starts from semi-universal  $\mathbb{C}^*$ -deformations of ADE surface singularities. Following [Slo80], there is an elegant description of the latter in terms of the associated simple complex Lie algebras, which is compatible with folding. In this section, we recall that construction, mainly following the original work [Slo80].

**Definition 2.5.1.** Let  $\Delta = \Delta_{h,\mathbf{C}}$  denote<sup>5</sup> an irreducible Dynkin diagram with  $\Delta_h$  an irreducible Dynkin diagram of type ADE and  $\mathbf{C} \subset \text{Aut}(\Delta_h)$  as before. A  $\Delta$ -singularity is a pair  $(Y, H)$  such that

- $Y = (Y, 0)$  is (a germ of) an isolated surface singularity of type  $\Delta_h$ ; in particular, the dual of the resolution graph of the minimal resolution  $\tilde{Y} \rightarrow Y$  coincides with  $\Delta_h$ ,
- $H \subset \text{Aut}(Y)$  is a finite subgroup with  $H \cong \mathbf{C}$  which acts freely on  $Y - \{0\}$ ,
- the induced action of  $H$  on the dual of the resolution graph coincides with the  $\mathbf{C}$ -action on  $\Delta_h$ .

If  $H = \{\text{id}\}$ , then we simply write  $Y$  instead of  $(Y, \{\text{id}\})$ .

Every  $\Delta$ -singularity is quasi-homogeneous, i.e. there is a natural  $\mathbb{C}^*$ -action on each  $\Delta$ -singularity  $(Y, H)$ . Moreover, every  $\Delta$ -singularity  $(Y, H)$  admits a semi-universal  $\mathbb{C}^*$ -deformation according to [Slo80, §2.4-2.7]. By this, following [Slo80, Definition 2.6], we mean a (formal)  $(\mathbf{C} \times \mathbb{C}^*)$ -equivariant deformation  $\zeta: \mathcal{Y} \rightarrow B$  of  $Y$  with trivial action of  $\mathbf{C}$  on the base such that every other deformation with these properties allows a  $(\mathbf{C} \times \mathbb{C}^*)$ -equivariant morphism to  $\zeta$  whose differential on the base is uniquely determined.

---

<sup>5</sup>This definition goes back to Slodowy [Slo80, §6.2], who however uses the dual notion, i.e.  $\mathbf{C}$ -invariants instead of  $\mathbf{C}$ -coinvariants to label the singularities. Our conventions are better adapted to the folding of simple complex Lie algebras and Hitchin systems.

If the isolated surface singularity  $Y = (Y, 0)$  is defined by a quasi-homogeneous polynomial  $f \in \mathbb{C}[x, y, z]$  of weights<sup>6</sup>  $(\mathbf{w}_x, \mathbf{w}_y, \mathbf{w}_z)$  and degree  $\deg(f) = \mathbf{w}_x + \mathbf{w}_y + \mathbf{w}_z$ , then the work of Schlessinger [Sch68], Elkik [Elk74] and Rim [Rim72, 4.14] gives a direct way to construct a semi-universal  $\mathbb{C}^*$ -deformation of  $Y$ . The construction uses the Jacobian ring  $J_f := \mathbb{C}[x, y, z]/(\partial_x f, \partial_y f, \partial_z f)$  of  $f$  as follows. In our situation,  $J_f$  is a finite dimensional vector space,  $J_f \cong \mathbb{C}^r$ . If  $(g_1, \dots, g_r)$  with quasi-homogeneous  $g_j \in \mathbb{C}[x, y, z]$  for  $j \in \{1, \dots, r\}$  represents a  $\mathbb{C}$ -basis of  $J_f$ , then

$$\mathcal{Y}_h := \left\{ ((x, y, z), b) \in \mathbb{C}^3 \times \mathbb{C}^r \mid f(x, y, z) + \sum_{j=1}^r b_j g_j(x, y, z) = 0 \right\}, \quad B_h := \mathbb{C}^r,$$

$$\zeta: \mathcal{Y}_h \longrightarrow B_h, \quad ((x, y, z), b) \mapsto b$$

defines a semi-universal  $\mathbb{C}^*$ -deformation of  $Y$ , cf. [Slo80, Theorem 2.4]. Note that  $B_h \cong J_f$  by construction. The  $\mathbb{C}^*$ -action on  $Y$  is naturally extended to  $\mathcal{Y}_h$  by assigning the weight  $\deg(f) - \deg(g_j)$  to  $b_j$  for  $j \in \{1, \dots, r\}$ . Then  $\mathcal{Y}_h \longrightarrow B_h$  is a semi-universal  $\mathbb{C}^*$ -deformation of  $Y$  by [Slo80, Theorem 2.5] and its proof. If  $(Y, H)$  is any  $\Delta$ -singularity, then the  $H$ -action on  $Y$  extends to an  $H$ -action on  $\mathcal{Y}_h$  and  $B_h$ . The restriction  $\zeta: \zeta^{-1}(B_h^H) \longrightarrow B_h^H$  is then a semi-universal  $\mathbb{C}^*$ -deformation of  $(Y, H)$ , see [Slo80, §2.6].

Let us illustrate this construction by carrying it out for our running example.

**Example 2.5.2.** We consider  $C_2 = A_{3, \mathbf{C}}$  for  $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ . The  $A_3$ -surface singularity is given by

$$(2.20) \quad Y = \{(x, y, z) \in \mathbb{C}^3 \mid x^4 - yz = 0\}.$$

The defining polynomial  $f(x, y, z) := x^4 - yz$  is quasi-homogeneous with respect to the  $\mathbb{C}^*$ -action  $(\lambda, (x, y, z)) \mapsto (\lambda x, \lambda^2 y, \lambda^2 z)$ . Consider the group  $H \cong \mathbf{C}$  generated by the automorphism  $(x, y, z) \mapsto (-x, z, y)$ . Then  $(Y, H)$  is a  $\Delta$ -singularity with  $\Delta = C_2$ , as one checks by an explicit calculation, see [Slo80, p. 77].

The Jacobian ring for this singularity is  $J_f = \mathbb{C}[x, y, z]/(x^3, y, z)$ , so  $(x^2, x, 1)$  represents a quasi-homogeneous basis. Hence a semi-universal  $\mathbb{C}^*$ -deformation of the  $A_3$ -singularity is given by

$$(2.21) \quad \mathcal{Y}_h = \{(x, y, z, b_2, b_3, b_4) \in \mathbb{C}^6 \mid x^4 - yz + b_2 x^2 + b_3 x + b_4 = 0\} \longrightarrow \mathbb{C}^3,$$

$$(x, y, z, b_2, b_3, b_4) \mapsto (b_2, b_3, b_4)$$

with  $\mathbb{C}^*$ -action

$$(2.22) \quad \forall \lambda \in \mathbb{C}^*: \quad \lambda \cdot (x, y, z, b_2, b_3, b_4) = (\lambda x, \lambda^2 y, \lambda^2 z, \lambda^2 b_2, \lambda^3 b_3, \lambda^4 b_4).$$

The action of  $H$  is extended to  $\mathcal{Y}_h$  by

$$(2.23) \quad (x, y, z, b_2, b_3, b_4) \longmapsto (-x, z, y, b_2, -b_3, b_4).$$

---

<sup>6</sup>Clearly, these weights are not unique and we fix one choice here. As we shall see, cf. Remark 2.5.3, the natural choice of coprime weights is in general not compatible with the Lie algebraic description of  $\Delta$ -singularities.

The preimage under  $\mathcal{Y}_h \rightarrow \mathbb{C}^4$  of the  $H$ -fixed point locus in  $\mathbb{C}^4$  is a semi-universal  $\mathbb{C}^*$ -deformation of the  $C_2$ -singularity  $(Y, H)$ , namely

$$\begin{aligned} \mathcal{Y} = \{x, y, z, b_2, b_4\} \in \mathbb{C}^5 \mid x^4 - yz + b_2x^2 + b_4 = 0\} &\longrightarrow \mathbb{C}^2, \\ (x, y, z, b_2, b_4) &\mapsto (b_2, b_4). \end{aligned}$$

The  $\mathbb{C}^*$ - and  $H$ -action are the restrictions of (2.22) and (2.23) respectively. By construction, the  $H$ -action is trivial on the base, so  $H$  acts on all fibers.

The direct construction of semi-universal  $\mathbb{C}^*$ -deformations is useful in explicit computations, see Examples 4.1.1 and 4.1.2. The Lie-theoretic construction due to Brieskorn [Bri71] (for  $\Delta_h$ -singularities) and Slodowy [Slo80, §8] (for all  $\Delta$ -singularities) is very convenient for our study of the relation between Calabi–Yau and Hitchin integrable systems, cf. Section 4. We emphasize that with appropriate choices of  $\mathbb{C}^*$ -actions (cf. Remark 2.5.3) both constructions give isomorphic  $\mathbb{C}^*$ -deformations by semi-universality, see [Sch68, Proposition 2.9]. In Appendix A we work out an explicit isomorphism for our running example.

To review Slodowy’s construction, let  $\mathfrak{g} = \mathfrak{g}(\Delta)$  denote the simple complex Lie algebra determined by  $\Delta$ , and let  $x \in \mathfrak{g}$  be a subregular nilpotent element. Choose an  $\mathfrak{sl}_2$ -triple  $(x, y, h)$  for  $x$  with semisimple element  $h \in \mathfrak{g}$  such that  $[h, x] = 2x$ ,  $[h, y] = -2y$ ,  $[x, y] = h$ , so  $\langle x, y, h \rangle_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$ . Then a so-called Slodowy slice through  $x$  is given by

$$S := x + \ker \operatorname{ad}_y \subset \mathfrak{g},$$

with the restriction  $\sigma := \chi|_S$  of the adjoint quotient,

$$\sigma: S \longrightarrow \mathfrak{t}/W.$$

We note that any two nilpotent subregular  $x \in \mathfrak{g}$  are conjugate under the adjoint action of  $G$  [Ste74, §3.10], as are any two  $\mathfrak{sl}_2$ -triples that contain  $x$  [Kos59, Corollary 3.6]. The Slodowy slice  $S$  carries actions by our cyclic group  $\mathbf{C}$  and by  $\mathbb{C}^*$ , which commute with each other, as follows. To obtain the  $\mathbb{C}^*$ -action, let  $\exp: \mathfrak{g} \rightarrow G$  denote the exponential map from the Lie algebra to the Lie group stemming from the construction of one-parameter subgroups. Observe that  $\operatorname{ad}_{th}(x) = 2tx$  and  $\lambda = e^t \in \mathbb{C}^*$  yield  $\operatorname{Ad}_{\exp(th)}(x) = \exp(\operatorname{ad}_{th})(x) = \lambda^2x$  because  $[h, x] = 2x$ . Thus there is a  $\mathbb{C}^*$ -action on  $S$  given by

$$(2.24) \quad \forall \lambda = e^t \in \mathbb{C}^*, v \in S: \quad \lambda \cdot v = \lambda^2 \operatorname{Ad}_{\exp(-th)}(v).$$

*Remark 2.5.3.* By construction, the adjoint quotient  $\sigma: S \rightarrow \mathfrak{t}/W$  is  $\mathbb{C}^*$ -equivariant with respect to the action (2.24) on  $S$  and the action with weights  $2d_j$  on the coordinate ring  $\mathbb{C}[\chi_1, \dots, \chi_r]$  of  $\mathfrak{t}/W$ , where  $\epsilon_j = d_j - 1$  are the exponents of  $\mathfrak{g}$  as before. In particular,  $\sigma$  is *not*  $\mathbb{C}^*$ -equivariant with respect to the standard  $\mathbb{C}^*$ -action on  $\mathfrak{t}/W$  with weights  $d_j$ .

As we shall see in greater detail in Theorem 2.5.5, equipped with this  $\mathbb{C}^*$ -action the Slodowy slice  $S$  is  $\mathbb{C}^*$ -equivariantly isomorphic to the corresponding unfolding constructed from the Jacobian ring  $J_f$ ,  $f \in \mathbb{C}[x, y, z]$ , of an isolated surface singularity of ADE-type as described before Example 2.5.2. For this to be true, the weights of  $f$  and hence the  $\mathbb{C}^*$ -action on  $J_f$  have to be chosen accordingly (see also [Slo80, §7.4, Proposition 2]).

Indeed, let  $J_f = \mathbb{C}[x, y, z]/(\partial_x f, \partial_y f, \partial_z f)$  as before, and denote by  $(w_x, w_y, w_z)$  coprime weights of  $x, y, z$  such that  $f$  is quasi-homogeneous. If  $\mathfrak{g}$  is of type  $A_{2k}$ ,  $k \in \mathbb{N}$ , then one

needs to use the weights  $(w_x, w_y, w_z)$  for  $x, y, z$ . In all other cases, weights  $(2w_x, 2w_y, 2w_z)$  must be used for  $x, y, z$ , as we confirm explicitly for our running example in Appendix A.

For the  $\mathbf{C}$ -action, consider the simple adjoint complex Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , and its subgroup

$$(2.25) \quad C(x, y) := \{g \in G \mid \text{Ad}_g(x) = x, \quad \text{Ad}_g(y) = y\} \subset G,$$

which acts on  $S$  by the adjoint representation, as one checks by a direct calculation. Then  $\text{Aut}(\Delta_h)$  admits an embedding  $\text{Aut}(\Delta_h) \hookrightarrow C(x, y)$  (cf. [Slo80, Proposition 7.5]), so  $\mathbf{C}$  acts on  $S$  via the adjoint action. Moreover, the induced map  $\text{Aut}(\Delta_h) \rightarrow C(x, y)/C(x, y)^\circ$  to the component group of  $C(x, y)$  is bijective by [Slo80, Proposition 7.5]. In summary,  $\mathbf{C}$  acts on  $S$  by inner automorphisms. Indeed, for  $\mathfrak{g} \cong \mathfrak{g}_h^{\mathbf{C}}$  with  $\mathbf{C} \neq \{\text{id}\}$ ,  $\mathfrak{g}$  does not have any non-trivial outer automorphisms.

*Remark 2.5.4.* If  $\Delta_h$  is a simply-laced Dynkin diagram and  $S_h = x_h + \ker \text{ad}_{y_h} \subset \mathfrak{g}_h$  is a Slodowy slice over  $\mathfrak{t}_h/W_h$ , then  $C(x_h, y_h)/C(x_h, y_h)^\circ$  is trivial by [Slo80, §7.5 Lemma 4]. In other words, one cannot choose an  $\mathfrak{sl}_2$ -triple  $(x_h, y_h, h_h)$  in  $\mathfrak{g}_h$  which is invariant under the natural action of any non-trivial Dynkin graph automorphisms of  $\Delta_h$ , lifted to the inner automorphisms of  $\mathfrak{g}_h$ . In fact, one cannot choose  $(x_h, y_h, h_h)$  to be invariant under *any* lift of  $\mathbf{C}$  to  $\text{Aut}(\mathfrak{g}_h)$  that yields  $\mathfrak{g}_h^{\mathbf{C}} \cong \mathfrak{g}$ , see Remark 2.5.6.

If  $\mathbf{C}$  is non-trivial, then the  $\mathbf{C}$ -action on  $S = x + \ker \text{ad}_y$  introduced through (2.25) must be carefully distinguished from the  $\mathbf{C}$ -action on  $\mathfrak{g}_h$  used for folding, even though  $S \subset \mathfrak{g}$  and  $\mathfrak{g} = \mathfrak{g}_h^{\mathbf{C}}$ , i.e.  $\mathfrak{g} \hookrightarrow \mathfrak{g}_h$ . Indeed, the relation to folding is more delicate: let

$$(2.26) \quad \text{CA}(x_h, y_h) := \{\phi \in \text{Aut}(\mathfrak{g}_h) \mid \phi(x_h) = x_h, \phi(y_h) = y_h\},$$

including all automorphisms of  $\mathfrak{g}_h$  that fix  $x_h$  and  $y_h$ , not just the inner automorphisms as in (2.25). Then by [Slo80, §7.6, Lemma 2] the group of connected components of  $\text{CA}(x_h, y_h)$  satisfies

$$(2.27) \quad \text{CA}(x_h, y_h)/\text{CA}(x_h, y_h)^\circ \cong \text{Aut}(\Delta_h),$$

and if the group  $\text{Aut}_D(\Delta_h)$  of Dynkin graph automorphisms (see footnote 2) is non-trivial, then (2.27) lifts uniquely to an embedding  $\text{Aut}(\Delta_h) \hookrightarrow \text{CA}(x_h, y_h)$ . In particular, if  $\mathbf{C} \subset \text{Aut}_D(\Delta_h)$  is a cyclic subgroup, then  $\mathbf{C}$  acts on  $S_h$  by outer automorphisms. Note that  $\mathbf{C}$  does not preserve all fibers of  $\sigma_h: S_h \rightarrow \mathfrak{t}_h/W_h$ , but it does preserve the fibers over  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$ , cf. the following Theorem 2.5.5 c).

For  $\Delta$ -singularities we now have:

**Theorem 2.5.5.** *Let  $\mathfrak{t} \subset \mathfrak{g}$  be a Cartan subalgebra with Weyl group  $W$  and adjoint quotient  $\chi: \mathfrak{g} \rightarrow \mathfrak{t}/W$ . Then the restriction  $\sigma := \chi|_S: S \rightarrow \mathfrak{t}/W$  satisfies the following:*

- a) [Slo80, §8.3, §8.4] *The pair  $(S_{\bar{0}}, x)$ , where  $S_{\bar{0}} = \sigma^{-1}(\bar{0})$ , together with the induced  $\mathbf{C}$ -action is a  $\Delta$ -singularity.*
- b) [Slo80, §8.7] *The morphism  $\sigma: S \rightarrow \mathfrak{t}/W$  with the above  $(\mathbb{C}^* \times \mathbf{C})$ -action is a semi-universal  $\mathbb{C}^*$ -deformation of the  $\Delta$ -singularity  $S_{\bar{0}}$ . For any  $\bar{t} \in \mathfrak{t}/W$  the singularities of the complex surface  $\sigma^{-1}(\bar{t})$  are precisely the finitely many non-regular elements in  $\sigma^{-1}(\bar{t}) \subset \mathfrak{g}$ , all of which are subregular.*



c) [Slo80, §8.8] Let  $\mathfrak{t}_h^{\mathbf{C}} \cong \mathfrak{t}$  and  $\mathfrak{g}_h^{\mathbf{C}} \cong \mathfrak{g}$  under folding. Moreover, let  $\sigma_h: S_h \rightarrow \mathfrak{t}_h/W_h$  be a Slodowy slice with the  $\mathbf{C}$ -action defined in Remark 2.5.4 and the  $\mathbf{C}^*$ -action defined above. Then these two actions commute, and  $\sigma_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}})$  is  $(\mathbf{C}^* \times \mathbf{C})$ -equivariantly isomorphic to  $S$  over  $(\mathfrak{t}_h/W_h)^{\mathbf{C}} \cong \mathfrak{t}/W$ .

For the respective bases of semi-universal  $\mathbf{C}^*$ -deformations, the above theorem implies the natural isomorphisms  $J_f \cong \mathfrak{t}_h/W_h$  and  $J_f^{\mathbf{C}} \cong (\mathfrak{t}_h/W_h)^{\mathbf{C}} \cong \mathfrak{t}/W$  as  $\mathbf{C}^*$ -spaces.

*Remark 2.5.6.* We emphasize that an isomorphism  $\sigma_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}}) \cong S$  as in Theorem 2.5.5 c) cannot be induced by a Lie algebra homomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}_h$  if  $\mathbf{C} \neq \{\text{id}\}$ , see [Slo80, §8.7 Remark 3)]. As was already mentioned in Remark 2.5.4, this is related to the fact that there is no  $\mathfrak{sl}_2$ -triple  $(x_h, y_h, h_h)$  in  $\mathfrak{g}_h$  which is invariant under  $\mathbf{C}$  and with subregular nilpotent  $x_h, y_h$ . Indeed, such  $x_h, y_h$  would yield Slodowy slices  $S_h = x_h + \ker \text{ad}_{y_h} \subset \mathfrak{g}_h$  and  $S = x_h + \ker(\text{ad}_{y_h})|_{\mathfrak{g}_h^{\mathbf{C}}} \subset \mathfrak{g}$  and a  $\mathbf{C}$ -equivariant map  $S \hookrightarrow S_h$  induced by a Lie algebra homomorphism  $\mathfrak{g} \hookrightarrow \mathfrak{g}_h$ , contradicting the above-mentioned Remark 3) of [Slo80, §8.7]. In fact, as we have seen in Remark 2.5.4,  $\mathbf{C}$  acts by inner automorphisms on  $S$  but by outer automorphisms on  $S_h$ . In this sense, folding of ADE-singularities is not compatible with folding of ADE-Lie algebras.

For any irreducible Dynkin diagram  $\Delta$  we will always work with a Slodowy slice  $S$  inside the simple Lie algebra  $\mathfrak{g}(\Delta)$ . In particular, the  $\mathbf{C}$ -action on  $S$  is always defined by lifting  $\mathbf{C}$  to  $G$  so that it acts by the adjoint action.

**Example 2.5.7.** To construct a Slodowy slice  $S$  for  $\mathfrak{g}(\mathbf{C}_2) = \mathfrak{sp}_4(\mathbb{C})$ , we take the  $\mathfrak{sl}_2$ -triple  $(x, y, h)$  with

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y = x^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad h = [x, y] = \text{diag}(1, 1, -1, -1).$$

Note that there does not exist any nilpotent subregular  $x \in \mathfrak{g}_h = \mathfrak{sl}_4(\mathbb{C})$  with  $x \in \mathfrak{sp}_4(\mathbb{C})$ , in accord with the above Remark 2.5.6. The element  $x$  chosen above is subregular in  $\mathfrak{sp}_4(\mathbb{C})$  but not in  $\mathfrak{sl}_4(\mathbb{C})$ . For the resulting Slodowy slice, by a direct computation we obtain from the definition of  $S$  that

$$S = x + \ker \text{ad}_y = \{s(v_1^-, v_2^-, v_1^+, v_2^+) \mid v_1^\pm, v_2^\pm \in \mathbb{C}\}$$

$$\text{with } s(v_1^-, v_2^-, v_1^+, v_2^+) := \begin{pmatrix} 0 & v_1^- & 1 & 0 \\ -v_1^- & 0 & 0 & 1 \\ v_2^+ + v_2^- & v_1^+ & 0 & v_1^- \\ v_1^+ & v_2^+ - v_2^- & -v_1^- & 0 \end{pmatrix}.$$

The  $\mathbf{C}$ -action on  $S$  is given by conjugation with an appropriate element  $M$  in the group  $C(x, y)$  defined in (2.25). We find

$$C(x, y) = \left\{ \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \mid K \in \text{Mat}_{2 \times 2}(\mathbb{C}), K \cdot K^T = \mathbf{1} \right\},$$

so  $C(x, y)$  has two connected components which are distinguished by  $\det(K) \in \{\pm 1\}$ . We choose

$$M = \begin{pmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q} \end{pmatrix} \quad \text{where} \quad \mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

cf. Example 2.4.2, as a representative of the generator of  $C(x, y)/C(x, y)^\circ \cong \mathbb{Z}/2\mathbb{Z}$ . The induced action on  $S$  in terms of the  $v_k^\pm$  is then  $v_k^+ \mapsto v_k^+$ ,  $v_k^- \mapsto -v_k^-$ . Continuing to use the same notations as in Examples 2.2.4 and 2.3.2, we find

$$\xi \circ \chi: S \longrightarrow \xi(\mathfrak{t}/W) = \mathbb{C}^2$$

$$s(v_1^-, v_2^-, v_1^+, v_2^+) \longmapsto (2(v_1^-)^2 - 2v_2^+, (v_1^-)^4 + 2(v_1^-)^2 v_2^+ + (v_2^+)^2 - (v_2^-)^2 - (v_1^+)^2).$$

The  $\mathbb{C}^*$ -action of  $\lambda \in \mathbb{C}^*$ ,  $\lambda = e^t$ , on  $S$  is obtained by conjugation with the matrix  $\exp(-th) = \text{diag}(\lambda^{-1}, \lambda^{-1}, \lambda, \lambda)$ , followed by multiplication by  $\lambda^2$ . For the  $v_k^\pm$  we thereby read off:

$$\lambda \cdot (v_1^-, v_2^-, v_1^+, v_2^+) = (\lambda^2 v_1^-, \lambda^4 v_2^-, \lambda^4 v_1^+, \lambda^4 v_2^+).$$

In Appendix A we construct a Slodowy slice  $S_h$  in  $\mathfrak{g}_h = \mathfrak{sl}_4(\mathbb{C})$  and construct an explicit isomorphism to  $S$  over  $(\mathfrak{t}_h/W_h)^\mathbb{C} \cong \mathfrak{t}/W$  in accordance with Theorem 2.5.5 c).

### 3. FOLDING OF HITCHIN SYSTEMS

From an algebraic-geometric point of view, Hitchin systems are a global analogue of the adjoint quotient as we explain in the next subsection. The aim of this section is to introduce the notion of folding of Hitchin systems and to globalize the commutative diagram (2.14) to Hitchin systems over the locus of smooth fibers, see Theorem 3.4.6.

To make this precise, we fix a Riemann surface  $\Sigma$  of genus  $g_\Sigma \geq 2$ . Given any reductive complex algebraic group  $H$ , by  $\mathcal{M}_0(\Sigma, H)$  we denote the *neutral component* of the moduli space of (S-equivalence classes of) semistable principal  $H$ -Higgs bundles or, equivalently, of polystable  $H$ -Higgs bundles of degree zero. In fact, since we restrict to the neutral component throughout this work<sup>7</sup>, to unclutter the notation, we drop the index 0 altogether and denote by  $\mathcal{M}(\Sigma, H) := \mathcal{M}_0(\Sigma, H)$  the neutral component of the moduli space, hoping that this will not lead to confusion. Then  $\mathcal{M}(\Sigma, H)$  is a quasi-projective variety [Hit87a, Nit91, Sim94a, Sim94b] whose complex structure is induced by that of  $\Sigma$  and  $H$ . It contains the cotangent bundle  $T^*\mathcal{N}(\Sigma, H)^{sm}$  of the smooth locus of the moduli space  $\mathcal{N}(\Sigma, H)$  of semistable/polystable  $H$ -bundles as an open and dense subset. The canonical symplectic structure on  $T^*\mathcal{N}(\Sigma, H)^{sm}$  extends to a holomorphic symplectic structure  $\omega_{\mathcal{M}}$  on the smooth locus  $\mathcal{M}(\Sigma, H)^{sm}$  of  $\mathcal{M}(\Sigma, H)$  by [BR94, Theorem 4.3].

We next recall the structure of an algebraic integrable system on the holomorphic symplectic manifold  $(\mathcal{M}(\Sigma, H)^{sm}, \omega_{\mathcal{M}})$ .

<sup>7</sup>We do so because so far, only this component has been related to Calabi–Yau integrable systems. Other components hopefully also arise geometrically, perhaps by G-flux.

**3.1. Hitchin systems.** As above, let  $H$  denote a reductive complex algebraic group. Its associated Lie algebra is  $\mathfrak{h}$ , and  $\mathfrak{t} \subset \mathfrak{h}$  denotes a Cartan subalgebra with Weyl group  $W$ . Then the variety  $\mathcal{M}(\Sigma, H)$  possesses the structure of an algebraic integrable system [Hit87a, Hit87b, BNR89, BR94, Fal93, DG02, ...]. Recall that an algebraic integrable system is a flat and surjective holomorphic map  $\pi: (M, \omega) \rightarrow B$  from a holomorphic symplectic manifold  $(M, \omega)$  to a complex manifold  $B$  with the following properties:

- the smooth loci of the fibers of  $\pi$  are Lagrangian in  $(M, \omega)$ ,
- there exists a Zariski-dense subset  $B^\circ \subset B$  such that the restriction of  $\pi$  to  $\pi^{-1}(B^\circ)$  has connected and compact fibers and admits a relative polarization, that is, there exists a line bundle  $\mathcal{L} \rightarrow M$  whose restriction to  $\pi^{-1}(b)$  is ample for every  $b \in B^\circ$ .

The holomorphic version of the Arnold–Liouville theorem implies that the fibers over  $B^\circ$  are torsors for abelian varieties. For the construction of an algebraic integrable system on  $\mathcal{M}(\Sigma, H)$ , recall from Section 2.2 that  $\mathfrak{t}/W$  carries a natural  $\mathbb{C}^*$ -action. Considering the canonical bundle  $K_\Sigma$  of  $\Sigma$  as a  $\mathbb{C}^*$ -bundle, we define the total space

$$U := \text{tot}(K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W)$$

of the bundle  $K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W$ . Its space of holomorphic sections is denoted by  $\mathbf{B}(\Sigma, H) := H^0(\Sigma, U)$ . The adjoint quotient  $\chi: \mathfrak{h} \rightarrow \mathfrak{t}/W$  induces the Hitchin map

$$\chi: \mathcal{M}(\Sigma, H) \rightarrow \mathbf{B}(\Sigma, H).$$

On the smooth locus equipped with the holomorphic symplectic structure, this is an algebraic integrable system, called the  $H$ -Hitchin system. By the above,  $\chi$  may be considered as global version of the adjoint quotient  $\chi$ . Many properties of  $\chi$  carry over to  $\chi$ . For example, the  $\mathbb{C}^*$ -equivariance of  $\chi$  is inherited by  $\chi$ . As mentioned in Section 2.2, one may choose homogeneous generators of  $\mathbb{C}[\mathfrak{t}]^W$  of degrees  $d_1, \dots, d_r$ , where  $d_1 - 1, \dots, d_r - 1$  are the exponents of  $\mathfrak{h}$ . One then obtains a convenient decomposition of  $\mathbf{B} := \mathbf{B}(\Sigma, H)$  as

$$(3.1) \quad \mathbf{B} \cong \bigoplus_{j=1}^r H^0(\Sigma, K_\Sigma^{d_j}).$$

Moreover,  $\chi$  admits (Lagrangian) sections [Hit92, §5], so-called Hitchin sections.

In accord with our conventions in Section 2, we write  $H = G$  if  $H = (G_h^{\mathbb{C}})^\circ$  arises from folding a simple *adjoint* complex Lie group  $G_h$  with Dynkin diagram of ADE type as in Proposition 2.4.1. If we choose  $H = G_h$ , then we decorate all our notations with the subscript  $h$ . Now the inclusion  $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbb{C}} \hookrightarrow \mathfrak{t}_h/W_h$  obtained in Corollary 2.2.3 induces the isomorphism

$$(3.2) \quad \iota: \mathbf{B} \xrightarrow{\cong} \mathbf{B}_h^{\mathbb{C}} \quad \text{with} \quad \mathbf{B} = \mathbf{B}(\Sigma, G), \quad \mathbf{B}_h = \mathbf{B}(\Sigma, G_h), \quad \mathbf{B}_h^{\mathbb{C}} \subset \mathbf{B}_h.$$

The Hitchin sections mentioned above are the analogues of the Kostant sections of  $\chi$  and  $\chi_h$ , respectively. In Theorem 3.4.6 below, we further extend this analogy and establish a global version of the commutative diagram (2.14), at least over the locus of smooth fibers of  $\chi$  and  $\chi_h$ , see equation (3.15).

**Example 3.1.1.** In our running example with  $G_h = PSL_4(\mathbb{C})$  and  $G = Sp_4(\mathbb{C})$  (see Example 2.4.2), it is convenient to first introduce  $SL_4(\mathbb{C})$ - and  $Sp_4(\mathbb{C})$ -Higgs bundles. A principal  $SL_4(\mathbb{C})$ -Higgs bundle is equivalent to a Higgs vector bundle  $(E, \varphi)$  consisting of a holomorphic vector bundle  $E$  over  $\Sigma$  of rank 4 with trivial determinant bundle  $\Lambda^4 E$  and a trace-free Higgs field  $\varphi \in H^0(\Sigma, K_\Sigma \otimes \text{End}_0(E))$ .

Analogously, a principal  $Sp_4(\mathbb{C})$ -Higgs bundle is equivalent to a pair  $((F, \langle \cdot, \cdot \rangle), \psi)$  of a holomorphic vector bundle  $F$  of rank 4 with symplectic pairing  $\langle \cdot, \cdot \rangle$  and a holomorphic section  $\psi \in H^0(\Sigma, K_\Sigma \otimes \text{End}(F))$  which is skew-symmetric with respect to  $\langle \cdot, \cdot \rangle$ . In particular,  $\psi$  is trace-free.

Then the moduli space of semistable principal  $PSL_4(\mathbb{C})$ -Higgs bundles is described in [HT03, §2]; the result generalizes to any classical reductive complex Lie group of adjoint type and reads

$$(3.3) \quad \begin{aligned} \mathcal{M}(\Sigma, PSL_4(\mathbb{C})) &\cong \mathcal{M}(\Sigma, SL_4(\mathbb{C}))/J(\Sigma)[4], \\ \mathcal{M}(\Sigma, Sp_4(\mathbb{C})) &\cong \mathcal{M}(\Sigma, Sp_4(\mathbb{C}))/J(\Sigma)[4]. \end{aligned}$$

Recall here that we are working with the neutral components  $\mathcal{M}(\Sigma, H) = \mathcal{M}_0(\Sigma, H)$  of the moduli spaces, throughout. In (3.3) the elements of  $J(\Sigma)[4]$  of order 4 in the Jacobian  $J(\Sigma)$  of  $\Sigma$  act on the respective moduli spaces by taking the tensor product.

Choosing the same generators of  $\mathbb{C}[t_h]^{W_h}$  and  $\mathbb{C}[t]^W$  as in Example 2.2.4, we obtain the isomorphisms

$$(3.4) \quad \mathbf{B}_h \cong H^0(\Sigma, K_\Sigma^2) \oplus H^0(\Sigma, K_\Sigma^3) \oplus H^0(\Sigma, K_\Sigma^4), \quad \mathbf{B} \cong H^0(\Sigma, K_\Sigma^2) \oplus H^0(\Sigma, K_\Sigma^4).$$

Hence we write elements of  $\mathbf{B}_h$  and  $\mathbf{B}$  as  $b^h = (b_2^h, b_3^h, b_4^h)$  and  $b = (b_2, b_4)$  respectively. Then the Hitchin maps are given by

$$\chi_h([(E, \varphi)]) = (\text{tr}(\Lambda^2 \varphi), \text{tr}(\Lambda^3 \varphi), \text{tr}(\Lambda^4 \varphi)), \quad \chi_h([(F, \psi)]) = (\text{tr}(\Lambda^2 \psi), \text{tr}(\Lambda^4 \psi)).$$

**3.2. Generic Hitchin fibers.** To give the isomorphism classes of the generic Hitchin fibers, i.e. the generic fibers of  $\chi$ , we introduce the total space

$$\tilde{U} := \text{tot}(K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t})$$

of the bundle associated to the standard  $\mathbb{C}^*$ -action on  $\mathfrak{t}$ . The quotient map  $q: \mathfrak{t} \rightarrow \mathfrak{t}/W$  globalizes to give

$$q: \tilde{U} \rightarrow U.$$

Then the cameral curve  $\tilde{\Sigma}_b$  associated to  $b \in \mathbf{B}$  is defined by the fiber product

$$\begin{array}{ccc} \tilde{\Sigma}_b & \longrightarrow & \tilde{U} \\ p_b \downarrow & & \downarrow q \\ \Sigma & \xrightarrow{b} & U \end{array}$$

If the fiber  $\chi^{-1}(b)$ ,  $b \in \mathbf{B}$ , is smooth, then it is a torsor for the abelian variety  $P_b$ . If  $\tilde{\Sigma}_b$  is smooth, then  $P_b$  is explicitly determined by  $\tilde{\Sigma}_b$ , see [DG02, §4] for the most general treatment. For our purposes we only need to consider the case where  $H$  is a simple complex Lie group of *adjoint type*, which we assume from now on.

Let  $\mathbf{A}$  be the cocharacter lattice of  $H$  and  $b \in \mathbf{B}$  such that  $\tilde{\Sigma}_b$  is smooth. Then the cocharacter lattice of  $P_b$  is given by

$$(3.5) \quad \text{cochar}(P_b) = H^1(\Sigma, (p_{b*}\mathbf{A})^W),$$

see [DP12, §3] and [Bec17, Corollary 3.3.2 and Proposition 3.4.1]. The complex structure is determined by the universal coverings

$$(3.6) \quad H^1(\tilde{\Sigma}_b, \mathfrak{t})^W$$

of the abelian varieties  $P_b$ . In summary, (3.5) completely determines the isomorphism class of  $P_b$  and hence of  $\chi^{-1}(b)$ .

**3.3. Folding of cameral curves.** As a first step towards folding Hitchin systems, in this section we show how to fold the cameral curves that we introduced in Section 3.2. In the following,  $G = (G_h^{\mathbf{C}})^{\circ}$  arises from folding a simple complex Lie group  $G_h$  of adjoint type. We further use the notation set up in (3.2).

**Proposition 3.3.1.** *Let  $b \in \mathbf{B}$  be transversal to the discriminant locus  $\text{discr}(\mathbf{q})$  of  $\mathbf{q}$ . Then  $\tilde{\Sigma}_b$  and  $\tilde{\Sigma}_{h,\iota(b)}$  are smooth  $W$ - and  $W_h$ -cameral curves respectively. They are related via*

$$(3.7) \quad \tilde{\Sigma}_{h,\iota(b)} \cong (\tilde{\Sigma}_b \times W_h)/W.$$

Here  $W \subset W_h$  acts diagonally on  $\tilde{\Sigma}_b \times W_h$ . Moreover, let  $\alpha \in R_h$ ,  $\beta = \alpha^O$ , so  $\beta \in R_h^{\mathbf{C}} = R^{\vee}$ , cf. equation (2.2). Then the local monodromies around the branch points of  $p_{\iota(b)}^h: \tilde{\Sigma}_{h,\iota(b)} \rightarrow \Sigma$  are given by the restrictions of  $s_{\beta} = \prod_{\gamma \in O(\alpha)} s_{\gamma}$ , cf. equation (2.6).

*Remark 3.3.2.* If  $b \in \mathbf{B}$  is transversal to  $\text{discr}(\mathbf{q})$ , then  $\iota(b) \in \mathbf{B}_h$  cannot be transversal to  $\text{discr}(\mathbf{q}_h)$  for  $\mathbf{C} \neq \{\text{id}\}$ . Indeed, by the compactness of  $\Sigma$ , it is easy to see that such  $b$  has to (transversally) intersect each hyperplane  $\mathbf{U}^{\alpha^O} = K_{\Sigma} \times_{\mathbb{C}^*} \mathfrak{t}^{\alpha^O}/W \subset \mathbf{U}_h$ ,  $\alpha^O \in R^{\vee}$ , with  $\mathfrak{t}^{\alpha^O}$  as in (2.6). Let  $\alpha \in R$  with  $|O(\alpha)| > 1$  and let  $x \in \Sigma$  be a point such that  $b(x) \in \mathbf{U}^{\alpha^O}$ . For any  $\tilde{u} \in \mathbf{q}^{-1}(b(x))$ , by (2.7) we have  $\dim \ker(d\mathbf{q}_{h,\tilde{u}}) = |O(\alpha)| > 1$  by assumption, so

$$\text{rk}(d\mathbf{q}_{h,\tilde{u}}) + 1 < \dim(\tilde{\mathbf{U}}_h) = \dim(\mathbf{U}_h).$$

Hence  $\iota(b)$  cannot intersect  $\text{discr}(\mathbf{q}_h)$  transversally at  $x$ . In particular, we cannot conclude from the general theory that  $\tilde{\Sigma}_{h,\iota(b)}$  is smooth.

*Proof of Proposition 3.3.1.* We first consider the local situation. Let  $D \subset \Sigma$  be a sufficiently small open subset and  $b: D \rightarrow \mathfrak{t}/W \subset \mathfrak{t}_h/W_h$  be a holomorphic map which is transversal to  $\text{discr}(\mathbf{q})$ . Further note that  $q_h^{-1}(\mathfrak{t}/W) = W_h(\mathfrak{t})$  fits into the following commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \hookrightarrow & W_h(\mathfrak{t}) \\ & \searrow q & \downarrow q_h \\ & & \mathfrak{t}/W. \end{array}$$

Since  $b$  is transversal to  $\text{discr}(q)$ , the commutativity of the above diagram implies that  $b$  is transversal to  $\text{discr}(q_h|_{W_h(\mathfrak{t})})$  as well. Hence the cameral curves are locally given by the smooth coverings  $\tilde{D}_b = D \times_{\mathfrak{t}/W} \mathfrak{t}$  and  $\tilde{D}_{h,\iota(b)} = D \times_{\mathfrak{t}/W} W_h(\mathfrak{t})$  of  $D$  and are therefore smooth. This proves the first statement of the proposition.

To prove (3.7) we next observe

$$(3.8) \quad W_h(\mathfrak{t}^\circ) \cong (\mathfrak{t}^\circ \times W_h)/W,$$

where  $\mathfrak{t}^\circ$  denotes the set of semisimple regular elements in  $\mathfrak{t}$  and  $W$  acts by  $w \cdot (t, w_h) = (w \cdot t, w_h w^{-1})$ . Indeed, by Lemma 2.2.2 the map

$$(3.9) \quad (\mathfrak{t}^\circ \times W_h)/W \longrightarrow W_h(\mathfrak{t}^\circ), \quad [(t, w)] \mapsto w \cdot t$$

is a well-defined isomorphism of  $W_h$ -coverings of  $\mathfrak{t}^\circ/W$ .

Next observe that the branch locus of  $p_{\iota(b)}^h: \tilde{\Sigma}_{h,\iota(b)} \rightarrow \Sigma$  in  $\Sigma$  coincides with that of  $p_b: \tilde{\Sigma}_b \rightarrow \Sigma$ . We denote by  $\Sigma^\circ$  its complement and by  $\tilde{\Sigma}_{h,\iota(b)}^\circ$  and  $\tilde{\Sigma}_b^\circ$  the complements of the ramifications divisors of  $p_{\iota(b)}^h$  and  $p_b$  respectively. The latter are the pullbacks of  $K_\Sigma \times_{\mathbb{C}^*} W_h(\mathfrak{t}^\circ)$  and  $K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}^\circ$  along  $\iota(b)$  and  $b$  respectively. Hence (3.8) implies

$$\tilde{\Sigma}_{h,\iota(b)}^\circ \cong (\tilde{\Sigma}_b^\circ \times W_h)/W$$

as  $W_h$ -coverings over  $\Sigma^\circ$ . Here  $W$  acts diagonally as before. Since the local monodromies around the points of  $\Sigma - \Sigma^\circ$  coincide and  $\tilde{\Sigma}_{h,\iota(b)}$  as well as  $\tilde{\Sigma}_b$  are smooth, the previous isomorphism extends to the isomorphism (3.7) of simply branched  $W_h$ -Galois coverings of  $\Sigma$ .

The claim about the local monodromies now follows from the prescription for folding  $W_h$  to  $W$ , see Proposition 2.2.1, in particular formula (2.6).  $\square$

In the setup of Proposition 3.3.1,  $\tilde{\Sigma}_{h,\iota(b)}$  decomposes non-canonically into  $[W_h : W]$  copies of  $\tilde{\Sigma}_b$ : fix a representative  $w_j \in W_h$  for each equivalence class in  $W_h/W$ , where we choose the canonical representative  $1 \in W$  for the class  $W$ . Then we obtain the inclusions

$$(3.10) \quad i_{w_j}: \tilde{\Sigma}_b \hookrightarrow \tilde{\Sigma}_{h,\iota(b)}, \quad x \mapsto [(x, w_j)]$$

under the isomorphism (3.7). In particular, with  $i := i_1$ , we have  $i_{w_j} = w_j \circ i$ . By the universal property of the coproduct, these inclusions induce the morphism

$$\coprod_{[w_j] \in W_h/W} \tilde{\Sigma}_b \longrightarrow \tilde{\Sigma}_{h,\iota(b)}$$

which is a  $W_h$ -equivariant isomorphism.

**Example 3.3.3.** The heart of the previous proof is (3.8) which is equivalent to the non-canonical decomposition

$$(3.11) \quad W_h(\mathfrak{t}^\circ) \cong \coprod_{[w_j] \in W_h/W} \mathfrak{t}^\circ.$$

In our running example, we have  $W_h = S_4$  and  $W = \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$  so that  $[W_h : W] = 3$ . In fact, with notations as in Example 2.2.4,  $w \in S_4$  acts on the elements of  $\mathfrak{t}_h$  by

$$\text{diag}(x_1, x_2, x_3, x_4) \longmapsto \text{diag}(x_{w(1)}, x_{w(2)}, x_{w(3)}, x_{w(4)}),$$

and  $W \subset W_h$  is generated by the reflections in  $(\beta_1^\vee)^\perp, (\beta_2^\vee)^\perp$  which act as  $w = (1, 4)(2, 3)$  and  $w = (2, 4)$ . By Example 2.2.4 we further know

$$\mathfrak{t}^\circ = \{\text{diag}(u, v, -u, -v) \mid u, v \in \mathbb{C}^*, u \neq \pm v\}.$$

It follows that  $W_h(\mathfrak{t}^\circ) \subset \mathfrak{t}_h$  decomposes into three connected components, according to (3.11),

$$\begin{aligned} & \{\text{diag}(u, v, -u, -v) \mid u, v \in \mathbb{C}^*, u \neq \pm v\} \coprod \{\text{diag}(u, v, -v, -u) \mid u, v \in \mathbb{C}^*, u \neq \pm v\} \\ & \coprod \{\text{diag}(u, -u, v, -v) \mid u, v \in \mathbb{C}^*, u \neq \pm v\}. \end{aligned}$$

This example also shows that the decomposition (3.11) ceases to be disjoint when  $\mathfrak{t}^\circ$  is replaced by a subset of  $\mathfrak{t}$  which contains non-regular elements.

**3.4. Folding of Hitchin systems.** For any homomorphism  $\sigma: H \rightarrow H'$  of reductive complex algebraic groups and any  $H$ -Higgs bundle  $(F, \theta)$ , we denote by  $(\sigma(F), \sigma_*(\theta))$  the associated  $H'$ -Higgs bundle obtained by the extension of the structure group. In particular, the automorphism group  $\text{Aut}(G_h) \cong \text{Aut}(\mathfrak{g}_h)$  acts naturally on  $\mathcal{M}(\Sigma, G_h)$  via

$$(3.12) \quad [(E, \varphi)] \mapsto [(\sigma(E), \sigma_*(\varphi))].$$

Inner automorphisms act trivially so that the  $\text{Aut}(G_h)$ -action descends to an  $\text{Aut}(\Delta_h)$ -action, in particular to an action by  $\mathbf{C} = \langle a \rangle \subset \text{Aut}(\Delta_h)$ .

**Proposition 3.4.1.** *The inclusion  $G \hookrightarrow G_h$  of simple adjoint complex Lie groups induces an injective holomorphic map*

$$\iota_{\mathcal{M}}: \mathcal{M}(\Sigma, G) \rightarrow \mathcal{M}(\Sigma, G_h)^{\mathbf{C}}$$

over  $\iota(\mathbf{B}) \subset \mathbf{B}_h$ .

As a preparation for our proof, we need the following lemma which seems to be known to the experts (see e.g. [GR16, §5]), but we include a proof for completeness.

**Lemma 3.4.2.** *Let  $i_H: H \hookrightarrow G_h$  be the inclusion of a reductive subgroup and  $i_N: H \hookrightarrow N_{G_h}(H)$  the inclusion into its normalizer  $N = N_{G_h}(H)$ . Further let  $(E_j, \varphi_j)$ ,  $j \in \{1, 2\}$ , be two  $H$ -Higgs bundles such that  $(i_H(E_j), i_{H,*}(\varphi_j))$  are isomorphic  $G_h$ -Higgs bundles. Then  $(i_N(E_j), i_{N,*}(\varphi_j))$  are isomorphic  $N$ -Higgs bundles.*

*Proof.* Let  $f: (i_H(E_1), i_{H,*}(\varphi_1)) \rightarrow (i_H(E_2), i_{H,*}(\varphi_2))$  be an isomorphism of  $G_h$ -Higgs bundles. It is proven in [GPPNR18, Lemma 5.11] that  $f$  restricts to an isomorphism  $f_N: i_N(E_1) \rightarrow i_N(E_2)$  of  $N$ -bundles. Since  $i_{H,*}(\varphi_j)$  is (locally) valued in  $\text{Lie}(H) \subset \text{Lie}(N) \subset \mathfrak{g}_h$ ,  $f_N: (i_N(E_1), i_{N,*}(\varphi_1)) \rightarrow (i_N(E_2), i_{N,*}(\varphi_2))$  is an isomorphism of  $N$ -Higgs bundles.  $\square$

*Proof of Proposition 3.4.1.* Let  $j: G \rightarrow G_h$  be the group monomorphism obtained by folding by  $\mathbf{C} = \langle a \rangle$ . We claim that  $(j(E), j_*(\varphi))$  is polystable as a  $G_h$ -Higgs bundle if  $(E, \varphi)$  is polystable as a  $G$ -Higgs bundle. To this end, recall that an  $H$ -Higgs bundle  $(F, \theta)$  is polystable iff the  $GL(\mathfrak{h})$ -Higgs bundle  $(\text{Ad}(F), \text{Ad}_*(\theta))$  associated to  $(F, \theta)$  via the adjoint representation  $\text{Ad}: H \rightarrow GL(\mathfrak{h})$  is polystable [AB01, Lemma 4.7]. Hence we may assume  $G \subset G_h \subset GL(n)$ . The Higgs vector bundle  $(E, \varphi)$  is polystable iff there exists a Hermitian metric on  $E$  whose Chern connection  $\nabla$  (taken with respect to the given holomorphic structure on  $E$ ) satisfies

$$(3.13) \quad F^\nabla + [\varphi, \varphi^\dagger] = 0, \quad \bar{\partial}^\nabla \varphi = 0,$$

see [Sim92, Theorem 1]. Since (3.13) is still satisfied after passing from the polystable Higgs vector bundle  $(E, \varphi)$  to the Higgs vector bundle  $(j(E), j_*(\varphi))$ , the latter is polystable as well. Therefore

$$(3.14) \quad \iota_{\mathcal{M}}: \mathcal{M}(\Sigma, G) \rightarrow \mathcal{M}(\Sigma, G_h), \quad [(E, \varphi)] \mapsto [(E_h, \varphi_h)]$$

is well-defined. The equality  $a \circ j = j$  implies that the image of  $\iota_{\mathcal{M}}$  lies in  $\mathcal{M}(\Sigma, G_h)^{\mathbf{C}}$ . Since  $\chi_h$  is  $\mathbf{C}$ -equivariant,  $\iota_{\mathcal{M}}$  is defined over  $\iota(\mathbf{B}) \subset \mathbf{B}_h$ .

We next show the injectivity of  $\iota_{\mathcal{M}}$ . In [GR16, Proposition 2.20] it is proven<sup>8</sup> that

$$N_{G_h}(G_h^{\mathbf{C}}) = \{g \in G_h \mid a(g) = c(g)g \text{ for some } c(g) \in Z(G_h)\}.$$

Since  $G \cong (G_h^{\mathbf{C}})^\circ$  by Proposition 2.4.1 and  $Z(G_h) = \{\text{id}\}$ , we conclude  $G = N_{G_h}(G)$ . Hence  $\iota_{\mathcal{M}}$  is injective by Lemma 3.4.2.  $\square$

**Example 3.4.3.** Following Example 3.1.1, we use (3.3) to represent isomorphism classes of Higgs vector bundles that correspond to semistable principal  $PSL_4(\mathbb{C})$ - and  $PSp_4(\mathbb{C})$ -Higgs bundles, respectively. Then the inclusion of Proposition 3.4.1 is given by

$$\iota_{\mathcal{M}}: \mathcal{M}(\Sigma, PSp_4(\mathbb{C})) \rightarrow \mathcal{M}(\Sigma, PSL_4(\mathbb{C})), \quad [(F, \langle \cdot, \cdot \rangle, \psi)] \mapsto [(F, \psi)].$$

Note that  $(F, \psi)$  is indeed an  $SL_4(\mathbb{C})$ -Higgs vector bundle because it is an  $Sp_4(\mathbb{C})$ -Higgs vector bundle, so in particular  $\langle \cdot, \cdot \rangle$  induces a trivialization of  $\Lambda^4 F$  and  $\psi$  is a trace-free Higgs field.

*Remark 3.4.4.* In [GR16, Theorem 6.3] it is shown that if  $\mathbf{C} \neq \{\text{id}\}$ , then  $\mathcal{M}(\Sigma, G) \subsetneq \mathcal{M}(\Sigma, G_h)^{\mathbf{C}}$ . For example if  $G_h = PSL_{2n}(\mathbb{C})$ , then

$$\mathcal{M}(\Sigma, PSO_{2n}(\mathbb{C})) \cup \mathcal{M}(\Sigma, PSp_{2n}(\mathbb{C})) \hookrightarrow \mathcal{M}(\Sigma, PSL_{2n}(\mathbb{C}))^{\mathbf{C}},$$

see [GR16, Section 9.2]. Observe that  $G = PSp_{2n}(\mathbb{C})$  is the group obtained by folding by the involution  $a(A) = J(A^T)^{-1}J^{-1}$ , where  $J$  is the standard symplectic matrix as in Example 2.4.2. On the other hand,  $PSO_{2n}(\mathbb{C})$  arises from folding by the involution  $\sigma(A) = (A^T)^{-1}$ . Note that both  $a$  and  $\sigma$  induce the same outer automorphism on the Dynkin diagram.

In the remainder of this section, we show that  $\mathcal{M}(\Sigma, G_h)^{\mathbf{C}} - \iota_{\mathcal{M}}(\mathcal{M}(\Sigma, G))$  is empty over a suitable open and dense subset of  $\iota(\mathbf{B})$ .

<sup>8</sup>This proposition is stated for  $\text{ord}(a) = 2$  but generalizes to arbitrary finite order.



**Proposition 3.4.5.** *Choose an arbitrary  $b \in \mathbf{B}$  which is transversal to  $\text{discr}(\mathbf{q})$ . Then the sheaves  $((p_{i(b)}^h)_* \mathbf{\Lambda}_h)^{W_h}$  and  $(p_{b*} \mathbf{\Lambda}_h)^W$  are naturally isomorphic to each other.*

*Proof.* By (3.10) we have  $p_b = p_{i(b)}^h \circ i$  and

$$(p_{i(b)}^h)_* (\mathbf{\Lambda}_h) \cong \bigoplus_{[w_j] \in W_h/W} p_{b*} (\mathbf{\Lambda}_h), \quad f \mapsto (f_j)_{[w_j] \in W_h/W},$$

Here the sections  $f_j$  are uniquely determined by

$$f_j \circ i = (w_j^{-1})^* (f_{|_{i_{w_j}(\tilde{\Sigma}_b)}}).$$

Restricting this isomorphism to  $W_h$ -equivariant sections  $f$ , one obtains  $f_j \circ i = f_{|_{i(\tilde{\Sigma}_b)}} \in (p_{b*} (\mathbf{\Lambda}_h))^W$  by the  $W_h$ -equivariance and  $i_{w_j} = w_j \circ i$ . Hence  $f \mapsto f_1$  yields the desired isomorphism  $((p_{i(b)}^h)_* \mathbf{\Lambda}_h)^{W_h} \cong (p_{b*} (\mathbf{\Lambda}_h))^W$ .  $\square$

**Theorem 3.4.6.** *Let  $\mathbf{B}^\circ \subset \mathbf{B}$  denote the open set of sections that are transversal to  $\text{discr}(\mathbf{q})$ . Then for any  $b \in \mathbf{B}^\circ$  the Hitchin fibers  $P = P_b$  and  $P_h = P_{h,b}$  are smooth. Moreover,*

$$(3.15) \quad \begin{array}{ccc} \mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ} & \longrightarrow & \mathcal{M}(\Sigma, G_h)|_{\mathbf{B}^\circ} \\ \chi \downarrow & & \downarrow \chi_h \\ \mathbf{B}^\circ & \longrightarrow & \mathbf{B}^\circ \end{array}$$

is an isomorphism of smooth algebraic integrable systems over  $\mathbf{B}^\circ$ . Here and in the following, we identify  $\mathbf{B}$  with its image  $\iota(\mathbf{B})$  in  $\mathbf{B}_h$  under the isomorphism  $\iota: \mathbf{B} \rightarrow \mathbf{B}_h^{\mathbb{C}}$  induced by  $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbb{C}}$  according to (3.2).

*Proof.* Proposition 3.3.1 implies that  $\chi^{-1}(b)$  and  $\chi_h^{-1}(b)$ ,  $b \in \mathbf{B}^\circ$ , are smooth since  $\chi^{-1}(b) \cong P_b$  and  $\chi_h^{-1}(b) \cong P_{h,b}$ , see Section 3.2.

To prove that (3.15) is an isomorphism of algebraic integrable systems, we first show that the top horizontal arrow of (3.15) is biholomorphic. Since both  $\mathcal{M}(\Sigma, G)$  and  $\mathcal{M}(\Sigma, G_h)$  are smooth over  $\mathbf{B}^\circ$  with fibers  $P_b$  and  $P_{h,b}$  over  $b \in \mathbf{B}^\circ$  respectively, it is sufficient to prove that the morphism

$$(3.16) \quad P_b \hookrightarrow P_{h,b}^{\mathbb{C}}$$

is an isomorphism. It is an inclusion by Proposition 3.4.1. Since the horizontal arrows of (3.15) are holomorphic by Proposition 3.4.1, it is sufficient to prove (3.16) on the level of cocharacter lattices. Both  $G$  and  $G_h$  are of adjoint type, so that

$$\text{cochar}(P_b) = H^1(\Sigma, (p_{*b} \mathbf{\Lambda})^W), \quad \text{cochar}(P_{h,b}) = H^1(\Sigma, (p_{b*}^h \mathbf{\Lambda}_h)^{W_h}),$$

see (3.5). By Proposition 3.4.5,  $H^1(\Sigma, p_{b*}^h \mathbf{\Lambda}_h)^{W_h} \cong H^1(\Sigma, (p_{b*} \mathbf{\Lambda}_h)^W)$ . Hence it remains to prove that the morphism

$$H^1(\Sigma, (p_{*b} \mathbf{\Lambda})^W) \longrightarrow H^1(\Sigma, (p_{b*} \mathbf{\Lambda}_h)^W)^{\mathbb{C}}$$

induced by the inclusion  $\mathbf{\Lambda} \hookrightarrow \mathbf{\Lambda}_h$  is an isomorphism. But this follows from [Bec17, Proposition 5.1.2]. Therefore the top horizontal arrow of (3.15) is a biholomorphism.

To see that it is a symplectomorphism, first note that both spaces in (3.15) are algebraic integrable systems over  $\mathbf{B}^\circ$  which admit Lagrangian sections, e.g. Hitchin sections, see [DP12, Lemma 4.1]. The holomorphic symplectic structures therefore induce symplectomorphisms

$$(3.17) \quad \mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ} \cong T^*\mathbf{B}^\circ/\Gamma, \quad \mathcal{M}(\Sigma, G_h)|_{\mathbf{B}^\circ}^{\mathbf{C}} \cong T^*\mathbf{B}^\circ/\Gamma',$$

see e.g. [Fre99, §3]. Here  $\Gamma$  and  $\Gamma'$  are complex submanifolds of  $T^*\mathbf{B}^\circ$  which restrict to lattices in the fibers, and the quotients are taken fiberwise. They are Lagrangian with respect to the canonical symplectic structure  $\omega_c$  on  $T^*\mathbf{B}^\circ$  so that  $\omega_c$  descends to the quotients  $T^*\mathbf{B}^\circ/\Gamma$  and  $T^*\mathbf{B}^\circ/\Gamma'$ . The isomorphisms (3.17) are compatible with (3.15) giving the commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ} & \xrightarrow{\cong} & \mathcal{M}(\Sigma, G_h)|_{\mathbf{B}^\circ}^{\mathbf{C}} \\ \cong \downarrow & & \downarrow \cong \\ T^*\mathbf{B}^\circ/\Gamma & \longrightarrow & T^*\mathbf{B}^\circ/\Gamma', \end{array}$$

where the vertical arrows are biholomorphic symplectomorphisms, while the upper horizontal arrow is biholomorphic. This implies that the lower horizontal arrow is biholomorphic as well, and thus  $\Gamma$  and  $\Gamma'$  are isomorphic. Since on both sides of this map, the symplectic structure is induced by  $\omega_c$ , it follows that this map is a symplectomorphism. Hence the upper horizontal arrow (3.15) is a symplectomorphism as well, concluding the proof.  $\square$

#### 4. CREPANT RESOLUTIONS AND HITCHIN SYSTEMS

Generalizing the results of [DDP07] for trivial  $\mathbf{C}$ , in [Bec19], a family of quasi-projective Gorenstein Calabi–Yau threefolds  $\mathcal{X} \rightarrow \mathbf{B}$ ,  $\mathbf{B} = \mathbf{B}(\Sigma, G)$ , with a  $\mathbf{C}$ -action was constructed for any cyclic group  $\mathbf{C}$  of Dynkin graph automorphisms. Moreover, there it was shown that the canonical line bundles of these threefolds admit global nowhere-vanishing and  $\mathbf{C}$ -invariant sections. We say that their canonical classes are  $\mathbf{C}$ -trivializable. In [Bec19, Theorem 6], the first author constructed an isomorphism

$$J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \cong \mathcal{M}(\Sigma, G)|_{\mathbf{B}^\circ}, \quad \mathbf{B}^\circ := \mathbf{B}^\circ(\Sigma, G), \quad \mathcal{X}^\circ := \mathcal{X}|_{\mathbf{B}^\circ},$$

of smooth algebraic integrable systems with section based on previous results from [Bec17]. Here, for the fiber  $X_b$  of  $\mathcal{X}$  over  $b \in \mathbf{B}^\circ$ ,

$$J_{\mathbf{C}}^2(X_b) := H^3(X_b, \mathbb{C})^{\mathbf{C}} / (F^2 H^3(X_b, \mathbb{C})^{\mathbf{C}} + H^3(X_b, \mathbb{Z})^{\mathbf{C}})$$

is isomorphic to the orbifold intermediate Jacobian of the Calabi–Yau orbifold stack  $[X_b/\mathbf{C}]$ , see [Bec19, §4] for more details.

In this section, we construct simultaneous crepant resolutions of the singular quotients  $X_b/\mathbf{C}$ ,  $b \in \mathbf{B}^\circ$ . Moreover, we show that the Griffiths intermediate Jacobians of these crepant resolutions contain algebraic integrable systems that are isogenous to the folded Hitchin systems of Theorem 3.4.6.

**4.1. Quasi-projective Calabi–Yau threefolds over Hitchin bases.** As before, let  $\mathfrak{g} = \mathfrak{g}(\Delta)$  denote the simple complex Lie algebra determined by  $\Delta = \Delta_{h,\mathbf{C}}$ , and  $\Sigma$  a compact Riemann surface of genus  $g_\Sigma \geq 2$ . We choose a Slodowy slice  $\sigma: S \rightarrow \mathfrak{t}/W$  and a theta characteristic  $L \rightarrow \Sigma$ , i.e.  $L^2 = K_\Sigma$ . Using the  $\mathbb{C}^*$ -action on  $S$  given in (2.24), we twist  $S$  by  $L$ , considered as a  $\mathbb{C}^*$ -bundle, to obtain the family of surfaces

$$(4.1) \quad \mathcal{S} := L \times_{\mathbb{C}^*} S, \quad \sigma: \mathcal{S} \rightarrow L \times_{\mathbb{C}^*} \mathfrak{t}/W.$$

Here  $\mathbb{C}^*$  acts with twice the usual weights on  $\mathfrak{t}/W$ , cf. Remark 2.5.3, so that  $L \times_{\mathbb{C}^*} \mathfrak{t}/W \cong U$ . Except for the case  $(\Delta_h, \mathbf{C}) = (A_{2k}, \{\text{id}\})$ ,  $k \in \mathbb{N}$ , we could alternatively work with a  $\mathbb{C}^*$ -action on  $S$  which squares to (2.24) and then twist by  $K_\Sigma$  instead of  $L$  in (4.1), as also follows from Remark 2.5.3.

Pulling back the family  $\mathcal{S} \rightarrow U$  of surfaces via the evaluation map  $ev: \Sigma \times \mathbf{B} \rightarrow U$  and projecting to the second factor yields a family

$$\begin{array}{ccc} & \pi & \\ & \curvearrowright & \\ \mathcal{X} := ev^* \mathcal{S} & \longrightarrow & \Sigma \times \mathbf{B} \longrightarrow \mathbf{B} \end{array}$$

of threefolds. By construction, the  $\mathbf{C}$ -action on  $S$  induces a  $\mathbf{C}$ -action on  $\mathcal{S}$  and therefore a  $\mathbf{C}$ -action on  $\mathcal{X}$  as well. Since  $\mathbf{C}$  acts trivially on the base  $\mathbf{B}$  (see (3.2)), the fibers  $X_b := \pi^{-1}(b)$ ,  $b \in \mathbf{B}$ , are preserved by the  $\mathbf{C}$ -action. In [Bec19, §3.3.], it is proven that  $\mathcal{X} \rightarrow \mathbf{B}$  is an algebraic family of quasi-projective Gorenstein threefolds with  $\mathbf{C}$ -trivializable canonical class. For trivial  $\mathbf{C}$ , this was first observed in [Sze04, DDP07].

By construction, each  $X_b$ ,  $b \in \mathbf{B}$ , admits a map  $X_b \rightarrow \Sigma$  which is affine. This allows us to realize each of the threefolds  $X_b$ ,  $b \in \mathbf{B}$ , as a hypersurface in the total space of a sum of powers of  $K_\Sigma$  which is affine over  $\Sigma$ . We next present the explicit equations for our running example and for  $\Delta = \Delta_h^{\mathbf{C}} = G_2$  with  $\Delta_h = D_4$  and  $\mathbf{C} = \mathbb{Z}/3\mathbb{Z}$  which will be useful for later computations.

**Example 4.1.1.** For  $\Delta = C_2$ , we choose  $b = (b_2, b_4) \in \mathbf{B}$  using  $\mathbf{B} \cong H^0(\Sigma, K_\Sigma^2) \oplus H^0(\Sigma, K_\Sigma^4)$ , cf. (3.4). By Example 2.5.2, using  $L^2 = K_\Sigma$  and writing tensor products as products, we have

$$X_b = \{(\alpha_1, \alpha_2, \alpha_3) \in \text{tot}(K_\Sigma \oplus K_\Sigma^2 \oplus K_\Sigma^2) \mid [\alpha_1^4 - \alpha_2 \alpha_3 + b_2 \alpha_1^2 + b_4](\sigma) = 0, \sigma \in \Sigma\}.$$

The  $\mathbf{C}$ -action is defined by  $(\alpha_1, \alpha_2, \alpha_3) \mapsto (-\alpha_1, \alpha_3, \alpha_2)$ . Hence the  $\mathbf{C}$ -fixed point locus is

$$X^{\mathbf{C}} = \{\alpha \in \text{tot}(K_\Sigma^2) \mid [\alpha^2 - b_4](\sigma) = 0, \sigma \in \Sigma\}$$

which has genus  $6g_\Sigma - 5$ . By varying  $b \in \mathbf{B}$ , we arrive at the family  $\mathcal{X} \rightarrow \mathbf{B}$  with fiberwise  $\mathbf{C}$ -action. Note that this is the global description of our family of Calabi–Yau threefolds mentioned in [DDP07, Proof of Proposition 2.3], which is worked out in detail in [Bec16].

**Example 4.1.2.** Let  $\Delta = \Delta_{h,\mathbf{C}} = G_2$  with  $\Delta_h = D_4$  and  $\mathbf{C} = \mathbb{Z}/3\mathbb{Z}$ . By [Slo80, §6.2(3)], the  $\Delta_h$ -singularity  $Y$  is determined by the equation

$$x^3 + y^3 + z^2 = 0$$

with  $\mathbb{C}^*$ -action  $(\lambda, (x, y, z)) \mapsto (\lambda^4 x, \lambda^4 y, \lambda^6 z)$ . Note that  $(x, y, z)$  have weights  $(2w_x, 2w_y, 2w_z)$  with coprime  $(w_x, w_y, w_z) = (2, 2, 3)$ , in accord with Remark 2.5.3. The Jacobian ring is  $\mathbb{C}[x, y, z]/(x^2, y^2, z)$ , a quasi-homogeneous basis of which is represented by  $(xy, x, y, 1)$ . Thus a semi-universal  $\mathbb{C}^*$ -deformation is given by

$$\mathcal{Y}_h := \left\{ (x, y, z, \beta_2, \beta_4, \tilde{\beta}_4, \beta_6) \in \mathbb{C}^3 \times \mathbb{C}^4 \mid x^3 + y^3 + z^2 + \beta_2 xy + \beta_4 x + \tilde{\beta}_4 y + \beta_6 = 0 \right\},$$

$$\mathcal{Y}_h \longrightarrow \mathbb{C}^4, \quad (x, y, z, \beta_2, \beta_4, \tilde{\beta}_4, \beta_6) \mapsto (\beta_2, \beta_4, \tilde{\beta}_4, \beta_6).$$

As  $\mathbb{C}^*$ -action we have

$$\lambda \cdot (x, y, z, \beta_2, \beta_4, \tilde{\beta}_4, \beta_6) = (\lambda^4 x, \lambda^4 y, \lambda^6 z, \lambda^4 \beta_2, \lambda^8 \beta_4, \lambda^8 \tilde{\beta}_4, \lambda^{12} \beta_6).$$

The  $\mathbf{C}$ -action is induced by  $(x, y, z) \mapsto (\mu x, \mu^2 y, z)$  for any primitive third root  $\mu$  of unity, which extends to  $(x, y, z, \beta_2, \beta_4, \tilde{\beta}_4, \beta_6) \mapsto (\mu x, \mu^2 y, z, \beta_2, \mu^2 \beta_4, \mu \tilde{\beta}_4, \beta_6)$ , yielding the following semi-universal  $\mathbb{C}^*$ -deformation of  $(Y, \mathbf{C})$ :

$$\left\{ (x, y, z, \beta_2, \beta_6) \in \mathbb{C}^3 \times \mathbb{C}^2 \mid x^3 + y^3 + z^2 + \beta_2 xy + \beta_6 = 0 \right\} \longrightarrow \mathbb{C}^2,$$

$$(x, y, z, \beta_2, \beta_6) \mapsto (\beta_2, \beta_6).$$

Using (3.1), one finds that the Hitchin base  $\mathbf{B}$  for  $\Delta$  is isomorphic to  $H^0(\Sigma, K_\Sigma^2) \oplus H^0(\Sigma, K_\Sigma^6)$ . Again fixing such an isomorphism we write any  $b \in \mathbf{B}$  as  $(b_2, b_6)$  for  $b_j \in H^0(\Sigma, K_\Sigma^{\otimes j})$ ,  $j \in \{2, 6\}$ . Arguing as in the previous example, we see that the corresponding threefold  $X_b$  is given by

$$X_b = \{(\alpha_1, \alpha_2, \alpha_3) \in \text{tot}(K_\Sigma^2 \oplus K_\Sigma^2 \oplus K_\Sigma^3) \mid [\alpha_1^3 + \alpha_2^3 + \alpha_3^2 + b_2 \alpha_1 \alpha_2 + b_6](\sigma) = 0, \sigma \in \Sigma\}.$$

The  $\mathbf{C}$ -action is induced by  $(\alpha_1, \alpha_2, \alpha_3) \mapsto (\mu \alpha_1, \mu^2 \alpha_2, \alpha_3)$  so that

$$X^{\mathbf{C}} \cong \{\gamma \in \text{tot}(K_\Sigma^3) \mid [\gamma^2 + b_6](\sigma) = 0, \sigma \in \Sigma\}.$$

Note that  $X^{\mathbf{C}}$  has genus  $8g_\Sigma - 7$ . Varying  $b \in \mathbf{B}$  yields the family  $\mathcal{X} \longrightarrow \mathbf{B}$  with  $\mathbf{C}$ -action.

*Remark 4.1.3.* While Slodowy slices allow a uniform construction of the families  $\pi: \mathcal{X} \longrightarrow \mathbf{B}$  and to prove some of their properties, e.g. that  $\pi$  is smooth over  $\mathbf{B}^\circ$  [Bec19, §3.3.], it is often useful to work with explicit equations along the lines of the above examples.

**4.2. Crepant resolutions and their intermediate Jacobians.** In this section, we construct unique crepant resolutions of the quotients  $X_b/\mathbf{C}$  by the action of  $\mathbf{C}$  for the family of Calabi–Yau threefolds  $\mathcal{X}$  introduced above, restricted to  $\mathbf{B}^\circ$ . To make sure that each quotient  $X_b/\mathbf{C}$ ,  $b \in \mathbf{B}^\circ$ , admits a crepant resolution, we need the following:

**Lemma 4.2.1.** *Let  $S = x + \ker \text{ad}_y \subset \mathfrak{g}$  be our choice of Slodowy slice with  $\mathbf{C}$ -action, and assume that  $\mathbf{C}$  is non-trivial. Then the fixed point locus  $S^{\mathbf{C}} \subset S$  is finite over  $\mathfrak{t}/W$ .*

*Proof.* Let  $\bar{t} \in \mathfrak{t}/W$  and  $Y := \sigma^{-1}(\bar{t})$  with smooth part  $Y^{sm} \subset Y$ . By Theorem 2.5.5,  $Y^{sm}$  coincides with the set of regular elements of  $\mathfrak{g}$  in  $Y$ , i.e.  $Y^{sm} = Y \cap \mathfrak{g}^{reg}$ . Moreover, the Kostant–Kirillov form on adjoint orbits, cf. [CG10, Chapter 1], is known to restrict to a symplectic form  $\omega$  on  $Y^{sm}$  by [GG02, §7]. As was explained in Section 2.5,  $\mathbf{C}$  is represented

on the Slodowy slice by the adjoint action, hence  $\omega$  is left invariant under the  $\mathbf{C}$ -action, i.e.  $a^*\omega = \omega$ .

Since  $Y - Y^{sm}$  is finite by Theorem 2.5.5 b), it suffices to consider  $Y^{sm}$ . Therefore the rest of the proof is a standard application of Cartan's linearization trick [Car57, Lemme 1], which allows us to replace the automorphism  $a \in \mathbf{C}$  by its linearization on the tangent space. Since  $a^*\omega = \omega$ , we obtain a special unitary map of finite order on  $\mathbb{C}^2$ , which is thus either the identity or has the origin as its unique fixed point. Hence by the assumption that  $\mathbf{C}$  is non-trivial, the fixed points of  $a$  in  $Y$  are isolated. But  $a$  is an algebraic automorphism so that  $Y^a \subset Y$  is finite.  $\square$

Lemma 4.2.1 implies that each  $X_b$ ,  $b \in \mathbf{B}^\circ$ , has a one-dimensional fixed point locus  $X_b^{\mathbf{C}} \subset X_b$  whose irreducible components descend to curves of singularities of type  $A_1$  or  $A_2$  in  $X_b/\mathbf{C}$ . Hence we may deduce

**Proposition 4.2.2.** *Let  $X = X_b$ ,  $b \in \mathbf{B}^\circ$ , be a smooth quasi-projective Calabi–Yau threefold with  $\mathbf{C}$ -action as constructed in Section 4.1, and view the fixed point locus  $X^{\mathbf{C}}$  as a subset of  $X/\mathbf{C}$ . Then the blow up  $\widetilde{X/\mathbf{C}} := Bl_{X^{\mathbf{C}}}(X/\mathbf{C})$  of  $X/\mathbf{C}$  along  $X^{\mathbf{C}}$  is the unique crepant resolution*

$$\widetilde{X/\mathbf{C}} \longrightarrow X/\mathbf{C}$$

of  $X/\mathbf{C}$ .

*Proof.* The following proof is mostly standard. We give some details which will prove useful in our constructions below.

To see that  $X/\mathbf{C}$  has a *unique* crepant resolution, we may analyse each fixed point  $x \in X^{\mathbf{C}}$  by applying Cartan's linearization trick analogously to the proof of Lemma 4.2.1. Here, by the results of Lemma 4.2.1, a non-trivial automorphism in  $\mathbf{C}$  is linearized at any fixed point by a special unitary map on  $\mathbb{C}^3$  with precisely one eigenvalue 1. Hence every singular point of  $X/\mathbf{C}$  possesses a local neighborhood of the form  $\mathbb{C} \times (\mathbb{C}^2/\mathbf{C})$  such that  $0 \in \mathbb{C}^2/\mathbf{C}$  is a singularity of type  $A_1$  or  $A_2$ . In these local neighborhoods, the blowup  $Bl_{X^{\mathbf{C}}}(X/\mathbf{C})$  is given by

$$Bl_{\mathbb{C} \times \{0\}}(\mathbb{C} \times (\mathbb{C}^2/\mathbf{C})) \cong \mathbb{C} \times Bl_0(\mathbb{C}^2/\mathbf{C}).$$

Since  $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$  or  $\mathbb{Z}/3\mathbb{Z}$ , this blowup is smooth. In fact, it is the unique crepant resolution of the local model  $\mathbb{C} \times (\mathbb{C}^2/\mathbf{C})$ . By uniqueness, the local crepant resolutions glue together to give the unique crepant resolution  $\widetilde{X/\mathbf{C}}$  of  $X/\mathbf{C}$ , and by construction, it coincides with  $Bl_{X^{\mathbf{C}}}(X/\mathbf{C})$ .  $\square$

The resolution described above is in fact a simultaneous resolution for  $\widetilde{\mathcal{X}^\circ/\mathbf{C}} \longrightarrow \mathbf{B}^\circ$ :

**Proposition 4.2.3.** *The family  $\mathcal{X}^\circ/\mathbf{C} \longrightarrow \mathbf{B}^\circ$  admits a simultaneous crepant resolution  $\widetilde{\mathcal{X}^\circ/\mathbf{C}} \longrightarrow \mathbf{B}^\circ$ .*

*Proof.* Since  $\mathcal{X}^\circ \longrightarrow \mathbf{B}^\circ$  is smooth over  $\mathbf{B}^\circ$  and  $\mathbf{C}$  is finite,  $\mathcal{X}^\circ$  is covered by affine  $\mathbf{C}$ -invariant open subsets  $D \cong D' \times D''$  with  $D' \subset \mathbf{B}^\circ$  and  $D'' \subset X_b$  for all  $b \in D'$ . By

Proposition 4.2.2, we may assume without loss of generality that each  $D/\mathbf{C}$  is either smooth or obeys

$$(4.2) \quad D/\mathbf{C} \cong D' \times (D''/\mathbf{C}) \cong D' \times \mathbb{C} \times (\mathbb{C}^2/\mathbf{C}),$$

where  $0 \in (\mathbb{C}^2/\mathbf{C})$  is a singularity of type  $A_1$  or  $A_2$ . By the uniqueness of these resolutions, as in the proof of Proposition 4.2.2, the resolutions  $\widetilde{D}/\mathbf{C} \rightarrow D/\mathbf{C}$  glue to

$$\widetilde{\mathcal{X}^\circ}/\mathbf{C} \rightarrow \mathcal{X}^\circ/\mathbf{C}$$

over  $\mathbf{B}^\circ$ . The restriction to each  $b \in \mathbf{B}^\circ$  is the crepant resolution  $\widetilde{X_b}/\mathbf{C} \rightarrow X_b/\mathbf{C}$  of Proposition 4.2.2 by construction.  $\square$

We next determine the rational mixed Hodge structure ( $\mathbb{Q}$ -MHS) of the crepant resolution in Proposition 4.2.2. As a preparation, we need to determine the exceptional divisor of the blowup.

**Proposition 4.2.4.** *For  $b \in \mathbf{B}^\circ$ , let  $X := X_b$  and  $Z := \widetilde{X}/\mathbf{C}$ . Moreover, let  $E \hookrightarrow Z$  denote the exceptional divisor of the resolution  $\rho: Z \rightarrow X/\mathbf{C}$ . Then  $\rho|_E: E \rightarrow X^\mathbf{C}$  is a  $\mathbb{P}^1$ -bundle if  $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ . If  $\mathbf{C} = \mathbb{Z}/3\mathbb{Z}$ , then  $E = E_1 \cup E_2$  decomposes into two  $\mathbb{P}^1$ -bundles  $\rho|_{E_j}: E_j \rightarrow X^\mathbf{C}$ ,  $j \in \{1, 2\}$ .*

*Proof.* By Proposition 4.2.2,  $Z = \text{Bl}_{X/\mathbf{C}}(X)$ . If  $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ , it follows that  $\rho|_E: E \rightarrow X^\mathbf{C}$  is a  $\mathbb{P}^1$ -bundle. If  $\mathbf{C} = \mathbb{Z}/3\mathbb{Z}$ , then the fibers of  $E \rightarrow X^\mathbf{C}$  consist of two transversally intersecting copies of  $\mathbb{P}^1$ . Using the fact that  $X \rightarrow \Sigma$  is an affine morphism, we next show that  $E$  decomposes globally into two  $\mathbb{P}^1$ -bundles over  $\Sigma$ .

We use the notations of Example 4.1.2 for fixed  $b = (b_2, b_6) \in \mathbf{B}^\circ \subset H^0(\Sigma, K_\Sigma^2) \oplus H^0(\Sigma, K_\Sigma^6)$ . Further we introduce the coordinates

$$(4.3) \quad \nu_1 := \alpha_1^3, \quad \nu_2 := \alpha_2^3, \quad \nu_3 := \alpha_1\alpha_2, \quad \nu_4 := \alpha_3.$$

on  $X/\mathbf{C}$ , such that with  $M := K_\Sigma^6 \oplus K_\Sigma^6 \oplus K_\Sigma^4 \oplus K_\Sigma^3$ ,

$$Z \cong \{(\nu_1, \nu_2, \nu_3, \nu_4) \in \text{tot}(M) \mid \nu_1 + \nu_2 + \nu_4^2 + b_2\nu_3 + b_6 = 0, \quad \nu_3^3 - \nu_1\nu_2 = 0\}.$$

Under this isomorphism, by what was said in Example 4.1.2, the singular locus  $X^\mathbf{C} \subset X/\mathbf{C}$  is given by  $\{(0, 0, 0, \nu_4) \in X/\mathbf{C} \mid \nu_4^2 + b_6 = 0\}$ . To compute the exceptional divisor  $E$  of  $Z = \text{Bl}_{X/\mathbf{C}}(X/\mathbf{C})$ , note that  $\text{Bl}_{X/\mathbf{C}}(X/\mathbf{C})$  is the proper transform of  $X/\mathbf{C} \subset \text{tot}(M)$  in  $\text{Bl}_{M'_0}(M)$ , where  $M'_0$  is the zero section of  $M' := K_\Sigma^6 \oplus K_\Sigma^6 \oplus K_\Sigma^4$  viewed as a submanifold of  $\text{tot}(M)$ . Hence we obtain  $E$  as the proper transform over  $X^\mathbf{C} \subset X/\mathbf{C}$ . Since  $X/\mathbf{C} \rightarrow \Sigma$  is affine, the calculation essentially works as in the affine case and we obtain

$$(4.4) \quad E = \{((0, 0, 0, \nu_4), [\gamma_1 : \gamma_2 : \gamma_3]) \in \text{tot}(M) \times \mathbb{P}(M') \mid \nu_4^2 + b_6 = 0, \quad \gamma_1\gamma_2 = 0\}.$$

In particular,  $E = E_1 \cup E_2$  and each  $E_j$ ,  $j \in \{1, 2\}$ , is a  $\mathbb{P}^1$ -bundle over  $X^\mathbf{C}$  as claimed.  $\square$

**Theorem 4.2.5.** *Let  $X = X_b$ ,  $b \in \mathbf{B}^\circ$ , and  $p: X \rightarrow X/\mathbf{C}$  its quotient under the action of  $\mathbf{C} = \langle a \rangle$ . Further let  $Z := \widetilde{X/\mathbf{C}}$  and  $\rho: Z \rightarrow X/\mathbf{C}$  be the crepant resolution of  $X/\mathbf{C}$ . The  $\mathbb{Q}$ -MHS on  $H^3(Z, \mathbb{Q})$  is pure of weight 3, and it is isomorphic to the direct sum*

$$(4.5) \quad H^3(Z, \mathbb{Q}) \cong H^3(X, \mathbb{Q})^{\mathbf{C}} \oplus H^1(X^{\mathbf{C}}, \mathbb{Q})(-1)^{|a|-1}$$

of pure  $\mathbb{Q}$ -Hodge structures of weight 3. Here  $H^1(X^{\mathbf{C}}, \mathbb{Q})(-1)$  denotes the Tate twist of  $H^1(X^{\mathbf{C}})$  by the  $\mathbb{Q}$ -Hodge structure  $\mathbb{Q}(-1)$  of weight 2, shifting all weights by 2.

*Proof.* All cohomology in this proof is  $\mathbb{Q}$ -valued, so to save notation, we suppress all  $\mathbb{Q}$ -coefficients in the following. By [PS08, Corollary-Definition 5.37], there is a long exact Mayer–Vietoris sequence of  $\mathbb{Q}$ -MHS, which reads

$$(4.6) \quad \begin{array}{ccccccc} \dots & \longrightarrow & H^2(E) & \xrightarrow{\delta_2} & H^3(X/\mathbf{C}) & \xrightarrow{(\rho^*, i^*)} & H^3(Z) \oplus H^3(X^{\mathbf{C}}) \\ & & & & & & \\ & & & & \xrightarrow{i_E^* - \rho^*} & H^3(E) & \xrightarrow{\delta_3} & H^4(X/\mathbf{C}) & \longrightarrow & \dots \end{array}$$

Here  $i_E: E \hookrightarrow Z$  is the inclusion of the exceptional divisor and  $i: X^{\mathbf{C}} \rightarrow X/\mathbf{C}$  is the inclusion of the fixed curve. We now analyze the above sequence term-by-term.

Firstly,  $H^3(X^{\mathbf{C}}) = 0$  because  $\dim_{\mathbb{R}} X^{\mathbf{C}} = 2$ . Secondly, since  $\mathbf{C}$  is a finite group and  $\mathbb{Q}$  is a field of characteristic 0,

$$(4.7) \quad p^*: H^k(X/\mathbf{C}) \longrightarrow H^k(X)^{\mathbf{C}}$$

is an isomorphism of  $\mathbb{Q}$ -vector spaces (see e.g. [Bor60, §3, Corollary 2.3.]). Furthermore  $p^*$  is a morphism of  $\mathbb{Q}$ -MHS. The category of  $\mathbb{Q}$ -MHS is abelian, so that  $p^*$  is an isomorphism of  $\mathbb{Q}$ -MHS. Moreover, since  $X$  is smooth (though non-compact), the weights of  $H^k(X)$  are bounded below by  $k$ , hence so are the weights of  $H^k(X/\mathbf{C}) \cong H^k(X)^{\mathbf{C}}$ .

Next we show that the  $\mathbb{Q}$ -MHS on  $H^k(E)$  is in fact pure of weight  $k$ . More precisely, we claim that there is an isomorphism

$$(4.8) \quad H^k(E) \cong \left[ H^{k-2}(X^{\mathbf{C}})(-1) \right]^{|a|-1} \quad \text{for } k \in \{2, 3\}$$

of  $\mathbb{Q}$ -MHS which is in fact an isomorphism of pure Hodge structures since the right hand side is pure. Indeed, if  $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ , then by Proposition 4.2.4 we know that  $E \rightarrow X^{\mathbf{C}}$  is a  $\mathbb{P}^1$ -bundle over  $X^{\mathbf{C}}$ . Hence the Leray–Hirsch theorem (see for example [Voi07, Theorem 7.33]) gives an isomorphism (4.8) of pure  $\mathbb{Q}$ -Hodge structures of weight  $k$  with  $|a| = 2$ . If  $\mathbf{C} = \mathbb{Z}/3\mathbb{Z}$ , then by Proposition 4.2.4 we know that  $E = E_1 \cup E_2$  in  $Z$  decomposes into two  $\mathbb{P}^1$ -bundles over  $X^{\mathbf{C}}$  with transverse intersection. The Mayer–Vietoris sequence for the triple  $(Z, E_1, E_2)$  yields the short exact sequence

$$(4.9) \quad 0 \longrightarrow \text{cok} \longrightarrow H^k(E) \longrightarrow \text{ker} \longrightarrow 0$$

of  $\mathbb{Q}$ -MHS. Here  $\text{cok}$  is the cokernel of  $(i_1^*, i_2^*): H^{k-1}(E_1) \oplus H^{k-1}(E_2) \rightarrow H^{k-1}(E_1 \cap E_2)$  where  $i_j: E_j \hookrightarrow Z$ ,  $j \in \{1, 2\}$ , is the inclusion. Moreover,  $\text{ker}$  is the kernel of  $i_1^* - i_2^*: H^k(E_1) \oplus H^k(E_2) \rightarrow H^k(E)$ . Now (4.9) implies that  $H^k(E)$  only has weights  $k - 1$

and  $k$ . If  $k \geq 2$ , then  $\text{cok} = 0$  because  $(i_1^*, i_2^*)$  is then surjective. Hence we again conclude by means of the Leray–Hirsch theorem that (4.8) holds, now with  $|a| = 3$ . The previous observations imply that the morphisms  $\delta_k: H^k(E) \rightarrow H^{k+1}(X/\mathbf{C})$ ,  $k \in \{2, 3\}$ , have to vanish due to the weights of the  $\mathbb{Q}$ -MHS. Combining this fact with the isomorphisms (4.7) and (4.8), the long exact sequence (4.6) yields the short exact sequence

$$(4.10) \quad 0 \longrightarrow H^3(X)^{\mathbf{C}} \longrightarrow H^3(Z) \longrightarrow H^1(X^{\mathbf{C}})(-1)^{|a|-1} \longrightarrow 0$$

of  $\mathbb{Q}$ -MHS. The  $\mathbb{Q}$ -MHS on  $H^3(X)^{\mathbf{C}}$  is known to be pure of weight 3 and polarizable (see [Bec17, Lemma 4.3.1], also [DDP07, Lemma 3.1] in the case where  $\mathbf{C} = \{\text{id}\}$ ). By choosing a polarization on each of the pure  $\mathbb{Q}$ -Hodge structures of weight 3 in (4.10), we arrive at an isomorphism (4.5).  $\square$

**Corollary 4.2.6.** *The intermediate Jacobian of the crepant resolution  $Z$  of  $X/\mathbf{C}$ ,  $X = X_b$ ,  $b \in \mathbf{B}^\circ$ , is an abelian variety and admits the non-canonical isogeny decomposition*

$$(4.11) \quad J^2(Z) \simeq J_{\mathbf{C}}^2(X) \times J(X^{\mathbf{C}})^{|a|-1}.$$

Here  $J(X^{\mathbf{C}})$  is the Jacobian of the curve  $X^{\mathbf{C}} \subset X$ .

*Proof.* Since the Mayer–Vietoris sequence is defined over the integers, the proof of Theorem 4.2.5 gives the short exact sequence

$$0 \longrightarrow H^3(X/\mathbf{C}, \mathbb{Z}) \longrightarrow H^3(Z, \mathbb{Z}) \longrightarrow H^1(X^{\mathbf{C}}, \mathbb{Z})(-1)^{|a|-1} \longrightarrow 0$$

of polarizable  $\mathbb{Z}$ -Hodge structures of weight 3. The  $\mathbb{Z}$ -Hodge structure  $H^3(X, \mathbb{Z})^{\mathbf{C}}$  is known to be effective<sup>9</sup> and pure of weight 1 up to a Tate twist by [DDP07, Lemma 3.1] for  $\mathbf{C} = \{\text{id}\}$  and [Bec17, Corollary 4.3.2] in general. Therefore both polarizable  $\mathbb{Z}$ -Hodge structures  $H^3(X/\mathbf{C}, \mathbb{Z})$  and  $H^3(Z, \mathbb{Z})$  are effective of weight 1 up to a Tate twist as well. In particular, the intermediate Jacobians  $J^2(X/\mathbf{C})$ ,  $J^2(Z)$  and  $J_{\mathbf{C}}^2(X)$  are abelian varieties. Since  $J^2(X/\mathbf{C})$  is isogenous to  $J_{\mathbf{C}}^2(X)$  by the previous proof (see (4.7)), Poincaré reducibility (e.g. [Deb05, Theorem 6.20]) implies a non-canonical isogeny decomposition

$$J^2(Z) \simeq J_{\mathbf{C}}^2(X) \times J(X^{\mathbf{C}})^{|a|-1}.$$

$\square$

*Remark 4.2.7.* In Examples 4.1.1 and 4.1.2 we have seen that the genus of  $X_b^{\mathbf{C}}$ ,  $b \in \mathbf{B}^\circ$ , is strictly larger than  $g_\Sigma \geq 2$ . In particular,  $X_b^{\mathbf{C}}$  is a non-trivial branched covering of  $\Sigma$  and  $J(X_b^{\mathbf{C}})$  varies non-trivially in  $b \in \mathbf{B}^\circ$ . By inspection of all semi-universal  $\mathbb{C}^*$ -deformations of ADE-type singularities one immediately checks that this always holds when  $\mathbf{C}$  is non-trivial.

<sup>9</sup>A pure  $\mathbb{Z}$ -Hodge structure of non-negative weight  $w$  is called effective if  $H^{p,q} = \{0\}$  for all  $(p, q) \notin \mathbb{N}^2$  with  $p + q = w$ .



**Corollary 4.2.8.** *Let  $\mathcal{Z} \rightarrow \mathcal{X}^\circ/\mathbf{C}$  be the simultaneous crepant resolution over  $\mathbf{B}^\circ$ . Then the intermediate Jacobian fibration  $J^2(\mathcal{Z}/\mathbf{B}^\circ)$  admits a non-canonical fiberwise isogeny decomposition*

$$(4.12) \quad J^2(\mathcal{Z}/\mathbf{B}^\circ) \simeq J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \times J(\mathcal{X}_{|\mathbf{B}^\circ}^{\mathbf{C}})^{|a|-1}$$

over  $\mathbf{B}^\circ$ .

*Proof.* The key observation is that Proposition 4.2.4 works over  $\mathbf{B}^\circ$  by arguing similarly to the proof of Proposition 4.2.3. Indeed, the proof of Theorem 4.2.5 globalizes to an isomorphism of polarizable rational variations of Hodge structures of weight 3 over  $\mathbf{B}^\circ$ . Then by the same reasoning as in the proof of Corollary 4.2.6 we have a fiberwise isogeny (4.12).  $\square$

In [Bec17, Theorem 5.2.1] it was proven that there is an isomorphism

$$J_{\mathbf{C}}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \cong \mathcal{M}(\Sigma, G)_{|\mathbf{B}^\circ}$$

of algebraic integrable systems over  $\mathbf{B}^\circ$ . These are in turn isomorphic to  $\mathcal{M}(\Sigma, G_h)_{|\mathbf{B}^\circ}^{\mathbf{C}}$  by Theorem 3.4.6. In particular,  $J^2(\mathcal{Z}/\mathbf{B}^\circ)$  contains  $\mathcal{M}(\Sigma, G)_{|\mathbf{B}^\circ}$  up to isogeny by Corollary 4.2.8. This corollary also implies that  $J^2(\mathcal{Z}/\mathbf{B}^\circ) \rightarrow \mathbf{B}^\circ$  cannot admit the structure of an algebraic integrable system for dimensional reasons when  $\mathbf{C}$  is non-trivial. It is an intriguing question if  $\mathcal{Z} \rightarrow \mathbf{B}^\circ$  embeds into a larger family  $\widehat{\mathcal{Z}} \rightarrow \widehat{\mathbf{B}}^\circ$  such that  $J^2(\widehat{\mathcal{Z}}/\widehat{\mathbf{B}}^\circ) \rightarrow \widehat{\mathbf{B}}^\circ$  has the structure of an algebraic integrable system. And if so, does passing to the crepant resolutions have an analogue for Hitchin systems? We hope to return to these questions in future work.

#### APPENDIX A. AN ISOMORPHISM $S \cong S_{h|_{(\mathfrak{t}_h/W_h)^{\mathbf{C}}}}$ IN OUR RUNNING EXAMPLE

In Example 2.5.7, we constructed a Slodowy slice  $S \subset \mathfrak{g}(\Delta)$  with a  $\mathbf{C}$ -action for our running example  $\Delta = \mathbf{C}_2 = \mathbf{A}_{3, \mathbf{C}}$ . In the following, we construct a Slodowy slice  $S_h \subset \mathfrak{g}(\Delta_h) = \mathfrak{sl}_4(\mathbb{C})$  with a  $\mathbf{C}$ -action and a  $(\mathbf{C}^* \times \mathbf{C})$ -equivariant inclusion  $S \hookrightarrow S_h$  which is an isomorphism over  $(\mathfrak{t}_h/W_h)^{\mathbf{C}} \cong \mathfrak{t}/W$ , in accordance with Theorem 2.5.5. We also construct an explicit  $(\mathbf{C}^* \times \mathbf{C})$ -equivariant isomorphism of  $S_h$  with the semi-universal  $\mathbf{C}^*$ -deformation obtained by means of the Jacobian ring for the surface singularity of type  $\mathbf{A}_3$ .

Consider the  $\mathfrak{sl}_2$ -triple  $(x_h, y_h, h_h)$  with

$$x_h = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y_h = x_h^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \quad h_h = [x_h, y_h] = \text{diag}(2, 0, 0, -2)$$

in  $\mathfrak{sl}_4(\mathbb{C})$ . For the corresponding Slodowy slice we obtain

$$S_h = x_h + \ker \text{ad}_{y_h} = \left\{ \left( \begin{array}{cccc} u_1^- & 1 & 1 & 0 \\ u_1^+ - u_2^- & -u_1^- & 2u_1^- & -1 \\ u_2^+ + u_2^- & 2u_1^- & -u_1^- & -1 \\ u_3^- & u_2^- - u_2^+ & -u_1^+ - u_2^- & u_1^- \end{array} \right) \middle| u_1^\pm, u_2^\pm, u_3^- \in \mathbb{C} \right\}.$$

To lift the  $\mathbf{C}$ -action to  $S_h$ , according to Remark 2.5.4 we need to pick some automorphism in  $\mathrm{CA}(x_h, y_h)$  as in (2.26) which does not belong to  $\mathrm{CA}(x_h, y_h)^\circ$ . One checks that setting

$$\forall m \in \mathbb{C}^*: M_m := \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \frac{1}{2}(m + m^{-3}) & \frac{1}{2}(m - m^{-3}) & 0 \\ 0 & \frac{1}{2}(m - m^{-3}) & \frac{1}{2}(m + m^{-3}) & 0 \\ 0 & 0 & 0 & m \end{pmatrix}, \quad \varphi_a: A \mapsto -A^{\tilde{T}},$$

we obtain

$$\mathrm{CA}(x_h, y_h) = \{\mathrm{Ad}_{M_m}, \mathrm{Ad}_{M_m} \circ \varphi_a \mid m \in \mathbb{C}^*\}.$$

The unique nontrivial automorphism in  $\mathrm{CA}(x_h, y_h)$  which induces  $\alpha_1^\vee \leftrightarrow \alpha_3^\vee$  on  $\mathfrak{t}_h$  is  $\varphi_a$ , which thus generates our choice of  $\mathbf{C}$ -action on  $\mathfrak{g}_h = \mathfrak{sl}_4(\mathbb{C})$ . Since  $\varphi_a$  induces the desired automorphism  $a$  on  $\Delta_h$ , by what was said in Remark 2.5.4, on  $S_h$  we have the correct induced action  $u_k^+ \mapsto u_k^+$ ,  $u_k^- \mapsto -u_k^-$ , yielding

$$S_h^{\mathbf{C}} = \left\{ \left( \begin{array}{cccc} 0 & 1 & 1 & 0 \\ u_1^+ & 0 & 0 & -1 \\ u_2^+ & 0 & 0 & -1 \\ 0 & -u_2^+ & -u_1^+ & 0 \end{array} \right) \middle| u_1^+, u_2^+ \in \mathbb{C} \right\}.$$

As explained in Example 2.3.2, with respect to the standard coordinates  $\xi_h$  on  $\mathfrak{t}_h/W_h$  we may calculate

$$\begin{aligned} \xi_h \circ \chi_h: & \begin{pmatrix} u_1^- & 1 & 1 & 0 \\ u_1^+ - u_2^- & -u_1^- & 2u_1^- & -1 \\ u_2^+ + u_2^- & 2u_1^- & -u_1^- & -1 \\ u_3^- & u_2^- - u_2^+ & -u_1^+ - u_2^- & u_1^- \end{pmatrix} \\ & \xrightarrow{(2.16)} \begin{pmatrix} -6(u_1^-)^2 - 2(u_1^+ + u_2^+) & & & \\ -8(u_1^-)^3 + 4u_1^-(u_1^+ + u_2^+) - 2u_3^- & & & \\ -3(u_1^-)^4 + 6(u_1^-)^2(u_1^+ + u_2^+) + 6u_1^-u_3^- + (u_1^+ - u_2^+)^2 - 4(u_2^-)^2 & & & \end{pmatrix}^T \end{aligned}$$

which as one checks is  $\mathbf{C}$ -equivariant.

This form allows us to construct an explicit isomorphism between  $S_h$  and the semi-universal  $\mathbb{C}^*$ -deformation  $\mathcal{Y}_h$  of the surface singularity of type  $A_3$  given in Example 2.5.2. Indeed, setting  $(b_2, b_3, b_4) := \xi_h \circ \chi_h(A)$  for  $A \in S_h$ , with parameters  $u_k^\pm$  as before, the above calculation yields

$$\begin{aligned} u_1^+ + u_2^+ &= -\frac{1}{2}b_2 - 3(u_1^-)^2, \\ u_3^- &= -\frac{1}{2}b_3 - 4(u_1^-)^3 + 2u_1^-(u_1^+ + u_2^+) = -\frac{1}{2}b_3 - 10(u_1^-)^3 - b_2u_1^-, \\ b_4 &= -81(u_1^-)^4 - 9(u_1^-)^2b_2 - 3u_1^-b_3 + (u_1^+ - u_2^+)^2 - 4(u_2^-)^2. \end{aligned}$$

Hence

$$(A.1) \quad (x, y, z) := (3u_1^-, u_1^+ - u_2^+ + 2u_2^-, u_1^+ - u_2^+ - 2u_2^-)$$

yields

$$S_h = \{(x, y, z, b_2, b_4, b_6) \in \mathbb{C}^6 \mid b_4 = -x^4 - b_2x^2 - b_3x + yz\}$$

in accord with (2.21). One immediately checks that (A.1) is  $\mathbf{C}$ -equivariant, since by the above,  $a$  acts by  $u_k^\pm \mapsto \pm u_k^\pm$ , in accord with (2.23).

Next we obtain

$$\xi_h \circ \chi_h (S_h^{\mathbf{C}}) = \{(-2(u_1^+ + u_2^+), 0, (u_1^+ - u_2^+)^2) \mid u_1^+, u_2^+ \in \mathbb{C}\} = \xi_h((\mathfrak{t}_h/W_h)^{\mathbf{C}})$$

by Example 2.3.2, and hence

$$s_h(u_1^-, u_2^-, u_1^+, u_2^+) := \begin{pmatrix} u_1^- & 1 & 1 & 0 \\ u_1^+ - u_2^- & -u_1^- & 2u_1^- & -1 \\ u_2^+ + u_2^- & 2u_1^- & -u_1^- & -1 \\ -4(u_1^-)^3 + 2u_1^-(u_1^+ + u_2^+) & u_2^- - u_2^+ & -u_1^+ - u_2^- & u_1^- \end{pmatrix}, \quad u_k^\pm \in \mathbb{C}$$

parametrizes  $\chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}})$ . Moreover, we have

$$\xi_h \circ \chi_h : \chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}}) \longrightarrow \xi_h((\mathfrak{t}_h/W_h)^{\mathbf{C}}) = \mathbb{C}^2 \quad (\text{A.2})$$

$$s_h(u_1^-, u_2^-, u_1^+, u_2^+) \longmapsto \begin{pmatrix} -6(u_1^-)^2 - 2(u_1^+ + u_2^+) \\ 0 \\ -27(u_1^-)^4 + 18(u_1^-)^2(u_1^+ + u_2^+) + (u_1^+ - u_2^+)^2 - 4(u_2^-)^2 \end{pmatrix}^T.$$

The  $\mathbb{C}^*$ -action of  $\lambda \in \mathbb{C}^*$ ,  $\lambda = e^t$ , on  $S_h$  is obtained by conjugation with the matrix  $\exp(-th_h) = \text{diag}(\lambda^{-2}, 1, 1, \lambda^2)$ , followed by multiplication by  $\lambda^2$ . For the  $u_k^\pm$  we thereby read off:

$$\lambda \cdot (u_1^-, u_2^-, u_3^-, u_1^+, u_2^+) = (\lambda^2 u_1^-, \lambda^4 u_2^-, \lambda^6 u_3^-, \lambda^4 u_1^+, \lambda^4 u_2^+).$$

One now immediately checks that (A.1) is also  $\mathbb{C}^*$ -equivariant, if in (2.22) one doubles all weights, as explained in Remark 2.5.3.

According to Theorem 2.5.5, there is a  $(\mathbb{C}^* \times \mathbf{C})$ -equivariant isomorphism over  $(\mathfrak{t}_h/W_h)^{\mathbf{C}} \cong \mathfrak{t}/W$  from  $S$  to  $\chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}})$ . This is best visible by choosing new parametrizations for the bases  $(\mathfrak{t}_h/W_h)^{\mathbf{C}}$  and  $\mathfrak{t}/W$  of our semi-universal  $\mathbb{C}^*$ -deformations. We let

$$\tilde{\xi}_h \circ \chi_h : \chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{C}}) \longrightarrow \mathbb{C}^2, \quad \tilde{\xi}_h \circ \xi_h^{-1}(b_2^h, 0, b_4^h) = (-\frac{1}{6}b_2^h, -b_4^h),$$

so by (A.2) we have

$$\tilde{\xi}_h \circ \chi_h (s_h(u_1^-, u_2^-, u_1^+, u_2^+)) = \begin{pmatrix} (u_1^-)^2 + \frac{1}{3}(u_1^+ + u_2^+) \\ 27(u_1^-)^4 - 18(u_1^-)^2(u_1^+ + u_2^+) - (u_1^+ - u_2^+)^2 + 4(u_2^-)^2 \end{pmatrix}^T.$$

In the notation of Example 2.5.7, we similarly let

$$\tilde{\xi} \circ \chi : S \longrightarrow \mathbb{C}^2, \quad \tilde{\xi} \circ \xi^{-1}(b_2, b_4) = (\frac{1}{2}b_2, 9(b_4 - \frac{1}{4}b_2^2)),$$

so by (2.28) we have

$$\tilde{\xi} \circ \chi (s(v_1^-, v_2^-, v_1^+, v_2^+)) = ((v_1^-)^2 - v_2^+, \quad 36(v_1^-)^2 v_2^+ - 9((v_2^-)^2 + (v_1^+)^2)).$$

One now checks that

$$\begin{aligned} \Phi: S &\longrightarrow \chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbb{C}}), \\ s(v_1^-, v_2^-, v_1^+, v_2^+) &\mapsto s_h\left(\sqrt{\frac{2}{3}}v_1^-, \frac{3i}{2}v_2^-, \frac{1}{2}(v_1^-)^2 + \frac{3}{2}(v_1^+ - v_2^+), \frac{1}{2}(v_1^-)^2 - \frac{3}{2}(v_1^+ + v_2^+)\right) \end{aligned}$$

yields a  $(\mathbb{C}^* \times \mathbb{C})$ -equivariant isomorphism such that the diagram

$$(A.3) \quad \begin{array}{ccc} S & \xrightarrow{\Phi} & \chi_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbb{C}}) \\ \downarrow & & \downarrow \\ \mathfrak{t}/W & \xrightarrow{\tilde{\xi}_h^{-1} \circ \tilde{\xi}} & (\mathfrak{t}_h/W_h)^{\mathbb{C}} \end{array}$$

commutes.

Some aspects of this example have already been worked out in [Bec19, Appendix B]. Beware that the triple  $(x_0, y_0, h_0)$  in  $\mathfrak{sl}_4(\mathbb{C})$  given there is not an  $\mathfrak{sl}_2$ -triple but only satisfies  $[x_0, y_0] = h_0$ . However, the corresponding slice still gives the correct semi-universal  $\mathbb{C}^*$ -deformation of a  $B_2$ -singularity. Moreover, there, the Lie algebra  $\mathfrak{so}_5(\mathbb{C})$  of type  $\Delta = B_2$  is used rather than the (isomorphic) Lie algebra  $\mathfrak{sp}_4(\mathbb{C})$  of type  $\Delta = C_2$ .

#### REFERENCES

- [AB01] Boudjemaa Anchouche and Indranil Biswas. Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. *Amer. J. Math.*, 123(2):207–228, 2001.
- [Bec16] Florian Beck. Hitchin and Calabi-Yau integrable systems. PhD thesis, Albrecht-Ludwigs Universität Freiburg, <http://dx.doi.org/10.6094/UNIFR/11668>, 2016.
- [Bec17] Florian Beck. Hitchin and Calabi-Yau integrable systems via variations of Hodge structures. 2017. arXiv:1707.05973v2. Under review.
- [Bec19] Florian Beck. Calabi–Yau orbifolds over Hitchin bases. *J. Geom. Phys.*, 136:14–30, 2019. arXiv:1807.05134.
- [Beh04] K. Behrend. Cohomology of stacks. In *Intersection theory and moduli*, ICTP Lect. Notes, XIX, pages 249–294. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.
- [BNR89] Arnaud Beauville, M. S. Narasimhan, and S. Ramanan. Spectral curves and the generalised theta divisor. *J. Reine Angew. Math.*, 398:169–179, 1989.
- [Bor60] Armand Borel. *Seminar on transformation groups*. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.
- [BR94] Indranil Biswas and S. Ramanan. An infinitesimal study of the moduli of Hitchin pairs. *J. London Math. Soc. (2)*, 49(2):219–231, 1994.
- [Bri71] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.
- [Car57] Henri Cartan. Quotient d’un espace analytique par un groupe d’automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz,.
- [CG10] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.
- [Che55] Claude Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

- [DDP07] Duiliu-Emanuel Diaconescu, Ron Donagi, and Tony Pantev. Intermediate Jacobians and *ADE* Hitchin systems. *Math. Res. Lett.*, 14(5):745–756, 2007.
- [Deb05] Olivier Debarre. *Complex tori and abelian varieties*, volume 11 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2005. Translated from the 1999 French edition by Philippe Mazaud.
- [DG02] Ron Donagi and Dennis Gaitsgory. The gerbe of Higgs bundles. *Transform. Groups*, 7(2):109–153, 2002.
- [DM96] Ron Donagi and Eyal Markman. Cubics, integrable systems, and Calabi-Yau threefolds. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, volume 9 of *Israel Math. Conf. Proc.*, pages 199–221. Bar-Ilan Univ., Ramat Gan, 1996.
- [DP12] Ron Donagi and Tony Pantev. Langlands duality for Hitchin systems. *Invent. Math.*, 189(3):653–735, 2012.
- [Elk74] Renée Elkik. Algébrisation du module formel d’une singularité isolée. In *Quelques problèmes de modules (Sém. Géom. Anal., École Norm. Sup., Paris, 1971-1972)*, pages 133–144. Astérisque, No. 16. 1974.
- [Fal93] Gerd Faltings. Stable  $G$ -bundles and projective connections. *J. Algebraic Geom.*, 2(3):507–568, 1993.
- [Fre99] Daniel S. Freed. Special Kähler manifolds. *Comm. Math. Phys.*, 203(1):31–52, 1999.
- [GG02] Wee Liang Gan and Victor Ginzburg. Quantization of Slodowy slices. *Int. Math. Res. Not.*, (5):243–255, 2002.
- [GPPNR18] Oscar García-Prada, Ana Peón-Nieto, and S. Ramanan. Higgs bundles for real groups and the Hitchin-Kostant-Rallis section. *Trans. Amer. Math. Soc.*, 370(4):2907–2953, 2018.
- [GR16] Oscar Garcia-Prada and S. Ramanan. Involutions and higher order automorphisms of Higgs bundle moduli spaces. *arXiv e-prints*, page arXiv:1605.05143, May 2016.
- [Hit87a] Nigel J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc.* (3), 55(1):59–126, 1987.
- [Hit87b] Nigel J. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.
- [Hit92] Nigel J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.
- [HT03] Tamás Hausel and Michael Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.
- [Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.
- [Kos59] Bertram Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.*, 81:973–1032, 1959.
- [Kri05] Igor Krichever. Algebraic versus Liouville integrability of the soliton systems. In *XIVth International Congress on Mathematical Physics*, pages 50–67. World Sci. Publ., Hackensack, NJ, 2005.
- [Nit91] Nitin Nitsure. Moduli space of semistable pairs on a curve. *Proc. London Math. Soc.* (3), 62(2):275–300, 1991.
- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.
- [Rim72] D.S. Rim. Exposé VI: Formal Deformation Theory. In *Groupes de monodromie en géométrie algébrique. I*, volume 288 of *Lecture Notes in Mathematics*, pages 32–132. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.
- [Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

- [Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.
- [Sim94a] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.
- [Sim94b] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. *Inst. Hautes Études Sci. Publ. Math.*, (80):5–79 (1995), 1994.
- [Slo80] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.
- [Ste] John R. Stembridge. Folding by Automorphisms. <http://www.math.lsa.umich.edu/~jrs/papers/folding.pdf>. Accessed: 2019-09-25.
- [Ste74] Robert Steinberg. *Conjugacy classes in algebraic groups*. Lecture Notes in Mathematics, Vol. 366. Springer-Verlag, Berlin-New York, 1974. Notes by Vinay V. Deodhar.
- [Sze04] Balázs Szendrői. Artin group actions on derived categories of threefolds. *J. Reine Angew. Math.*, 572:139–166, 2004.
- [Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.