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Classes of regular Sobolev-mappings

Let Ω be an open subset of $\mathbb{R}^n$, and let $m \geq 1$. It is well known that, for $n > 1$, the Sobolev space $W^{1,n}_{loc}(\Omega, \mathbb{R}^m)$ contains mappings unbounded in a neighborhood of each $x \in \Omega$. The problem of finding classes of regular mappings in $W^{1,n}_{loc}(\Omega, \mathbb{R}^m)$ was studied, among others, by L. Cesari [5], A.P. Calderon [4], T.Rado - P.V. Reichelderfer [13], E. Stein [14].

In 1999 J. Malý [12] introduced the following notion of n-absolute continuity:

Definition 1 A mapping f from an open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be *n-absolutely continuous* (briefly $f \in AC^n_M(\Omega, \mathbb{R}^m)$) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i \operatorname{osc}^n(f, B(\mathbf{x}_i, r_i)) < \varepsilon,$$

for each nonoverlapping finite family $\{B(\mathbf{x}_i, r_i)\}$ of closed balls in Ω with

$$\sum_i \mathcal{L}^n(B(\mathbf{x}_i, r_i)) < \delta.$$

The family $AC^n_M(\Omega, \mathbb{R}^m)$ is a regular subclass of $W^{1,n}_{loc}(\Omega, \mathbb{R}^m)$ ([12, Theorems 3.2 - 3.5]) containing properly the class of all mappings f with distributional gradient ∇f in the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^m)$ (see [11] and [8]) and satisfying the area and coarea formulas.

A variant of the above definition in which balls are replaced by cubes produces a new class of mappings denoted by $\mathcal{Q}$-$AC^n_M(\Omega, \mathbb{R}^m)$. The fact that $\mathcal{Q}$-$AC^n_M(\Omega, \mathbb{R}^m) \neq AC^n_M(\Omega, \mathbb{R}^m)$ is proved in [6], [10] and [1].

In [7] S. Hencl introduced a new variant of the Definition 1, called n, λ-absolute continuity, in which, for a fixed $0 < \lambda < 1$, the oscillations are taken on the balls $B(\mathbf{x}_i, \lambda r_i)$, for each i.

The resulting class, here denoted by $AC^n_H(\Omega,\mathbb{R}^m)$, doesn't depend by λ. It contains properly both $AC^n_M(\Omega,\mathbb{R}^m)$ and $\mathcal{Q}$-$AC^n_M(\Omega,\mathbb{R}^m)$, it is a regular subclass of $W^{1,n}_{loc}(\Omega,\mathbb{R}^m)$ and it is stable under quasiconformal mappings (see [10] and [9]).

Moreover, in the definition of $AC^n_H(\Omega,\mathbb{R}^m)$, it is possible to change from balls into cubes without affecting the resulting class of functions.

In [2] we introduced a new notion of absolute continuity (Definition 2, bellow) in which the increments on the extreme points of n-dimensional intervals are used, instead of the oscillations on cubes or balls.

Given $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$ in $\mathbb{R}^n$ such that

$$x_1 < y_1, x_2 < y_2, \dots, x_n < y_n,$$

we set

$$[\mathbf{x},\mathbf{y}] = \{\mathbf{t} = (t_1,t_2,\dots,t_n) \in \mathbb{R}^n : x_h \le t_h \le y_h, h = 1,\dots,n\},$$

to denote the n-dimensional interval having $\mathbf{x}$ and $\mathbf{y}$ as extreme points; and for $f\colon \mathbb{R}^n \to \mathbb{R}^m$ we set $f([\mathbf{x},\mathbf{y}]) = f(\mathbf{y}) - f(\mathbf{x})$.
Moreover we set

$$r[\mathbf{x},\mathbf{y}] = \frac{\mathcal{L}^n([\mathbf{x},\mathbf{y}])}{(\max_h |x_h - y_h|)^n},$$

and whenever $r[\mathbf{x},\mathbf{y}] \ge \alpha$ we say that the interval $[\mathbf{x},\mathbf{y}]$ is α-regular.

Definition 2 Let $0 < \alpha < 1$. A mapping $f\colon \Omega \to \mathbb{R}^m$ is said to be α-*absolutely continuous* (briefly $f \in \alpha$-$AC^{(n)}(\Omega,\mathbb{R}^m)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i |f(\mathbf{b}_i) - f(\mathbf{a}_i)|^n < \varepsilon,$$

for each disjoint finite family of α-regular intervals $\{[\mathbf{a}_i,\mathbf{b}_i] \subset \Omega\}$ with

$$\sum_i \mathcal{L}^n([\mathbf{a}_i,\mathbf{b}_i]) < \delta.$$

In [2] we proved that the class α-$AC^{(n)}(\Omega,\mathbb{R}^m)$ is properly between $\mathcal{Q}$-$AC^n_M(\Omega,\mathbb{R}^m)$ and $AC^n_H(\Omega,\mathbb{R}^m)$.

In [3], denoted by ${}^{\lambda}[\mathbf{x},\mathbf{y}]$ the interval with center $(\mathbf{x}+\mathbf{y})/2$ and sides of length $\lambda(y_i - x_i)$, $i = 1,2,\dots,n$, we gave the following Hencl-type variant of the notion of α-absolute continuity:

Definition 3 Let $0 < \alpha < 1$ and let $0 < \lambda \leq 1$. A mapping $f : \Omega \to \mathbb{R}^m$ is said to be (α, λ)-*absolutely continuous* (briefly $f \in \alpha\text{-}AC_\lambda^{(n)}(\Omega, \mathbb{R}^m)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i |f(^\lambda[\mathbf{a}_i, \mathbf{b}_i])|^n < \varepsilon,$$

for each disjoint finite family of α-regular intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}_i$ with

$$\sum_i \mathcal{L}^n([\mathbf{a}_i, \mathbf{b}_i]) < \delta.$$

The $(\alpha, 1)$-absolute continuity coincide with the α-absolute continuity.

Theorem 4 ([3, Theorem 3]) *A mapping $f : \Omega \to \mathbb{R}^m$ is absolutely continuous in the sense of Hencl if and only if f is (α, λ)-absolutely continuous (in the sense of Definition 3), for some $0 < \alpha, \lambda < 1$.*

In other words

$$AC_H^n(\Omega, \mathbb{R}^m) = \alpha\text{-}AC_\lambda^{(n)}(\Omega, \mathbb{R}^m), \quad \forall\ 0 < \alpha, \lambda < 1.$$

This implies that the family $\alpha\text{-}AC_\lambda^{(n)}(\Omega, \mathbb{R}^m)$ is independent by α and λ, and

- It is a regular subclass of the Sobolev space $W_{loc}^{1,n}(\Omega, \mathbb{R}^m)$;
- It contains properly the class of all mappings f with distributional gradient ∇f in the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^m)$;
- It is stable under quasiconformal mappings.

Moreover

- area and coarea formulas hold for each $f \in \alpha\text{-}AC_\lambda^{(n)}(\Omega, \mathbb{R}^m)$.

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