Improvements in the algorithms for the determination of the shear stress components in critical plane class fatigue criteria

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ABSTRACT. *This paper presents two improvements in the procedure for the evaluation of the amplitude and mean values of the shear stress in critical plane class fatigue criteria, when the method of the Minimum Circumscribed Circle (MCC) to the path of the tip of the* **τ** *vector is used. In particular, the paper shows how it is possible to reduce the number of material planes passing through the point of interest in which the MCC has to be determined and the number of points of each curve that has to be considered for the determination of the MCC, noticeably reducing the computational time.*

INTRODUCTION

In multiaxial high cycle fatigue criteria based on the critical plane approach, the amplitude and mean values of the shear stress (τ_a and τ_m) acting on the material planes at a point have to be evaluated [1-4]. In fig.1 the stress vector $\mathbf{p}_n$ acting on a generic plane F centred at point O of the material is shown. The component of $\mathbf{p}_n$ parallel to F, i.e. $\boldsymbol{\tau}_n$, can be identified by its components in a reference system with orthogonal axis *u*-*v* parallel to F, centred in O (fig.1b). In general, the length and the direction of $\boldsymbol{\tau}_n$ change with time (fig.1a) and the tip of the vector describes a path represented by a plane curve T, like the example in fig.1c. Also in the case of periodic stress histories, for which the T curve is closed, the determination of τ_a and τ_m is not straightforward and various definitions exist [4,5]. In particular, the preferred definition [4,5] is related to the Minimum Circumscribed Circle(MCC) of T, in which τ_a is assumed to be equal to the

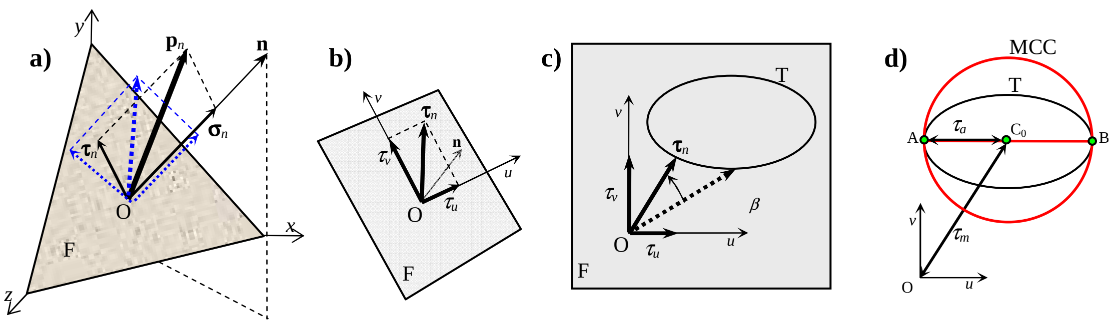

Figure 1 – a) Stress vector $\mathbf{p}_n$ acting at point O on plane F, b) components of vector $\boldsymbol{\tau}_n$, c) example of T curve, d) minimum circumscribed circle of T, τ_a and τ_m components.

radius of the MCC and τ_m to the modulus of the vector that points to the centre of the MCC, as shown in fig.1d.

Usually the path of the tip of the vector $\boldsymbol{\tau}$ acting on the j-th plane passing through the material point (the T curve) is known in discrete form as a sequence of N points τ_k of coordinates u_k, v_k, with k=1,2…N, so that the determination of the MCC requires the application of proper algorithms. Furthermore, the determination of τ_a and τ_m has to be carried out in a discrete set of planes F_i, i=1,2..N_F, passing through the considered point. In particular, for various method of multiaxial fatigue analysis, the plane in which the maximum value of a *certain function* of τ_a is reached has to be determined. In this case, a great number of planes should be analyzed to obtain a good estimation [6].

Various algorithms for the determination of the MCC to a curve known in discrete form have been applied or especially developed for the case of the path of the tip of the vector $\boldsymbol{\tau}$, in order to obtain the maximum speed of execution and precision [5-10]. The most important can be resumed in the following categories [5]: points combination algorithms [6,7], incremental algorithms [4], optimization algorithms [8,9], randomised algorithms [10]. A comparison carried out in [5] has shown that when the number of points is less than about 40, the methods proposed in [6] and [10] are the most effective, otherwise the method proposed in [10] is preferable. In particular, the method [10] has different execution times depending on the sequence of points to be analyzed, but the average time on a number of cases is the lower. The dependency of the execution time by the number of points is approximately linear for the method [10] and the quadratic for method [6].

In this paper two improvements in the procedures for the determination of the τ_a and τ_m components acting at a point of the material are proposed:

1. a reduction of the number of planes that have to be fully analyzed, for the case of the methods based on the critical plane approach,
2. a reduction of the number of points that has to be considered to determine the MCC for each T curve analyzed.

THE DETERMINATION OF THE MCC

In theory, the centre of the MCC to a curve is the point of the plane for which the maximum of the distances between the point itself and the points of the curve is minimum. In short, the problem can be summarized in the following expression:

$$(U,V) = \min_{U_g,V_g}\left\{ \max_{k=1\div N}\sqrt{\left(u_k - U_g\right)^2 + \left(v_k - V_g\right)^2} \right\} \tag{1}$$

where U, V are the coordinates of the centre of the MCC, U_g, V_g are the coordinates of the generic point, u_k, v_k are the coordinates of the curve. The distances between the points of the curve and the centre are less or equal to the radius R of the MCC:

$$D_{C,k} = \sqrt{\left(u_k - U\right)^2 + \left(v_k - V\right)^2} \le R \tag{2}$$

In the determination of the MCC is useful to refer to the pair of points of the curve between which there is the *maximum relative distance*. In particular, with reference to fig.1d, we define:

- A,B points of the curve having the maximum relative distance,
- d_0 the distance between points A, B,
- C_0 the midpoint of segment AB joining the two points,
- R_0 the radius of the circle having as diameter the segment AB ($R_0=d_0/2$).

The determination of these quantities can be carried out by properly using a relationship like this:

$$d_0 = \max_{k=1\div N}\left\{\max_{j=1\div N}\left\{\sqrt{\left(u_k-u_j\right)^2+\left(v_k-v_j\right)^2}\right\}\right\} \tag{3}$$

In general, the MCC to a curve passes through at least two points of the curve. If it passes only through two points (fig.1d), they are certainly points A and B and the AB segment coincides with one of the diameters of the circle. In this case the centre of the MCC coincides with point C_0 and the length of the radius of the MCC is $R=R_0$.

When there are points of the curve farther than R_0 from C_0 (fig.2a), the MCC can be identified as the circle of minimum radius among those passing through *at least* 3 points of the curve and verifying eq.(2). Obviously, in these cases, the length of radius of the circle is $R>R_0$ and, in general, the three points do not necessarily include A and/or B.

Being d_0 the maximum distance between 2 points of the curve, any point of the curve is considered, the others must lie within an arc of radius d_0 centred in it. This fact implies various limitations to the configuration of the curve with respect to its segment AB. First, all points of the curve should lie between 2 arcs of radius d_0 centred in A and B, named a_A and a_B, which intersect at points U and D (fig.2, dashed line). Further limitations in the configuration of the curve can be observed if we identify the 2 most distant points from the centre C_0 on opposite sides with respect to segment AB.

For example, with reference to Fig.2b, P_U is the point of the curve farthest from the centre C_0: in this case the points of the curve must also lie inside a circle with radius

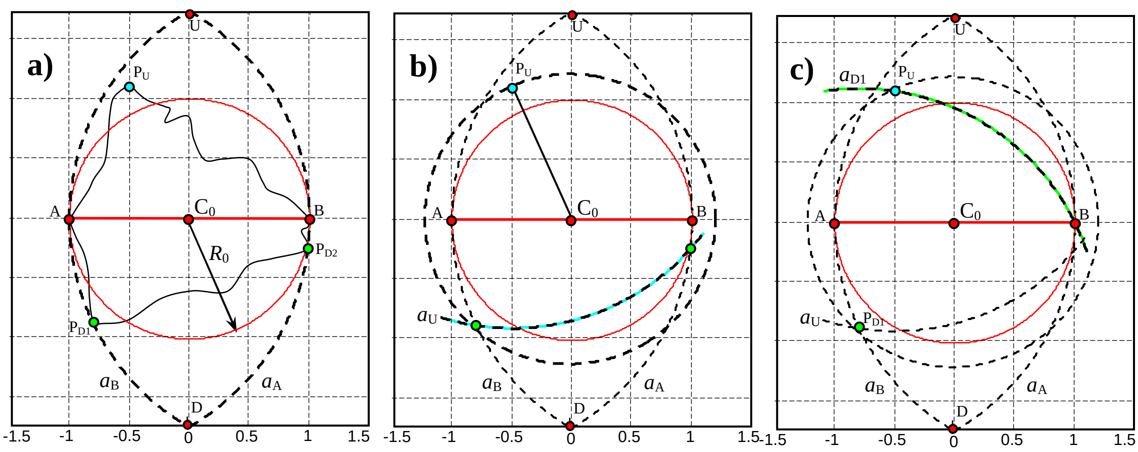

Figure 2 – Geometrical limits of T curves according to AB segment and P_u, P_d points.

equal to the distance C_0P_U centred in C_0, and within the arc a_U, with radius d_0, centred in

P_U. Point P_{D1} that lies at the intersection between arcs a_U and a_B (fig.2c), coincides with the most distant position from C_0, on the opposite side of P_U with respect to the segment AB, where could lie a point of a curve that includes the points A, B e P_U. If this point really belongs to the curve, the other points must lie within the arc a_{D1}, with radius d_0 and centred in P_{D1}.

These arcs, here called *boundary arcs*, limit the field of existence of a curve T with respect to some points belonging to it. It is important to notice that the radius of the MCC of T is greater if there are points of the curve at elevated distances from C_0, in opposite position with respect to the segment AB. In particular, for fixed value of d_0, the radius R is maximum if some points of T correspond to the points of intersection of the boundary arches, such as points P_{D1} and P_{D2} in Fig.2a.

REDUCTION OF THE NUMBER OF PLANES

The limitations on the geometry of the curve T with respect to segment d_0 enable to determine a limit to the length of the radius of the MCC in comparison to the length of the segment d_0 itself. In particular, it is possible to affirm that the radius R of the MCC to a curve T whose length of the segment AB is equal to d_0=2 R_0 respects the following relationship

$$R_0 \leq R \leq 1.1547\, R_0 \tag{4}$$

This feature has relevance in cases in which the maximum alternate shear stress at a point of the material has to be determined, as in the case of the critical plane approach [1-4]. In this case, it is possible to reduce the number of plane orientations where the MCC to the T curve has to be determined. To this aim, it suffices to determine R_0 in all the planes of interest (F_i, i=1,2..N_F), then to determine the greatest among them, $R_{0,\max}$

$$R_{0,\max} = \max_i \left\{ R_{0,i} \right\} \tag{5}$$

finally it is possible to determine the MCC *only* in planes where it is:

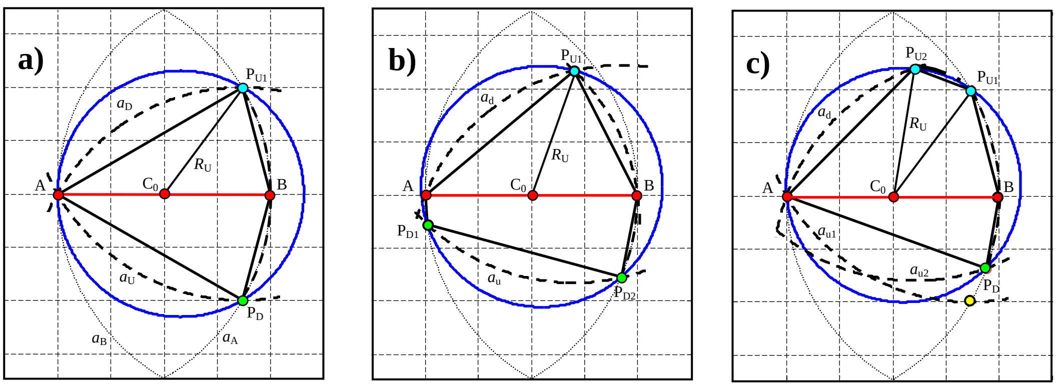

Figure 3 – MCC for various cases of positions of the points P_U.

$$R_{0,i} \geq \frac{R_{0,\max}}{1.1547} = 0.866\, R_{0,\max} \tag{6}$$

since, for the other planes, the largest circle can be smaller or at most equal to $R_{0,\max}$.

This feature can be verified by a simulation in which, once the points A, B and C_0 have been determined, different positions of the point on the curve whose distance from C_0 is the maximum are considered. Let us suppose that the curve T is represented by a polygon with the following characteristics:

1. the vertex whose distance $R_U>R_0$ from C_0 is the maximum coincides with a point belonging to one of the a_A or a_B arcs, for example P_{U1} in Fig.3a
2. the other vertices of the polygon are located on the opposite side of P_{U1} with respect to segment AB, for example, point P_D in Fig.3a.

In this case, as shown above, the curve T has to be included between the arcs a_A, a_B and a_U (fig.3a) and must lie within a circle of radius R_U centred in C_0. If a vertex of T coincides with the intersection of the arcs a_U and a_A, i.e. the point P_D (fig.3a), the MCC passes through points A, P_{U1} and P_D and is the largest possible for a polygon with a single upper vertex that coincides with the point P_{U1}. In particular, the radius of the MCC is R=1.1547 R_0.

If the point P_{U1} is at the same distance from C_0 of the previous case, but is moved angularly within the arcs a_A and a_B, as shown in Fig.3b, the arcs a_U intersect a_A and a_B at points P_{D1} and P_{D2}. If two vertices of the polygon T coincide with those points, as in fig.3b, the MCC passes through the points P_{U1}, P_{D1} and P_{D2} and is the largest possible for a polygon with a single upper vertex that coincides with the point P_{U1}. It is important to note that the MCC of this case is smaller than the previous one, being R<1.1547R_0. In general, the radius of the largest MCC decreases when point P_{U1} moves away from arcs a_A and a_B.

Although no example is reported in this paper, it is possible to observe that, whatever the position of P_{U1} along the arc a_B, when a vertex of T coincides with the point of intersection between the arcs a_U and a_A, the MCC has the same length of the radius R=1.1547 R_0 or, in a case not described here, a lower value.

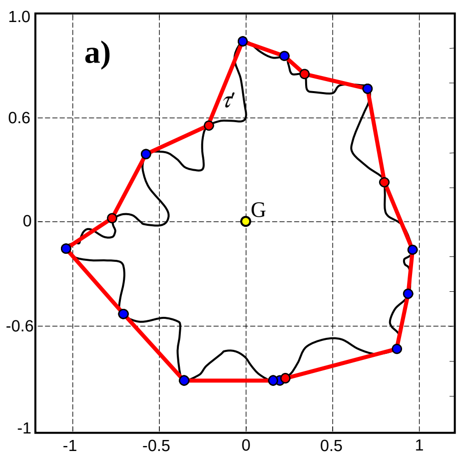

Figure 4 – Example of T curve and corresponding T′ polygon.

In a more general case, other points of the T curve lie on the same side of the point P_{U1}, at distances from C_0 comparable to R_U, as the point P_{U2} in fig.3c. It is therefore possible that the MCC passes through more than one of those points. However, it is clear that, in this case, the curve T is limited by *several* boundary arcs a_U, each relative to its point P_U, as a_{U1} and a_{U2} of fig.3c. In these cases, the points of intersection between the boundary arcs are *closer* to the segment AB, so the maximum MCC is smaller than that of the other analyzed cases and eq.(3) is verified.

Various simulations have shown that, by taking into account eq.6, the number of material planes that need to be fully analyzed to determine the MCC at a point is reduced to about *one quarter*.

REDUCTION OF THE NUMBER OF POINTS OF T CURVES

As mentioned, the number of calculations for the determination of the MCC to a curve is proportional to the number of points *N* where the curve is known. In this section it is shown how it is possible to reduce the number of points that has to be elaborated, keeping almost unchanged the values of the radius and the coordinates of the center of the MCC. In particular, it is possible to determine the polygon T′ made by points of maximum relative distance from the centroid of the curve T (fig.4, dotted line), then to determine the MCC of T′.

The determination of the polygon T′ is carried out by a simple procedure. Firstly the coordinates of the centroid G of the curve T (see fig.4) are calculated. These coordinates can be obtained according to two definitions. The first one, that is the simplest and fastest, considers the centroids of the points of the T curve, whose coordinates can be obtained as:

$$U_G = \frac{1}{N}\sum_{i=1}^{N} u_k \qquad\qquad V_G = \frac{1}{N}\sum_{i=1}^{N} v_k \qquad (8a,b)$$

The second definition considers the centroids of the segments comprised between couple of points of the curve. Defining:

- L the length of the curve T,
- L_k the lengths of the *segments of curve T*, each comprise between points τ_k and τ_{k+1},
- U_{Gk}, V_{Gk} the coordinates of the *centroids of such segments*.

The coordinates of the centroid of T can be determined by the following equations

$$U_G = \frac{1}{L}\sum_{k=1}^{N} U_{G_k} L_k \qquad\qquad V_G = \frac{1}{L}\sum_{k=1}^{N} V_{G_k} L_k \qquad (9a,b)$$

being L_k, L ed U_{Gk} e V_{Gk} given by the following relationships

$$L_k = \sqrt{\left(u_k - u_{k+1}\right)^2 + \left(v_k - v_{k+1}\right)^2} \qquad\qquad L = \sum_{k=1}^{N} L_k \qquad (10,11)$$

$$U_{G_k} = \frac{u_k + u_{k+1}}{2} \qquad\qquad V_{G_k} = \frac{v_k + v_{k+1}}{2} \tag{12,13}$$

The relationships (9) and (10) lead to slightly different results in the determination of the MCC as it will be described in the following discussion.

Once the coodinates of the centroids are calculated, the distances D_{Gk} between each point τ_k and the centroid are determined by the following equation:

$$D_{G_k} = \sqrt{(U_G - u_k)^2 + (V_G - v_k)^2} \tag{15}$$

Finally, the relative maxima of the function D_G, defined τ'_i, are determined. They are the points for which the condition $D_{Gk-1} \leq D_{Gk} \leq D_{Gk+1}$, is verified, i.e.

$$D_{G_{k-1}} \leq D_{G_k} \leq D_{G_{k+1}} \;\rightarrow\; \tau_k \in T' \tag{16}$$

In general $N' \ll N$ maxima are obtained, each corresponding to a point u_k, v_k of the curve T (fig.4). The ensemble of these points is the polygon T′.

For comparison, the standard procedure proposed in [6] was applied to 64 T curves made of $72 \leq N \leq 460$ points and to the corresponding T′ polygons obtained by the proposed procedure. Some of the analyzed curves and polygons are shown in fig.5.

The algorithms have been written in the MATLAB® programming language.

The differences between the radius of the MCC obtained by the standard procedure and that obtained determining the T′ polygons proved to be negligible, as can be observed in fig.5, where the original T curves, the corresponding T′ polygons and the MCC obtained by the polygons are shown. In particular, the average percentage errors in the determination of the radius of the MCC using eqs.(9) and eqs.(10) were $E_m = -0.025\%$ and $E_m = -0.05\%$ respectively, while the maximum percentage errors were $E_M = -0.33\%$ and $E_M = -0.41\%$.

Regarding the calculation time, 30 T curves made of $N=256$ points were considered. The total time needed to evaluate 100 hundred times all the curves was recorded for the standard procedure (T_0) and the proposed technique, considering both eq.(9) and (10) (T_1, T_2). The calculations were repeated by reducing the number of points of each curve by proper factors in order to consider its effect in the calculation time. The results reported in Table 1, that show that, if $N \geq 128$, the computational time in the determination of the MCC can be noticeably reduced by using the proposed procedure.

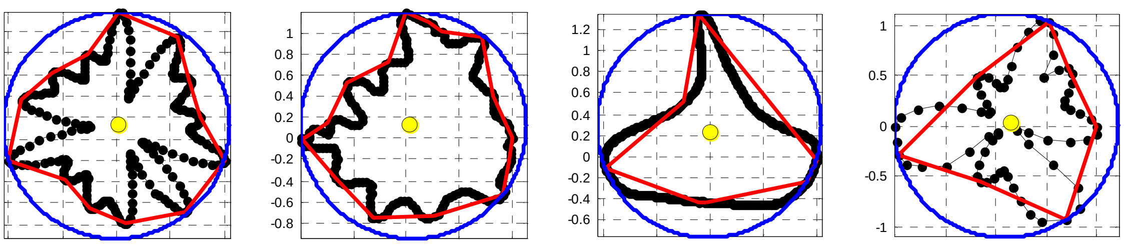

Figure 5 – Various examples of T curves (black points), corresponding T′ polygons (thick red line) and MCC obtained by the T′ polygons (blu line).

The procedure based on eqs.(10) proved to be a bit more precise, but slower. It can be considered if the points of the curve are spaced in a very irregular manner.

N	T_0 [sec]	T_1 [sec]	T_2 [sec]
256	8.6	3.8	4.2
205	6.7	3.5	4.0
171	5.7	3.0	3.4
128	4.1	2.9	3.4

Table 1 – Comparison of computational times in determining 100 times the MCC of 30 curves. N number of points, T_i computational times.

CONCLUSIONS

In this paper two improvements in the procedure for the determination of the amplitude and mean values of the shear stress acting at a material point in multiaxial high cycle fatigue analysis have been proposed.

In particular, the proposed procedures enables to obtain a reduction of the number of planes that have to be fully analyzed, for the case of the methods based on the critical plane approach, and a reduction of the number of points that has to be considered to determine the MCC for each T curve analyzed.

The reduction of calculation speed is noticeably, while the decrement of precision in the determination of the MCC is negligible. The proposed procedures are also easy to be implemented.

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