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UNCERTAINTIES IN LCA

# Toward a solution of allocation in life cycle inventories: the use of least-squares techniques

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**Abstract**
*Purpose* The matrix method for the solution of the so-called *inventory problem* in LCA generally determines the *inventory vector* related to a specific system of processes by solving a system of linear equations. The paper proposes a new approach to deal with systems characterized by a rectangular (and thus non-invertible) coefficients matrix. The approach, based on the application of regression techniques, allows solving the system without using computational expedients such as the *allocation* procedure.
*Methods* The regression techniques used in the paper are (besides the ordinary least squares, OLS) total least squares (TLS) and data least squares (DLS). In this paper, the authors present the application of TLS and DLS to a case study related to the production of bricks, showing the differences between the results accomplished by the traditional matrix approach and those obtained with these techniques. The system boundaries were chosen such that the resulting technology matrix was not too big and thus easy to display, but at the same time complex enough to provide a valid demonstrative example for analyzing the results of the application of the above-described techniques.
*Results and discussion* The results obtained for the case study taken into consideration showed an obvious but not overwhelming difference between the inventory vectors obtained by using the least-squares techniques and those obtained with the solutions based upon allocation. The inventory vectors obtained with the DLS and TLS techniques are closer to those obtained with the physical rather than with the economic allocation. However, this finding most probably cannot be generalized to every inventory problem.
*Conclusions* Since the solution of the inventory problem in life cycle inventory (LCI) is not a standard forecasting problem because the real solution (the real inventory vector related to the investigated functional unit) is unknown, we are not able to compute a proper performance indicator for the implemented algorithms. However, considering that the obtained least squares solutions are unique and their differences from the traditional solutions are not overwhelming, this methodology is worthy of further investigation.
*Recommendations* In order to make TLS and DLS techniques a valuable alternative to the traditional allocation procedures, there is a need to optimize them for the very particular systems that commonly occur in LCI, i.e., systems with sparse coefficients matrices and a vector of constants whose entries are almost always all null but one. This optimization is crucial for their applicability in the LCI context.

**Keywords** Allocation · Discrepancy vector · Least squares · Multi-functional process · Orthogonal regression · Over-determined system

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## 1 Introduction

The computational structure of life cycle inventory analysis (LCI), although being of paramount importance for a deep comprehension of the validity and reliability of the results obtained in the inventory analysis phase, is rarely tackled in scientific publications. Concerning the computational aspects, practitioners generally limit themselves to rely on the commercial LCA software used to accomplish the study and rarely delve into the mathematical details.

The matrix method, commonly used to solve the inventory problem in LCA, generally determines the inventory table related to a specific single (or composite) product system by solving a system of linear equations using matrix algebra. To be specific, one has $\mathbf{A} \cdot \mathbf{s} = \mathbf{f}$, where the technology matrix **A** represents the flows within the economic system; the vector **f**, called final demand vector or external demand vector, is an exogenously defined set of economic flows whose amount is imposed by the LCA analyst; the solution vector **s** is called scaling vector (Heijungs and Suh 2002; Ardente et al. 2004). After computing the scaling vector, it is possible to determine the inventory table **g** (that is the vector of the environmental flows associated with the reference flow under consideration) as $\mathbf{g} = \mathbf{B} \cdot \mathbf{s}$, where **B** (which is called intervention matrix) represents the environmental interventions of the system of unit processes.

The computation of the inventory table **g** associated with the studied product is the aim of LCI. The scaling vector **s** is only an intermediate result. Most LCA guidelines, software, and case studies do not provide detail on the value of **s** obtained, and on the method employed to calculate it. This is a strange affair, since the calculation of **g** requires the calculation of **s**, and this in turn requires the solution of a system of equations $\mathbf{A} \cdot \mathbf{s} = \mathbf{f}$, with **A** and **f** given.

The underlying hypothesis of the matrix method for the solution of the inventory problem is that the technology matrix **A** is square and invertible, so that it is possible to find a unique scaling vector **s** which solves the linear system $\mathbf{A} \cdot \mathbf{s} = \mathbf{f}$ by means of a matrix inverse $A^{-1}$ using the formula $\mathbf{s} = \mathbf{A}^{-1} \cdot \mathbf{f}$. However, when **A** is not square but rectangular, the matrix method fails to give a solution, because the inverse is only defined for square matrices.

A rectangular technology matrix is very common in LCA studies. For instance, the unallocated version of the ecoinvent v1.3 (http://www.ecoinvent.ch) data yields a matrix **A** of 2,632 rows and 2,471 columns. It can accordingly not be submitted to the matrix methods for LCI. In order to understand why this is so, we have to briefly discuss the architecture of the matrices and vectors.

A set of $n$ unit processes can be connected by means of $m$ economic flows. For every unit process, coefficients specify the quantity of each economic flow involved. For instance, if process 4 (electricity production) requires 10 MJ of economic flow 6 (heavy fuel) for producing 1 kWh of economic flow 13 (electricity), the coefficient at row 13 and column 4 (hence $\mathbf{a}_{13,4}$) is 1 and the coefficient at row 6 and column 4 (hence $\mathbf{a}_{6,4}$) is –10, where the convention is that negatives refer to inputs. Asking a reference flow of 1,000 kWh economic flow 13 (electricity) means putting $f_{13}$=1,000. Intuitively, it is clear that we have to scale process 4 (electricity production) by a factor 1,000 to accomplish this. Hence, we will find $s_4$=1,000. For the full system of connected processes, intuition does not suffice, and a more formal method must be used to calculate the scaling vector **s**.

However, it is well known that we can only (uniquely) solve a set of equations when the number of equations is equal to the number of unknowns. In a system of $n$ unit processes and $m$ economic flows, we have $m$ equations (a balance for every flow) and $n$ unknowns (a scaling factor for every process). So when $m \neq n$, the matrix method will not provide a (unique) solution.

There are various situations in which the number of rows and columns may be unequal:

- we may have specified economic flows for which no producing processes are available;
- we may have specified more than one process to produce the same economic flow;
- we may have specified processes that produce more than one economic flow.

Such cases have been identified and various solutions have been discussed (Heijungs and Suh 2002). The first one is commonly solved with cut-off, the second one by re-specifying economic flows so that they match uniquely with a process, and the third one by allocation. Such approaches effectively add or remove rows and/or columns so that, finally, a square system appears. The allocated and cut-off version of the ecoinvent v1.3 data yields a matrix **A** of 2,630 rows and columns (hence, removing two rows and adding 159 columns), and can be processed by the matrix method.

Technically, the tricks mentioned manage to reduce a rectangular system to a square system. However, some major controversies remain with the implementation of these tricks, especially with the third one: allocation of processes that produce more than one function.

In this paper, we will start to review briefly some of the main issues involved, and then proceed to introduce a novel approach to avoid allocation. This new approach, based on the regression techniques called total least squares (TLS) and data least squares (DLS), is a generalization of the well-known ordinary least squares (OLS) technique for fitting lines and curves to data. The assumed advantage of such least-squares techniques is that they seem to provide a neutral approach and circumvent the subjective choice between physical-based allocation, economic-based allocation, substitution, etc.

TLS and DLS are two of the several linear parameter estimation techniques that have been devised to compensate for data errors. The problem of linear parameter estimation arises in a variety of scientific disciplines such as signal processing, automatic control, system theory, and in general in engineering, statistics, physics, economy, biology, medicine, etc.

From the point of view of the system identification, very interesting applications can be found in:

- Time-domain system identification and parameter estimation: deconvolution techniques in *renography* (Van Huffel et al. 1987): estimates of the autoregressive parameters of an autoregressive moving average model from noisy measurements (Stoian et al. 1990), structural identification (Beghelli et al. 1987), modeling of industrial engines (Jakubek et al. 2008), parameter estimation and control of induction motors and machine drives (Cirrincione et al. 2003; Cirrincione et al. 2006; Cirrincione et al. 2007), modeling of proton exchange membrane fuel cells systems (Blunier et al. 2008), chaotic time series prediction (Li and Yu 2008), and parameter estimation for statistical probability density functions (Marković and Jukić 2010);
- Identification of state-space models from noisy input-output measurements: examples, including the identification of an industrial plant can be found in (Moonen et al. 1989; Moonen and Vandewalle 1990; De Moor 1990)
- *Signal processing*: classical *harmonic retrieval problem* (Rahman and Yu 1987; Roy and Kailath 1987; Zoltowski and Stavrinides 1989); general class of practical signal processing problems (Roy and Kailath 1989; Swindlehurst et al. 1992); *minimum variance distortionless response* (MVDR) *beamforming* problem (Zoltowski 1987); *adaptive infinite impulse response* filtering problems (Dunne and Williamson 2003); *sensor array signal processing* and *high-resolution frequency estimation* (Zoltowski 1988); *channel equalization* (Lim 2008; DeGroat and Dowling 1993). An interesting array of environmental signal processing problems based on the utilization of TLS and related methods is presented in (Ramos 2007);
- Experimental modal analysis: estimation of *frequency response functions* from measured input forces and response signals applied to mechanical structures (Rost and Leuridan 1985);
- Acoustic radiation: computation of the acoustic pressure surface (Hall and Bernhard 1989);
- Geology: interpretation of metamorphic mineral assemblages (Fisher 1989);
- Inverse scattering: inference of the shape, size, and constitutive properties of an object from scattering measurements that result from seismic, acoustic, or electromagnetic probes (Silvia and Tacker 1982); *Geophysical termographic imaging*: geophysical monitoring of hydrocarbon reservoirs (Justice and Vassiliou 1990).

*The allocation problem* Allocation is a recognized methodological step in LCA. It is the procedure of assigning to each of the processes of a multi-functional system only those environmental burdens and impacts that were generated by it (Azapagic and Clift 1999).

When a unit process provides more than one product, the question arises on how the economic flows and environmental burdens should be partitioned and distributed among the multiple products. This has been one of the most controversial issues in the development of LCA (Klöpffer and Rebitzer 2000). In this case, the ISO standards on LCA suggest a stepwise procedure consisting of three consecutive steps (International Standard ISO 14044 2006).

Step 1 actually aims to avoid allocation "wherever possible", by either of two options: division of multifunctional processes into two or more mono-functional subprocesses each of which contributes only to one functional output (step 1a), or expansion of the product system to include the additional functions related to the co-products (step 1b). However, the former step rarely avoids allocation completely because most multiple-function systems include processes which are common for some or all of its functional outputs so that some kind of allocation will still be necessary. The latter step essentially means redefining the functional unit and the system boundaries to include the additional functions related to the co-products (Ekvall 1994; Heijungs 1994; Heintz and Baisnee 1992; Tillman et al. 1994). An equivalent approach is the already mentioned "avoided burdens" or "avoided impacts" method that consists in subtracting burdens arising from an alternative way of producing a function (i.e., a stand-alone process for the production of a sub-product of a main process) from the main process (Heijungs and Suh 2002; Azapagic and Clift 1999). When the allocation is unavoidable, ISO prescribes one of the following alternatives.

Step 2 states that system inputs and outputs should be "partitioned on the basis of the underlying physical relationships between them", i.e., reflect the way in which the inputs and outputs are affected by quantitative changes in the products or functions delivered by the system.

If allocation based on physical, causal relationship is not feasible or does not provide a full solution, step 3 of ISO 14044 is to be followed. According to this option, the exchanges between the products and functions have to be partitioned "in a way which reflects other relationships between them. For example, input and output data might be allocated between co-products in proportion to the economic value of the products". However, it has to be borne in mind that the market prices of every function can be subject to significant fluctuations in time, thus affecting the validity and credibility of the results of the LCA study. As a consequence, it is of paramount importance to have correct information about the relative prices of the functional flows at stake. A way to handle fluctuations consists into basing the allocation procedure on the shares in proceeds and not on the absolute values. In fact, prices may fluctuate considerably but, often, shares remain quite constant, particularly in the longer term. However, for flows with missing or distorted markets, it may also be difficult to determine proceeds and shares. In Guinée et al. (2004), a summary of the solutions that could be adopted to find prices of products with missing or distorted markets was shown: the authors establish a decision tree for the economic allocation.

Generally speaking, the last option recommended in ISO 14044 allows also an allocation based on relationships that are not causal. This includes allocation in proportion to an arbitrary physical property of the products, such as mass, volume, or energy content.

Allocation based on physical properties of the products is the predominant allocation method in LCI practice because data on these properties are generally readily available and easily interpreted. However, when the allocation is not based on an accurate model of causal relationships, it will not provide accurate results.

The ISO hierarchy on allocation can be criticized on providing too much freedom. For instance, what means "wherever possible", or when do we say that allocation really "cannot be avoided"?

The matrix method provides insight into the mechanisms of how multi-functional processes lead to problems (Heijungs and Frischknecht 1998). But this comes at a price, there are cases where the matrix method does not work while there is in fact no real problem. As an example, consider the case of closed-loop recycling, the situation that a waste flow is processed and reused by the same system. According to the logic of ISO, such a system produces just one product, so there is nothing to be allocated. However, the matrix method will fail, as the matrix $\mathbf{A}$ has more rows than columns. The concept of pseudo-inverse of a matrix (Golub and Van Loan 1989; Stewart and Sun 1990) has been introduced to deal with this situation (Heijungs and Suh 2002; Heijungs and Frischknecht 1998). It is based on the generalization of the matrix inverse for non-square $\mathbf{A}$. Avoiding the formal mathematical details, it is possible do define a matrix $\mathbf{A}^{\dagger} = \left(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$. Using the pseudo-inverse of $\mathbf{A}$, we can calculate $\mathbf{s} = \mathbf{A}^{\dagger} \cdot \mathbf{f}$, also when $\mathbf{A}$ is rectangular. There is a caveat, however. Although the pseudo-inverse gives an answer to the inventory problem, it does not necessarily provide an exact solution. That is, if we substitute the obtained value of $\mathbf{s}$ back into the original system of equations, we do not necessarily obtain the original final demand $\mathbf{f}$, but we may obtain a different result. This can be interpreted as the final supply $\widetilde{\mathbf{f}}$ (Heijungs and Suh 2002). The element-wise difference between what we ask (final demand $\mathbf{f}$) and what we get (final supply $\widetilde{\mathbf{f}}$), has been referred to as the discrepancy vector $\mathbf{d} = \widetilde{\mathbf{f}} - \mathbf{f}$ (Heijungs and Suh 2002). In the ideal situation, all elements of $\mathbf{d}$ are equal to zero, indicating an exact solution, without discrepancy. When we apply the pseudo-inverse to a case with strict closed-loop recycling, we also find an exact solution, with $\mathbf{d}=\mathbf{0}$. When we apply it to another rectangular system, e.g., with co-production or open-loop recycling, the method gives an answer but with a non-zero discrepancy. Some products will be produced in excess, some in a too small amount.

The situation is summarized in Heijungs and Frischknecht (1998):

- If the normal inverse of the technology matrix exists, there is never an allocation problem;
- If the normal inverse of the technology matrix does not exist because the matrix has more rows than columns, the pseudo-inverse may in a substantial number of cases provide an exact solution (e.g., in the case of internal recycling, also called closed-loop recycling). If the procedure with the pseudo-inverse does not produce an exact solution, there is an allocation problem.

As a consequence, the allocation procedure should be used only when strictly required, that is when an exact solution cannot be found by using the pseudo-inverse of the technology matrix. Moreover, it should be checked if the solution obtained is satisfactory, in the sense of being exact, i.e., yielding a zero discrepancy. But, and this is a crucial observation, the use of the pseudo-inverse is able to solve a rectangular LCI without allocation.

This observation forms the starting point of the new approach of this paper. In the next section, we will further generalize the pseudo-inverse, and show how these generalizations can tackle an unallocated multi-functional system.

Allocation constitutes a crucial step in the LCI phase, and the choices made in this phase of the study deeply influence the obtained results and consequently the comparability between replaceable alternatives on the basis of their inventory tables. It follows that, if the inventory table of a product (good or service) can be found without allocation, the influence of subjective choices has decreased, thus contributing to make the results of the study more transparent. To a certain extent, the use of least-square techniques might be seen as a kind of "consensus allocation", which can be considered objective, and thus is more prone to be accepted as "fair". One could argue that solving the inventory problem directly in its rectangular form (without making the system matrix square) intrinsically leaves unallocated the single functions of a multi-functional process without actually solving the problem of allocation, but just hiding it. However, in our view, making resort to allocation has no other usefulness than allowing the transformation of a rectangular system into a square one.

## 2 Methods

### 2.1 The solution of an over-determined system of equations

As noted in the introductory section, multi-functional processes are associated with systems of equations that are over-determined in the sense that there are more equations than unknowns, reflected by a matrix with more rows than columns. In this section, we will develop strategies to deal with such over-determined systems in a systematic way.

There is an interesting connection between the pseudo-inverse and the ordinary least squares regression, where a straight line is found that optimally fits a relationship between empirical data. We will study this situation by going to the very simple case of one process and many economic flows. Suppose that we have coefficients $a_1, a_2, \ldots, a_m$, symbolizing the connection of $m$ flows to the only process, and that the reference flows for these $m$ flows has been specified as $f_1, f_2, \ldots, f_m$. There is only one unknown: the scaling factor $s$ for the only process. A regression analysis determines the value of $s$ that provides an optimal fit (see Fig. 1).

Mathematically, we determine the coefficient $s$ in the equation $f = s \cdot a$. We do this in such a way that the sum of squares of errors in the fit is minimal. That is, we determine

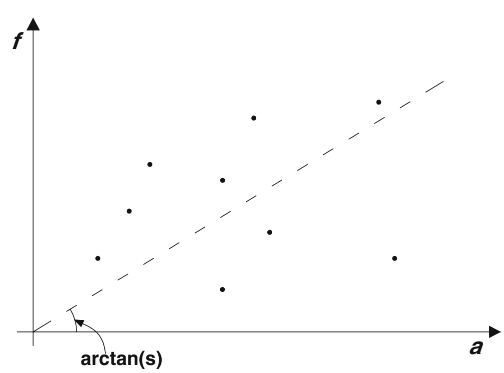


**Fig. 1** Interpretation of the inventory problem for one unit process as fit of a straight line

$s$ such that $\sum_{i=1}^{m} (f_i - s \cdot a_i)^2$ is minimal. The estimated value $s$ provides an approximate fit, but the best fit, in a least squares sense. Once the problem has been mathematically formulated (the equations are defined and a set of unit of measurements has been chosen), the method provides a neutral (or, if one prefers, "fair") answer. This is also the reason why regression analysis is routinely applied for fitting model parameters: it provides an optimal (neutral, "fair") fit.

One of the assumptions of this so-called OLS method is that the independent variable(s), $a$ in this example, is assumed to be known without error, and that it is in fact the dependent variable, $f$, which is subject to uncertainty. Thus, adjustments in $f$ are made so as to find the best line. In the end, we can calculate what we have found: $\tilde{f}_i = a_i \cdot s$ instead of $f_i = a_i \cdot s$, thus changing the final demand $f$ into the final supply $\tilde{f}$.

In a more general setting, multiple regression, there is more than one independent variable (more than one $a$ variable), and there is more than one unknown (more than one $s$). The assumed equation is $\mathbf{f} = \mathbf{A} \cdot \mathbf{s}$. The estimation procedure for finding the optimal values of $\mathbf{s}$ is by means of the equation $\mathbf{s} = \left( \left( \mathbf{A}^{\mathrm{T}} \cdot \mathbf{A} \right)^{-1} \cdot \mathbf{A}^{\mathrm{T}} \right) \cdot \mathbf{f}$ (Dobson 1983).

For any $m \times n$ matrix $\mathbf{A}$ of rank $n$ (i.e., of full column rank) this is identical to $\mathbf{s} = \mathbf{A}^{\dagger} \cdot \mathbf{f}$(see Harville 1997, p. 495). Thus, we see that the use of the pseudo-inverse is identical to interpreting the inventory problem as an exercise in multiple regression, where a least-squares minimization of the discrepancy between final demand and final supply provides the criterion of best fit.

As may be clear from the word "ordinary" in "ordinary least squares" regression, there are more ways of solving an over-determined system of equations by least-squares techniques. Below, we will discuss DLS and TLS and contrast them with OLS.

In the OLS approach, a solution is sought for the system $\mathbf{A} \cdot \mathbf{s} = (\mathbf{f} + \Delta\mathbf{f})$, where $\Delta\mathbf{f}$ is the residual error vector (or discrepancy vector in Heijungs and Suh 2002) corresponding to a perturbation in $\mathbf{f}$. The OLS solution vector is chosen so that the Euclidean norm $\|\Delta\mathbf{f}\|_2 = \sum_{i=1}^{m} (\Delta f_i)^2$ is minimized. In the classical OLS approach, there is the underlying assumption that only the vector $\mathbf{f}$ is affected by noise or errors. Furthermore, the OLS solution is optimal only if these errors have a zero mean Gaussian distribution. In this case, the OLS solution is identical to the maximum likelihood one (Mendel 1987). If the errors have a different distribution, this approach leads to a biased estimate of the solution vector (Levin 1964). The use of the Euclidean norm relates to the S in OLS: the sum of squares of residuals is minimized. There is no fundamental objection to using a different norm than the Euclidean one in OLS. This norm, however, makes the problem more tractable, and also insensitive to an orthogonal transformation.

In the DLS approach, the error is assumed to lie in the data matrix **A**, not in the vector **f**. The problem is thus converted into the solution of the system $(\mathbf{A}+\Delta\mathbf{A})\mathbf{s}=\mathbf{f}$, where $\Delta\mathbf{A}$ is the noise portion of the matrix **A**. The DLS solution vector is usually chosen so that the Frobenius norm $\|\Delta\mathbf{A}\|_F=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta a_{i,j}^2}$ of $\Delta\mathbf{A}$ is minimized. Again, a different norm could be used for DLS.

The TLS approach, finally, combines the assumptions of OLS and DLS. It is a linear parameter estimation technique that has been devised to compensate for data errors (Golub and Van Loan 1980). It is a natural generalization of the OLS approximation method and it is used when the data both in **A** and **f** are allowed to be perturbed. The classical TLS problem searches the minimal corrections $\Delta\mathbf{A}$ and $\Delta\mathbf{f}$ on the given data **A** and **f** that make the corrected system of equations $(\mathbf{A}+\Delta\mathbf{A})\mathbf{s}=(\mathbf{f}+\Delta\mathbf{f})$solvable, i.e.,:

$$\{\mathbf{s}_{TLS},\Delta\mathbf{A}_{TLS},\Delta\mathbf{f}_{TLS}\} = \arg\min_{\mathbf{s},\Delta\mathbf{A},\Delta\mathbf{f}}\|[\Delta\mathbf{A}|\Delta\mathbf{f}]\|_F \text{ s.t.}(\mathbf{A}+\Delta\mathbf{A})\mathbf{s}=(\mathbf{f}+\Delta\mathbf{f}) \quad (1)$$

where, $[\Delta\mathbf{A}|\Delta\mathbf{f}]$ is the correction matrix $\Delta\mathbf{A}$ augmented by the column vector $\Delta\mathbf{f}$. The Frobenius norm of this augmented matrix is given by

$$\|[\Delta\mathbf{A}|\Delta f]\|_F=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n+1}[\Delta a|\Delta f]_{i,j}^2} \quad (2)$$

The OLS, TLS, and DLS regression problems have been formulated as minimization problems. Algorithms to carry out these minimizations and obtain the optimal solutions have been described in Van Huffel and Vandewalle (1991). The appendix provides more details on these algorithms.

As described in Rao (1997) and resumed in Paige and Strakos (2002), these approaches can be unified by using the Frobenius norm and considering a very general Scaled TLS problem.

Figure 2 shows a graphical interpretation of the differences between the three least-squares techniques in the simplified case of only one independent variable. The three methods assess the fitting accuracy in different ways. Being based on the assumption that the errors are confined to the *observation vector* (the vector of the known terms), OLS minimizes the sum of squared vertical distances from the data points to the fitting line (Fig. 2a). Whereas, since in DLS, the errors are supposed to lie only in the matrix **A**, it minimizes the sum of squares in the **A** direction (Fig. 2b). Finally, TLS minimizes the sum of squares in the direction orthogonal to the line (this means that the errors are supposed to be both in **A** and **f**) and, for this reason, TLS is also called orthogonal regression (Fig.2c).

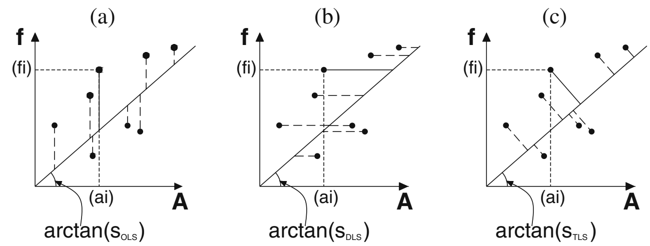


**Fig. 2** Representation of the OLS (**a**), DLS (**b**), and TLS (**c**) techniques in the mono-dimensional case

In the classical inventory problem which typically occurs in LCA, the external demand vector **f** is fixed by the operator and thus it could be considered not to be affected by uncertainty or noise, whereas the technology matrix **A**, that represents the flows within the economic systems, is the result of an estimation process or a measurement campaign and it is consequently affected by several uncertainties. As a consequence, if we accept the hypothesis that the external demand vector is noise-free (that means accepting that when the economic flows of the system are exactly the ones described by the technology matrix, the outputs of the system are exactly the ones described by the chosen vector **f**) then the problem at hand can be treated as a DLS problem. However, if we suppose that, due to any kind of error, the output of the system can be different than the one described by the chosen external demand vector, the problem at hand becomes a TLS problem.

## 2.2 Illustrative case study

In this paper, the authors present the application of the above mentioned least-squares techniques to an illustrative LCA case study investigating the process of production of bricks in a Sicilian factory. The product system is modeled in SimaPro; least squares calculations have been made in Matlab.

In the first step, the assumption was made that the errors lied only in the technology matrix and thus the problem was tackled as a DLS one. In a second step, the authors also made the assumption that both the technology matrix and the final demand vector were affected by noise, and the problem was solved by the TLS method.

It is important to underline that not all of the entries of the technology matrix are realistic; the matrix was built mostly to test the applicability and the numerical stability of the investigated algorithms from the mathematical point of view. The matrix was built starting from a real industrial case study, subsequently adding some *ad hoc* processes as well as modifying some realistic processes, such that the resulting technology matrix was not too big and thus easy to display, but at the same time, it could provide a valid demonstrative example for analyzing the results of the techniques proposed in this paper.

The system consists of eight processes connected by 14 economic flows, and there are 21 environmental flows. A flow chart is given in Fig. 3. Table 1 shows the process data with respect to the economic flows (matrix **A**), Table 2 with respect to the environmental flows (matrix **B**).

In the overall system, there are six multi-functional processes:

1. Production of electricity and heat: it is a process of co-generation, in which electricity and heat are produced at the same time;
2. Production of clay (white and red): two kinds of clay are produced in the same factory and both are used within the bricks production process;
3. Production of sand and gravel: the sand production process also produces gravel, that is one of the input materials used to produce bricks;
4. Production of oil derivatives: the process of crude oil refinery produces different co-products (namely oil derivatives) some of which are used in the process of bricks production;
5. Supply of biomass: two different kinds of biomass (olive cake and straw) are used to produce bricks.
6. Production of bricks and inerts: along with bricks, some low value recycled inerts are co-produced.

The final demand vector (or functional unit) **f** is defined as:

$$f_i = \begin{cases} 0 \text{ for } i = 1 \ldots 13 \\ 1 \text{ for } i = 14 \end{cases} \quad (3)$$

In the calculations below, least-squares techniques will be used next to traditional allocation techniques. For these latter, two alternative allocation principles have been tested. First, a physical allocation has been applied, using the choices of Table 3. Alternatively, economic allocation has been used, based on price information in Table 4. In addition, the economic allocation applies the substitution of inerts recycling from Table 3.

Altogether, we do this case study according to five calculation principles as summarized in Table 5.

## 3 Results

### 3.1 Case I, the original data

The physical allocation (substitution for the recycled inerts, energy-based for the oil derivatives and "expert"-based for the other processes) transforms the matrix **A** into a square one $\mathbf{A}'$ (see Table 6). The economic allocation (still including the substitution for recycled inerts) transforms the matrix **A** into a different square one $\mathbf{A}''$ (see Table 7).

The environmental matrices $\mathbf{B}'$ and $\mathbf{B}''$ obtained after physical-based allocation and economic allocation are showed in Tables 8 and 9, respectively.

The solution vectors obtained by applying the OLS method and the above mentioned DLS and TLS techniques are showed in Table 10 along with the solutions obtained with the two allocation methods.

Once the solution vector **s** has been computed, it is possible to determine the inventory table $\mathbf{g} = \mathbf{B} \cdot \mathbf{s}$ of the

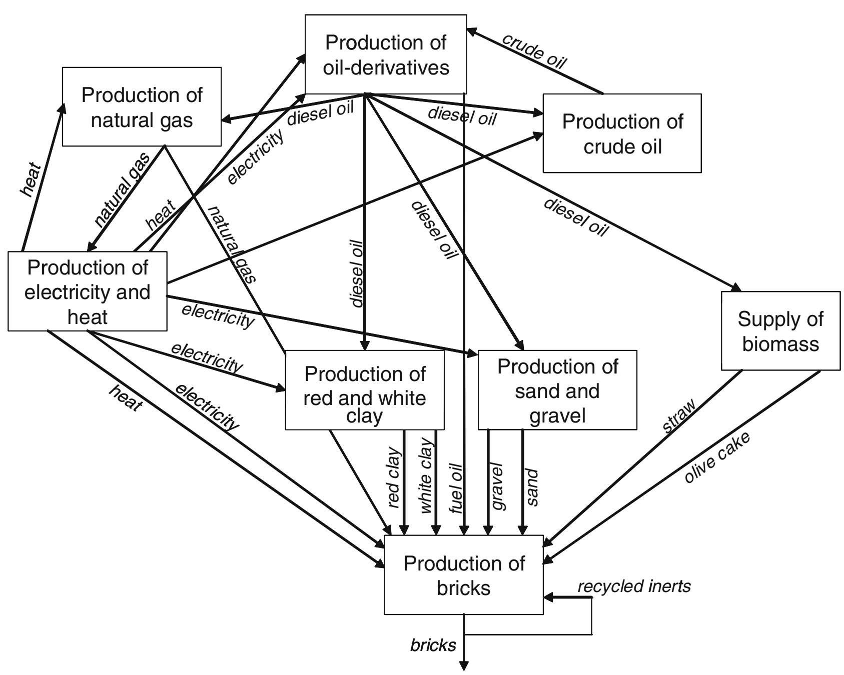


**Fig. 3** Flow chart of the case study on bricks production

**Table 1** Process data (economic flows, i.e., only the technology matrix $\mathbf{A} \in \Re^{14\times8}$) for the case study

| | Production of electricity and heat | Production of clay (white and red) | Production of sand and gravel | Production of crude oil | Production of oil derivatives | Production of natural gas | Supply of biomass | Production of bricks and inerts |
|---|---|---|---|---|---|---|---|---|
| MJ of electricity | 1 | −7.20E−03 | −1.80E−02 | 0 | −3.20E−02 | 0 | 0 | −3.69E+02 |
| MJ of heat | 2.48E+00 | 0 | 0 | −4.87E−02 | −1.13E+00 | −4.87E−02 | 0 | −1.01E+03 |
| kg of white clay | 0 | 1 | 0 | 0 | 0 | 0 | 0 | −1.37E+03 |
| kg of red clay | 0 | 1 | 0 | 0 | 0 | 0 | 0 | −8.51E+02 |
| kg of recycled inerts | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6.90E+01 |
| kg of sand | 0 | 0 | 1 | 0 | 0 | 0 | 0 | −5.00E+02 |
| kg of gravel | 0 | 0 | 1 | 0 | 0 | 0 | 0 | −3.47E+02 |
| kg of olive cake | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.53E+02 |
| kg of straw | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.92E+01 |
| MJ of crude oil | 0 | 0 | 0 | 1 | −2.34E+00 | 0 | 0 | 0 |
| MJ of diesel oil | 0 | −4.54E−03 | −3.60E−02 | −5.29E−03 | 1 | −3.61E+01 | −4.06E−01 | −1.41E+03 |
| MJ of fuel oil | 0 | 0 | 0 | 0 | 1 | 0 | 0 | −1.11E+03 |
| MJ of natural gas | −4.27E+00 | 0 | 0 | 0 | 0 | 1 | 0 | −5.52E+03 |
| Ton of bricks | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

The unit of measurement in which each economic flow in the table is expressed is defined at the beginning of the corresponding row

**Table 2** Process data (environmental flows, i.e., only the intervention matrix $\mathbf{B} \in \Re^{21\times8}$) for the case study

| | Production of electricity and heat | Production of clay (white and red) | Production of sand and gravel | Production of crude oil | Production of oil derivatives | Production of natural gas | Supply of biomass | Production of bricks and inerts |
|---|---|---|---|---|---|---|---|---|
| Resources and raw materials | | | | | | | | |
| MJ of coal | −1.49E−03 | −5.16E−03 | −1.36E−02 | −7.46E−07 | −5.77E−02 | −7.46E−07 | −1.40E−03 | 0 |
| MJ of lignite | −2.74E−04 | −6.46E−03 | −1.63E−02 | −1.77E−09 | −2.68E−04 | −1.77E−09 | −8.32E−04 | 0 |
| MJ of hydropower | 0 | −3.82E−04 | −1.04E−03 | −9.58E−08 | −7.66E−03 | −9.58E−08 | −4.54E−04 | 0 |
| MJ of geothermal energy | 0 | −2.80E−08 | −1.17E−07 | −6.08E−15 | −1.05E−04 | −6.08E−15 | 0 | 0 |
| kg of water | 0 | −7.80E−03 | −2.40E−02 | −1.49E−02 | −7.26E−03 | −1.49E−02 | 0 | −1.03E+03 |
| kg of ores (sand, gravel, etc.) | −2.20E−05 | −2.00E+00 | −2.00E+00 | −3.51E−05 | −4.55E−04 | −3.51E−05 | 0 | 0 |
| MJ of crude oil | −1.21E−02 | −1.82E−02 | −1.43E−01 | −1.02E+00 | −2.34E+00 | −1.02E+00 | −4.15E−01 | −1.27E+03 |
| kg of other ores (iron, copper, etc.) | −1.25E−04 | −7.37E−06 | −3.95E−05 | −5.60E−10 | −2.40E−04 | −5.60E−10 | 0 | 0 |
| Emissions to air | | | | | | | | |
| kg of CO2 | 8.32E−02 | 2.52E−03 | 1.33E−02 | 4.17E−03 | 2.88E−02 | 4.17E−03 | 4.93E+00 | 6.02E+02 |
| kg of CO | 1.63E−04 | 3.83E−06 | 2.79E−05 | 1.14E−05 | 3.34E−05 | 1.14E−05 | 2.70E−02 | 1.10E+00 |
| kg of CH4 | 3.68E−04 | 2.97E−06 | 1.09E−05 | 7.13E−05 | 1.02E−04 | 7.13E−05 | 6.00E−03 | 1.18E+00 |
| kg of SO2 | 2.96E−05 | 2.61E−06 | 1.64E−05 | 5.31E−06 | 2.69E−04 | 5.31E−06 | 7.43E−03 | 2.20E+00 |
| kg of NMVOC | 3.70E−05 | 4.20E−07 | 2.87E−06 | 1.93E−05 | 4.23E−05 | 1.93E−05 | 3.07E−02 | 1.13E+00 |
| Emissions to water | | | | | | | | |
| kg of COD | 6.25E−07 | 2.00E−07 | 1.11E−06 | 1.57E−11 | 6.75E−06 | 1.57E−11 | 2.21E−04 | 2.38E−01 |
| kg of BOD | 4.36E−08 | 6.11E−09 | 3.51E−08 | 4.41E−13 | 1.89E−07 | 4.41E−13 | 6.75E−06 | 2.25E−01 |
| kg of P | 3.27E−10 | 4.95E−11 | 3.91E−10 | 3.73E−18 | 9.84E−13 | 3.73E−18 | 0.00E+00 | 1.26E−03 |
| kg of N | 3.11E−08 | 2.91E−09 | 2.29E−08 | 2.19E−16 | 1.08E−10 | 2.19E−16 | 1.61E−04 | 6.00E−03 |
| kg of AOX | 5.78E−11 | 3.69E−12 | 2.90E−11 | 4.64E−18 | 2.01E−12 | 4.64E−18 | 2.95E−07 | 1.43E−05 |
| Solid wastes | | | | | | | | |
| kg of ash | 0 | 1.11E−04 | 2.83E−04 | 1.48E−08 | 2.56E−04 | 1.48E−08 | 0 | 0 |
| kg of sludge | 0 | 2.46E−07 | 1.93E−06 | 5.30E−14 | 2.10E−05 | 5.30E−14 | 0 | 0 |
| kg of nuclear waste | 0 | 2.54E−09 | 6.49E−09 | 8.82E−17 | 7.70E−10 | 8.82E−17 | 0 | 0 |

The unit of measurement in which each environmental flow in the table is expressed is defined at the beginning of the corresponding row

**Table 3** Allocation principles for the six multi-functional processes, used for the physical allocation

| Process | Allocation |
|---|---|
| Production of electricity and heat | 0.8 for electricity and 0.2 for heat |
| Production of clay (white and red) | 0.5 for with clay and 0.5 for red clay |
| Production of sand and gravel | 0.5 for sand and 0.5 for gravel |
| Production of oil derivatives | Allocation following the energy criterion |
| Supply of biomass | 0.8 for olive cake and 0.2 for straw |
| Production of bricks and inerts | The output of recycled inerts is treated with the substitution method: it is considered as equivalent to sand, but obviously only a certain percentage of the inert materials produced by the mining activity can be really used as sand. For this reason an estimated correction factor of 0.85 was applied in order to account for the difference in quality between the mass of inert materials and an equivalent mass of sand obtained from them |

product. The inventory tables obtained with the three least-squares methods are reported in Table 11, where the ones obtained with the two allocation methods are also showed.

The rightmost columns of Table 11 also shows the percentage differences between the inventory vectors obtained with the DLS and TLS solutions and those obtained with the allocation-based solutions. The percentage differences related to the OLS solution are very close to 100% for every environmental intervention, so they were not showed into the table.

By taking the absolute values of the entries in the last four columns on the right of Table 11 and computing the mean value of each column, one obtains the following mean absolute values of the percentage differences: DLS-ECN=91.9%; DLS-PHY=53.2%; TLS-ECN=91.0%; TLS-PHY=54.0%.

Each of the three least squares methods produces an array of economic flows that does not fully agree with the final demand vector **f**. The discrepancy vectors obtained with the five explored methods are listed in Table 12 along with their Euclidean norms.

As we can observe in Table 12, the discrepancies associated with the OLS method are quite low, except for the last component (i.e., the one corresponding to the flow associated to the chosen functional unit), that is equal to −1. It means that the last element of the final supply vector found with the OLS method is equal to 0 (instead of the desired value 1). In contrast, the discrepancies associated with the DLS and TLS methods are much higher for all the components, except for the last one. It means that these last two methods (in particular DLS) deliver a flow of the investigated product (bricks) which is very close (exactly equal in this case for the DLS solution) to the quantity established in the choice of the functional unit. However, they also deliver undesired amounts of the other flows (electricity, heat, and so forth). The discrepancies related to the traditional solutions based on allocation are instead all very low and can be explained as artifacts due to round-off (Heijungs and Suh 2002).

On the basis of the observation of the discrepancy vectors alone, the traditional allocation-based solutions may therefore appear to be preferred to the solutions obtained using the regression techniques. However, a complete judgment should also take into account the observation of the differences among the inventory vectors since the inventory vector is the final goal of the LCI phase. From the observation of the last four columns on the right of Table 11, we can infer that the difference between the inventory vectors of the DLS and TLS solutions and the inventory vectors of the traditional solutions is not overwhelming., It is worth noting that while the solutions based on allocation can vary significantly with the choice of the allocation factors, the solutions based on the presented least-squares techniques have the advantage of being unique. Clearly, this calls for more work on the use of regression techniques vis-à-vis allocation in LCI.

**Table 4** Prices of the commodities used for the economic allocation

| Product | Price | Unit |
|---|---|---|
| Electricity | 0.034 | €/MJ |
| Heat | 0.013 | €/MJ |
| White clay | 2.2 | €/kg |
| Red clay | 2.0 | €/kg |
| Sand | 0.015 | €/kg |
| Gravel | 0.011 | €/kg |
| Olive cake | 0.16 | €/kg |
| Straw | 0.065 | €/kg |
| Diesel oil | 0.025 | €/MJ |
| Fuel oil | 0.4 | €/MJ |

## 3.2 Case II, a sensitivity analysis on changing the units of measurement

Minimization of a sum of squares is a procedure that may be sensitive to the coordinate system selected. In particular, the choice of units may affect the result (Heijungs and Suh 2002). We will study this below.

Changing the unit of the 14th product flow from tons of bricks into kg of bricks will change the 14th row of **A** and **f** by a factor 1,000.

In Table 13, the scaling factors now obtained applying allocation is compared with those obtained applying the least-squares techniques to the rectangular matrix. Note that the scaling vector obtained with the DLS technique

**Table 5** Overview of the main characteristics of the five different LCI techniques in the case study on bricks

| Aspect | LCI technique | | | | |
|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS |
| Allocation of co-products | Economic; see table 4 | Physical; see table 3 | – | – | – |
| Allocation of recycling | Substitution | Substitution | – | – | – |
| Allow changes in | – | – | **f** | **A** | **f** and **A** |
| Size of A | 13×13 | 13×13 | 14×8 | 14×8 | 14×8 |
| Size of B | 21×13 | 21×13 | 21×8 | 21×8 | 21×8 |

for this new system is identical to the one obtained with the original system (see Table 10). Moreover, for this particular choice of the technology matrix and the final demand vector, the DLS solution is also identical to the TLS solution. This is just a particular case and does not have to be considered as a general behavior. In fact, even for this case, if the normalization procedure showed in Section 3.3 is applied, the obtained DLS and TLS solutions differ from each other (the results are not showed in the paper because we are here keeping separate the effects of changing the unit of measurement and rescaling the entries of **A** and **f**). A general mathematical condition for the equality of the DLS and TLS solutions could be derived from the expression obtained putting Eq. 10 equal to Eq. 12 in Electronic Supplementary Material.

The inventory tables now obtained with the five methods applied are listed in Table 14. The right hand side of Table 14 also shows the percentage differences between the inventory vectors obtained with the OLS and TLS solutions and those obtained with the allocation-based solutions. Because of the equality of the DLS and TLS scaling vectors, their inventory vectors (and the consequent differences from the traditional solutions) are also equal. Also in this case, the percentage differences related to the OLS solution are very close to 100% for every environmental intervention and thus they were not showed into the table.

The mean absolute values of the percentage differences are in this case: DLS-ECN=91.9%; DLS-PHY=53.2%; TLS-ECN=91.9%; TLS-PHY=53.2%.

TThe discrepancy vectors obtained are reported in Table 15. Like before, also in this case, the last element of the discrepancy vector obtained with the three least squares methods is different than 0 (i.e., the last value of the final supply vector is different than the desired value, namely 1,000 in this case). Differently than for the previous case, now, also the other elements of the discrepancy vector related to the OLS solution are substantially higher than the corresponding elements of the discrepancy vectors obtained applying the allocation procedures.

### 3.3 Case III, sensitivity analysis on changing the rescaling

TRelated to a change in the unit is the issue of rescaling or normalization (Heijungs and Suh 2002). In physical problems influenced by variables of different nature, these variables are, in the most general case, expressed by numbers whose range of variation can differ by several orders of magnitude. For this reason, when any approach is used to solve a multiple-input problem, the solution could be polarized by those variables expressed by extreme values (the highest or the lowest ones). In order to prevent this, normalization is often used to reduce all numbers to the same range of variation.

In our case, an experiment was tried by applying a suitable normalization algorithm both to the technology matrix and the final supply vector. The normalization procedure used is explained in the following.

Let us consider the linear system $\mathbf{A} \cdot \mathbf{s} = \mathbf{f}$, with $\mathbf{A} \in \Re^{m \times n}$ and $\mathbf{f} \in \Re^{m \times 1}$. Let $\mathbf{H}$ be the diagonal square matrix defined as:

$$\mathbf{H} = \begin{pmatrix} \frac{1}{\max|a_{i1}|, i=1,\ldots,m} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\max|a_{in}|, i=1,\ldots,m} \end{pmatrix} \quad (4)$$

Let $\mathbf{A}_n$ be the normalized matrix $\mathbf{A}$. It is defined as:

$$\mathbf{A}_n = \mathbf{A} \cdot \mathbf{H} \quad (5)$$

Let $\mathbf{f}_n$ be the normalized vector $\mathbf{f}$, obtained by dividing each element of $\mathbf{f}$ by the maximum absolute element:

$$\mathbf{f}_n = \frac{\mathbf{f}}{\max\{|f_i|, \forall i = 1, \ldots, m\}} \quad (6)$$

If $\mathbf{A}_n$.is invertible, the de-normalized solution $X_{\text{den}}$ obtained as follows:

$$\mathbf{s}_n = \mathbf{A}_n^{-1} \cdot \mathbf{f}_n \quad (7)$$

$$\mathbf{s}_{\text{den}} = \mathbf{H} \cdot \mathbf{s}_n \cdot \max\{|f_i|, \forall i = 1, \ldots, m\} \quad (8)$$

**Table 6** Square technology matrix ($\mathbf{A}' \in \Re^{13\times13}$) obtained after applying physical allocation to the original **A** of Table 1

| | Production of electricity | Production of heat | Production of white clay | Production of red clay | Production of sand | Production of gravel | Production of crude oil | Production of diesel oil | Production of fuel oil | Production of natural gas | Supply of olive cake | Supply of straw | Production of bricks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| MJ of electricity | 1 | 0 | −3.60E−03 | −3.60E−03 | −9.00E−03 | −9.00E−03 | 0 | −1.60E−02 | −1.60E−02 | 0 | 0 | 0 | −3.69E+02 |
| MJ of heat | 0 | 2.48E+00 | 0 | 0 | 0 | 0 | −4.87E−02 | −5.66E−01 | −5.66E−01 | −4.87E−02 | 0 | 0 | −1.01E+03 |
| kg of white clay | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1.37E+03 |
| kg of red clay | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8.51E+02 |
| kg of sand | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −4.41E+02 |
| kg of gravel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | −3.47E+02 |
| kg of olive cake | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | −1.53E+02 |
| kg of straw | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.92E+01 |
| MJ of crude oil | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.17E+00 | −1.17E+00 | 0 | 0 | 0 | 0.00E+00 |
| MJ of diesel oil | 0 | 0 | −2.27E−03 | −2.27E−03 | −1.80E−02 | −1.80E−02 | −5.29E−03 | 1 | 0 | −3.61E+01 | −3.25E−01 | −8.12E−02 | −1.41E+03 |
| MJ of fuel oil | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1.11E+03 |
| MJ of natural gas | −3.42E+00 | −8.54E−01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | −5.52E+03 |
| Ton of bricks | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.00E+00 |

The unit of measurement in which each economic flow in the table is expressed is defined at the beginning of the corresponding row

**Table 7** Square technology matrix ($\mathbf{A}'' \in \Re^{13\times13}$) obtained after applying economic allocation to the original **A** of Table 1

| | Production of electricity | Production of heat | Production of white clay | Production of red clay | Production of sand | Production of gravel | Production of crude oil | Production of diesel oil | Production of fuel oil | Production of natural gas | Supply of olive cake | Supply of straw | Production of bricks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| MJ of electricity | 1 | 0 | −3.77E−03 | −3.43E−03 | −1.04E−02 | −7.62E−03 | 0 | −1.85E−03 | −3.01E−02 | 0 | 0 | 0 | −3.69E+02 |
| MJ of heat | 0 | 2.48E+00 | 0 | 0 | 0 | 0 | −4.87E−02 | −6.55E−02 | −1.07E+00 | −4.87E−02 | 0 | 0 | −1.01E+03 |
| kg of white clay | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1.37E+03 |
| kg of red clay | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8.51E+02 |
| kg of sand | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −4.41E+02 |
| kg of gravel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | −3.47E+02 |
| kg of olive cake | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | −1.53E+02 |
| kg of straw | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.92E+01 |
| MJ of crude oil | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1.35E−01 | −2.20E+00 | 0 | 0 | 0 | 0.00E+00 |
| MJ of diesel oil | 0 | 0 | −2.38E−03 | −2.16E−03 | −2.08E−02 | −1.52E−02 | −5.29E−03 | 1 | 0 | −3.61E+01 | −2.89E−01 | −1.17E−01 | −1.41E+03 |
| MJ of fuel oil | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1.11E+03 |
| MJ of natural gas | −2.21E+00 | −2.06E+00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | −5.52E+03 |
| t of bricks | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.00E+00 |

The unit of measurement in which each economic flow in the table is expressed is defined at the beginning of the corresponding row

**Table 8** Environmental matrix ($\mathbf{B}' \in \Re^{21\times13}$) obtained applying the physical allocation to the matrix showed in Table 2

| | Production of electricity | Production of heat | Production of white clay | Production of red clay | Production of sand | Production of gravel | Production of crude oil | Production of diesel oil | Production of fuel oil | Production of natural gas | Supply of olive cake | Supply of straw | Production of bricks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Resources and raw materials | | | | | | | | | | | | | |
| MJ of coal | −1.20E−03 | −2.99E−04 | −2.58E−03 | −2.58E−03 | −6.82E−03 | −6.82E−03 | −7.46E−07 | −2.89E−02 | −2.89E−02 | −7.46E−07 | −1.12E−03 | −2.81E−04 | 0 |
| MJ of lignite | −2.19E−04 | −5.47E−05 | −3.23E−03 | −3.23E−03 | −8.14E−03 | −8.14E−03 | −1.77E−09 | −1.34E−04 | −1.34E−04 | −1.77E−09 | −6.66E−04 | −1.66E−04 | 0 |
| MJ of hydropower | 0 | 0 | −1.91E−04 | −1.91E−04 | −5.19E−04 | −5.19E−04 | −9.58E−08 | −3.83E−03 | −3.83E−03 | −9.58E−08 | −3.63E−04 | −9.08E−05 | 0 |
| MJ of geothermal energy | 0 | 0 | −1.40E−08 | −1.40E−08 | −5.84E−08 | −5.84E−08 | −6.08E−15 | −5.25E−05 | −5.25E−05 | −6.08E−15 | 0 | 0 | 0 |
| kg of water | 0 | 0 | −3.90E−03 | −3.90E−03 | −1.20E−02 | −1.20E−02 | −1.49E−02 | −3.63E−03 | −3.63E−03 | −1.49E−02 | 0 | 0 | −1.03E+03 |
| kg of ores (sand, gravel, etc.) | −1.76E−05 | −4.39E−06 | −1.00E+00 | −1.00E+00 | −1.00E+00 | −1.00E+00 | −3.51E−05 | −2.27E−04 | −2.27E−04 | −3.51E−05 | 0 | 0 | 0 |
| MJ of crude oil | −9.71E−03 | −2.43E−03 | −9.09E−03 | −9.09E−03 | −7.14E−02 | −7.14E−02 | −1.02E+00 | −1.17E+00 | −1.17E+00 | −1.02E+00 | −3.32E−01 | −8.31E−02 | −1.27E+03 |
| kg of other ores (iron, copper, etc.) | −1.00E−04 | −2.50E−05 | −3.68E−06 | −3.68E−06 | −1.97E−05 | −1.97E−05 | −5.60E−10 | −1.20E−04 | −1.20E−04 | −5.60E−10 | 0 | 0 | 0 |
| Emissions to air | | | | | | | | | | | | | |
| kg of CO2 | 6.66E−02 | 1.66E−02 | 1.26E−03 | 1.26E−03 | 6.65E−03 | 6.65E−03 | 4.17E−03 | 1.44E−02 | 1.44E−02 | 4.17E−03 | 3.94E+00 | 9.86E−01 | 6.02E+02 |
| kg of CO | 1.30E−04 | 3.26E−05 | 1.91E−06 | 1.91E−06 | 1.40E−05 | 1.40E−05 | 1.14E−05 | 1.67E−05 | 1.67E−05 | 1.14E−05 | 2.16E−02 | 5.40E−03 | 1.10E+00 |
| kg of CH4 | 2.94E−04 | 7.36E−05 | 1.48E−06 | 1.48E−06 | 5.44E−06 | 5.44E−06 | 7.13E−05 | 5.10E−05 | 5.10E−05 | 7.13E−05 | 4.80E−03 | 1.20E−03 | 1.18E+00 |
| kg of SO2 | 2.37E−05 | 5.92E−06 | 1.31E−06 | 1.31E−06 | 8.21E−06 | 8.21E−06 | 5.31E−06 | 1.34E−04 | 1.34E−04 | 5.31E−06 | 5.94E−03 | 1.49E−03 | 2.20E+00 |
| kg of NMVOC | 2.96E−05 | 7.40E−06 | 2.10E−07 | 2.10E−07 | 1.43E−06 | 1.43E−06 | 1.93E−05 | 2.12E−05 | 2.12E−05 | 1.93E−05 | 2.46E−02 | 6.14E−03 | 1.13E+00 |
| Emissions to water | | | | | | | | | | | | | |
| kg of COD | 5.00E−07 | 1.25E−07 | 9.98E−08 | 9.98E−08 | 5.57E−07 | 5.57E−07 | 1.57E−11 | 3.37E−06 | 3.37E−06 | 1.57E−11 | 1.77E−04 | 4.42E−05 | 2.38E−01 |
| kg of BOD | 3.49E−08 | 8.72E−09 | 3.05E−09 | 3.05E−09 | 1.76E−08 | 1.76E−08 | 4.41E−13 | 9.47E−08 | 9.47E−08 | 4.41E−13 | 5.40E−06 | 1.35E−06 | 2.25E−01 |
| kg of P | 2.62E−10 | 6.54E−11 | 2.48E−11 | 2.48E−11 | 1.95E−10 | 1.95E−10 | 3.73E−18 | 4.92E−13 | 4.92E−13 | 3.73E−18 | 0 | 0 | 1.26E−03 |
| kg of N | 2.49E−08 | 6.22E−09 | 1.45E−09 | 1.45E−09 | 1.15E−08 | 1.15E−08 | 2.19E−16 | 5.40E−11 | 5.40E−11 | 2.19E−16 | 1.29E−04 | 3.22E−05 | 6.00E−03 |
| kg of AOX | 4.62E−11 | 1.16E−11 | 1.84E−12 | 1.84E−12 | 1.45E−11 | 1.45E−11 | 4.64E−18 | 1.00E−12 | 1.00E−12 | 4.64E−18 | 2.36E−07 | 5.90E−08 | 1.43E−05 |
| Solid wastes | | | | | | | | | | | | | |
| kg of ash | 0 | 0 | 5.6E−05 | 5.6E−05 | 1.4E−04 | 1.4E−04 | 1.5E−08 | 1.28E−04 | 1.28E−04 | 1.5E−08 | 0 | 0 | 0 |
| kg of sludge | 0 | 0 | 1.2E−07 | 1.2E−07 | 9.6E−07 | 9.6E−07 | 5.3E−14 | 1.05E−05 | 1.05E−05 | 5.3E−14 | 0 | 0 | 0 |
| kg of nuclear waste | 0 | 0 | 1.3E−09 | 1.3E−09 | 3.2E−09 | 3.2E−09 | 8.8E−17 | 3.85E−10 | 3.85E−10 | 8.8E−17 | 0 | 0 | 0 |

The unit of measurement in which each environmental flow in the table is expressed is defined at the beginning of the corresponding row

**Table 9** Environmental matrix ($\mathbf{B}'' \in \Re^{21\times13}$) obtained applying the economic allocation to the matrix showed in Table 2

| | Production of electricity | Production of heat | Production of white clay | Production of red clay | Production of sand | Production of gravel | Production of crude oil | Production of fuel oil | Production of diesel oil | Production of natural gas | Supply of olive cake | Supply of straw | Production of bricks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Resources and raw materials | | | | | | | | | | | | | |
| MJ of coal | −7.74E−04 | −7.20E−04 | −2.70E−03 | −2.46E−03 | −7.87E−03 | −5.77E−03 | −7.46E−07 | −5.44E−02 | −3.34E−03 | −7.46E−07 | −1.00E−03 | −4.03E−04 | 0 |
| MJ of lignite | −1.42E−04 | −1.32E−04 | −3.39E−03 | −3.08E−03 | −9.39E−03 | −6.89E−03 | −1.77E−09 | −2.53E−04 | −1.55E−05 | −1.77E−09 | −5.93E−04 | −2.39E−04 | 0 |
| MJ of hydropower | 0 | 0 | −2.00E−04 | −1.82E−04 | −5.99E−04 | −4.40E−04 | −9.58E−08 | −7.22E−03 | −4.44E−04 | −9.58E−08 | −3.24E−04 | −1.30E−04 | 0 |
| MJ of geothermal energy | 0 | 0 | −1.47E−08 | −1.33E−08 | −6.74E−08 | −4.94E−08 | −6.08E−15 | −9.90E−05 | −6.08E−06 | −6.08E−15 | 0 | 0 | 0 |
| kg of water | 0 | 0 | −4.08E−03 | −3.71E−03 | −1.38E−02 | −1.01E−02 | −1.49E−02 | −6.84E−03 | −4.20E−04 | −1.49E−02 | 0 | 0 | −1.03E+03 |
| kg of ores (sand, gravel, etc.) | −1.14E−05 | −1.06E−05 | −1.05E+00 | −9.52E−01 | −1.15E+00 | −8.46E−01 | −3.51E−05 | −4.28E−04 | −2.63E−05 | −3.51E−05 | 0 | 0 | 0 |
| MJ of crude oil | −6.29E−03 | −5.85E−03 | −9.53E−03 | −8.66E−03 | −8.24E−02 | −6.04E−02 | −1.02E+00 | −2.21E+00 | −1.36E−01 | −1.02E+00 | −2.96E−01 | −1.19E−01 | −1.27E+03 |
| kg of other ores (iron, copper, etc.) | −6.47E−05 | −6.03E−05 | −3.86E−06 | −3.51E−06 | −2.28E−05 | −1.67E−05 | −5.60E−10 | −2.26E−04 | −1.39E−05 | −5.60E−10 | 0 | 0 | 0 |
| Emissions to air | | | | | | | | | | | | | |
| kg of CO2 | 4.31E−02 | 4.01E−02 | 1.32E−03 | 1.20E−03 | 7.67E−03 | 5.63E−03 | 4.17E−03 | 2.71E−02 | 1.67E−03 | 4.17E−03 | 3.51E+00 | 1.42E+00 | 6.02E+02 |
| kg of CO | 8.44E−05 | 7.86E−05 | 2.00E−06 | 1.82E−06 | 1.61E−05 | 1.18E−05 | 1.14E−05 | 3.15E−05 | 1.93E−06 | 1.14E−05 | 1.92E−02 | 7.76E−03 | 1.10E+00 |
| kg of CH4 | 1.91E−04 | 1.77E−04 | 1.56E−06 | 1.41E−06 | 6.28E−06 | 4.61E−06 | 7.13E−05 | 9.61E−05 | 5.90E−06 | 7.13E−05 | 4.28E−03 | 1.72E−03 | 1.18E+00 |
| kg of SO2 | 1.53E−05 | 1.43E−05 | 1.37E−06 | 1.24E−06 | 9.47E−06 | 6.95E−06 | 5.31E−06 | 2.53E−04 | 1.56E−05 | 5.31E−06 | 5.30E−03 | 2.13E−03 | 2.20E+00 |
| kg of NMVOC | 1.92E−05 | 1.78E−05 | 2.20E−07 | 2.00E−07 | 1.66E−06 | 1.21E−06 | 1.93E−05 | 3.99E−05 | 2.45E−06 | 1.93E−05 | 2.19E−02 | 8.82E−03 | 1.13E+00 |
| Emissions to water | | | | | | | | | | | | | |
| kg of COD | 3.24E−07 | 3.01E−07 | 1.05E−07 | 9.51E−08 | 6.42E−07 | 4.71E−07 | 1.57E−11 | 6.36E−06 | 3.91E−07 | 1.57E−11 | 1.58E−04 | 6.35E−05 | 2.38E−01 |
| kg of BOD | 2.26E−08 | 2.10E−08 | 3.20E−09 | 2.91E−09 | 2.03E−08 | 1.49E−08 | 4.41E−13 | 1.79E−07 | 1.10E−08 | 4.41E−13 | 4.81E−06 | 1.94E−06 | 2.25E−01 |
| kg of P | 1.69E−10 | 1.58E−10 | 2.59E−11 | 2.36E−11 | 2.25E−10 | 1.65E−10 | 3.73E−18 | 9.27E−13 | 5.70E−14 | 3.73E−18 | 0 | 0 | 1.26E−03 |
| kg of N | 1.61E−08 | 1.50E−08 | 1.52E−09 | 1.38E−09 | 1.32E−08 | 9.70E−09 | 2.19E−16 | 1.02E−10 | 6.26E−12 | 2.19E−16 | 1.15E−04 | 4.63E−05 | 6.00E−03 |
| kg of AOX | 2.99E−11 | 2.79E−11 | 1.93E−12 | 1.76E−12 | 1.67E−11 | 1.23E−11 | 4.64E−18 | 1.89E−12 | 1.16E−13 | 4.64E−18 | 2.10E−07 | 8.48E−08 | 1.43E−05 |
| Solid wastes | | | | | | | | | | | | | |
| kg of ash | 0 | 0 | 5.83E−05 | 5.30E−05 | 1.63E−04 | 1.20E−04 | 1.5E−08 | 2.41E−04 | 1.48E−05 | 1.5E−08 | 0 | 0 | 0 |
| kg of sludge | 0 | 0 | 1.29E−07 | 1.17E−07 | 1.11E−06 | 8.16E−07 | 5.3E−14 | 1.98E−05 | 1.22E−06 | 5.3E−14 | 0 | 0 | 0 |
| kg of nuclear waste | 0 | 0 | 1.33E−09 | 1.21E−09 | 3.74E−09 | 2.74E−09 | 8.8E−17 | 7.25E−10 | 4.46E−11 | 8.8E−17 | 0 | 0 | 0 |

The unit of measurement in which each environmental flow in the table is expressed is defined at the beginning of the corresponding row

**Table 10** Scaling factors **s** for the five different methods of calculating the inventory for the production of 1 ton of bricks

| Process | LCI technique | | | | |
|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS |
| Production of electricity | 1.47E+00 | −9.52E+01 | −7.55E−05 | −1.58E+03 | −1.24E+03 |
| Production of heat | −5.73E+03 | −7.14E+03 | | | |
| Production of white clay | 1.37E+03 | 1.37E+03 | 8.51E−05 | 1.24E+03 | 9.72E+02 |
| Production of red clay | 8.51E+02 | 8.51E+02 | | | |
| Production of sand | 4.41E+02 | 4.41E+02 | 3.16E−05 | 4.57E+02 | 3.59E+02 |
| Production of gravel | 3.47E+02 | 3.47E+02 | | | |
| Production of crude oil | −2.80E+04 | −3.51E+04 | −2.31E−04 | −1.68E+04 | −1.32E+04 |
| Production of diesel oil | −2.25E+05 | −3.11E+04 | −9.59E−05 | −5.69E+03 | −4.47E+03 |
| Production of fuel oil | 1.11E+03 | 1.11E+03 | | | |
| Production of natural gas | −6.26E+03 | −8.97E+02 | −5.66E−06 | −1.97E+02 | −1.55E+02 |
| Supply of olive cake | 1.53E+02 | 1.53E+02 | 6.07E−06 | 1.03E+02 | 8.07E+01 |
| Supply of straw | 1.92E+01 | 1.92E+01 | | | |
| Production of bricks | 1.00E+00 | 1.00E+00 | 7.70E−08 | 1.00E+00 | 7.86E−01 |

is the same of the solution obtained as $\mathbf{s} = \mathbf{A}^{-1} \cdot \mathbf{f}$. In practice, however, $\mathbf{A}_n$ may be a more stable matrix to invert than $\mathbf{A}$. For this reason, this normalization procedure is suitable for the application that we want to realize.

We applied this normalization algorithm to our case study. The traditional solutions (i.e., those obtained with the physical-based allocation and the economic value-based allocation) and the OLS solution found after de-normalization resulted to be

**Table 11** Inventory table **g** of the production of 1 ton of bricks for the five different methods of calculating the inventory and percentage differences between the inventory vectors obtained with the DLS and TLS solutions and those obtained with the allocation-based solutions

| Elementary flow | LCI technique | | | | | % difference | | | |
|---|---|---|---|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS | DLS-ECN | DLS-PHY | TLS-ECN | TLS-PHY |
| Resources and raw materials | | | | | | | | | |
| MJ of coal | 1.22E+04 | 8.56E+02 | 4.77E−06 | 3.18E+02 | 2.50E+02 | −97.39 | −62.85 | −98.0 | −70.8 |
| MJ of lignite | 4.36E+01 | −9.28E+00 | −1.02E−06 | −1.36E+01 | −1.07E+01 | −131.19 | 46.55 | −124.5 | 15.3 |
| MJ of hydropower | 1.62E+03 | 1.14E+02 | 6.67E−07 | 4.26E+01 | 3.35E+01 | −97.37 | −62.63 | −97.9 | −70.6 |
| MJ of geothermal energy | 2.22E+01 | 1.57E+00 | 1.01E−08 | 5.97E−01 | 4.69E−01 | −97.31 | −61.97 | −97.9 | −70.1 |
| kg of water | 1.00E+03 | −4.03E+02 | −7.63E−05 | −7.55E+02 | −5.93E+02 | −175.50 | 87.34 | −159.3 | 47.1 |
| kg of ores (sand, gravel, etc.) | −2.95E+03 | −3.00E+03 | −2.33E−04 | −3.38E+03 | −2.66E+03 | 14.58 | 12.67 | −9.8 | −11.3 |
| MJ of crude oil | 5.30E+05 | 7.05E+04 | 3.61E−04 | 2.93E+04 | 2.30E+04 | −94.47 | −58.44 | −95.7 | −67.4 |
| kg of other ores (iron, copper, etc.) | 5.12E+01 | 3.76E+00 | 3.06E−08 | 1.54E+00 | 1.21E+00 | −96.99 | −59.04 | −97.6 | −67.8 |
| Emissions to air | | | | | | | | | |
| kg of CO2 | −5.30E+03 | 5.27E+02 | 6.68E−05 | 7.51E+02 | 5.90E+02 | −114.17 | 42.50 | −111.1 | 12.0 |
| kg of CO | −3.70E+00 | 3.37E+00 | 2.31E−07 | 3.24E+00 | 2.55E+00 | −187.57 | −3.86 | −168.9 | −24.3 |
| kg of CH4 | −2.32E+01 | −2.70E+00 | 7.30E−08 | −5.71E−01 | −4.49E−01 | −97.54 | −78.85 | −98.1 | −83.4 |
| kg of SO2 | −5.41E+01 | −1.11E+00 | 1.86E−07 | 1.31E+00 | 1.03E+00 | −102.42 | −218.02 | −101.9 | −192.8 |
| kg of NMVOC | −5.06E+00 | 3.63E+00 | 2.62E−07 | 3.66E+00 | 2.87E+00 | −172.33 | 0.83 | −156.7 | −20.9 |
| Emissions to water | | | | | | | | | |
| kg of COD | −1.17E+00 | 1.64E−01 | 1.90E−08 | 2.22E−01 | 1.74E−01 | −118.97 | 35.37 | −114.9 | 6.1 |
| kg of BOD | 1.86E−01 | 2.23E−01 | 1.74E−08 | 2.25E−01 | 1.77E−01 | 20.97 | 0.90 | −4.8 | −20.6 |
| kg of P | 1.26E−03 | 1.26E−03 | 9.67E−11 | 1.26E−03 | 9.87E−04 | 0 | 0 | −21.7 | −21.7 |
| kg of N | 2.44E−02 | 2.64E−02 | 1.44E−09 | 2.25E−02 | 1.77E−02 | −7.79 | −14.77 | −27.5 | −33.0 |
| kg of AOX | 4.76E−05 | 5.15E−05 | 2.89E−12 | 4.45E−05 | 3.49E−05 | −6.51 | −13.59 | −26.7 | −32.2 |
| Solid wastes | | | | | | | | | |
| kg of ash | −5.40E+01 | −3.60E+00 | −6.12E−09 | −1.19E+00 | −9.35E−01 | −97.80 | −66.94 | −98.3 | −74.0 |
| kg of sludge | −4.45E+00 | −3.14E−01 | −1.93E−09 | −1.18E−01 | −9.30E−02 | −97.35 | −62.42 | −97.9 | −70.4 |
| kg of nuclear waste | −1.58E−04 | −6.15E−06 | 3.48E−13 | 1.73E−06 | 1.36E−06 | −101.09 | −128.13 | −100.9 | −122.1 |

The unit of measurement in which each elementary flow in the table is expressed is defined at the beginning of the corresponding row

**Table 12** Discrepancy vectors (and corresponding Euclidean norms) for the five different methods of calculating the inventory for the production of 1 ton of bricks

| Product flow | LCI technique | | | | |
|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS |
| 1 MJ of electricity | −1.14E−13 | 2.27E−13 | −1.02E−04 | −1.78E+03 | −1.40E+03 |
| 1 MJ of heat | 1.14E−13 | −1.71E−12 | −1.45E−04 | 2.34E+03 | 1.84E+03 |
| 1 kg of white clay | 0 | 0 | −2.04E−05 | −1.34E+02 | −1.05E+02 |
| 1 kg of red clay | 0 | 0 | 1.96E−05 | 3.86E+02 | 3.03E+02 |
| 1 kg of recycled inerts | - | - | 5.31E−06 | 6.90E+01 | 5.42E+01 |
| 1 kg of sand | 0 | 0 | −6.86E−06 | −4.32E+01 | −3.39E+01 |
| 1 kg of gravel | 0 | 0 | 4.92E−06 | 1.10E+02 | 8.63E+01 |
| 1 kg of olive cake | 0 | 0 | −5.74E−06 | −5.08E+01 | −3.99E+01 |
| 1 kg of straw | 0 | 0 | 4.60E−06 | 8.35E+01 | 6.56E+01 |
| 1 MJ of crude oil | 1.36E−12 | 6.82E−13 | −7.08E−06 | −3.48E+03 | −2.74E+03 |
| 1 MJ of diesel oil | −3.21E−11 | 5.46E−12 | −2.80E−06 | 2.63E+01 | 2.07E+01 |
| 1 MJ of fuel oil | 0 | 0 | −1.81E−04 | −6.80E+03 | −5.34E+03 |
| 1 MJ of natural gas | 2.73E−12 | −9.09E−13 | −1.08E−04 | 1.02E+03 | 8.02E+02 |
| 1 ton of bricks | 0 | 0 | −1.00E+00 | 0 | −2.14E−01 |
| Euclidean norm | 3.22E−011 | 5.83E−012 | 1.00E+00 | 8.26E+03 | 6.49E+03 |

The unit of measurement in which each product flow in the table is expressed is defined at the beginning of the corresponding row

the same as those obtained without using any normalization procedure, whereas the TLS and DLS solutions are slightly different than those that had been obtained without normalization. This behavior of the solutions holds both for the system in which the amount of bricks is expressed in tons (case I) and for the system in which it is expressed in kilograms (case II). The scaling factors obtained with the TLS and DLS methods in both cases are showed in Table 16. The two DLS solutions are identical and the two TLS solutions are nearly identical. The observation of Table 16 thus suggests that DLS and TLS regression techniques, if applied after the rescaling of the technology matrix and the final demand vector, are insensitive to this change of unit of measurement.

The corresponding inventory tables are reported in Table 17. The discrepancy vectors obtained are reported in Table 18.

## 4 Discussion

The solution of the inventory problem in LCA is a very complicated task, especially in presence of multi-functional

**Table 13** Scaling factors for the five different methods of calculating the inventory after changing the units of measurement (1,000 kg of bricks)

| Process | LCI technique | | | | |
|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS |
| Production of electricity | 1.47E+00 | −9.52E+01 | −7.01E+01 | −1.58E+03 | −1.58E+03 |
| Production of heat | −5.73E+03 | −7.14E+03 | | | |
| Production of white clay | 1.37E+03 | 1.37E+03 | 7.90E+01 | 1.24E+03 | 1.24E+03 |
| Production of red clay | 8.51E+02 | 8.51E+02 | | | |
| Production of sand | 4.41E+02 | 4.41E+02 | 2.94E+01 | 4.57E+02 | 4.57E+02 |
| Production of gravel | 3.47E+02 | 3.47E+02 | | | |
| Production of crude oil | −2.80E+04 | −3.51E+04 | −2.15E+02 | −1.68E+04 | −1.68E+04 |
| Production of diesel oil | −2.25E+05 | −3.11E+04 | −8.90E+01 | −5.69E+03 | −5.69E+03 |
| Production of fuel oil | 1.11E+03 | 1.11E+03 | | | |
| Production of natural gas | −6.26E+03 | −8.97E+02 | −5.25E+00 | −1.97E+02 | −1.97E+02 |
| Supply of olive cake | 1.53E+02 | 1.53E+02 | 5.64E+00 | 1.03E+02 | 1.03E+02 |
| Supply of straw | 1.92E+01 | 1.92E+01 | | | |
| Production of bricks | 1.00E+00 | 1.00E+00 | 7.15E−02 | 1.00E+00 | 1.00E+00 |

**Table 14** Inventory table for the five different methods of calculating the inventory after changing the units of measurement (1,000 kg of bricks) and percentage differences between the inventory vectors obtained with the DLS and TLS solutions and those obtained with the allocation-based solutions

| Elementary flow | LCI technique | | | | | % difference | | | |
|---|---|---|---|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS | DLS-ECN | DLS-PHY | TLS-ECN | TLS-PHY |
| Resources and raw materials | | | | | | | | | |
| MJ of coal | 1.22E+04 | 8.56E+02 | 4.43E+00 | 3.18E+02 | 3.18E+02 | −97.39 | −62.85 | −97.39 | −62.85 |
| MJ of lignite | 4.36E+01 | −9.28E+00 | −9.51E−01 | −1.36E+01 | −1.36E+01 | −131.19 | 46.55 | −131.19 | 46.55 |
| MJ of hydropower | 1.62E+03 | 1.14E+02 | 6.19E−01 | 4.26E+01 | 4.26E+01 | −97.37 | −62.63 | −97.37 | −62.63 |
| MJ of geothermal energy | 2.22E+01 | 1.57E+00 | 9.35E−03 | 5.97E−01 | 5.97E−01 | −97.31 | −61.97 | −97.31 | −61.97 |
| kg of water | 1.00E+03 | −4.03E+02 | −7.09E+01 | −7.55E+02 | −7.55E+02 | −175.50 | 87.34 | −175.50 | 87.34 |
| kg of ores (sand, gravel, etc.) | −2.95E+03 | −3.00E+03 | −2.17E+02 | −3.38E+03 | −3.38E+03 | 14.58 | 12.67 | 14.58 | 12.67 |
| MJ of crude oil | 5.30E+05 | 7.05E+04 | 3.36E+02 | 2.93E+04 | 2.93E+04 | −94.47 | −58.44 | −94.47 | −58.44 |
| kg of other ores (iron, copper, etc.) | 5.12E+01 | 3.76E+00 | 2.84E−02 | 1.54E+00 | 1.54E+00 | −96.99 | −59.04 | −96.99 | −59.04 |
| Emissions to air | | | | | | | | | |
| kg of CO2 | −5.30E+03 | 5.27E+02 | 6.21E+01 | 7.51E+02 | 7.51E+02 | −114.17 | 42.50 | −114.17 | 42.50 |
| kg of CO | −3.70E+00 | 3.37E+00 | 2.15E−01 | 3.24E+00 | 3.24E+00 | −187.57 | −3.86 | −187.57 | −3.86 |
| kg of CH4 | −2.32E+01 | −2.70E+00 | 6.78E−02 | −5.71E−01 | −5.71E−01 | −97.54 | −78.85 | −97.54 | −78.85 |
| kg of SO2 | −5.41E+01 | −1.11E+00 | 1.73E−01 | 1.31E+00 | 1.31E+00 | −102.42 | −218.02 | −102.42 | −218.02 |
| kg of NMVOC | −5.06E+00 | 3.63E+00 | 2.43E−01 | 3.66E+00 | 3.66E+00 | −172.33 | 0.83 | −172.33 | 0.83 |
| Emissions to water | | | | | | | | | |
| kg of COD | −1.17E+00 | 1.64E−01 | 1.76E−02 | 2.22E−01 | 2.22E−01 | −118.97 | 35.37 | −118.97 | 35.37 |
| kg of BOD | 1.86E−01 | 2.23E−01 | 1.61E−02 | 2.25E−01 | 2.25E−01 | 20.97 | 0.90 | 20.97 | 0.90 |
| kg of P | 1.26E−03 | 1.26E−03 | 8.98E−05 | 1.26E−03 | 1.26E−03 | 0 | 0 | 0 | 0 |
| kg of N | 2.44E−02 | 2.64E−02 | 1.34E−03 | 2.25E−02 | 2.25E−02 | −7.79 | −14.77 | −7.79 | −14.77 |
| kg of AOX | 4.76E−05 | 5.15E−05 | 2.68E−06 | 4.45E−05 | 4.45E−05 | −6.51 | −13.59 | −6.51 | −13.59 |
| Solid wastes | | | | | | | | | |
| kg of ash | −5.40E+01 | −3.60E+00 | −5.69E−03 | −1.19E+00 | −1.19E+00 | −97.80 | −66.94 | −97.80 | −66.94 |
| kg of sludge | −4.45E+00 | −3.14E−01 | −1.79E−03 | −1.18E−01 | −1.18E−01 | −97.35 | −62.42 | −97.35 | −62.42 |
| kg of nuclear waste | −1.58E−04 | −6.15E−06 | 3.23E−07 | 1.73E−06 | 1.73E−06 | −101.09 | −128.13 | −101.09 | −128.13 |

The unit of measurement in which each elementary flow in the table is expressed is defined at the beginning of the corresponding row

**Table 15** Discrepancy vectors (and corresponding Euclidean norms) for the five different methods of calculating the inventory after changing the units of measurement (1,000 kg of bricks)

| Product flow | LCI technique | | | | |
|---|---|---|---|---|---|
| | ECN | PHY | OLS | DLS | TLS |
| MJ of electricity | −1.65E−12 | 2.27E−13 | −9.48E+01 | −1.78E+03 | −1.78E+03 |
| MJ of heat | −2.05E−12 | −1.71E−12 | −1.35E+02 | 2.34E+03 | 2.34E+03 |
| kg of white clay | 0 | 0 | −1.89E+01 | −1.34E+02 | −1.34E+02 |
| kg of red clay | 0 | 0 | 1.82E+01 | 3.86E+02 | 3.86E+02 |
| 1 kg of recycled inerts | - | - | 4.93E+00 | 6.90E+01 | 6.90E+01 |
| kg of sand | 0 | 0 | −6.37E+00 | −4.32E+01 | −4.32E+01 |
| kg of gravel | 5.68E−14 | 5.68E−14 | 4.57E+00 | 1.10E+02 | 1.10E+02 |
| kg of olive cake | 0 | 0 | −5.33E+00 | −5.08E+01 | −5.08E+01 |
| kg of straw | 0 | 0 | 4.27E+00 | 8.35E+01 | 8.35E+01 |
| MJ of crude oil | −6.37E−12 | 6.82E−13 | −6.57E+00 | −3.48E+03 | −3.48E+03 |
| MJ of diesel oil | 2.89E−11 | 5.46E−12 | −2.60E+00 | 2.63E+01 | 2.63E+01 |
| MJ of fuel oil | 0 | 0 | −1.68E+02 | −6.80E+03 | −6.80E+03 |
| MJ of natural gas | 9.09E−13 | −9.09E−13 | −1.00E+02 | 1.02E+03 | 1.02E+03 |
| t of bricks | 0 | 0 | −9.29E+02 | 0 | 0 |
| Euclidean norm | 2.97E−011 | 5.83E−012 | 2.57E+02 | 8.26E+03 | 8.26E+03 |

processes or open-loop recycling, where the system is characterized by a rectangular and thus, non-invertible technology matrix. In these cases, the LCA analyst has to make many subjective assumptions, for example the choice of the allocation factors. Even though different criteria can be followed to make this choice, there is no precise rule for it and this increases the subjectivity in LCA studies. It therefore represents one of the Achilles' heels of the LCA methodology.

The application of the DLS and TLS methods to the solution of the inventory problem, as described for an illustrative case study concerning the process of bricks production, could represent an alternative way to compute the inventory vector of any product/process. It can be useful in presence of multi-functional processes and in all those cases in which the technology matrix is rectangular but the pseudo-inverse method does not provide an acceptable solution. The substantial advantage in using these techniques lies into the possibility to circumvent the drawback of the traditional solution of the inventory problem, which needs the use of some computational expedients (such as substitution and allocation) to transform the rectangular technology matrix into a square and invertible matrix.

The case study tackled in this paper shows that the scaling factors (and, consequently, the inventory vectors) obtained applying the DLS and TLS methods to the 14×

**Table 16** Scaling factors for the DLS and TLS cases applied after rescaling of the original matrix (case I) and of the matrix with changed unit of measurement (case II)

| Process | LCI technique | | | |
|---|---|---|---|---|
| | Case I | | Case II | |
| | DLS | TLS | DLS | TLS |
| Production of electricity<br>Production of heat | −1.41E+03 | −1.26E+03 | −1.41E+03 | −1.26E+03 |
| Production of white clay<br>Production of red clay | 1.17E+03 | 1.05E+03 | 1.17E+03 | 1.05E+03 |
| Production of sand<br>Production of gravel | 4.42E+02 | 3.97E+02 | 4.42E+02 | 3.97E+02 |
| Production of crude oil | −9.67E+03 | −8.68E+03 | −9.67E+03 | −8.61E+03 |
| Production of fuel oil<br>Production of diesel oil | −3.71E+03 | −3.33E+03 | −3.71E+03 | −3.31E+03 |
| Production of natural gas | −1.60E+02 | −1.44E+02 | −1.60E+02 | −1.43E+02 |
| Supply of olive cake<br>Supply of straw | 2.20E+02 | 1.97E+02 | 2.20E+02 | 1.96E+02 |
| Production of bricks | 1.00E+00 | 8.98E−01 | 1.00E+00 | 8.98E−01 |

**Table 17** Inventory tables for the DLS and TLS cases applied after rescaling of the original matrix (case I) and of the matrix with changed unit of measurement (case II)

| Elementary flow | LCI technique | | | |
|---|---|---|---|---|
| | Case I | | Case II | |
| | DLS | TLS | DLS | TLS |
| Resources and raw materials | | | | |
| MJ of coal | 2.04E+02 | 1.83E+02 | 2.04E+02 | 1.82E+02 |
| MJ of lignite | −1.35E+01 | −1.21E+01 | −1.35E+01 | −1.22E+01 |
| MJ of hydropower | 2.75E+01 | 2.46E+01 | 2.75E+01 | 2.45E+01 |
| MJ of geothermal energy | 3.90E−01 | 3.50E−01 | 3.90E−01 | 3.48E−01 |
| kg of water | −8.75E+02 | −7.85E+02 | −8.75E+02 | −7.87E+02 |
| kg of ores (sand, gravel, etc.) | −3.21E+03 | −2.88E+03 | −3.21E+03 | −2.89E+03 |
| MJ of crude oil | 1.73E+04 | 1.55E+04 | 1.73E+04 | 1.54E+04 |
| kg of other ores (iron, copper, etc.) | 1.04E+00 | 9.35E−01 | 1.04E+00 | 9.29E−01 |
| Emissions to air | | | | |
| kg of CO2 | 1.43E+03 | 1.28E+03 | 1.43E+03 | 1.28E+03 |
| kg of CO | 6.58E+00 | 5.90E+00 | 6.58E+00 | 5.87E+00 |
| kg of CH4 | 9.04E−01 | 8.12E−01 | 9.04E−01 | 8.13E−01 |
| kg of SO2 | 2.75E+00 | 2.47E+00 | 2.75E+00 | 2.47E+00 |
| kg of NMVOC | 7.47E+00 | 6.71E+00 | 7.47E+00 | 6.67E+00 |
| Emissions to water | | | | |
| kg of COD | 2.61E−01 | 2.34E−01 | 2.61E−01 | 2.34E−01 |
| kg of BOD | 2.26E−01 | 2.03E−01 | 2.26E−01 | 2.03E−01 |
| kg of P | 1.26E−03 | 1.13E−03 | 1.26E−03 | 1.13E−03 |
| kg of N | 4.13E−02 | 3.71E−02 | 4.13E−02 | 3.69E−02 |
| kg of AOX | 7.90E−05 | 7.09E−05 | 7.90E−05 | 7.05E−05 |
| Solid wastes | | | | |
| kg of ash | −6.96E−01 | −6.25E−01 | −6.96E−01 | −6.19E−01 |
| kg of sludge | −7.69E−02 | −6.90E−02 | −7.69E−02 | −6.85E−02 |
| kg of nuclear waste | 2.97E−06 | 2.67E−06 | 2.97E−06 | 2.68E−06 |

The unit of measurement in which each elementary flow in the table is expressed is defined at the beginning of the corresponding row

**Table 18** Discrepancy vectors (and corresponding Euclidean norms) for the DLS and TLS cases applied after rescaling of the original matrix (case I) and of the matrix with changed unit of measurement (case II)

| Product flow | LCI technique | | | |
|---|---|---|---|---|
| | Case I | | Case II | |
| | DLS | TLS | DLS | TLS |
| 1 MJ of electricity | −1.67E+03 | −1.50E+03 | −1.67E+03 | −1.50E+03 |
| 1 MJ of heat | 1.80E+02 | 1.62E+02 | 1.80E+02 | 1.40E+02 |
| 1 kg of white clay | −2.05E+02 | −1.84E+02 | −2.05E+02 | −1.84E+02 |
| 1 kg of red clay | 3.15E+02 | 2.83E+02 | 3.15E+02 | 2.83E+02 |
| 1 kg of sand | 6.90E+01 | 6.19E+01 | 6.90E+01 | 6.20E+01 |
| 1 kg of gravel | −5.81E+01 | −5.22E+01 | −5.81E+01 | −5.24E+01 |
| 1 kg of olive cake | 9.49E+01 | 8.52E+01 | 9.49E+01 | 8.50E+01 |
| 1 kg of straw | 6.61E+01 | 5.93E+01 | 6.61E+01 | 5.81E+01 |
| 1 MJ of crude oil | 2.00E+02 | 1.80E+02 | 2.00E+02 | 1.79E+02 |
| 1 MJ of diesel oil | −9.78E+02 | −8.78E+02 | −9.78E+02 | −8.69E+02 |
| 1 MJ of fuel oil | 6.01E+02 | 5.39E+02 | 6.01E+02 | 5.34E+02 |
| 1 MJ of natural gas | −4.82E+03 | −4.33E+03 | −4.82E+03 | −4.30E+03 |
| 1 t of bricks | 3.28E+02 | 2.94E+02 | 3.28E+02 | 2.78E+02 |
| 1 MJ of electricity | −1.11E−16 | −1.02E−01 | −9.99E+02 | −1.02E+02 |
| Euclidean norm | 5.26E+03 | 4.73E+03 | 5.36E+03 | 4.70E+03 |

The unit of measurement in which each product flow in the table is expressed is defined at the beginning of the corresponding row

8 economic matrix which describes the system at hand are very close to each other, but they differ substantially from the scaling factors obtained applying the OLS regression method. A direct comparison with the scaling vectors obtained with the allocation procedures is not possible, because these latter have a different number of elements, since allocation leads to a square matrix with a higher number of columns. A direct comparison can instead be made with the inventory vectors. From this case study, it resulted that the inventory vectors obtained with the DLS and TLS techniques are closer to the inventory vectors obtained with the physical allocation than to those obtained using the economic allocation. However, this last property cannot be generalized to every inventory problem in LCA, also taking into account that the system tackled in this work was intentionally over-simplified since our main concern was more on the investigation of the numerical stability and applicability of the investigated algorithms.

It is worth noting that in all the experiments accomplished, there is a large difference between the scaling factors obtained by using TLS and those obtained by using OLS. This is not surprising taking into consideration the theoretical analysis of the TLS problems (see Van Huffel and Vandewalle 1989).

Let $\mathbf{A} = \mathbf{U}'\mathbf{\Sigma}'\mathbf{V}'^T$ be a SVD of $\mathbf{A}$, where: $\mathbf{A} \in \Re^{m \times n}$; $\mathbf{\Sigma}' = diag(\sigma'_1, \ldots \sigma'_n)$; $\sigma'_1 \geq \ldots \geq \sigma'_n$ are the singular values of $\mathbf{A}$; $\mathbf{U}' = [\mathbf{u}'_1, \ldots, \mathbf{u}'_m]$ (with $\mathbf{u}_i \in \Re^m$) and $\mathbf{V}' = [v'_1, \ldots, v'_n]$ (with $\mathbf{v}'_i \in \Re^n$) are the matrices of its left and right singular vectors, respectively. As it is proved in Van Huffel and Vandewalle (1989), the closer $\sigma_{n+1}$ (see Appendix) is to zero and the larger is the difference between $\sigma_{n+1}$ and $\sigma'_n$, the smaller the distance between the OLS and the TLS solution.

As underlined in Van Huffel and Vandewalle (1988), when $\sigma_{n+1}$ is close to $\sigma'_\text{n}$ the TLS problem becomes close-to-nongeneric. Nongeneric TLS problems occur whenever $\mathbf{A}$ is (nearly) rank deficient ($\sigma'_\text{n} \approx 0$) or when the system $\mathbf{A} \cdot \mathbf{s} \approx \mathbf{f}$ is highly incompatible (large $\sigma_{n+1}$; Van Huffel and Vandewalle 1989). In these cases, the generic TLS solution can still be computed, but it becomes instable and very sensitive to data errors (Golub and Van Loan 1980). Whereas, identifying the problem as nongeneric and computing the nongeneric TLS solution (see Van Huffel and Vandewalle 1991, pages 66-75) reduces the sensitivity considerably, "the solution error is kept small and the stability is maintained" (Van Huffel and Vandewalle 1991, page 210).

From the geometrical point of view, when the difference $\sigma'_\text{n} - \sigma_{n+1}$ becomes very small (occurrence of the nongeneric TLS case) $\mathbf{f}$ is nearly orthogonal to $\mathbf{u}_{n+1}$ and $\mathbf{u}_{n+1} \approx \mathbf{u}'_\text{n}$ (see Appendix).

As a consequence, the difference $\sigma'_\text{n} - \sigma_{n+1}$, or equivalently the difference $\sigma'^2_\text{n} - \sigma^2_{\text{n}+1}$, can be used as a measure of how close the system $\mathbf{A} \cdot \mathbf{s} \approx \mathbf{f}$ is to the class of nongeneric TLS problems.

The expected distance between the *standard* (i.e., not nongeneric) TLS solution and the OLS solution can instead be inspected through the ratio $\sigma_n/\sigma'_n$. In this case, the expected accuracy of the TLS solution with respect to the OLS solution increases as this ratio increases.

Resuming:

- The difference $\sigma'_\text{n} - \sigma_{n+1}$ (or $\sigma'^2_\text{n} - \sigma^2_{\text{n}+1}$) is used to assess whether or not the problem at hand is to be considered as a nongeneric TLS problem. The smaller is this difference, the closer the problem at hand is to the class of nongeneric TLS problems, for which the specific nongeneric TLS solution (described in Van Huffel and Vandewalle 1991, Theorem 3.12, pag. 72) must be applied;
- The ratio $\sigma_n/\sigma'_n$ is the indicator to be used to assess the distance between the *standard* TLS solution and the OLS solution. The larger this ratio, the higher the expected accuracy of the TLS solution with respect to the OLS one;

In our case, $\sigma_{n+1}$ is equal to 0.46 in both cases when the amount of bricks is expressed in tons (case I) and in kilograms (case II) and $\sigma'_\text{n}$ is equal to 0.46 for case I and to 0.47 for case II. As a consequence, the difference between $\sigma'_\text{n}$ and $\sigma_{n+1}$ is exactly zero for case I and is 0.01 for case II and the problem is close to nongeneric. This finding, on one hand, justifies the significant differences between the TLS and the OLS solutions; on the other hand, it opens interesting perspectives for a further investigation of the problem and the search for its solution as a nongeneric TLS problem.

A last observation concerns the consistency of the TLS solution, related to the number of rows of the matrix $\mathbf{A}$. The TLS solution vector $\widetilde{\mathbf{s}}$ of the over-determined system $\mathbf{A} \cdot \mathbf{s} = \mathbf{f}$ computes a strongly consistent estimate of the true but unknown parameter $\mathbf{s}$, i.e., it converges to s with probability one as the number of equations $m$ tends to infinity (Van Huffel and Vandewalle 1991; Markovsky and Van Huffel 2007). In our case, the number of equations is as low as 14, that is very far from justifying the assumption that $m \to \infty$. In those cases in which the inventory matrices has a high number of rows (like the aforementioned case of the unallocated version of the ecoinvent v1.3, with 2632 rows) the use of the TLS technique is fully justified by its higher accuracy with respect to the OLS method.

Finally, if one rescales the economic matrix applying the normalization procedure described in Section 3.3, the traditional solutions (i.e., those obtained with the allocation based on physical relations among products or the allocation based on the economic value of the products) and the OLS solution found after de-normalization result to be the same as the corresponding solutions obtained without using any normalization procedure, whereas the DLS and TLS solutions are different from those that were obtained without normalization. This behavior of the solutions holds both for case I and case II.

The conclusion that is possible to draw from the interpretation of the results obtained in our paper is that the observation of the discrepancy vector is important but it should not be used as the unique inspection criterion for the assessment of a solution method. Of course, when the application of a least-square technique yields an acceptable solution (with low discrepancies between the desired and the final demand vector), allocation can be avoided. However, for those cases in which very high discrepancies are obtained, allocation procedures are still necessary.

If one decides to make resort to the least squares solutions, the choice of the most suitable one can be driven by the observation of some "quality" indicators like the ratio $\sigma_n/\sigma'_n$ and the difference $\sigma'_\mathrm{n} - \sigma_{n+1}$ (or, equivalently, the difference $\sigma'^2_\mathrm{n} - \sigma^2_\mathrm{n+1}$; Van Huffel and Vandewalle 1989).

The application to LCA of the least-squares techniques described in the paper is absolutely innovative and liable to further developments.

## 5 Conclusions and recommendations

In order for the least-squares techniques (in particular TLS and DLS) to became a valuable alternative to the traditional allocation procedures, there is a need for an optimization of these techniques for the very particular systems that commonly occur in LCI, i.e., systems with sparse coefficients matrices and with a vector of the constants **f** which is always defined as a vector of null entries, except for the one referring to the investigated reference flow. Furthermore, for those TLS problems which are identified as nongeneric, the application of a nongeneric TLS solution could significantly reduce the sensitivity to data errors (Van Huffel 1987) and thus is worth of being investigated also in the field of LCI.

The future optimization of the above mentioned algorithms could mark the route for a novel approach to the solution of the problem of allocation in LCA. In fact, the described approach requires only a minimal intervention by the LCA analyst. He/she has only to apply a codified (and thus repeatable) mathematical procedure which does not allow enough *degrees of freedom* to overwhelmingly affect the final results according to his/her personal choices. This also makes it easier to accomplish an eventual sensitivity analysis.

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