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# Filtering of the Navier-Stokes equations in the context of time-dependent flows

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## Abstract

The concepts of *filtering* and *decomposition* as applied to the analysis and modelling of turbulent flows are reviewed, starting from the seminal work of Reynolds. The problems arising from the use of finite spatial or time filters are discussed, and the recent approach based on the concepts of scale-similarity and multi-level filtering are presented. Finally, the possible extension of these ideas to finite *time* filtering is considered, with a view to developing turbulence models for time-dependent flows; a simple two-dimensional direct simulation of a turbulent free convective flow is used as an example.

## 1 Introduction

Dating from the time of Professor Osborne Reynolds's inspired analysis[1], the application of the Navier-Stokes equations to turbulent flows has conventionally applied 'Reynolds averaging' to filter out the turbulent eddying motions. This allows gross 'steady' flows to be described by laminar-like equations, with the addition of Reynolds stresses which have to be measured or modelled. The approach has not only stood the test of a century's advance in understanding, but more than successfully weathered the introduction, post World War II, of digital computing methods and sophisticated experimental techniques. Certainly, while state-of-the-art computing was limited to steady two-dimensional turbulent flows, turbulence models (typically the two-equation k-ε model) proved useful even in engineering contexts.

However, it was always recognised that the three-dimensional, time-dependent behaviour of turbulence (eddy motion) implied limitations to the validity of such modelling, and, when computing reached the stage of (just) permitting treatment of the three-dimensional transient form of the Navier-Stokes equations, Large-Eddy and Direct Simulations began to appear. With the grid resolution then available, the latter could treat transitional flows and possibly even low turbulence, but for 'high' turbulence space-related averaging was essential, resulting in sub-grid models, and hence resolution of only the large eddy behaviour.

Since the advent of LES and DS two historical stages may be defined. Firstly, 'time-average' and LES approaches continued in parallel, the latter always requiring specially-written codes and the largest available computers; the former was always cheaper, and had the security of being used by the majority of Computational Fluid Dynamics specialists. LES eventually began to demonstrate the capability of better fundamental understanding of some of the turbulent processes. The second historical stage was the development of highly sophisticated commercial codes, now able to treat three-dimensional transient flows in arbitrary geometries. Not only are the codes readily available but local workstations may be exploited for predictions. The whole approach gives the individual CFD worker great potential in treating complex turbulent flows, even for engineering purposes.

The current situation in the context of *time-dependent* flows will probably evolve towards a convergence in the two modelling approaches. Whether time- or space-averaged, both will predict gross eddying behaviour of transient or periodic nature.

## 2 Osborne Reynolds

We first focus our attention on O. Reynolds's analysis of 1895[1]. The distinction between laminar and turbulent flows (respectively termed 'direct' and 'sinuous'), whether for flow in pipes or in Stokes's prior studies on pendulums[2], was seen as due to eddying, and *not* to any difference in the basic theoretical equations. In fact, Reynolds interpreted the laminar/turbulent distinction in terms of the 'periods' of the fluid motion, in the context of conversion of fluid energy into heat (viscous dissipation). We will see how these concepts of 'period' and 'energy conversion' lead to the decomposition of the equations.

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Starting from the concepts of separate periods of overall fluid and thermal behaviour (respectively termed 'mean-motion' and 'heat-motion'), the laminar flow is interpreted as allowing for continuous conversion of fluid energy into heat 'without passing through any intermediate stage' (p.129). However, for higher speed flows, the periods of the main flow and thermal fields become increasingly separate, and there exists an intermediate eddying field with correspondingly intermediate periods - hence turbulent flow. In the latter, energy drains from the main field to both the eddying and thermal fields, and from the eddying field to the thermal field. The flow field ('mean motion') may be decomposed into an overall field ('mean-mean motion') and an eddying field ('relative-mean motion'). In turbulent flow, then, the overall velocities u, v, w being continuous functions of space and time, averages $\bar{u}, \bar{v}$ and $\bar{w}$ may be seen as functions of *space* for a given small time interval, or as functions of *time* for a given small space interval (pp. 134/5). Reynolds also allowed for a mean flow varying with time, so what is termed 'Reynolds averaging' does not wholly reflect his understanding.

# 3 The effect of finite filters

Historically, 'Reynolds' averaging has been assumed to be over all time periods; most turbulence models have attempted to predict ensemble averages, identified, for statistically stationary (ergodic) flows, with the infinite-time statistical averages:

$$\langle\Phi\rangle = \lim\ (\theta \to \infty)\ \frac{1}{\theta}\int_0^\theta \Phi(t)dt \tag{1}$$

In fact the concept of averaging over only a small *space* interval did not occur until the advent of Large-Eddy Simulation., the term proposed in 1974 by Leonard[3]. Finite *time* filters have received so far little attention in the literature.

Let us briefly review the concept of filtering, starting from the incompressible Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2}$$

in which $\sigma_{ij}$ is the traceless part of the viscous stress tensor. Let us now apply to the original variables a *filter* $\langle\cdot\rangle$ (which can be, in general, either spatial or temporal) possessing only the two properties of:

i) being linear: $\langle\alpha f+\beta h\rangle = \alpha\langle f\rangle + \beta\langle h\rangle$;

ii)commuting with time/space derivatives: $\langle\partial f/\partial t\rangle = \partial\langle f\rangle/\partial t$, $\langle\partial f/\partial x_i\rangle = \partial\langle f\rangle/\partial x_i$.

By filtering the whole Eqn. (2), one obtains:

$$\frac{\partial\langle u_i\rangle}{\partial t} + \frac{\partial\langle u_i\rangle\langle u_j\rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\langle\sigma_{ij}\rangle - \tau(u_i,u_j)\right] \tag{3}$$

in which the terms $\tau(u_i,u_j)$ are the central moments of the variables $u_i$, $u_j$ with respect to the filter chosen, i.e., $\tau(u_i,u_j) \equiv \langle u_i\ u_j\rangle - \langle u_i\rangle\langle u_j\rangle$. Eqn. (3) is formally identical to the original NS equation, save that (a) primitive quantities are now replaced by filtered quantities, and (b), the viscous stresses are augmented by the extra terms $-\tau(u_i,u_j)$, which have to be modelled in order to close the system of flow equations.

Thus, in principle, flow equations for the filtered quantities can be derived without any explicit reference to the *decomposition* of $u_i$ into filtered and residual components $\langle u_i\rangle$, $u_i'$. Thus, we will not go into details here but will just mention that, following such decomposition, also the central moments are decomposed into terms depending only on the unresolved fluctuations, or small-scale, fields (Reynolds stresses proper); terms depending on the resolved, or large-scale, fields (Leonard terms); and terms containing both, expressing the interaction between large and small scales (cross terms). Considerable effort has been dedicated to the modelling of all the above terms, either treating them separately or grouping them in different ways.

A great formal simplification is achieved if filters possessing also the independent properties:

iii) $\langle\langle f\rangle\rangle = \langle f\rangle$ and, iv) $\langle\langle f\rangle h\rangle = \langle f\rangle\langle h\rangle$

(among which the statistical filter of Eqn. (1)) are used. Most smooth filters, however, do not possess properties (iii) and (iv) and give rise to Leonard and cross terms.

# 4 The scale similarity 'dynamic' approach

A considerable breakthrough in 'subgrid' modelling has been the introduction of the *scale similarity* concept. Basically, it consists of sampling the smallest resolved scales to deduce the unresolved ('subgrid') terms arising from filtering. The idea was firstly introduced by Bardina[4] and was further developed by Horiuti[5]; recently, the same basic ideas have been given a more rigorous and coherent formulation by Germano[6]. He observed that, if a second-level, or 'test' filter $G = \overline{\cdot}$ is introduced besides the first-level, or 'grid' filter $F = \langle\cdot\rangle$, then a simple algebraic identity occurs between the following quantities:

- central moment at F-level, $\tau_f(u_i, u_j) \equiv \langle u_i u_j\rangle - \langle u_i\rangle\langle u_j\rangle$;
- central moment at FG level, $\tau_{fg}(u_i, u_j) \equiv \overline{\langle u_i u_j\rangle} - \overline{\langle u_i\rangle}\,\overline{\langle u_j\rangle}$;

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- resolved central moment, $\tau_g(\langle u_i\rangle, \langle u_j\rangle) \equiv \overline{\langle u_i\rangle \langle u_j\rangle} - \overline{\langle u_i\rangle}\,\overline{\langle u_j\rangle}$

(this last being extracted from the resolved scales F, and thus explicitly computable). This identity is:

$$\overline{\tau_f(u_i, u_j)} + \tau_g(\langle u_i\rangle, \langle u_j\rangle) = \tau_{fg}(u_i, u_j) \qquad (4)$$

Eqn. (4) itself does *not* solve the closure problem, because now we have a second unknown, namely, the FG-level moment $\tau_{fg}$, but it provides a rigorous link between resolved and unresolved scales which may be exploited in several ways. So far, most applications have consisted of expressing both $\tau_f$ and $\tau_{fg}$ by the Smagorinsky model and using Eqn. (4) to compute locally and instantaneously its 'free' constant $c_S$ ('dynamic' subgrid modelling); however, alternative ways of exploiting the 'Germano identity' may be devised. For example, if - using the scale similarity concept - the FG-level and F-level central moments are simply assumed to be proportional, i.e. $\tau_{fg}(u_i, u_j) = K\ \overline{\tau_f(u_i, u_j)}$ (the proportionality constant K being related to the turbulence spectrum and to the ratio of the F and G filter widths), then Eqn. (4) yields $\overline{\tau_f(u_i, u_j)} = \tau_g(\langle u_i\rangle, \langle u_j\rangle)/(K-1)$, so that the unresolved terms $\tau_f$ (at least as G-level averages) can be computed from resolved quantities.

# 5 Time filtering and transient turbulent flows

Coming back to Reynolds's original ideas, since rigorous formulations of the filtering treatment of turbulence, as formulated by Germano[6], are equally applicable to time as to space, it is natural to ask whether an approach based on finite *time* filtering can be developed (Germano himself suggests such an approach in the closing lines of his paper).

An obvious field of application would be the computation of *turbulent flows with transient mean*, a class of problems which have been somehow embarassing for conventional turbulence models based on the statistical (infinite-time average) approach of Eqn. (1). The applicability of models like the k-ε to such flows is questionable, since the necessary substitution of finite for infinite time filtering would lead to the appearance in the residual stresses of Leonard and cross terms which the model is *not* equipped to deal with. The only 'safe' cases would be those in which large and small time scales were well separate.

In some problems, a large-time-scale transient behaviour depends on intrinsic instabilities of the mean flow and not by external influences. We cite but a few examples, starting from the classic problem of the *backward-facing step*. Here, both experiments[7] and direct numerical simulations[8] indicate that the separated shear layer undergoes a flapping motion, with a Strouhal number (based on free-stream velocity and step height) of ~0.06, so that the (spanwise averaged) reattachment location oscillates from ~5.5 to ~7 step heights downstream of the step.

A second example is given by *turbulent vortex shedding* past a bluff body, e.g. a cylinder or a square prism. For this problem (which has been a paradigm of the existence of coherent spanwise structures also in highly turbulent flows) Bègue et al.[9] presented time-dependent 2-D solutions obtained also with the standard k-ε model, and blamed insufficient grid resolution for previously reported failures in obtaining transient solutions. On the other hand, the Fluent group[10] reported that transient solutions could be obtained only by the RNG k-ε, while the standard version of the model lead to fluctuation damping.

In other instances, time dependence is imposed by an external (e.g., time periodic) forcing function. An example is the prediction of flow in *cylindrical vessels stirred by rotating impellers*, a problem of great importance in chemical engineering[11]. Eulerian time spectra of turbulence at a generic location exhibit low-frequency peaks associated with the periodic passage of the impeller blades, superimposed on conventional turbulence. The former component is quite significant, and even dominant, close to the impeller periphery[12]. A steady-state or a time-dependent approach can be used to simulate the flow, but in either case it is not clear how the low frequencies ought to be dealt with.

# 6 An example: turbulent convection in a cavity

In the following, attention will be focussed on the *high-Rayleigh number natural convection* in a liquid metal-filled slender rectangular enclosure with volumetric heating, a problem studied in connection with the design of the breeder blanket for the future nuclear fusion reactors.

For a similar problem Farouk[13], using a low-Reynolds number k-ε model, obtained time-dependent predictions exhibiting symmetry breaking and large-time scale oscillations. Here, we post-process the results of *two-dimensional direct simulations* in order to discuss the applicability of the scale-similarity concept to time filtering.

The enclosure, filled with a low Prandtl number fluid, had height H=0.1 m and width 2δ=H/4 and was volumetrically heated with a power density of 15 MW/m$^3$, giving a Rayleigh number (based on H) of ~$10^8$. Top and bottom walls were adiabatic, with T=0 on side walls. Predictions were obtained by a finite-volume code using Crank-Nicolson time stepping, central differencing in space and the SIMPLEC pressure-velocity coupling algorithm. The computational grid had

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64x115 nodes, and substantial grid-independence was verified for this choice. Simulations, starting from u=v=T=0 everywhere, were protracted up to ~100 s with a time step Δt=0.02 s.

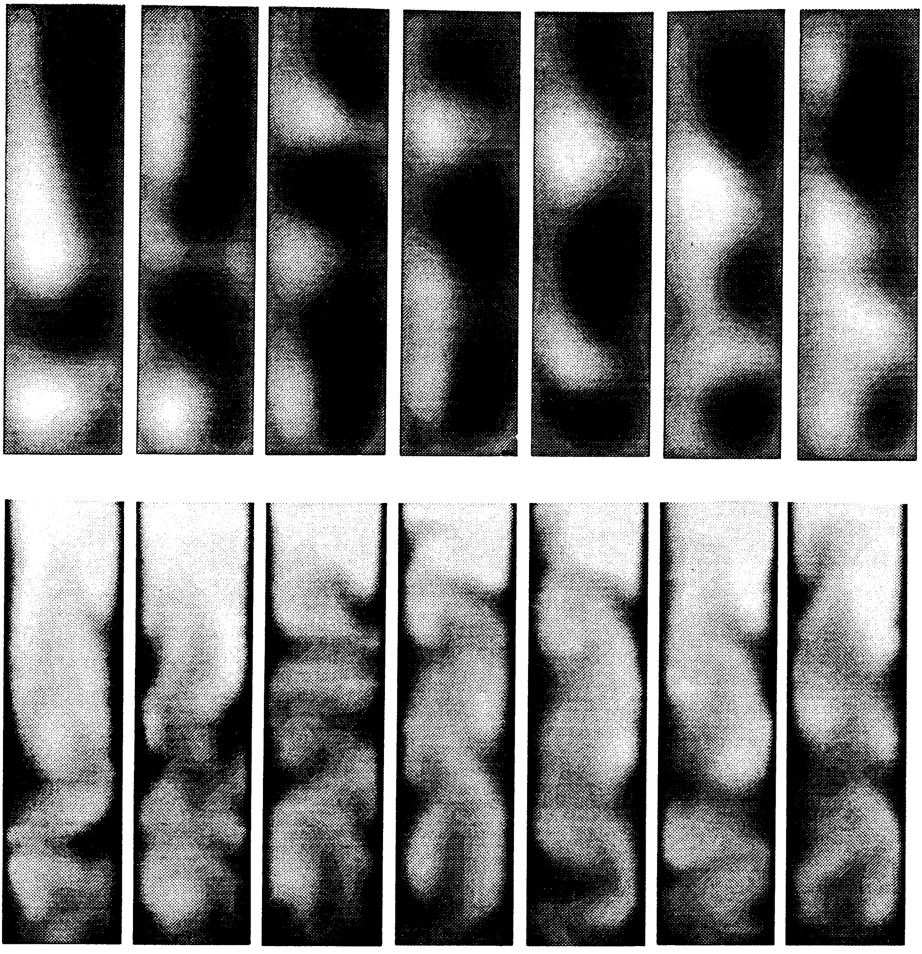

Figure 1: Instantaneous stream function ψ (top) and temperature T (bottom). Plots are at 1 s intervals from 43 to 49 s.

Figure 1 reports shade plots of instantaneous stream function ψ and temperature T at 1 s intervals to illustrate the basic features of the flow. Most of the fluid rises in the central region along a 'meandering' path, turns downwards near the top of the enclosure, where the temperature maximum is attained, and then gets cooled as it comes down along the cold side walls. Secondary circulation cells ('cat's eyes') develop in the enclosure, whose number, location and intensity fluctuate in time.

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Figure 2 reports the time behaviour of the instantaneous vertical velocity v at three distinct locations. Points 1 and 2 are symmetric with respect to the cavity vertical centerline; symmetry breaking, and the development of turbulent fluctuations, are observed for t ≈ 10 s.

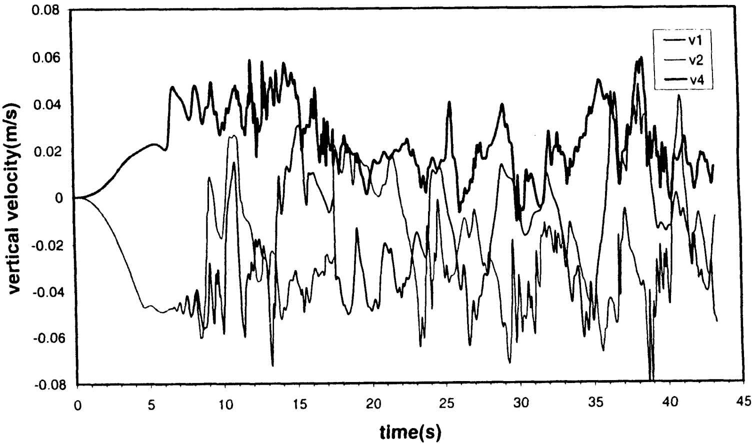


Figure 2: Time behaviour of the instantaneous vertical velocity v at three locations in the enclosure.

Figure 3 is the power spectrum (squared Fourier transform of the one-point dimensionless Eulerian time correlation) for point 1.

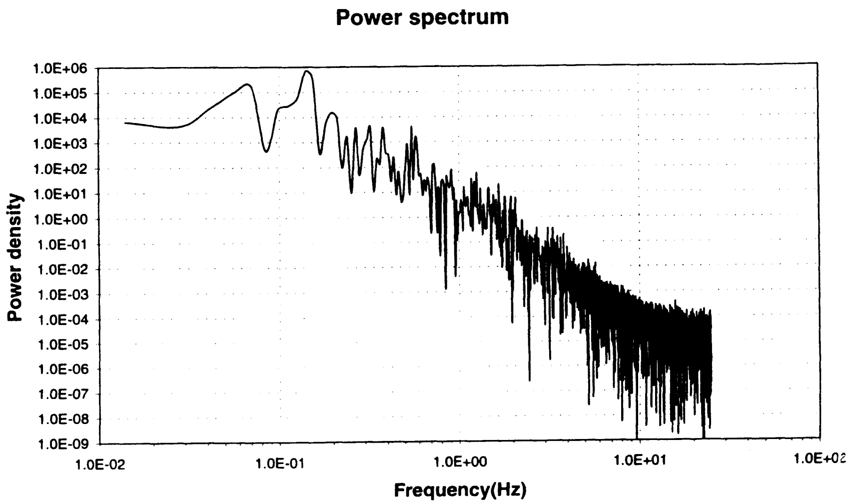


Figure 3: Power spectrum of vertical velocity for point 1

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A marked peak can be observed for a frequency f of ~0.14 $s^{-1}$ (period ~7 s) with a second, broader peak at one half the above frequency. Fluctuations are significant up to ~1 $s^{-1}$ and then decay rapidly. Of course, the spectrum becomes meaningless at frequencies close to one half that of the time step (25 $s^{-1}$).

The above results show that the flow is indeed turbulent, although the instantaneous spatial distribution of a generic quantity does not exhibit the fine structures characteristic of high turbulence and time sequences present a dominant low frequency (quasi-periodic flow). Conventional turbulence models give disappointing results for this problem. For example, Fig. 4 compares the long-term averages of $\psi$ and T from the direct simulation (left) with RNG k-ε steady-state predictions (right): clearly, the turbulence model is not able to predict even the statistical mean structure of the flow and temperature fields, notably giving a single recirculation cell on each side of the enclosure (with just a hint of 'cat's eyes').

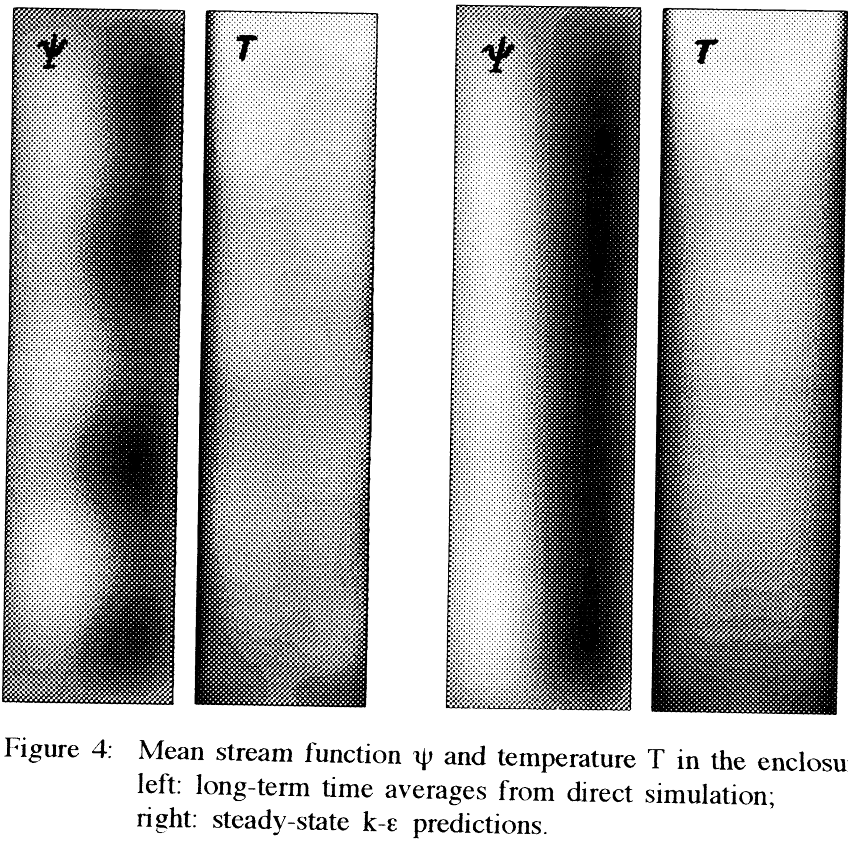


Figure 4: Mean stream function $\psi$ and temperature T in the enclosure. left: long-term time averages from direct simulation; right: steady-state k-ε predictions.

It may be argued that a time-dependent turbulence model based on *finite time filtering*, which resolved the main time-dependent modes up to a fraction of a Hz while modelling the higher-frequency fluctuations, would perform better than the k-ε without requiring the small time steps necessary for direct simulation. In the following, we report some preliminary results suggesting that such a model can probably be based on the concepts of *dynamic modelling* and *scale-similarity*.

Fig. 5 compares the following quantities computed by *time filtering*:

- moment $\tau_f(u, v) \equiv \langle uv\rangle - \langle u\rangle\langle v\rangle$, obtained by filtering the direct simulation results u, v with a top-hat, or running average, filter having a width of 0.28 s (F-level filter, denoted by $\langle \cdot \rangle$);
- moment $\tau_g(\langle u_i\rangle, \langle u_j\rangle) \equiv \overline{\langle u_i\rangle\langle u_j\rangle} - \overline{\langle u_i\rangle}\,\overline{\langle u_j\rangle}$, obtained by filtering the F-level averages $\langle u\rangle, \langle v\rangle$ with a second top-hat filter having a width of 2 s (G-level filter, denoted by $\overline{\cdot}$).

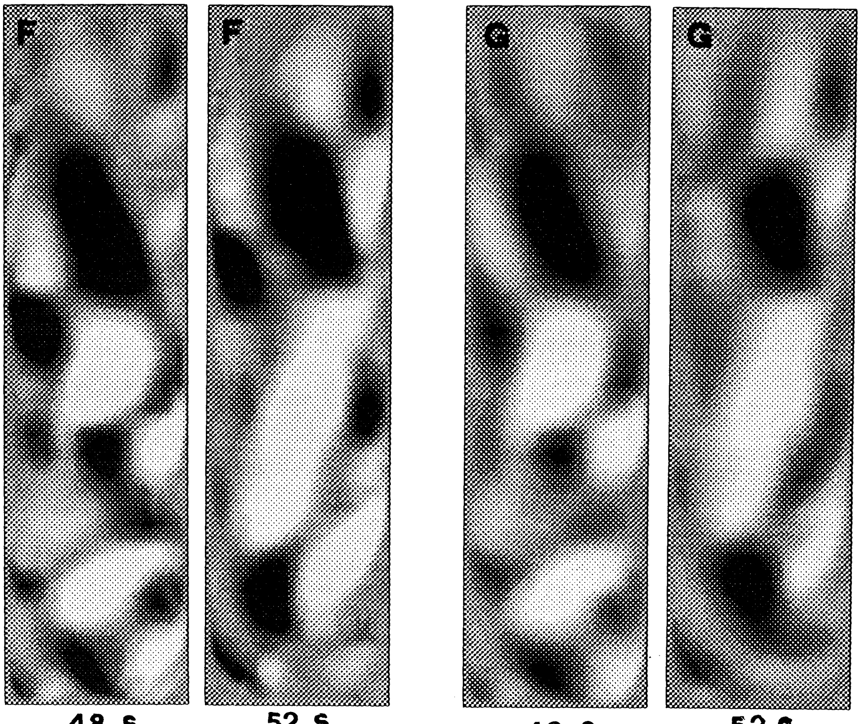


Figure 5: Generalized central moments τ(u,v) at two filtering levels. left: F-level (filter width=0.28 s); right: G-level (width=2 s).

Note that $\tau_g$ coincides with the resolved generalized central moment if the F-level quantities are regarded as the 'resolved' quantities. Of both $\tau_f$ and $\tau_g$ the distributions over the whole computational domain are shown at two instants (t=48 and 52 s). The similarity of the two terms is quite evident.

Generalized central moments akin to those described above can be built also for (u,T) and (v,T), giving rise to generalized *turbulent heat fluxes* **q** at each filtering level. Fig. 6 reports on the left the vector plot of the F-level turbulent heat flux for t=48 s, together with maps of the F-level filtered stream function and temperature and, on the right, a comparison between the modules of **q** at F and G levels.

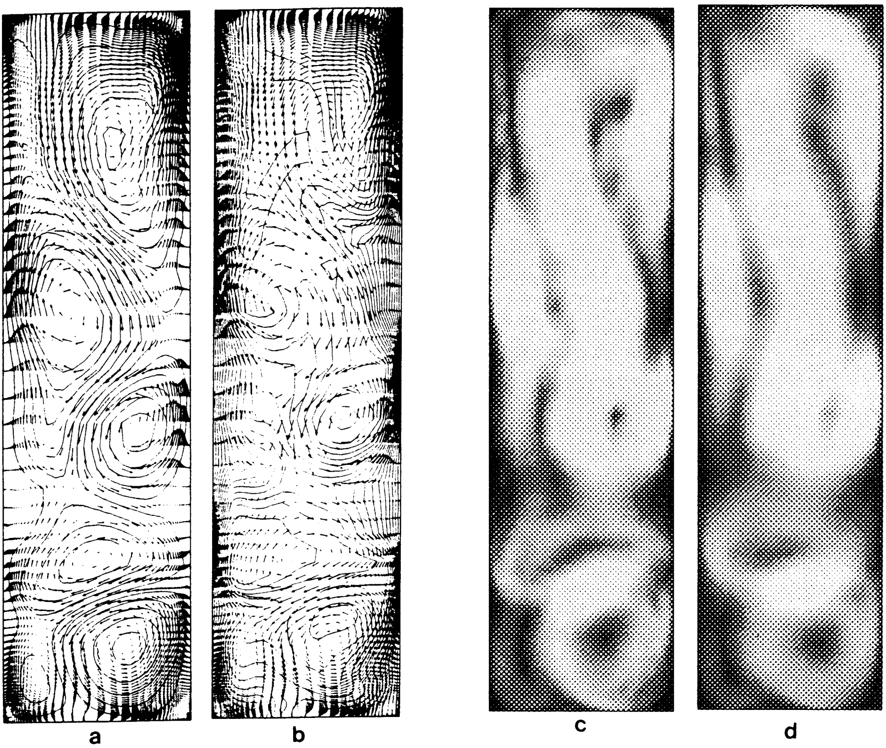


Figure 6: Generalized turbulent heat flux **q** for t=48 s.
a, b) vector plots of **q** superimposed on F-level stream function and temperature distributions;
c, d): comparison of the modulus of **q** at F- and G-levels.

A striking feature of the vector plots reported above is that **q** is far from following a gradient-diffusion behaviour. As graph (b) shows, it is rather aligned with the streamlines (a detailed analysis shows that this is

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a consequence of the Leonard and cross terms contributing to **q**). Graphs (c) and (d) confirm also for the generalized turbulent heat flux the scale-similarity behaviour observed above for the generalized turbulent stresses.

# Conclusions

The concept of filtering as applied to the Navier-Stokes equations was reviewed. It was shown that even in the seminal work of Osborne Reynolds both the concepts of spatial and time filtering were present, although the former found concrete application only with the advent of Large Eddy Simulation, and the latter has been used up to now only in the form of infinite-time average.

Using an accurately time-resolved direct two-dimensional simulation of an unsteady flow (free convection in a volumetrically heated enclosure), an assessment was made of the possibility of extending to the time domain the concepts of filtering, scale similarity and dynamic modelling of the unresolved terms, which have been common for some years with reference to spatial filtering (Large Eddy Simulation) following the work of Bardina and Germano.

Preliminary results indicate that, indeed, the terms arising from time filtering (generalized Reynolds stresses and turbulent heat fluxes) exhibit a striking similarity in their spatial and time distributions at two, well separate, filtering levels. This may be taken as a starting point for the development of novel turbulence models which should find their natural field of application in time-dependent turbulent flows.

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