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# INVERSION FORMULAE FOR THE INTEGRAL TRANSFORM ON A LOCALLY COMPACT ZERO-DIMENSIONAL GROUP

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**ABSTRACT.** Generalized inversion formulae for multiplicative integral transform with a kernel defined by characters of a locally compact zero-dimensional abelian group are obtained using a Kurzweil-Henstock type integral.

## 1. Introduction

In this paper, we consider integral transforms with kernels defined by characters of a locally compact zero-dimensional abelian group. Transforms of this kind are usually called multiplicative transforms (see [1]).

The problem of getting an inversion formula for integral transform is a continual analogue of the one of recovering the coefficients of a convergent series with respect to characters of a compact zero-dimensional abelian group considered in [6]. So, we use those results on series to obtain the correspondent results on transforms. As in [5] and [6], we will use here the Kurzweil-Henstock method of integration.

Our first result is a locally compact version of [6, Theorem 4.2]. The second result is a generalization of [5, Theorem 5.1], where we are weakening the conditions on the way of convergence of the transform.

In comparison with [4], we consider here transforms directly on the group instead of using a mapping of this group on the real line. That mapping was connected with introduction of a certain ordering in this group which can be avoided here.

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## 2. Preliminaries

Let  $G$  be a zero-dimensional locally compact abelian group which satisfies the second countability axiom. We also suppose that the group  $G$  is periodic. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$\dots \supset G_{-n} \supset \dots \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 \dots \supset G_n \supset \dots \quad (1)$$

with  $G = \bigcup_{n=-\infty}^{+\infty} G_n$  and  $\{0\} = \bigcap_{n=-\infty}^{+\infty} G_n$ . The subgroups  $G_n$  are clopen sets with respect to this topology. As  $G$  is periodic, the factor group  $G_n/G_{n+1}$  is finite for each  $n$  and this implies that  $G_n$  (and so, also all its cosets) is compact. Note that the factor group  $G_n/G_0$  is also finite for any  $n < 0$ , and so, the factor group  $G/G_0$  is countable. We will use the notation given in [6], in particular, we denote by  $K_n$  any coset of the subgroup  $G_n$  and by  $K_n(g)$  the coset of the subgroup  $G_n$  which contains the element  $g$ . For each  $g \in G$  the sequence  $\{K_n(g)\}$  is decreasing and  $\{g\} = \bigcap_n K_n(g)$ .

Now, for each coset  $K_n$  of  $G_n$ , we choose and fix an element  $g_{K_n}$  for the rest of the paper. Then, for each  $n \in \mathbb{Z}$ , we can represent any element  $g \in G$  in the form

$$g = g_{K_n} + \{g\}_n, \quad (2)$$

where  $\{g\}_n \in G_n$ . Indeed let  $g \in K_n$ , for some  $n$ , then  $g = g_{K_n} + g - g_{K_n}$  and we can put  $\{g\}_n := g - g_{K_n}$ . We agree to put  $g_{G_n} = 0$ , so that  $g = \{g\}_n$  if  $g \in G_n$ .

Let  $\Gamma$  denotes the dual group of  $G$ , i.e., the group of characters of the group  $G$ . It is known (see [1]) that under the assumption imposed on  $G$ , the group  $\Gamma$  is also a periodic locally compact zero-dimensional abelian group (with respect to the pointwise multiplication of characters) and we can represent it as a sum of increasing sequence of subgroups

$$\dots \supset \Gamma_{-n} \supset \dots \supset \Gamma_{-2} \supset \Gamma_{-1} \supset \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_n \supset \dots \quad (3)$$

introducing a topology in  $\Gamma$ . Then  $\Gamma = \bigcup_{i=-\infty}^{+\infty} \Gamma_i$  and  $\bigcap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ , where  $(g, \gamma^{(0)}) = 1$  for all  $g \in G$  (here and below,  $(g, \gamma)$  denote the value of a character  $\gamma$  at a point  $g$ ). For each  $n \in \mathbb{Z}$  the group  $\Gamma_{-n}$  is the annihilator of  $G_n$ , i.e.,

$$\Gamma_{-n} = G_n^\perp := \{\gamma \in \Gamma : (g, \gamma) = 1 \text{ for all } g \in G_n\}.$$

The representation (2), the properties of a character and of the annihilator imply

$$(g, \gamma) = (g_{K_n}, \gamma) \cdot (\{g\}_n, \gamma) = (g_{K_n}, \gamma).$$

So, with a fixed element  $g_{K_n}$ , the value  $(g, \gamma)$  is constant for all  $g \in K_n$ .

The factor groups  $\Gamma_{-n-1}/\Gamma_{-n} = G_{n+1}^\perp/G_n^\perp$  and  $G_n/G_{n+1}$  are isomorphic (see [1]) and so they are of a finite order for each  $n \in \mathbb{Z}$ . This implies that the group  $\Gamma_{-n}/\Gamma_0$  is also finite for any  $n > 0$ , and  $\Gamma/\Gamma_0$  is countable.

Now, as we have done above for the group  $G$ , we choose and fix an element  $\gamma_J \in J$  for each coset  $J$  of  $\Gamma_0$ . Then, using multiplication as the group operation on the group  $\Gamma$ , we can represent any element  $\gamma \in \Gamma$  in the form:

$$\gamma = \gamma_J \cdot \{\gamma\}, \tag{4}$$

where  $\{\gamma\} \in \Gamma_0$ . We agree to put  $\gamma_{\Gamma_0} = \gamma^{(0)}$ , so that  $\gamma = \{\gamma\}$  if  $\gamma \in \Gamma_0$ .

We denote by  $\mu_G$  and  $\mu_\Gamma$  the Haar measures on the groups  $G$  and  $\Gamma$ , respectively, and normalize them so that  $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$ . We can make these measures complete by including all the subsets of the sets of measure zero into the respective class of measurable sets.

### 3. Results related to the compact case

We remind here some definitions and results given in the papers [5] and [6]. We consider derivation basis  $\mathcal{B}_G$  constituted by the family of basis sets

$$\beta_\nu = \{(I, g) : g \in G, I = K_n(g), n \geq \nu(g)\},$$

where  $\nu$  runs over the set of all integer-valued functions on  $G$ .

In the terminology of the derivation basis theory, any coset  $K_n$ ,  $n \in \mathbb{Z}$ , can be called  $\mathcal{B}_G$ -interval.

This basis has all the usual properties of a general derivation basis (see [2], [7]).

A  $\beta_\nu$ -partition is a finite collection  $\pi$  of elements of  $\beta_\nu$ , where the distinct elements  $(I', g')$  and  $(I'', g'')$  in  $\pi$  have  $I'$  and  $I''$  disjoint. If  $L$  is a  $\mathcal{B}_G$ -interval and  $\bigcup_{(I,g) \in \pi} I = L$ , then  $\pi$  is called  $\beta_\nu$ -partition of  $L$ .

For each  $\mathcal{B}_G$ -interval  $L$  and for any  $\beta_\nu \in \mathcal{B}_G$  there exists a  $\beta_\nu$ -partition of  $L$ .

The following Kurzweil-Henstock type integral was defined in [5]:

**DEFINITION 3.1.** Let  $L$  be a  $\mathcal{B}_G$ -interval. A complex-valued function  $f$  on  $L$  is said to be *Kurzweil-Henstock integrable with respect to basis  $\mathcal{B}_G$*  (or  *$H_G$ -integrable*) on  $L$ , with  $H_G$ -integral  $A$ , if for every  $\varepsilon > 0$ , there exists a function  $\nu : L \mapsto \mathbb{Z}$  such that for any  $\beta_\nu$ -partition  $\pi$  of  $L$  we have:

$$\left| \sum_{(I,g) \in \pi} f(g)\mu_G(I) - A \right| < \varepsilon.$$

We denote the integral value  $A$  by  $(H_G) \int_L f d\mu_G$ .

**Remark 3.1.** We note that all the above definitions depend on the structure of the sequence of subgroups (1). So, if we consider for the group  $\Gamma$  the definitions of the  $\mathcal{B}_\Gamma$ -basis and the  $H_\Gamma$ -integral, then we should use the sequence (3) in our construction.

**Remark 3.2.** It is easy to check that  $H_G$ -integral is invariant under translation given by some element  $g \in G$ .

We also need the following extension of Definition 3.1 to the case of functions defined only almost everywhere on  $L$ .

**DEFINITION 3.2.** A complex valued function  $f$  defined almost everywhere on a  $\mathcal{B}_G$ -interval  $L$  is said to be  $H_G$ -integrable on  $L$ , with integral value  $A$ , if the function

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is  $H_G$ -integrable on  $L$  to  $A$  in the sense of Definition 3.1.

It is clear that a complex-valued function is  $H_G$ -integrable if and only if both its real and imaginary parts are  $H_G$ -integrable.

If the group  $G$  is compact, as it is in [6], the chain (1) is reduced to the one-side sequence

$$G = G_0 \supset G_1 \supset G_2 \dots \supset G_n \supset \dots$$

In this case the  $H_G$ -integral is defined on the whole group  $G$ . Moreover, the group  $\Gamma$  of characters of the group  $G$  is discrete now (see [1]) and can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_{-1} \subset \Gamma_{-2} \subset \dots \subset \Gamma_{-n} \subset \dots,$$

where  $\Gamma_0 = \{\gamma^{(0)}\}$  with  $(g, \gamma^{(0)}) = 1$  for all  $g \in G$ .

Characters  $\gamma$  constitute a countable orthonormal system on  $G$  with respect to the normalized measure  $\mu_G$  (see [5]) and we can consider a series

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \tag{5}$$

with respect to this system. The convergence of this series at a point  $g$  is defined in [6] as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma \in \Gamma_{-n}} a_\gamma(g, \gamma) \tag{6}$$

when  $n$  tends to infinity.

With the series (5) we associate a function  $F$  defined on each coset  $K_n$  by

$$F(K_n) := \int_{K_n} S_n(g) d\mu_G. \tag{7}$$

The above integral can be understood in the Lebesgue sense.  $F$  is known to be an additive function on the family of all  $\mathcal{B}_G$ -intervals.

The next theorem is proved in [6].

**THEOREM 3.1.** *Suppose that the partial sums  $S_n(g)$  of a series  $\sum_{\gamma \in \Gamma} a_\gamma \gamma$  and a  $H_G$ -integrable function  $f$  satisfy the inequalities*

$$\liminf_{n \rightarrow \infty} \operatorname{Re} S_n(g) \leq \operatorname{Re} f(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Re} S_n(g), \quad (8)$$

$$\liminf_{n \rightarrow \infty} \operatorname{Im} S_n(g) \leq \operatorname{Im} f(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Im} S_n(g) \quad (9)$$

everywhere on  $G$  except on a countable set  $S$ , where

$$\liminf_{n \rightarrow \infty} \mu(K^n) \operatorname{Re} S_n(g) \leq 0 \leq \limsup_{n \rightarrow \infty} \mu(K^n) \operatorname{Re} S_n(g), \quad (10)$$

$$\liminf_{n \rightarrow \infty} \mu(K^n) \operatorname{Im} S_n(g) \leq 0 \leq \limsup_{n \rightarrow \infty} \mu(K^n) \operatorname{Im} S_n(g) \quad (11)$$

hold. Then the series  $\sum_{\gamma \in \Gamma} a_\gamma \gamma$  is convergent to  $f$  a.e. and it is the  $H_G$ -Fourier series of  $f$ .

Another version of the theorem on the recovering coefficients was proved in [5]:

**THEOREM 3.2.** *Suppose that the partial sums  $S_n(g)$  of a series  $\sum_{\gamma \in \Gamma} a_\gamma \gamma$  converge almost everywhere on  $G$  to a function  $f$  and satisfy the conditions*

$$-\infty < \liminf_{n \rightarrow \infty} \operatorname{Re} S_n(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Re} S_n(g) < +\infty, \quad (12)$$

$$-\infty < \liminf_{n \rightarrow \infty} \operatorname{Im} S_n(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Im} S_n(g) < +\infty \quad (13)$$

everywhere on  $G$  except on a countable set  $S$ , where

$$S_n(g) = o\left(\frac{1}{\mu_G(K_n(g))}\right) \quad (14)$$

holds. Then  $f$  is  $H_G$ -integrable in the sense of Definition 3.2 and the series  $\sum_{\gamma \in \Gamma} a_\gamma \gamma$  is the  $H_G$ -Fourier series of  $f$ .

We need also the following theorem (see [5]).

**THEOREM 3.3.** *The partial sums  $S_n(f, g)$  of the  $H_G$ -Fourier series of a  $H_G$ -integrable on  $G$  function  $f$  are convergent to  $f$  almost everywhere on  $G$ .*

## 4. Inversion formula for transform in the locally compact case

To simplify our notation, in this section we will put  $K = K_0$ ,  $[g] := g_K$ ,  $\{g\} := \{g\}_0$ , so that representation (2) with  $n = 0$  for any element  $g$  of some coset  $K$  of  $G_0$  can be rewritten in the form  $g = [g] + \{g\}$ , where  $[g]$  is a fixed element of  $K$  and  $\{g\} \in G_0$ . Similarly, we will sometimes use the notation

$[\gamma] := \gamma_J$  to underline duality, so the representation (4) for any element  $\gamma$  of some coset  $J$  of  $\Gamma_0$  can be rewritten in the form  $\gamma = [\gamma] \cdot \{\gamma\}$ , where  $[\gamma]$  is a fixed element of  $J$  and  $\{\gamma\} \in \Gamma_0$ .

Using this notation and the properties of a character  $\gamma$ , we can write

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot (\{g\}, \{\gamma\}) \cdot ([g], \{\gamma\}). \quad (15)$$

Now we observe that:

- 1)  $\{g\} \in G_0$  and  $\{\gamma\} \in \Gamma_0 = G_0^\perp$ . So  $(\{g\}, \{\gamma\}) = 1$ , and we can eliminate  $(\{g\}, \{\gamma\})$  from representation (15) getting

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}). \quad (16)$$

- 2)  $[\gamma] \in \Gamma_{-m(\gamma)} = G_{m(\gamma)}^\perp$  where  $m(\gamma) \geq 0$  and  $[\gamma]|_{G_0}$  is a character of the subgroup  $G_0$ .
- 3)  $([g], [\gamma])$  is constant if  $g$  belongs to a fixed coset of  $G_0$  and  $\gamma$  belongs to a fixed coset of  $\Gamma_0$ .
- 4) Using the duality between  $G$  and  $\Gamma$  we can state that  $g$  represents a character of  $\Gamma$  and, similarly to the property 2),  $[g]|_{\Gamma_0}$  is a character of  $\Gamma_0$ . So  $([g], \{\gamma\})$  is a value of this character at the point  $\{\gamma\}$ .

Therefore, according to (16), if  $g$  belongs to a fixed coset of  $G_0$  and  $\gamma$  belongs to a fixed coset of  $\Gamma_0$ , we can represent  $(g, \gamma)$ , up to a constant multiplier  $([g], [\gamma])$ , as a product of  $(\{g\}, [\gamma])$  considered as a value of the character  $[\gamma]$  at  $\{g\}$ , and  $([g], \{\gamma\})$  considered as a value of the character  $[g]$  at  $\{\gamma\}$ .

We now generalize Theorem 3.1 and Theorem 3.2 to a locally compact case. In what follows, all the integrals are either  $H_G$ - or  $H_\Gamma$ -integrals. The measure  $\mu_G$  or  $\mu_\Gamma$ , written under the integral sign, will indicate which of the integrals is used.

**THEOREM 4.1.** *Assume that  $G$  is the group described in Section 2,  $\Gamma$  being its dual group. Let  $a(\gamma)$  be a locally  $H_\Gamma$ -integrable function and let for some locally  $H_G$ -integrable function  $\phi$  the following inequalities hold:*

$$\liminf_{n \rightarrow \infty} \operatorname{Re} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \leq \operatorname{Re} \phi(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Re} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma, \quad (17)$$

$$\liminf_{n \rightarrow \infty} \operatorname{Im} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \leq \operatorname{Im} \phi(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Im} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \quad (18)$$

INVERSION FORMULAE FOR THE INTEGRAL TRANSFORM ON A GROUP

everywhere on  $G$  except on a countable set  $T$ , where we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mu_G(K_n) \operatorname{Re} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \\ & \leq 0 \leq \limsup_{n \rightarrow \infty} \mu_G(K_n) \operatorname{Re} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma, \end{aligned} \tag{19}$$

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mu_G(K_n) \operatorname{Im} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \\ & \leq 0 \leq \limsup_{n \rightarrow \infty} \mu_G(K_n) \operatorname{Im} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma. \end{aligned} \tag{20}$$

Then the function  $a(\gamma)$  can be recovered from  $\phi$  by the following inversion formula:

$$a(\gamma) = \lim_{n \rightarrow \infty} \int_{G_{-n}} \phi(g) \overline{(g, \gamma)} d\mu_G \quad \text{a.e. on } \Gamma.$$

A particular case of the previous theorem is the following

**COROLLARY 4.2.** *With the same assumptions on  $G$ ,  $\Gamma$  and  $\alpha(\gamma)$ , let*

$$\lim_{n \rightarrow \infty} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma = \phi(g)$$

*a.e. on  $G$ , where  $\phi$  is a locally  $H_G$ -integrable function on  $G$ . Moreover, everywhere on  $G$  except on a countable set  $T$ , we have*

$$\limsup_{n \rightarrow \infty} \left| \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \right| < +\infty$$

*and for  $g \in T$ , we have*

$$\lim_{n \rightarrow \infty} \mu(K_n(g)) \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu = 0.$$

Then the function  $a(\gamma)$  can be recovered from  $\phi$  by the following inversion formula:

$$a(\gamma) = \lim_{n \rightarrow \infty} \int_{G_{-n}} \phi(g) \overline{(g, \gamma)} d\mu_G \quad \text{a.e. on } \Gamma.$$

**THEOREM 4.3.** *Assume that  $G$  is a group described above,  $\Gamma$  being its dual group. Let  $a(\gamma)$  be a locally  $H_\Gamma$ -integrable function and*

$$\lim_{n \rightarrow \infty} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma = \phi(g) \quad (21)$$

*a.e. on  $G$ . Moreover, everywhere on  $G$  except on a countable set  $T$ , we have*

$$\limsup_{n \rightarrow \infty} \left| \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma \right| < +\infty \quad (22)$$

*and for  $g \in T$  we have*

$$\lim_{n \rightarrow \infty} \mu_G(K_n(g)) \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma = 0. \quad (23)$$

*Then  $\phi$  is locally  $H_G$ -integrable and the function  $a(\gamma)$  can be recovered from  $\phi$  by the following inversion formula:*

$$a(\gamma) = \lim_{n \rightarrow \infty} \int_{G_{-n}} \phi(g) \overline{(g, \gamma)} d\mu_G \quad \text{a.e. on } \Gamma. \quad (24)$$

**Proof of Theorem 4.1 and Theorem 4.3.** We sketch the proof following the lines of the proof of a similar result given in [5] indicating the step where we use different theorems on recovering the coefficients of series.

Having fixed a coset  $K$ , suppose that  $g \in K$ , and let  $J$  denote any coset of  $\Gamma_0$ . Then, by (16),

$$\begin{aligned} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) d\mu_\Gamma &= \sum_{J \subset \Gamma_{-n}} \int_J a(\gamma)(\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}) d\mu_\Gamma \\ &= \sum_{J \subset \Gamma_{-n}} (\{g\}, \gamma_J) \cdot \int_J a(\gamma)([g], [\gamma]) \cdot ([g], \{\gamma\}) d\mu_\Gamma. \end{aligned} \quad (25)$$

The latter sum can be considered as a partial sum

$$\sum_{J \subset \Gamma_{-n}} b_J^{(K)}(\{g\}, \gamma_J) \quad (26)$$

of the series with respect to the system of characters  $\{\gamma_J\}_J$ , at the point  $\{g\}$ , with the coefficients

$$b_J^{(K)} = \int_J a(\gamma)([g], [\gamma]) (g_K, \{\gamma\}) d\mu_\Gamma.$$

Now, we are going to apply Theorem 3.1 or Theorem 3.2 to this series.



INVERSION FORMULAE FOR THE INTEGRAL TRANSFORM ON A GROUP

In the case of Theorem 4.1, we obtain, according to the inequalities (17), (18), (19), (20) and the equality (25), that for the last series the following inequalities hold

$$\liminf_{n \rightarrow \infty} \operatorname{Re} S_n(g) \leq \operatorname{Re} \phi(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Re} S_n(g), \quad (27)$$

$$\liminf_{n \rightarrow \infty} \operatorname{Im} S_n(g) \leq \operatorname{Im} \phi(g) \leq \limsup_{n \rightarrow \infty} \operatorname{Im} S_n(g) \quad (28)$$

except on a countable set  $T$  where (10) and (11) hold.

We now introduce the variable  $t = \{g\} \in G_0$  and we can consider the above inequalities to hold on  $G_0$  for the function  $f(t) = \phi(g_K + t)$ . This means that the partial sums of our series and the function  $f$  satisfy on  $G_0$  all the conditions of Theorem 3.1. Applying this theorem we get that the coefficients  $b_J^{(K)}$  are the  $H_G$ -Fourier coefficients of  $f(t)$ , with respect to characters  $\gamma_J$ , i.e.,

$$\begin{aligned} b_J^{(K)} &= \int_J a(\gamma)([g], [\gamma])(g_K, \{g\}) d\mu_\Gamma \\ &= \int_{G_0} f(t) \overline{(\{g\}, \gamma_J)} d\mu_G \\ &= \int_K \phi(g) \overline{(\{g\}, \gamma_J)} d\mu_G \end{aligned} \quad (29)$$

(in the last equality we use Remark 3.2)

In the case of Theorem 4.3, we obtain, according to the assumption (21) and the equality (25) that partial sums (26) are convergent almost everywhere on  $K$  to a function  $\phi(g)$ .

Introducing the variable  $t = \{g\} \in G_0$  once again, we can consider this series to be convergent almost everywhere on  $G_0$  to the function  $f(t) = \phi(g_K + t)$ . The partial sums (26) are bounded, according to (22) and (25), except on a countable set  $S = \{t \in G_0 : g_K + t \in T\}$ , where (23) holds corresponding to the condition (14) applied to  $t \in S$ .

Therefore, by Theorem 3.2, the function  $f$  is  $H_G$ -integrable on  $G_0$ , and so,  $\phi$  is  $H_G$ -integrable on  $K$  and the coefficients  $b_J^{(K)}$  are the  $H_G$ -Fourier coefficients of  $f(t)$ , with respect to characters  $\gamma_J$  getting again the equality (29).

Now, we can finish the proof of both theorems. By observation 3),  $([g], [\gamma])$  is constant when  $g \in K$  and  $\gamma \in J$  with  $|( [g], [\gamma] )| = 1$ . Hence (29) implies

$$\int_J a(\gamma)(g_K, \{g\}) d\mu_\Gamma = \int_K \phi(g) \overline{([g], [\gamma]) (\{g\}, \gamma_J)} d\mu_G. \quad (30)$$

Now, we notice that for each fixed  $J$ , the value  $\int_J a(\gamma)(g_K, \{g\}) d\mu_\Gamma$  is the  $H_G$ -Fourier coefficient, with respect to the character  $\overline{g_K}$ , of the  $H_G$ -integrable

function  $a(\gamma) = a([\gamma] \cdot \{\gamma\})$  considered as a function of  $\{\gamma\} \in \Gamma_0$ . Applying Theorem 3.3 to this  $H_G$ -Fourier series, we get

$$\lim_{n \rightarrow \infty} \sum_{K \subset G_{-n}} \int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} = a([\gamma] \cdot \{\gamma\}) = a(\gamma)$$

for almost all values of  $\{\gamma\}$  on  $\Gamma_0$ , i.e., a.e. on  $J$ . Hence using (30) and then (16), we compute

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{K \subset G_{-n}} \int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} \\ &= \lim_{n \rightarrow \infty} \sum_{K \subset G_{-n}} \int_K \phi(g)([g], [\gamma]) (\{g\}, \gamma_J) d\mu_G \cdot \overline{(g_K, \{\gamma\})} \\ &= \lim_{n \rightarrow \infty} \int_{G_{-n}} \phi(g)(\{g\}, \gamma_J) \cdot \overline{(g_K, \{\gamma\})} \cdot ([g], [\gamma]) d\mu_G \\ &= \lim_{n \rightarrow \infty} \int_{G_{-n}} \phi(g) \overline{(g, \gamma)} d\mu_G = a(\gamma) \quad \text{a.e. on } J. \end{aligned}$$

The last equality is true for any  $J$ , so we get (24), completing the proof.  $\square$

We remark that Theorems 4.1 is not true if we use Denjoy-Khintchine integrable function  $\phi$  (see [3] for the compact case), but it becomes true with this last integral if we put some additional hypothesis on the group and on the type of convergence (see [8]).

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INVERSION FORMULAE FOR THE INTEGRAL TRANSFORM ON A GROUP

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