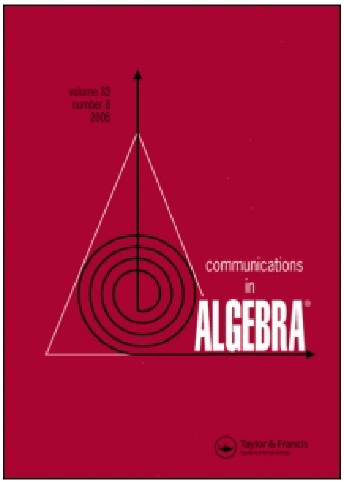

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Characterizing Varieties of Colength 4

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CHARACTERIZING VARIETIES OF COLENGTH ≤4

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Let A be an associative algebra over a field F of characteristic zero, and let $\chi_n(A)$, $n = 1, 2, \ldots$, be the sequence of cocharacters of A. For every $n \geq 1$, let $l_n(A)$ denote the nth colength of A, counting the number of S_n-irreducibles appearing in $\chi_n(A)$. In this article, we classify the algebras A such that the sequence of colengths $l_n(A)$, $n = 1, 2, \ldots$, is bounded by four. Moreover we construct a finite number of algebras $A_1, \ldots, A_d$, such that $l_n(A) \leq 4$ if and only if $A_1, \ldots, A_d \notin var(A)$.

Key Words: Codimensions; Colengths; Polynomial identity; Variety.

2000 Mathematics Subject Classification: 16R10.

1. INTRODUCTION

Let A be an associative algebra over a field F of characteristic zero, $F\langle X\rangle$ the free associative algebra on a countable set X over F, and $\mathrm{Id}(A) \subseteq F\langle X\rangle$ the T-ideal of polynomial identities of A. An effective way of studying such an ideal is that of determining some numerical invariants allowing to give a quantitative description. A very useful numerical invariant that can be attached to $\mathrm{Id}(A)$ is given by the sequence of codimensions of A denoted by $c_n(A)$, $n = 1, 2, \ldots$. In general $c_n(A)$ is bounded from above by $n!$, but in case A is a PI-algebra, i.e., satisfies a nontrivial polynomial identity, a celebrated theorem of Regev asserts that $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded (Regev, 1972). Later Kemer (1978, 1979) showed that, given any PI-algebra A, $c_n(A)$, $n = 1, 2, \ldots$, cannot have intermediate growth, i.e., either is polynomially bounded or grows exponentially. For general PI-algebras the exponential rate of growth was computed in Giambruno and Zaicev (1998, 1999) and it turns out to be a non-negative integer.

In case the codimensions are polynomially bounded, Kemer (1979) gave the following characterization. Let G be the infinite dimensional Grassmann algebra over F, and let UT_2 be the algebra of 2×2 upper triangular matrices. Then $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $G, UT_2 \notin \mathrm{var}(A)$, where $\mathrm{var}(A)$ denotes the variety of algebras generated by A.

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Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially. Recently in La Mattina (2007) the author determined a complete list of finite dimensional algebras generating the subvarieties of $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$.

A classification of T-ideals of polynomial growth was started in Giambruno and La Mattina (2005) and in Giambruno et al. (2007). More precisely the authors gave a complete list of finite dimensional algebras generating varieties of at most linear growth and, in the unitary case, of at most cubic growth.

An equivalent formulation of Kemer's result can be given as follows. Let V_n be the vector space of multilinear polynomials in n variables. The permutation action of S_n on the space V_n induces a structure of S_n-module on $\frac{V_n}{V_n \cap \mathrm{Id}(A)}$, and let $\chi_n(A)$ be its character. By complete reducibility, we can write $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ where χ_λ is the irreducible S_n-character associated to the partition λ of n and $m_\lambda \geq 0$ is the corresponding multiplicity. Then $l_n(A) = \sum_{\lambda \vdash n} m_\lambda$ is the nth colength of A. Now Kemer's result can be stated as follows (Mishchenko et al., 1999): $c_n(A)$ is polynomially bounded if and only if the sequence of colengths is bounded by a constant, i.e., $l_n(A) \leq k$, for some $k \geq 0$ and for all $n \geq 1$.

A finer classification depending on the value of the constant k was started in Giambruno and La Mattina (2005). There the authors completely classified, up to PI-equivalence, the algebras A such that $l_n(A) \leq 2$ for n large enough.

In this article, we classify the algebras A such that $l_n(A) \leq 4$. Moreover, we show that if $l_n(A) \leq 4$, then for n large enough, $l_n(A)$ is always constant. Moreover, we exhibit a finite number of finite dimensional algebras $A_1, \ldots, A_d$, such that $l_n(A) \leq 4$ if and only if $A_1, \ldots, A_d \notin \mathrm{var}(A)$.

2. GENERALITIES

Throughout this article, we shall denote by F a field of characteristic zero, by A an associative algebra over F and by $\mathrm{var}(A)$ the variety of algebras generated by A.

Let $F\langle X\rangle$ be the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over F and $\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ the T-ideal of $F\langle X\rangle$ of polynomial identities of A.

It is well known that in characteristic zero $\mathrm{Id}(A)$ is completely determined by its multilinear polynomials. We denote by V_n the vector space of multilinear polynomials in the variables $x_1, \ldots, x_n$ and by $c_n(A) = \dim_F \frac{V_n}{V_n \cap \mathrm{Id}(A)}$ the nth codimension of A.

In case A is an algebra with 1, $\mathrm{Id}(A)$ is completely determined by its multilinear proper polynomials (Drensky, 2000). Recall that $f(x_1, \ldots, x_n) \in V_n$ is a proper polynomial if it is a linear combination of products of (long) Lie commutators.

Let Γ_n be the subspace of V_n of proper polynomials in $x_1, \ldots, x_n$. Then, the sequence of proper codimensions is defined as $c_n^p(A) = \dim \frac{\Gamma_n}{\Gamma_n \cap \mathrm{Id}(A)}$, $n = 0, 1, 2, \ldots$.

For a unitary algebra A, the relation between ordinary codimensions and proper codimensions (see for instance Drensky and Regev, 1996), is given by the formula

$$c_n(A) = \sum_{i=0}^{n} \binom{n}{i} c_i^p(A), \qquad n = 1, 2, \ldots. \tag{1}$$

One of the main tools in the study of the T-ideals is given by the representation theory of the symmetric group. Recall that the symmetric group S_n acts on the left on the space V_n by permuting the variables: if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in V_n$, $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This action is very useful since T-ideals are invariant under renaming of the variables. Hence $\frac{V_n}{V_n \cap \mathrm{Id}(A)}$ becomes an S_n-module. The S_n-character of $V_n(A) = \frac{V_n}{V_n \cap \mathrm{Id}(A)}$, denoted by $\chi_n(A)$, is called the nth *cocharacter of* A.

By complete reducibility, we can write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition λ and m_λ is the corresponding multiplicity. Also

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the nth *colength of* A. Sometimes we shall also write $m_\lambda = m_\lambda(A)$.

In the next section, we shall use also the representation theory of the general linear group in order to study the sequences of cocharacters and colengths of some algebras. For this reason, we introduce the space of homogeneous polynomials in a given set of variables. Let $F_m\langle X\rangle = F\langle x_1, \ldots, x_m\rangle$ denote the free associative algebra in m variables and let $U = \mathrm{span}_F\{x_1, \ldots, x_m\}$. The group $GL(U) \cong GL_m$ acts naturally on the left on the space U and we can extend this action diagonally to get an action on $F_m\langle X\rangle$.

The space $F_m\langle X\rangle \cap \mathrm{Id}(A)$ is invariant under this action, hence

$$F_m(A) = \frac{F_m\langle X\rangle}{F_m\langle X\rangle \cap Id(A)}$$

inherits a structure of left GL_m-module. If F_m^n denotes the space of homogeneous polynomials of degree n in the variables $x_1, \ldots, x_m$

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap \mathrm{Id}(A)}$$

is a GL_m-submodule of $F_m(A)$ whose character is denoted by $\psi_n(A)$. Write

$$\psi_n(A) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda,$$

where ψ_λ is the irreducible GL_m-character associated to the partition λ and $\bar{m}_\lambda$ is the corresponding multiplicity.

The S_n-module structure of $V_n/(V_n \cap \mathrm{Id}(A))$ and the GL_m-module structure of $F_m^n(A)$ are related by the following: if $\chi_n(A) = \sum m_\lambda \chi_\lambda$ is the decomposition of the nth cocharacter of A, then $m_\lambda = \bar{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most m (see for instance Drensky, 2000).

It is also well known that any irreducible submodule of $F_m^n(A)$ corresponding to λ is generated by a nonzero polynomial f_λ, called *highest weight vector*, of the form

$$f_\lambda = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1, \ldots, x_{h_i(\lambda)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \tag{2}$$

where $\alpha_\sigma \in F$, the right action of S_n on $F_m^n(A)$ is defined by place permutation, $h_i(\lambda)$ is the height of the ith column of the diagram of λ and $St_r(x_1, \ldots, x_r) = \sum_{\tau \in S_r} (\operatorname{sgn} \tau) x_{\tau(1)} \cdots x_{\tau(r)}$ is the standard polynomial of degree r.

For a Young tableau T_λ, denote by f_{T_λ} the highest weight vector obtained from (2) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \ldots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\sigma(h_1(\lambda)+1), \ldots, \sigma(h_1(\lambda)+h_2(\lambda))$ the second column of T_λ, etc.

Recall that if

$$\psi_n(A) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

is the GL_m-character of $F_m^n(A)$, then $\bar{m}_\lambda \neq 0$ if and only if there exists a tableau T_λ such that the corresponding highest weight vector f_{T_λ} is not a polynomial identity for A. Moreover, $\bar{m}_\lambda$ is equal to the maximal number of linearly independent highest weight vectors f_{T_λ} in $F_m^n(A)$.

3. SOME PI-ALGEBRAS

The purpose of this section is to state some results concerning the colengths, the cocharacters, the codimensions, and the T-ideals of some PI-algebras that will play a basic role in the next section.

Most of the algebras treated here are direct sums of subalgebras of the algebra of $n \times n$ upper triangular matrices $UT_n = UT_n(F)$, $n \leq 4$.

Given $A \subseteq UT_n$, we shall denote by A^* the subalgebra of UT_n obtained by flipping A along its secondary diagonal.

Notice that given a polynomial $f \in F\langle X \rangle$ if we denote by f^* the polynomial obtained by reversing the order of the variables in each monomial of f, then f is a polynomial identity of A if and only if f^* is a polynomial identity of A^*.

Given polynomials $f_1, \ldots, f_n \in F\langle X \rangle$, let us denote by $\langle f_1, \ldots, f_n \rangle_T$ the T-ideal generated by $f_1, \ldots, f_n$. Also, we shall use the left-normed notation for Lie commutators; hence we write $[[\cdots[[x_1, x_2], x_3], \ldots], x_k] = [x_1, \ldots, x_k]$.

In what follows we shall use the following two lemmas from Giambruno and La Mattina (2005) which we fix in our notation. Here the e_{ij}'s denote the usual matrix units.

Lemma 1. *Let $A_1 = Fe_{11} + Fe_{12}$, $A_1^* = Fe_{12} + Fe_{22} \subseteq UT_2$, and $A = A_1 \oplus A_1^*$. Then for all $n > 2$:*

(i) $\mathrm{Id}(A_1) = \langle [x_1, x_2]x_3 \rangle_T$, $\mathrm{Id}(A_1^*) = \langle x_3[x_1, x_2] \rangle_T$, *and* $\mathrm{Id}(A) = \langle St_3(x_1, x_2, x_3), x_1[x_2, x_3]x_4, [x_1, x_2][x_3, x_4] \rangle_T$;
(ii) $\chi_n(A_1) = \chi_n(A_1^*) = \chi_{(n)} + \chi_{(n-1,1)}$ *and* $\chi_n(A) = \chi_{(n)} + 2\chi_{(n-1,1)}$.

Hence

(iii) $l_n(A_1) = l_n(A_1^*) = 2$, $l_n(A) = 3$ *and* $c_n(A_1) = c_n(A_1^*) = n$, $c_n(A) = 2n - 1$.

Lemma 2. *Let $A_2 = F(e_{11} + e_{22} + e_{33}) + Fe_{12} + Fe_{13} + Fe_{23} \subseteq UT_3$. Then for all $n > 3$:*

(i) $\mathrm{Id}(A_2) = \langle [x_1, x_2, x_3], [x_1, x_2][x_3, x_4] \rangle_T$;
(ii) $\chi_n(A_2) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)}$.

Hence

(iii) $l_n(A_2) = 3$ *and* $c_n(A_2) = \frac{n(n-1)+2}{2}$.

Lemma 3. *Let $A = A_1 \oplus A_2$ and $B = A_1^* \oplus A_2$. Then for all $n > 3$:*

1. $\mathrm{Id}(A) = \langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3]x_4 \rangle_T$, *and* $\mathrm{Id}(B) = \langle [x_1, x_2][x_3, x_4], x_4[x_1, x_2, x_3] \rangle_T$;
2. $\chi_n(A) = \chi_n(B) = \chi_{(n)} + 2\chi_{(n-1,1)} + \chi_{(n-2,1,1)}$;
3. $l_n(A) = l_n(B) = 4$;
4. $c_n(A) = c_n(B) = \frac{n(n+1)}{2}$.

Proof. By La Mattina (2007, Lemma 3.2), the statements 1 and 4 hold. We now determine the decomposition of the nth cocharacter of A. A similar proof will give the decomposition of the nth cocharacter of B. Since $A_1, A_2 \in \mathrm{var}(A)$, and $A = A_1 \oplus A_2$, we have that

$$m_\lambda(A_1), m_\lambda(A_2) \leq m_\lambda(A) \leq m_\lambda(A_1) + m_\lambda(A_2),$$

for any $\lambda \vdash n$ with $n > 3$. Hence, by the previous lemmas, $m_{(n)}(A) = 1$, $1 \leq m_{(n-1,1)}(A) \leq 2$, and $m_{(n-2,1,1)}(A) = 1$.

Since $\deg \chi_{(n)} + 2\deg \chi_{(n-1,1)} + \deg \chi_{(n-2,1,1)} = 1 + 2(n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n+1)}{2} = c_n(A)$, it follows that $m_{(n-1,1)}(A) = 2$. Thus the nth cocharacter of A has the wished decomposition and $l_n(A) = 4$. □

We shall see in the next section that the above algebras allow us to classify completely the varieties of colength ≤ 4.

Now consider the direct sum of A_1, A_1^*, and A_2.

Lemma 4. *Let $A_3 = A_1 \oplus A_1^* \oplus A_2$. Then for all $n > 3$:*

1. $\mathrm{Id}(A_3) = \langle [x_1, x_2][x_3, x_4], x_1[x_2, x_3, x_4]x_5 \rangle_T$;
2. $\chi_n(A_3) = \chi_{(n)} + 3\chi_{(n-1,1)} + \chi_{(n-2,1,1)}$;

3. $l_n(A_3) = 5$;
4. $c_n(A_3) = \frac{n^2+3n-2}{2}$.

Proof. By La Mattina (2007, Lemma 3.4), $\mathrm{Id}(A_3) = \langle [x_1, x_2][x_3, x_4], x_1[x_2, x_3, x_4]x_5\rangle_T$. In order to determine the decomposition of the nth cocharacter of A_3, we proceed as in the proof of the previous lemma. Since $A_1, A_1^* \oplus A_2 \in \mathrm{var}(A_3)$, we have that, for any $\lambda \vdash n$, $n > 3$,

$$m_\lambda(A_1), m_\lambda(A_1^* \oplus A_2) \leq m_\lambda(A_3) \leq m_\lambda(A_1) + m_\lambda(A_1^* \oplus A_2).$$

Hence, $m_{(n)}(A_3) = 1$, $2 \leq m_{(n-1,1)}(A_3) \leq 3$, and $m_{(n-2,1,1)}(A_3) = 1$.

If $m_{(n-1,1)}(A_3) = 2$, then $V_n(A_3) = \frac{V_n}{V_n \cap Id(A_3)}$ would have the same decomposition in irreducibles as $V_n(A_1^* \oplus A_2)$ and so, since $\mathrm{Id}(A_3) \subseteq \mathrm{Id}(A_1^* \oplus A_2)$, it would follow $\mathrm{Id}(A_3) = \mathrm{Id}(A_1^* \oplus A_2)$. This is a contradiction, because $x_4[x_1, x_2, x_3] \in \mathrm{Id}(A_1^* \oplus A_2)$ but $x_4[x_1, x_2, x_3] \notin \mathrm{Id}(A_3)$. This proves that

$$\chi_n(A_3) = \chi_{(n)} + 3\chi_{(n-1,1)} + \chi_{(n-2,1,1)}.$$

Hence $c_n(A_3) = \sum_{\lambda \vdash n} m_\lambda \deg \chi_\lambda = \frac{n^2+3n-2}{2}$ and $l_n(A_3) = \sum_{\lambda \vdash n} m_\lambda = 5$. □

In the following two lemmas, we fix some results about some more algebras whose colengths are equal to 5.

Lemma 5 (Giambruno and La Mattina, 2005, Lemma 6). *Let* $A_4 = Fe_{11} + Fe_{12} + Fe_{13} + Fe_{23}$, $A_5 = Fe_{22} + Fe_{12} + Fe_{13} + Fe_{23} \subseteq UT_3$.

Then for all $n > 3$:

1. $\mathrm{Id}(A_4) = \langle [x_1, x_2]x_3x_4\rangle_T$ *and* $\mathrm{Id}(A_5) = \langle x_1[x_2, x_3]x_4\rangle_T$;
2. $\chi_n(A_4) = \chi_n(A_5) = \chi_{(n)} + 2\chi_{(n-1,1)} + \chi_{(n-2,2)} + \chi_{(n-2,1,1)}$;
3. $l_n(A_4) = l_n(A_5) = 5$ *and* $c_n(A_4) = c_n(A_5) = n(n-1)$.

Hence $A_4^* = Fe_{33} + Fe_{12} + Fe_{13} + Fe_{23}$, $\mathrm{Id}(A_4^*) = \langle x_1x_2[x_3, x_4]\rangle_T$, *and* $l_n(A_4^*) = 5$.

Lemma 6 (Vieira and Alves Jorge, 2006, Theorem 3.1). *Let* $A_6 = F(e_{11} + e_{33}) + Fe_{12} + Fe_{13} + Fe_{23} \subseteq UT_3$. *Then for all* $n > 3$:

(i) $\chi_n(A_6) = \chi_{(n)} + 2\chi_{(n-1,1)} + \chi_{(n-2,2)} + \chi_{(n-2,1,1)}$.

Hence $l_n(A_6) = 5$ *and* $c_n(A_6) = n(n-1)$.

Recall that the above subalgebras of UT_n were introduced in Giambruno and La Mattina (2005) in order to classify the algebras with linear codimension growth.

Now in order to classify the algebras with colength sequence bounded from above by 4, we have to consider some more algebras.

Lemma 7. *Let* $A_7 = F(e_{11} + e_{22} + e_{33} + e_{44}) + Fe_{12} + Fe_{13} + Fe_{14} + Fe_{23} + Fe_{24} + Fe_{34} \subseteq UT_4$. *Then for all* $n > 4$:

1. $\mathrm{Id}(A_7) = \langle [x_1, x_2, x_3, x_4], [x_1, x_2][x_3, x_4]\rangle_T$;
2. $\chi_n(A_7) = \chi_{(n)} + 2\chi_{(n-1,1)} + 2\chi_{(n-2,1,1)} + \chi_{(n-2,2)} + \chi_{(n-3,2,1)}$;

3. $l_n(A_7) = 7$;
4. $c_n(A_7) = 1 + \binom{n}{2} + 2\binom{n}{3}$.

Proof. The Properties 1 and 4 follow from Giambruno et al. (2007, Theorem 3.1).

We next determine the decomposition of the nth cocharacter of A_7. Since $c_n(A_7)$ is polynomially bounded and $J(A_7)^4 = 0$, where $J(A_7)$ denotes the Jacobson radical of A_7, by Giambruno and Zaicev (2000, Theorem 3) we have that

$$\chi_n(A_7) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < 4}} m_\lambda \chi_\lambda.$$

Moreover, since $A_2 \in \mathrm{var}(A_7)$, $\chi_n(A_2) \subseteq \chi_n(A_7)$ and, so, $m_{(n)}(A_7)$, $m_{(n-1,1)}(A_7)$, $m_{(n-2,1,1)}(A_7) > 0$. Hence

$$\chi_n(A_7) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)} + \cdots .$$

Let $\lambda = (n-1, 1)$, and denote by $T_\lambda^{(2)}$ and $T_\lambda^{(n)}$ the standard tableaux containing the integers 2 and n, respectively, in the only box of the second row. Then

$$f_{T_\lambda^{(2)}} = [x_1, x_2]x_1^{n-2} \qquad \text{and} \qquad f_{T_\lambda^{(n)}} = x_1^{n-1}x_2 - x_2x_1^{n-1}$$

are the corresponding highest weight vectors. By making the evaluation $x_1 = I + e_{12} + e_{34}$, where I denotes the identity matrix, and $x_2 = e_{23}$, we get that

$$f_{T_\lambda^{(2)}} = e_{13} - e_{24} + (n-2)e_{14} \qquad \text{and} \qquad f_{T_\lambda^{(n)}} = (n-1)(e_{13} - e_{24}).$$

This says that $f_{T_\lambda^{(2)}}$ and $f_{T_\lambda^{(n)}}$ are not identities of A_7, and they are linearly independent (mod $\mathrm{Id}(A_7)$). Thus $m_{(n-1,1)} \geq 2$.

Now consider $f_{T_\lambda} = \sum_{\sigma,\tau \in S_2} (\mathrm{sgn}\, \sigma\tau) x_{\sigma(1)}x_{\tau(1)}x_{\sigma(2)}x_{\tau(2)}x_1^{n-4}$ the highest weight vector corresponding to the standard tableau

1	2	5	$\cdots$	n
3	4			

.

By making the evaluation $x_1 = I + e_{12}$ and $x_2 = e_{23} + e_{34}$, we get that $f_{T_\lambda} = e_{14}$ and, so, $m_{(n-2,2)} > 0$.

For $\lambda = (n-3, 2, 1)$ consider the following tableau

2	1	6	$\cdots$	n
3	5			
4				

and the corresponding highest weight vector

$$f_{T_\lambda} = \sum_{\sigma \in S_2, \tau \in S_3} (\mathrm{sgn}\, \sigma\tau) x_{\sigma(1)}x_{\tau(1)}x_{\tau(2)}x_{\tau(3)}x_{\sigma(2)}x_1^{n-5}.$$

Evaluating $x_1 = I$, $x_2 = e_{12} + e_{23}$ and $x_3 = e_{34}$, we get $f_{T_\lambda} = -e_{14}$, i.e, f_{T_λ} is not an identity of A_7 and, so, $m_{(n-3,2,1)} > 0$. Hence,

$$\begin{aligned} c_n(A_7) &= 1 + \binom{n}{2} + \binom{n}{3} 2 \\ &\geq \deg \chi_{(n)} + 2\deg \chi_{(n-1,1)} + \deg \chi_{(n-2,1,1)} + \deg \chi_{(n-2,2)} + \deg \chi_{(n-3,2,1)} \\ &= 1 + 2(n-1) + \frac{(n-1)(n-2)}{2} + \frac{n(n-3)}{2} + \frac{n(n-2)(n-4)}{3} \\ &= 1 + \binom{n}{2} + 2\binom{n}{3} - \frac{(n-1)(n-2)}{2} = c_n(A_7) - \deg \chi_{(n-2,1,1)}. \end{aligned}$$

Hence, since the multiplicities m_λ in $\chi_n(A_7)$ are bounded by a constant (see Mishchenko et al., 1999), it follows that $m_{(n-2,1,1)} = 2$, $\chi_n(A_7)$ has the wished decomposition and $l_n(A_7) = 7$. □

In the sequel, we shall also use the following notation.

Definition 8. Given two algebras A and B we say that A is PI-equivalent to B and we write $A \sim_{PI} B$ if $\mathrm{Id}(A) = \mathrm{Id}(B)$.

Lemma 9. *Let* $A_8 = F(e_{11} + e_{22} + e_{33}) + Fe_{12} + Fe_{13} + Fe_{14} + Fe_{23} + Fe_{24} + Fe_{34} \subseteq UT_4$. *Then for all* $n > 4$:

1. $\mathrm{Id}(A_8) = \langle [x_1, x_2][x_3, x_4]x_5, [x_1, x_2, x_3]x_4 \rangle_T$;
2. $\chi_n(A_8) = \chi_{(n)} + 2\chi_{(n-1,1)} + 2\chi_{(n-2,1,1)} + \chi_{(n-2,2)} + \chi_{(n-3,2,1)} + \chi_{(n-3,1,1,1)}$;
3. $l_n(A_8) = 8$;
4. $c_n(A_8) = n + \binom{n}{2}(n-2) = \frac{n^3-3n^2+4n}{2}$;
5. $\{x_1 \cdots x_n, x_{i_1} \cdots x_{i_{n-2}}[x_n, x_t],\ x_{i_1} \cdots x_{i_{n-3}}[x_i, x_j]x_k,\ i > j, i_1 < \cdots < i_{n-2}\}$ *is a basis of* V_n *(mod* $V_n \cap \mathrm{Id}(A_8)$*).*

Hence $A_8^* = F(e_{22} + e_{33} + e_{44}) + Fe_{12} + Fe_{13} + Fe_{14} + Fe_{23} + Fe_{24} + Fe_{34}$, $\mathrm{Id}(A_8^*) = \langle x_5[x_1, x_2][x_3, x_4], x_4[x_1, x_2, x_3] \rangle$, *and* $l_n(A_8^*) = 8$.

Proof. Notice that $A_8 \sim_{PI} \begin{pmatrix} A_2 & A_2 \\ 0 & 0 \end{pmatrix}$. Hence, by Guterman and Regev (2000), $\mathrm{Id}(A_8) = \langle \mathrm{Id}(A_2)x \rangle_T = \langle [x_1, x_2][x_3, x_4]x_5, [x_1, x_2, x_3]x_4 \rangle_T$ and $c_n(A_8) = nc_{n-1}(A_2) = \frac{n(n-1)(n-2)+2n}{2} = \frac{n^3-3n^2+4n}{2}$.

We are going to prove that the following polynomials in V_n

$$x_1 \cdots x_n, x_{i_1} \cdots x_{i_{n-2}}[x_n, x_t], \qquad x_{i_1} \cdots x_{i_{n-3}}[x_i, x_j]x_k, \quad i > j, \ i_1 < \cdots < i_{n-2}, \tag{3}$$

are linearly independent module $\mathrm{Id}(A_8)$.

Let $f \in \mathrm{Id}(A_8)$ be a linear combination of the elements in (3):

$$f = \alpha x_1 \cdots x_n + \sum_{\substack{t=1 \\ i_1 < \cdots < i_{n-2}}}^{n-1} \alpha_t x_{i_1} \cdots x_{i_{n-2}}[x_n, x_t] + \sum_{\substack{i>j \\ i_1 < \cdots < i_{n-3}}} \alpha_{ijk} x_{i_1} \cdots x_{i_{n-3}}[x_i, x_j]x_k.$$

By making the evaluation $x_1 = \cdots = x_n = e_{11} + e_{22} + e_{33}$, we get $\alpha = 0$. Also, for fixed t the evaluation $x_t = e_{34}$, $x_k = e_{11} + e_{22} + e_{33}$, $k \neq t$ gives $\alpha_t = 0$. Finally, for fixed i, j, k the evaluation $x_i = e_{12}$, $x_j = e_{23}$ $x_k = e_{34}$, $x_l = e_{11} + e_{22} + e_{33}$, $l \notin \{i, j, k\}$ gives $\alpha_{ijk} = 0$. Therefore, the element in (3) is linearly independent module $V_n \cap \mathrm{Id}(A_8)$ and, since their number equals the nth codimension of A_8, we may conclude that they are a basis of V_n (mod $V_n \cap \mathrm{Id}(A_8)$).

Next we determine the decomposition of the nth cocharacter of A_8. Since $A_2 \in \mathrm{var}(A_8)$, then $\chi_n(A_2) \subseteq \chi_n(A_8)$. Hence

$$\chi_n(A_8) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)} + \cdots .$$

Let $\lambda = (n-1, 1)$. Consider the following highest weight vectors

$$f_{T'_\lambda} = [x_1, x_2]x_1^{n-2} \qquad \text{and} \qquad f_{T_\lambda} = x_1^{n-2}[x_1, x_2].$$

By making the evaluation $x_1 = e_{11} + e_{22} + e_{33} + e_{12}$ and $x_2 = e_{23} + e_{34}$, we get that

$$f_{T'_\lambda} = e_{13} \qquad \text{and} \qquad f_{T_\lambda} = e_{13} + e_{34}.$$

This says that $f_{T'_\lambda}$ and f_{T_λ} are not identities of A_8, and they are linearly independent (mod $\mathrm{Id}(A_8)$), and so $m_{(n-1,1)} \geq 2$.

Now consider the tableau

$$\begin{array}{|c|c|c|c|c|}\hline n-3 & n-2 & 1 & \cdots & n-4 \\ \hline n-1 & n \\ \cline{1-2}\end{array}$$

and the corresponding highest weight vector

$$f_{T_\lambda} = \sum_{\sigma,\tau \in S_2} (\mathrm{sgn}\, \sigma\tau) x_1^{n-4} x_{\sigma(1)} x_{\tau(1)} x_{\sigma(2)} x_{\tau(2)}.$$

By making the evaluation $x_1 = e_{11} + e_{22} + e_{33} + e_{12}$ and $x_2 = e_{23} + e_{34}$, we get that $f_{T_\lambda} = 2e_{14}$ and so $m_{(n-2,2)} > 0$.

For $\lambda = (n-3, 2, 1)$, consider the following tableau

$$\begin{array}{|c|c|c|c|c|}\hline n-3 & n-4 & 1 & \cdots & n-5 \\ \hline n-2 & n \\ \cline{1-2} n-1 \\ \cline{1-1}\end{array}$$

and the corresponding highest weight vector

$$f_{T_\lambda} = \sum_{\sigma \in S_2, \tau \in S_3} (\mathrm{sgn}\, \sigma\tau) x_1^{n-5} x_{\sigma(1)} x_{\tau(1)} x_{\tau(2)} x_{\tau(3)} x_{\sigma(2)}.$$

Evaluating $x_1 = e_{11} + e_{22} + e_{33}$, $x_2 = e_{12} + e_{34}$ and $x_3 = e_{23}$, we get $f_{T_\lambda} = e_{14}$, i.e, f_{T_λ} is not an identity of A_8, and, so, $m_{(n-3,2,1)} > 0$.

For $\lambda = (n-3, 1, 1, 1)$ consider the following tableau

$$\begin{array}{|c|c|c|c|}\hline n-3 & 1 & \cdots & n-4 \\ \hline n-2 \\ \cline{1-1} n-1 \\ \cline{1-1} n \\ \cline{1-1}\end{array}$$

and the corresponding highest weight vector

$$f_{T_\lambda} = \sum_{\sigma \in S_4} (\operatorname{sgn}\sigma) x_1^{n-4} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}.$$

Evaluating $x_1 = e_{11} + e_{22} + e_{33}$, $x_2 = e_{12}$, $x_3 = e_{23}$, and $x_4 = e_{34}$, we get $f_{T_\lambda} = e_{14}$, i.e, f_{T_λ} is not an identity of A_8, and, so, $m_{(n-3,1,1,1)} > 0$. Hence $c_n(A_8) \geq \deg \chi_{(n)} + 2\deg \chi_{(n-1,1)} + \deg \chi_{(n-2,1,1)} + \deg \chi_{(n-2,2)} + \deg \chi_{(n-3,2,1)} + \deg \chi_{(n-3,1,1,1)} = 1 + 2(n-1) + \frac{(n-1)(n-2)}{2} + \frac{n(n-3)}{2} + \frac{n(n-2)(n-4)}{3} + \frac{(n-1)(n-2)(n-3)}{6} = c_n(A_8) - \deg \chi_{(n-2,1,1)}$. Hence $m_{(n-2,1,1)} = 2$, $\chi_n(A_8)$ has the wished decomposition and $l_n(A_8) = 8$. $\square$

Finally, let $G_{2k} \subseteq G$ denote the Grassmann algebra with 1 on a $2k$-dimensional vector space over F. Recall that $G_{2k} = \langle 1, e_1, \ldots, e_{2k} \mid e_i e_j = -e_j e_i \rangle$.

Notice that $G_2 \sim_{PI} A_2$.

In the following lemma, we determine the cocharacters, the codimensions and the T-ideal of G_{2k} in case $k = 2$. We refer the reader to Giambruno et al. (2007) and La Mattina (2007) for the properties of G_{2k}, for $k \geq 1$.

Lemma 10. *For the algebra G_4, the following holds*:

1. $\mathrm{Id}(G_4) = \langle [x_1, x_2, x_3], [x_1, x_2][x_3, x_4][x_5, x_6] \rangle_T$;
2. $\chi_n(G_4) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)} + \chi_{(n-3,1,1,1)} + \chi_{(n-4,1,1,1,1)}$;
3. $l_n(G_4) = 5$;
4. $c_n(G_4) = \sum_{j=0}^{2} \binom{n}{2j}$.

Proof. The Properties 1 and 4 follow from Giambruno et al. (2007, Theorem 3.5). Since $G_4 \subseteq G$, the multiplicities in the cocharacter of G_4 are bounded by the corresponding multiplicities in the cocharacter of G. By Giambruno and Zaicev (2005),

$$\chi_n(G) = \sum_{i=0}^{n-1} \chi_{(n-i,1^i)}.$$

Hence, since $J(G_4)^5 = 0$, by Giambruno and Zaicev (2000, Theorem 3), we obtain that

$$\chi_n(G_4) = m_1 \chi_{(n)} + m_2 \chi_{(n-1,1)} + m_3 \chi_{(n-2,1,1)} + m_4 \chi_{(n-3,1,1,1)} + m_5 \chi_{(n-4,1,1,1,1)}$$

and $m_i \leq 1$. Hence

$$\begin{aligned} c_n(G_4) &\leq \deg \chi_{(n)} + \deg \chi_{(n-1,1)} + \deg \chi_{(n-2,1,1)} + \deg \chi_{(n-3,1,1,1)} + \deg \chi_{(n-4,1,1,1,1)} \\ &= \sum_{j=0}^{2} \binom{n}{2j} = c_n(G_4). \end{aligned}$$

This implies that the nth cocharacter of G_4 is decomposed as in 2 and $l_n(G_4) = 5$. □

4. CHARACTERIZING VARIETIES OF COLENGTH ≤4

In this section, we classify up to PI-equivalence the algebras A such that $l_n(A) \leq 4$ for n large enough.

In what follows, we shall use a particular decomposition of the Jacobson radical of a finite dimensional algebra.

Let $A = F + J$ be a finite dimensional algebra over F, where J is the Jacobson radical of A, then J can be decomposed into the direct sum of F-bimodules (see for instance Giambruno and Zaicev, 2003), i.e., $J = J_{00} + J_{01} + J_{10} + J_{11}$, where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta and $J_{11} = FN$ for some nilpotent subalgebra N of A commuting with F.

Lemma 11. *Let $A = F + J$ be an F-algebra. If $A_4, A_4^*, A_5, A_6 \notin \mathrm{var}(A)$, then $J_{10}J_{00} = J_{00}J_{01} = J_{01}J_{10} = J_{10}J_{01} = 0$.*

Proof. The result follow from Giambruno and La Mattina (2005, Lemmas 15, 20, 21). □

In what follows, we shall use the following result from Giambruno et al. (2007), which we state in our notation.

Theorem 12 (Giambruno et al., 2007, Theorem 3.6). *Let A be an algebra with 1 over a field F of characteristic zero. If $c_n(A) \approx an^k$, for some $a \geq 1$, $k \leq 3$, then either $A \sim_{PI} F$ or $A \sim_{PI} A_2$ or $A \sim_{PI} A_7$.*

This says that either $l_n(A) = 1$ or $l_n(A) = 3$ or $l_n(A) = 7$ for all $n > 4$.

Lemma 13. *Let $A = F + J$. If $[J_{11}, J_{11}, J_{11}] \neq 0$, then $A_7 \in \mathrm{var}(A)$.*

Proof. Since $F + J_{11}$ is a subalgebra of A, without loss of generality, we may assume that $A = F + J$ and $J = J_{11}$.

Hence A is a noncommutative algebra with 1 which does not satisfy the polynomial identity $[x_1, x_2, x_3] \equiv 0$. So, being $c_n(A)$ polynomially bounded, by (1) and the previous result,

$$c_n(A) = \sum_{i=0}^{k} \binom{n}{i} c_i^p(A) \approx qn^k, \qquad q > 0$$

for some $k \geq 3$. If $k = 3$, then, by the same result, A is PI-equivalent to A_7, and we are done. So we may assume that $k \geq 4$. Since $[x_1, x_2, x_3]$ is not an identity of A, A does not satisfy any identity of degree ≤ 3 (see for instance Giambruno et al., 2007). Hence, being $\mathrm{Id}(A)$ generated by proper polynomials, $\mathrm{Id}(A) \subseteq \langle [x_1, x_2, x_3, x_4], [x_1, x_2][x_3, x_4] \rangle_T = \mathrm{Id}(A_7)$, i.e, $A_7 \in \mathrm{var}(A)$. □

Lemma 14. *Let $A = F + J$ with $[J_{11}, J_{11}, J_{11}] = [J_{11}, J_{11}][J_{11}, J_{11}] = 0$.*

1. *If $[J_{11}, J_{11}]J_{10} \neq 0$, then $A_8 \in \mathrm{var}(A)$.*
2. *If $J_{01}[J_{11}, J_{11}] \neq 0$, then $A_8^* \in \mathrm{var}(A)$.*

Proof. Suppose that $[J_{11}, J_{11}]J_{10} \neq 0$. Being $F + J_{11} + J_{10}$ a subalgebra of A, we may assume, without loss of generality, that $A = F + J_{11} + J_{10}$. Since $[J_{11}, J_{11}, J_{11}] = [J_{11}, J_{11}][J_{11}, J_{11}] = 0$, we have that $[A, A, A], [A, A][A, A] \subseteq J_{10}$. Hence, it is immediate that $[x_1, x_2, x_3]x_4 \equiv 0$ and $[x_1, x_2][x_3, x_4]x_5 \equiv 0$ are polynomial identities of A and, so, by Lemma 9, $\mathrm{Id}(A_8) \subseteq \mathrm{Id}(A)$.

Conversely, let $f \in \mathrm{Id}(A)$ be a multilinear polynomial of degree n. By Lemma 9, we can write $f \pmod{\mathrm{Id}(A_8)}$ as

$$f = \alpha x_1 \cdots x_n + \sum_{\substack{t=1 \\ i_1 < \cdots < i_{n-2}}}^{n-1} \alpha_t x_{i_1} \cdots x_{i_{n-2}} [x_n, x_t] + \sum_{\substack{i>j \\ i_1 < \cdots < i_{n-3}}} \alpha_{ijk} x_{i_1} \cdots x_{i_{n-3}} [x_i, x_j] x_k.$$

Choosing $x_i = 1_F$ for all $i = 1, \ldots, n$, we get $\alpha = 0$. For fixed $1 \leq t \leq n-1$, by evaluating $x_t = a \in J_{10}$, $a \neq 0$, and $x_k = 1_F$, $k \neq t$, we get $\alpha_t = 0$. Since $[J_{11}, J_{11}]J_{10} \neq 0$, there exist $a \in J_{10}$, $b, c \in J_{11}$ such that $bca \neq 0$. Therefore, for fixed i, j, k, the evaluation $x_i = b$, $x_j = c$ $x_k = a$, $x_l = 1_F$, $l \notin \{i, j, k\}$ gives $\alpha_{ijk} = 0$. This says that $f \in \mathrm{Id}(A_8)$, and, so, $\mathrm{Id}(A) = \mathrm{Id}(A_8)$.

The property 2 is proved similarly. □

In La Mattina (2007), it was proved that if $A \in \mathrm{var}(G)$ is a noncommutative algebra with 1, then either $A \sim_{PI} G$ or $A \sim_{PI} G_{2k}$ for some $k \geq 1$. From this, by recalling that $\mathrm{Id}(G) = \langle [x_1, x_2, x_3] \rangle_T$ (see Krakowski and Regev, 1973), the following remark follows easily.

Remark 15. If $A = F + J_{11}$ is an algebra satisfying the identity $[x_1, x_2, x_3] \equiv 0$, then either $A \sim_{PI} F$ or $A \sim_{PI} A_2$ or $G_4 \in \mathrm{var}(A)$.

Recall that, by Maltsev (1971), $\mathrm{Id}(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_T$.

Lemma 16. *Let $A = F + J$ be such that $G_4, A_7, A_8, A_8^* \notin var(A)$ and $J_{00} = J_{10}J_{01} = 0$. Then $A \in var(UT_2)$.*

Proof. Since $A_7 \notin \mathrm{var}(A)$, by Lemma 13, $[J_{11}, J_{11}, J_{11}] = 0$. So, $F + J_{11}$ is a subalgebra of A satisfying the identity $[x_1, x_2, x_3] \equiv 0$. Hence, since $G_4 \notin \mathrm{var}(A)$, we must have, by Remark 15, $[J_{11}, J_{11}][J_{11}, J_{11}] = 0$. Also, by Lemma 14, since $A_8, A_8^* \notin \mathrm{var}(A)$, we have that $[J_{11}, J_{11}]J_{10} = J_{01}[J_{11}, J_{11}] = 0$. Hence, since evaluating $f = [x_1, x_2][x_3, x_4]$ in $F + J_{11}$, we get a zero value of f, at least one variable x_i must be evaluated in J_{10} or J_{01}. But $[J_{11}, J_{11}][F, J_{10}] = [J_{11}, J_{11}]J_{10} = [J_{01}, F][J_{11}, J_{11}] = J_{01}[J_{11}, J_{11}] = [F, J_{10}][J_{01}, F] = J_{10}J_{01} = 0$ implies that $f \equiv 0$ is a polynomial identity for A and, so, $A \in \mathrm{var}(UT_2)$. □

In what follows, we shall use the following result from La Mattina (2007), which we state in our notation.

Theorem 17. *Let* $A = F + J_{10} + J_{01} + J_{11} \in \mathrm{var}(UT_2)$.

1) *If* J_{11} *is commutative* A *is PI-equivalent to one of the following algebras*: $F, A_1, A_1^*, A_1 \oplus A_1^*$;
2) *If* J_{11} *is not commutative and* $[J_{11}, J_{11}, J_{11}] = 0$, A *is PI-equivalent to one of the following algebras*: $A_2, A_1 \oplus A_2, A_1^* \oplus A_2, A_1 \oplus A_1^* \oplus A_2$.

Now we are in a position to prove the main result of this article, which allows us to classify all algebras whose colength sequence is bounded by 4. We remark that the case $l_n(A) \leq 2$ was proved in Giambruno and La Mattina (2005).

Theorem 18. *Let A be an F-algebra. Then the following conditions are equivalent*:

1. $l_n(A) = k$, $k \leq 4$, *for n large enough*;
2. $A_3, A_4, A_4^*, A_5, A_6, A_7, A_8, A_8^*, G_4 \notin \mathrm{var}(A)$;
3. *A is PI-equivalent to one of the following algebras*: N, $C \oplus N$, $A_1 \oplus N$, $A_1^* \oplus N$, $A_1 \oplus A_1^* \oplus N$ $A_2 \oplus N$, $A_1 \oplus A_2 \oplus N$, $A_1^* \oplus A_2 \oplus N$, *where N is a nilpotent algebra and C is a commutative non-nilpotent algebra.*

Proof. $1 \Rightarrow 2$ Since for $n > 4$, $A_3, A_4, A_4^*, A_5, A_6, A_7, A_8, A_8^*, G_4$ have colength sequence bounded from below by 5, it is clear that they do not belong to the variety generated by A.

$2 \Rightarrow 3$ Since $A_7 \in \mathrm{var}(UT_2)$ and $G_4 \in \mathrm{var}(G)$, by the hypotheses, $UT_2, G \notin \mathrm{var}(A)$. Hence, by Giambruno and Zaicev (2005, Theorem 7.2.12), we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional algebras such that $\dim B_i/J(A_i) \leq 1$ and $J(B_i)$ denotes the Jacobson radical of B_i, $1 \leq i \leq m$. Notice that this says that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent algebra.

If $B_i = J(B_i)$ is nilpotent for all i, then A is a nilpotent algebra, and we are done in this case.

Suppose that for some i, B_i is not a nilpotent algebra, and let $J = J_{11} + J_{10} + J_{01} + J_{00} = J(B_i)$. Since $A_4, A_4^*, A_5, A_6, A_7 \notin \mathrm{var}(A)$, by Lemmas 11 and 13,

$$J_{10}J_{00} = J_{00}J_{01} = J_{01}J_{10} = J_{10}J_{01} = [J_{11}, J_{11}, J_{11}] = 0.$$

Under these conditions J_{00} is a two-sided nilpotent ideal of B_i and $B_i = F + J_{01} + J_{10} + J_{11} \oplus J_{00}$. Let $D = F + J_{01} + J_{10} + J_{11}$. By the hypotheses, $G_4, A_7, A_8, A_8^* \notin \mathrm{var}(D)$ and, so, by Lemma 16, $D \in \mathrm{var}(UT_2)$ and, by Theorem 17, $B_i = D \oplus N$ is PI-equivalent to one of the following algebras: $C \oplus N$, $A_1 \oplus N$, $A_1^* \oplus N$, $A_1 \oplus A_1^* \oplus N$, $A_2 \oplus N$, $A_1 \oplus A_2 \oplus N$, $A_1^* \oplus A_2 \oplus N$, where $N = J_{00}$ is a nilpotent algebra, and C is a commutative algebra.

Since $A = B_1 \oplus \cdots \oplus B_m$, and $A_3 \notin \mathrm{var}(A)$, the desired conclusion follows easily.

$3 \Rightarrow 1$ Since each of the algebras in 3, for n large enough, has colength sequence constant and bounded from above by 4, then Property 3 implies Property 1, and we are done. □

In conclusion, we have the following classification: for any algebra A and n large enough:

1. $l_n(A) = 0$ if and only if $A \sim_{PI} N$;
2. $l_n(A) = 1$ if and only if $A \sim_{PI} C \oplus N$;
3. $l_n(A) = 2$ if and only if either $A \sim_{PI} A_1 \oplus N$ or $A \sim_{PI} A_1^* \oplus N$;
4. $l_n(A) = 3$ if and only if either $A \sim_{PI} A_1 \oplus A_1^* \oplus N$ or $A \sim_{PI} A_2 \oplus N$;
5. $l_n(A) = 4$ if and only if either $A \sim_{PI} A_1 \oplus A_2 \oplus N$ or $A \sim_{PI} A_1^* \oplus A_2 \oplus N$,;

where N denotes a nilpotent algebra and C a commutative non-nilpotent algebra.

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