

Tatra Mt. Math. Publ. 42 (2009), 27–37
DOI: 10.2478/v10127-009-0003-z

**TATRA
MOUNTAINS**
Mathematical Publications

DENJOY AND \mathcal{P} -PATH INTEGRALS ON COMPACT GROUPS IN AN INVERSION FORMULA FOR MULTIPLICATIVE TRANSFORMS

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ABSTRACT. Denjoy and \mathcal{P} -path Kurzweil-Henstock type integrals are defined on compact subsets of some locally compact zero-dimensional abelian groups. Those integrals are applied to obtain an inversion formula for the multiplicative integral transform.

1. Introduction

In [8], an inversion formula for the multiplicative integral transform was obtained for the case of any locally compact zero-dimensional abelian periodic group. This result does not cover the case of transforms convergent to Denjoy-Khintchine integrable functions because this integral is incompatible with \mathcal{P} -adic integral used in the above result (see [2]).

To overcome this difficulty, we consider here a little bit less general class of zero-dimensional groups and another Kurzweil-Henstock type integral on them. This gives us an opportunity to get an inversion formula for the multiplicative integral transform convergent to a Denjoy-Khintchine integrable function in the case of a class of groups considered here.

This problem is a generalization of that of recovering the coefficients of a convergent series with respect to characters of a compact zero dimensional abelian group which was considered in [7]. A similar problem related to some special groups was considered in [6].

2000 Mathematics Subject Classification: 43A70, 43A25, 26A39, 42C10.

Keywords: locally compact zero-dimensional abelian group, characters of a group, Denjoy integral, Kurzweil-Henstock \mathcal{P} -path integral, multiplicative integral transform, inversion formula.

2. Preliminaries

Let G be a zero-dimensional locally compact abelian periodic group which satisfies the second countability axiom. We can introduce (see [1]) a topology in such a group using a chain of subgroups

$$\dots \supset G_{-n} \supset \dots \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$$

with $G = \cup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \cap_{n=-\infty}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is periodic, the factor group G_n/G_{n+1} is finite for each n and this implies that G_n is compact. Let p_{n+1} be an order of G_n/G_{n+1} if $n \geq 0$ and p_{-n} an order of G_{-n}/G_{-n+1} if $n \geq 1$. We can suppose that all p_n are prime numbers. So, the group G defines a sequence

$$\mathcal{P} = \{p_j\}_{j=-\infty}^{+\infty}, \quad j \neq 0. \quad (1)$$

We also consider the reverse sequence

$$\mathcal{P}' = \{p'_j\}_{j=-\infty}^{+\infty}, \quad j \neq 0, \quad (2)$$

where $p'_j = p_{-j}$ for $j \in \mathbb{Z} \setminus \{0\}$. Moreover, we set $m_0 = 1$, $m_j = \prod_{s=1}^j p_s$, $m_{-j} = \prod_{s=-1}^{-j} p_s$.

We remind that a complex function χ on a locally compact abelian group G is called a *character* of G if $|\chi(g)| = 1$ for all $x \in G$ and if a functional equation

$$\chi(g_1 + g_2) = \chi(g_1) + \chi(g_2)$$

is satisfied for all $g_1, g_2 \in G$. The set of all continuous characters of G forms a group X , the *dual group* of G , if the addition is defined by

$$(\chi_1 + \chi_2)(g) = \chi_1(g) \cdot \chi_2(g),$$

where $x \in G$ and $\chi_1, \chi_2 \in X$. In view of the duality between G and X , see ([10]), it is customary to write (g, χ) in place of $\chi(g)$.

In what follows, X will denote the dual group of the group G described above. It is known (see [1]) that under the assumption imposed on G the group X is also a periodic locally compact zero-dimensional abelian group and we can represent it to be a sum of increasing sequence of subgroups:

$$\dots \supset X_n \supset \dots \supset X_2 \supset X_1 \supset X_0 \supset X_{-1} \supset X_{-2} \supset \dots \supset X_{-n} \supset \dots \quad (3)$$

introducing a topology in X . Then, $X = \cup_{i=-\infty}^{+\infty} X_i$ and $\cap_{i=-\infty}^{+\infty} X_i = \{\chi^{(0)}\}$ where $(g, \chi^{(0)}) = 1$ for all $g \in G$. For each $n \in \mathbb{Z}$, the group X_n is the annihilator of G_n , i.e.,

$$X_n = G_n^\perp = \{\chi \in X : (g, \chi) = 1 \quad \text{for all } g \in G_n\}.$$

The factor groups $X_{n+1}/X_n = G_{n+1}^\perp/G_n^\perp$ and G_n/G_{n+1} are isomorphic (see [1]). So, the order of X_{n+1}/X_n is p_{n+1} if $n \geq 0$ and p_n if $n \leq -1$.

It can be shown (see [8]) that any element $g \in G$ can be represented in the form

$$g = [g] + \{g\} \quad (4)$$

and any element $\chi \in X$ in the form

$$\chi = [\chi] \cdot \{\chi\} \quad (5)$$

in such a way that the following properties hold:

- 1) $\{g\} \in G_0$ and $\{\chi\} \in X_0 = G_0^\perp$. So, $(\{g\}, \{\chi\}) = 1$, and

$$(g, \chi) = (\{g\}, [\chi]) \cdot ([g], [\chi]) \cdot ([g], \{\chi\}). \quad (6)$$

- 2) $[\chi] \in G_m^\perp$ for some $m = m(\chi) \in \mathbb{N}$ and $[\chi] \setminus_{G_0}$ is a character of the subgroup G_0 .

- 3) $([g], [\chi])$ is constant if g belongs to a fixed coset of G_0 and χ belongs to a fixed coset of X_0 .

Using the duality between G and X , we can state that g represents a character of X and similarly to property 2), $[g] \setminus_{X_0}$ is a character of X_0 . So, $([g], \{\chi\})$ is a value of this character at the point $\{\chi\}$.

Therefore, according to 6), if g belongs to a fixed coset of G_0 and χ belongs to a fixed coset of X_0 , we can represent (g, χ) , up to a constant $([g], [\chi])$, as a product of $(\{g\}, [\chi])$ considered as a value of the character $[\chi]$ at $\{g\}$, and $([g], \{\chi\})$ considered as a value of the character $[g]$ at $\{\chi\}$.

It is also shown (see [8]) that we can map the groups G and X to the interval $[0, +\infty)$ by mappings

$$\varphi(g) = x = \{x\} + [x] \quad (7)$$

and

$$\psi(\chi) = x' = \{x'\} + [x'], \quad (8)$$

where $\{a\}$ is the fractional part of a and $[a]$ the integer part of a . Moreover, the preimage of $[x]$, under the mapping φ , is $[g]$ and the one of $\{x\}$ is $\{g\}$. This explains the notation in (4). In the same way, $\psi([\chi]) = [x']$ and $\psi(\{\chi\}) = \{x'\}$.

In this way, the subgroup $G_n, n \geq 0$, is mapped onto the interval $[0, \frac{1}{m_n}]$ and the respective cosets are mapped onto intervals $I_n^k = [\frac{k}{m_n}, \frac{k+1}{m_n}]$ with $k = 0, 1, \dots$. We denote these cosets by G_n^k with $G_n^0 = G_n$. In particular, for $n = 0$ we map G_0 onto $[0, 1]$ and respective cosets onto intervals $[k, k + 1]$. In case $n < 0$, the group G_n is mapped onto interval $[0, m_n]$ and respective cosets G_n^k are mapped onto intervals $[km_n, (k+1)m_n]$. We call all the above mentioned images of cosets the \mathcal{P} -adic intervals. If n is fixed, we refer to I_n^k as to intervals of a rank n . The set of all \mathcal{P} -adic intervals is denoted by $\mathcal{I}_{\mathcal{P}}$.

In a similar way, we can define \mathcal{P}' -adic intervals associated with the group X and the sequence \mathcal{P}' (see (2)). They are images, under mapping (8), of the respective cosets of the subgroup X_n denoted by X_n^s .

Note that the mappings (7) and (8) are one-one correspondences between the interval $[0, +\infty)$ and the groups G and X , respectively, up to a countable number of \mathcal{P} -adic (\mathcal{P}' -adic) rational points, i.e., points $\frac{t}{m_k}$ (or $\frac{t}{m_{-k}}$) with $t, k = 0, 1, 2, \dots$. Such a point x has two preimages corresponding to the finite and to the infinite expansion of x , respectively. We denote by $Q_{\mathcal{P}}$ the set of all \mathcal{P} -adic rational points and by $Q_{\mathcal{P}'}$ the set of all \mathcal{P}' -adic rational points. We agree to use only finite expansions for \mathcal{P} -adic (and \mathcal{P}' -adic) rational points so that the inverse mappings φ^{-1} and ψ^{-1} make sense on $[0, \infty)$.

We consider the Haar measure μ_G on the group G and we normalize it so that $\mu_G(G_0) = 1$. Then the measure of any coset G_n^k of G_n coincides with the length of the \mathcal{P} -adic interval which is the image of G_n^k , i.e.,

$$\mu_G(G_n^k) = \frac{1}{m_n} \quad \text{if } n \geq 0 \quad \text{and} \quad \mu_G(G_n^k) = m_n \quad \text{if } n < 0.$$

In the same way, we introduce the measure μ_X on X so that $\mu_X(X_0) = 1$ and

$$\mu_X(X_n^s) = \frac{1}{m_n} \quad \text{if } n \leq 0 \quad \text{and} \quad \mu_X(X_n^s) = m_n \quad \text{if } n > 0.$$

Note that, under the above mentioned mappings φ and ψ , the image of each set of Haar measure zero on the group is a set of Lebesgue measure zero on $[0, +\infty)$.

We denote by \mathcal{R}_G the ring generated by the family of all cosets G_n^k , $n \in \mathbb{Z}$, $k = 0, 1, 2, \dots$, and by $\mathcal{R}_{\mathcal{P}}$ the ring generated by $\mathcal{I}_{\mathcal{P}}$. Note that for each $g \in G$ there exists a decreasing sequence $\{G_n^{k(n)}\}_n$ of cosets such that $g \in \bigcap_n G_n^{k(n)}$.

3. \mathcal{P} -path Kurzweil-Henstock integral and Denjoy-Khintchine integral on the group G

Now, we introduce a Henstock-Kurzweil type integral with respect to the system of \mathcal{P} -paths and also Denjoy-Khintchine integral on compact subgroups of the considered group G and X .

With each \mathcal{P} -adic irrational point $x \in [0, \infty)$, i.e., a point $x \in [0, \infty) \setminus Q_{\mathcal{P}}$, we associate a unique nested sequence

$$\left\{ \varphi \left(G_n^{k(n)} \right) \right\}_n = \left\{ I_n^{k(n)} \right\} = \left\{ [a_n(x), b_n(x)] \right\}_{n=0}^{\infty} \quad (9)$$

of \mathcal{P} -adic intervals converging to x so that

$$\{x\} = \bigcap_{n=0}^{\infty} [a_n(x), b_n(x)].$$

If x is a \mathcal{P} -adic rational point, then there exist two nested sequences (9) — the *left* one and the *right* one — for which x , starting with some n , is the common end-point.

Using the notation given by (9), we define for a \mathcal{P} -adic irrational point x , the sequences $\mathcal{P}_x^- = \{a_n(x)\}$ and $\mathcal{P}_x^+ = \{b_n(x)\}$. If x is a \mathcal{P} -adic rational point, we use the same notation \mathcal{P}_x^+ and \mathcal{P}_x^- for sequences with $a_n(x)$ being the left end-point of interval of rank n of the above mentioned left nested sequence associated with x , and $b_n(x)$ being the right end-point of interval of rank n of the right nested sequence.

DEFINITION 3.1. The set $\mathcal{P}_x = \mathcal{P}_x^+ \cup \mathcal{P}_x^- \cup \{x\}$ is called the \mathcal{P} -path leading to x . If $E \in \mathcal{R}_{\mathcal{P}}$, the collection $\{\mathcal{P}_x : x \in E\}$ is called the *system of \mathcal{P} -paths on E* .

The continuity and the derivative at a point x with respect to the set \mathcal{P}_x are called *\mathcal{P} -path continuity* and *\mathcal{P} -path derivative*. In the same way, we define *\mathcal{P} -path upper* and *lower derivatives*.

DEFINITION 3.2. Let δ be a positive function defined on $E \in \mathcal{R}_{\mathcal{P}}$. A collection of interval-point pairs $\{([u_j, v_j], x_j)\}_{j=1}^k$ is called *δ -fine \mathcal{P} -partition of E* if the intervals $[u_j, v_j]$ are non-overlapping, $E = \cup_{j=1}^k [u_j, v_j]$ and for each j we have $u_j, v_j \in \mathcal{P}_x$, $u_j \leq x_j \leq v_j$ and $\max\{v_j - x_j, x_j - u_j\} < \delta(x_j)$.

It is not difficult to check that for any $E \in \mathcal{R}_{\mathcal{P}}$, any system of \mathcal{P} -paths on E , and for any positive function δ defined on E there exists a δ -fine \mathcal{P} -partition of E .

DEFINITION 3.3. A complex valued function f on $E \in \mathcal{R}_{\mathcal{P}}$ is said to be *Kurzweil-Henstock integrable with respect to the system of \mathcal{P} -paths* or, in brief, *$H_{\mathcal{P}}$ -integrable on E* , with integral value A , if for every $\varepsilon > 0$, there exists a positive function δ on E such that for any δ -fine \mathcal{P} -partition of E we have

$$\left| \sum_{j=1}^k f(x_j)(v_j - u_j) - A \right| < \varepsilon.$$

Then we write $(H_{\mathcal{P}}) \int_E f = A$.

It is easy to check that a function which is equal to zero almost everywhere on $E \in \mathcal{R}_{\mathcal{P}}$, is $H_{\mathcal{P}}$ -integrable to zero on E . This justifies the following extension of Definition 3.3 to the case of functions defined only almost everywhere on E (for short, a.e.).

DEFINITION 3.4. A complex valued function f defined a.e. on $E \in \mathcal{R}_{\mathcal{P}}$ is said to be *$H_{\mathcal{P}}$ -integrable on E* with integral value A if the function

$$f_1(x) = \begin{cases} f(x), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is $H_{\mathcal{P}}$ -integrable on E to A in the sense of Definition 3.3.

We can use this definition to introduce the respective H_G -integral on certain subsets of the considered group G .

DEFINITION 3.5. Let $E \in \mathcal{R}_{\mathcal{P}}$ be the image of $T \in \mathcal{R}_G$ under the mapping φ (see (7)). Then a complex-valued function f defined almost everywhere on T is H_G -integrable on T , with the value A , if the function $F(x) = f(\varphi^{-1}(x))$ is $H_{\mathcal{P}}$ -integrable on E with the value A in the sense of Definition 3.4, and we write $(H_G) \int_T f d\mu_G = A$.

Remark 3.1. We note that the above definition depends on the sequence \mathcal{P} defined by the group G . So, if we consider a similar definition of the H_X -integral on a subset of the group X , then we should use a sequence \mathcal{P}' defined by X and the respective $H_{\mathcal{P}'}$ -integral.

Now, we shall give a definition of *Denjoy-Khintchine integral* on compact subgroups of a locally compact group.

DEFINITION 3.6. A real function Φ defined on $[a, b]$ is an *ACG-function* if $[a, b] = \cup_{i=1}^{\infty} B_i$ and Φ is absolutely continuous on each B_i .

DEFINITION 3.7. A real function f defined almost everywhere on $[a, b]$ is *Denjoy-Khintchine integrable* on $[a, b]$, briefly, *D-integrable*, if there exists an ACG function Φ such that $\Phi'_{ap} = f$ almost everywhere, where Φ'_{ap} denotes the approximate derivative (see [4]). The value of this integral is defined as $(D) \int_a^b f = \Phi(b) - \Phi(a)$.

DEFINITION 3.8. A real function f defined almost everywhere on $E = \cup_{i=1}^n [\alpha_i, \beta_i]$ is *D-integrable on E* if f is *D-integrable* on each $[\alpha_i, \beta_i]$ and the value of integral is $(D) \int_E f = \sum_{i=1}^n (D) \int_{\alpha_i}^{\beta_i} f$. In the case of a complex-valued function f , it is *D-integrable on E* if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are *D-integrable* with integrals A and B , respectively, and the *D-integral* of f is $A + iB$.

DEFINITION 3.9. Let $E \in \mathcal{R}_{\mathcal{P}}$ be the image of $T \in \mathcal{R}_G$ under the mapping φ (see (7)). Then a complex-valued function f defined almost everywhere on T is *D-integrable on T*, with the value A , if the function $F(x) = f(\varphi^{-1}(x))$ is *D-integrable on E* with the value A in the sense of Definition 3.8, and we write $(D) \int_T f d\mu_G = A$.

Remark 3.2. It is easy to check out that the *D-integral* (in the sense of the above definition) is invariant under translation given by some “integer part” $[g]$ of some element $g \in G$.

We need the following theorem (see [9]).

THEOREM 3.1. *Suppose that the sequence (1) is bounded and a function f defined on $[a, b]$ is both $H_{\mathcal{P}}$ -integrable and *D-integrable* on $[a, b]$. If Φ is the indefinite*

$H_{\mathcal{P}}$ -integral of f and the upper and lower \mathcal{P} -path derivatives of $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ are bounded at each point of $[a, b]$, except a countable set, then the values of $H_{\mathcal{P}}$ - and D -integrals of f on $[a, b]$ coincide.

The next result is of great importance for our further consideration. It is a direct consequence of the above theorem, related to the real line setting, and of Definitions 3.5 and 3.9.

THEOREM 3.2. *Suppose that the sequence (1) is bounded and $E \in \mathcal{R}_{\mathcal{P}}$ is the image of $T \in \mathcal{R}_G$ under the mapping φ (see(7)). If a H_G -integrable on $T \in R_G$ function f and its indefinite H_G -integral Φ are such that $f(\varphi^{-1}(x))$ and $\Phi(\varphi^{-1}(x))$ satisfy the conditions of the previous theorem, then the values of H_G - and D -integrals of f on T coincide.*

4. An inversion formula for integral transforms convergent to a D -integrable function

Using the notations for the cosets of X_0 and G_0 introduced in Section 2, we denote $[\chi] \setminus_{G_0}$ by h_s if $[\chi] \in X_0^s$. In the same way, if $[g] \in G_0^k$, we denote $[g] \setminus_{X_0}$ by h'_k . It can be easily checked that the sequence $\{h_s\}_{s=0}^{\infty}$ includes all the characters of G_0 and $\{h'_k\}_{k=0}^{\infty}$ includes all the characters of X_0 . This sequence $\{h_s\}$ having been translated on $[0, 1]$ using the mapping (7), forms an orthonormal system $\{\xi_s\}$ on $[0, 1]$, where $\xi_s(x) = h_s(\varphi^{-1}(x))$ (see [1]).

The following theorem is proved in [7].

THEOREM 4.1. *Suppose that the sequence (1) is bounded, the series $\sum a_s \xi_s$ with respect to the above system $\{\xi_s\}$ converges a.e. on $[0, 1]$ to a function f and everywhere on $[0, 1] \setminus Q_{\mathcal{P}}$, we have*

$$\limsup_{n \rightarrow \infty} |S_n(x)| < +\infty, \quad (10)$$

where $S_n = \sum_{s=0}^{n-1} a_s \xi_s$. Then f is $H_{\mathcal{P}}$ -integrable on $[0, 1]$ (in the sense of Definition 3.4 and $\sum a_s \xi_s$ is $H_{\mathcal{P}}$ -Fourier series of f).

We can reformulate this result in terms of the system $\{h_s\}$ defined on a group.

THEOREM 4.2. *Let G_0 be a compact abelian zero-dimensional group such that the sequence (1) is bounded and let $\{h_s\}_s$ be a system defined as above. If the series $\sum_{s=0}^{\infty} a_s h_s$ converges a.e. on G_0 to a function f and everywhere on $G_0 \setminus \varphi^{-1}(Q_{\mathcal{P}})$ we have*

$$\limsup_{n \rightarrow \infty} |\sigma_n(g)| < +\infty, \quad (11)$$

where $\sigma_n = \sum_{s=0}^{n-1} a_s h_s$, then f is H_{G_0} -integrable on G_0 and $\sum a_s h_s$ is the Fourier series of f in the sense of integral defined in Definition 3.5.

It is known (see [7, Lemma 3.2]) that the condition (10) implies the boundedness of upper and lower derivatives of $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$, where Φ is an indefinite $H_{\mathcal{P}}$ -integral of f .

Combining this with the last two theorems and with Theorem 3.2, we get.

THEOREM 4.3. *Let G_0 and the system $\{h_s\}_s$ be as in Theorem 4.2. If the series $\sum_{s=0}^{\infty} a_s h_s$ converges a.e. on G_0 to a D -integrable function f and everywhere on $G_0 \setminus \varphi^{-1}(Q_{\mathcal{P}})$ the inequality (11) holds, then $\sum a_s h_s$ is the Denjoy-Fourier series of f (in the sense of the integral defined in Definition 3.9).*

This theorem is a generalization of a similar result related to some special case of zero-dimensional group (see [5, Theorem 2]).

We also need the following theorem, which can be established using the same arguments as in [8, Theorem 8]:

THEOREM 4.4. *The partial sums $\sigma_{m_n}(f, g)$ of the H_G -Fourier series (with respect to the system of characters of G_0) of a H_G -integrable on G_0 function f are convergent to f almost everywhere on G_0 .*

A continuum analog of the series $\sum_{s=0}^{\infty} a_s h_s$ is the integral transform

$$\int_X a(\chi)(g, \chi) d\mu_X$$

with appropriately defined improper integral on X . So, the next theorem, which gives an inversion formula for this transform, can be considered as a generalization of Theorem 4.3.

THEOREM 4.5. *Assume that G is a group described in Section 2 such that the sequence (1) is bounded, X is its dual group. Let $a(\chi)$ be a locally H_X -integrable function and*

$$\lim_{s \rightarrow \infty} (H_X) \int_{\cup_{i=0}^s X_0^i} a(\chi)(g, \chi) d\mu_X = f(g) \quad (12)$$

a.e. on G , where f is a D -integrable function on each $T \in \mathcal{R}_G$. Moreover, let us have everywhere on $G \setminus \varphi^{-1}(Q_{\mathcal{P}})$

$$\limsup_{s \rightarrow \infty} \left| (H_X) \int_{\cup_{i=0}^s X_0^i} a(\chi)(g, \chi) d\mu_X \right| < +\infty. \quad (13)$$

Then, the function $a(x)$ can be recovered from f by the following inversion formula:

$$a(\chi) = \lim_{n \rightarrow \infty} (D) \int_{G_{-n}} f(g) \overline{(g, \chi)} d\mu_G \quad \text{a.e. on } X.$$

Proof. Suppose that $g \in G_0^k$. Then, for each $s = 1, 2, \dots$, we have, according to (6),

$$\begin{aligned}
 & (H_X) \int_{\cup_{i=0}^s X_0^i} a(\chi)(g, \chi) \, d\mu_X \\
 &= (H_X) \int_{\cup_{i=0}^s X_0^i} a(\chi)(\{g\}, [\chi]) \cdot ([g], [\chi]) \cdot ([g], \{\chi\}) \, d\mu_X \\
 &= \sum_{i=0}^s (H_X) \int_{X_0^i} a(\chi)(\{g\}, [\chi]) \cdot ([g], [\chi]) \cdot ([g], \{\chi\}) \, d\mu_X \quad (14) \\
 &= \sum_{i=0}^s h_i(\{g\}) (H_X) \int_{X_0^i} a(\chi)([g], [\chi]) h'_k(\{\chi\}) \, d\mu_X.
 \end{aligned}$$

Now, for any $g \in G_0^k$ for which the limit (12) exists, we get:

$$f(g) = \lim_{s \rightarrow \infty} \sum_{i=0}^s h_i(\{g\}) (H_X) \int_{X_0^i} a(\chi)([g], [\chi]) h'_k(\{\chi\}) \, d\mu_X. \quad (15)$$

So, for such $g \in G_0^k$, the function $f(g)$ is the sum of series with respect to the system $\{h_i\}$ with coefficients

$$b_i = (H_X) \int_{X_0^i} a(\chi)([g], [\chi]) h'_k(\{\chi\}) \, d\mu_X$$

and this series is convergent almost everywhere on G_0^k .

In the same way, (13) and (15) imply, for any $g \in G_0^k \setminus \varphi^{-1}(Q_{\mathcal{P}})$

$$\limsup_{s \rightarrow \infty} \left| \sum_{i=0}^s h_i(\{g\}) (H_X) \int_{X_0^i} a(\chi)([g], [\chi]) h'_k(\{\chi\}) \, d\mu_X \right| < \infty. \quad (16)$$

Then, by Theorem 4.3, the coefficients b_i are the Denjoy-Fourier coefficients of the D -integrable function $p(t) = f([g] + t)$ on G_0 , i.e.,

$$\begin{aligned}
 b_i &= (H_X) \int_{X_0^i} a(\chi)([g], [\chi]) h'_k(\{\chi\}) \, d\mu_X = (D) \int_{G_0} p(t) \overline{h_i(t)} \, d\mu_G \\
 &= (D) \int_{G_0^k} f(g) \overline{h_i(\{g\})} \, d\mu_G \quad (17)
 \end{aligned}$$

(the last equality is justified by Remark 3.2). By property 3) of Section 2, $([g], [\chi])$ is constant when $g \in G_0^k$ and $\chi \in X_0^i$. Hence, (17) implies

$$(H_X) \int_{X_0^i} a(\chi) h'_k(\{\chi\}) \, d\mu_X = (D) \int_{G_0^k} f(g) \overline{([g], [\chi]) h_i(\{g\})} \, d\mu_G. \quad (18)$$

The rest of the proof follows the lines of the proof of Theorem 9 of [8].

For a fixed i , the value

$$(H_X) \int_{X_0^i} a(\chi) h'_k(\{\chi\}) \, d\mu_X$$

is the Fourier coefficient in the system $\{\overline{h'_k}\}$ of the H_X -integrable function $a(\chi) = a([\chi] + \{\chi\})$ considered as a function of $\{\chi\} \in X_0$. Therefore, Theorem 4.4 being applied to the above system and the appropriate partial sums, implies

$$\sum_{k=0}^{m-n} (H_X) \int_{X_0^i} a(\chi) h'_k(\{\chi\}) \, d\mu_X \cdot \overline{h'_k(\{\chi\})} \rightarrow a(\chi) \quad \text{a. e. on } X_0^i.$$

Then, using (18) and (6), we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=0}^{m-n} (H_X) \int_{X_0^i} a(\chi) h'_k(\{\chi\}) \, d\mu_X \cdot \overline{h'_k(\{\chi\})} \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{m-n} (D) \int_{G_0^k} f(g) \overline{([g], [\chi]) h_i(\{g\})} \, d\mu_G \cdot \overline{h'_k(\{\chi\})} \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{m-n} (D) \int_{G_0^k} f(g) \overline{h_i(\{g\}) h'_k(\{\chi\}) ([g], [\chi])} \, d\mu_G \\ &= \lim_{n \rightarrow \infty} (D) \int_{\cup_{k=0}^{m-n} G_0^k} f(g) \overline{(\{g\}, [\chi]) ([g], \{\chi\}) ([g], [\chi])} \, d\mu_G \\ &= \lim_{n \rightarrow \infty} (D) \int_{G_{-n}} f(g) \overline{(g, \chi)} \, d\mu_G = a(\chi) \quad \text{a. e. on } X_0^i. \end{aligned}$$

The last equality is true for any i . Therefore, we can write

$$\lim_{n \rightarrow \infty} (D) \int_{G_{-n}} f(g) \overline{(g, \chi)} \, d\mu_G = a(\chi) \quad \text{a. e. on } X.$$

This completes the proof. \square

Remark 4.1. As both D -integral and H_X -integral obviously include the Lebesgue integral, the last theorem can be formulated, in particular, for functions a and f being locally Lebesgue integrable on the groups X and G , respectively.

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Received September 4, 2007

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