

Edge Partitions of Complete Geometric Graphs

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Abstract

In this paper, we disprove the long-standing conjecture that any complete geometric graph on $2n$ vertices can be partitioned into n plane spanning trees. Our construction is based on so-called bumpy wheel sets. We fully characterize which bumpy wheels can and in particular which *cannot* be partitioned into plane spanning trees (or even into arbitrary plane *subgraphs*).

Furthermore, we show a sufficient condition for *generalized wheels* to not admit a partition into plane spanning trees, and give a complete characterization when they admit a partition into plane spanning double stars.

Finally, we initiate the study of partitions into beyond planar subgraphs, namely into k -planar and k -quasi-planar subgraphs and obtain first bounds on the number of subgraphs required in this setting.

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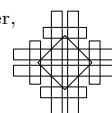
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1 Introduction

A geometric graph $G = G(P, E)$ is a drawing of a graph in the plane where the vertex set is drawn as a point set P in general position (that is, no three points are collinear) and each edge of E is drawn as a straight-line segment between its vertices. A geometric graph G is *plane* if no two of its edges *cross* (that is, share a point in their relative interior). A *partition* (also called edge partition) of a graph G is a set of edge-disjoint subgraphs of G whose union is G . A subgraph of (a connected graph) G is *spanning* if it is connected and its vertex set is the same as the one of G . In 2003, Ferran Hurtado shared the following long-standing open question, which has commonly been conjectured to have a positive answer (see [9, 6]):

► **Question 1** ([6]). *Can every complete geometric graph on $2n$ vertices be partitioned into n plane spanning trees?*

Note that with $2n > 0$ vertices, the complete graph has exactly the right number of edges to admit a partition into n spanning trees, while this is not the case for $2n + 1$ vertices. In the following, we consider complete geometric graphs to have $2n$ vertices unless stated otherwise. Further, we denote the complete geometric graph on a point set P as $K(P)$.

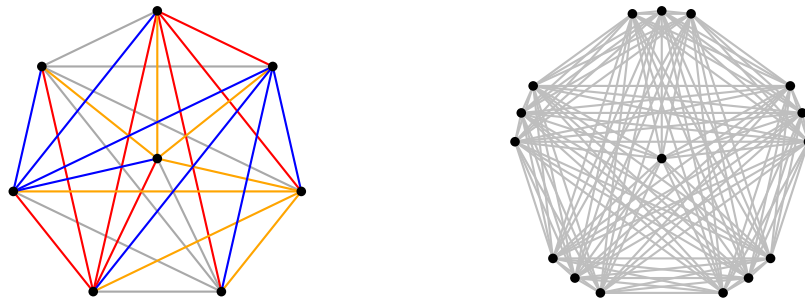
Related work. Several approaches have been made to answer Question 1. When P is in convex position it follows from a result of Bernhart and Kainen [4] that $K(P)$ can be partitioned into plane spanning paths, implying a positive answer. Further, Bose et al. [6] gave a complete characterization of all possible partitions into plane spanning trees for convex point sets. Similarly, when $P = W_{2n}$ is a *regular wheel set* (the vertex set of a regular $(2n - 1)$ -gon plus its center), Aichholzer et al. [2] showed how to partition $K(P)$ into plane spanning *double stars* (trees with at most two vertices of degree ≥ 2), and Trao et al. [14] recently characterized all possible partitions (into arbitrary plane spanning trees). Further, Aichholzer et al. [2] provide a positive answer to Question 1 for all point sets of (even) cardinality at most 10, obtained by exhaustive computations.

Relaxing the requirement that the trees must be spanning, Bose et al. [6] showed that if for a general point set P , there exists an arrangement of k lines in which every cell contains at least one point from P , then the complete geometric graph on P admits a partition into $2n - k$ plane trees, k of which are plane double stars. This result implies that Question 1 has a positive answer if P contains n pairwise crossing segments, which is the case if and only if P has exactly n *halving lines* [10] (a line through two points of P is called *halving line* if it has exactly $n - 1$ points of P on either side and the corresponding edge is called *halving edge*).

For the related *packing* problem where not all edges of the underlying graphs must be covered, Biniáz and García [5] showed that $\lfloor n/3 \rfloor$ plane spanning trees can be packed in any complete geometric graph on n vertices, which is currently the best lower bound. Further, in [1] and [2], packing plane spanning graphs with short edges and spanning paths, respectively, have been considered.

Contribution. In this work, we provide a negative answer to Question 1 (refuting the prevalent conjecture). We even provide a negative answer to the following weaker question:

► **Question 2.** *Can every complete geometric graph on $2n$ vertices be partitioned into n plane subgraphs?*



■ **Figure 1** *Left:* A partition of W_8 into $n = 4$ plane spanning trees. *Right:* The bumpy wheel $BW_{5,3}$.

Note that the problem of partitioning a geometric graph into plane subgraphs is equivalent to a classic edge coloring problem, where each edge should be assigned a color in such a way that no two edges of the same color cross (of course using as few colors as possible). This problem received considerable attention from a variety of perspectives (see for example [11] and references therein) and is also the topic of the CG:SHOP challenge 2022 [7].

The point sets in our construction, so-called *bumpy wheel sets*, have been introduced in [12, 13]. For positive odd¹ integers k and ℓ , the *bumpy wheel* $BW_{k,\ell}$ is derived from the regular wheel W_{k+1} by replacing each of the k hull vertices by a *group* of ℓ vertices as follows. All vertices (except the center) lie on the convex hull and the vertices within each group are ε -close for some (small enough) $\varepsilon > 0$. In particular, the convex hull of any $\frac{k+1}{2}$ consecutive groups does not contain the center vertex (see Figure 1 for an illustration). Slightly abusing notation, $BW_{k,\ell}$ refers to the underlying point set as well as the complete geometric graph interchangeably. Note that for $\ell = 1$ we obtain a regular wheel set and for $k = 1$ a point set in convex position and hence we assume $k, \ell \geq 3$ in the following.

Our motivation to study bumpy wheels stemmed from the fact that Schnider [12] showed that $BW_{3,3}$ cannot be partitioned into plane double stars. In contrast, this is always possible for complete geometric graphs on regular wheel sets [2], as well as complete geometric graphs on point sets admitting n pairwise crossing edges [6] (which also includes convex point sets).

Our first main contribution in this work is to fully characterize for which (odd) parameters k and ℓ , the bumpy wheel $BW_{k,\ell}$ can and in particular *cannot* be partitioned into plane spanning trees or plane *subgraphs* (note that also in the setting of partitioning into plane subgraphs we are only interested in partitions into n subgraphs). Surprisingly, allowing arbitrary subgraphs instead of spanning trees does not help much, as it turns out that $BW_{3,5}$ is the only bumpy wheel that can be partitioned into plane subgraphs but not into plane spanning trees.

► **Theorem 3.** *For odd parameters $k, \ell \geq 3$, the edges of $BW_{k,\ell}$ cannot be partitioned into $n = \frac{k\ell+1}{2}$ plane spanning trees if and only if $\ell > 3$.*

► **Theorem 4.** *For odd parameters $k, \ell \geq 3$, the edges of $BW_{k,\ell}$ cannot be partitioned into $n = \frac{k\ell+1}{2}$ plane subgraphs if and only if $\ell > 5$ or ($\ell = 5$ and $k > 3$).*

¹ We require k and ℓ to be odd for an even number of vertices in total (k has to be odd anyway, since otherwise W_{k+1} would not be in general position).

We further consider the more general case of complete geometric graphs on point sets with exactly one point inside the convex hull. In this generalized setting, we show a sufficient condition for the non-existence of a partition into plane spanning trees (Theorem 16), and give a complete characterization for partitions into plane double stars (Theorem 17). As both results need more notation, their statements are deferred to their section (the same holds for the remaining results).

Given the negative answers to Questions 1 and 2, a natural generalization is to study partitions into beyond planar subgraphs, that is, subgraphs in which certain restricted crossing patterns are allowed. We initiate this study for two important classes of beyond planar graphs, namely, *k-planar subgraphs* (where every edge is crossed by at most k other edges) and *k-quasi-planar subgraphs* (in which no k edges pairwise cross). For the former, we show bounds on the number of subgraphs required for partitioning $K(P)$ for P in convex position (Proposition 19 and Theorem 20). For the latter, we show that a partition into 3-quasi-planar spanning trees is possible for any P with $|P|$ even (Lemma 23). This is best possible, as 2-quasi-planar graphs are plane. We further present bounds on the partition of any $K(P)$ into k -quasi-planar subgraphs for general k (Theorem 25).

We remark that it is straightforward to model the problem of partitioning into (plane) subgraphs as an integer linear program (ILP), which easily computes solutions for point sets up to roughly 25 points. None of the proofs in this paper rely on the computer assisted ILP, but it served as a great source of inspiration (see the full version [8] for further details).

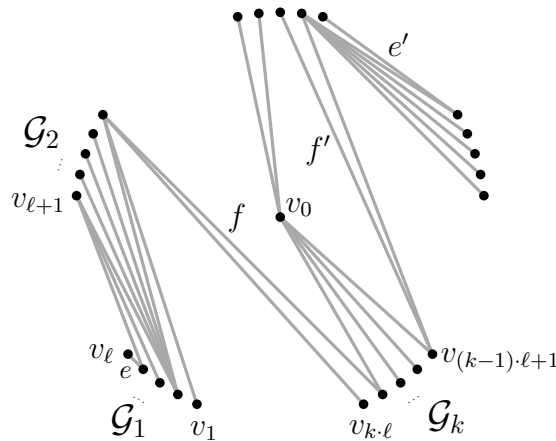
Organization of the paper. In Section 2, we prove Theorem 3 and Theorem 4, where we focus on the part showing the non-existence of partitions. In Section 3, we generalize our ideas from Section 2 about regular bumpy wheels to general wheel sets, proving Theorem 16 and Theorem 17. Finally, Sections 4 and 5 are dedicated to the more general setting of partitioning into k -planar and k -quasi-planar subgraphs, respectively.

2 Bumpy wheels

For a graph in (bumpy) wheel configuration we denote the center vertex by v_0 and the remaining vertices by v_1, \dots, v_{2n-1} in clockwise order. We also enumerate the groups in clockwise order: for $i \in \{1, \dots, k\}$, \mathcal{G}_i denotes the i 'th group (\mathcal{G}_1 contains v_1 , \mathcal{G}_k contains v_{2n-1})². An edge having v_0 as an endpoint is called a *radial edge*, an edge on the convex hull is called a *boundary edge* and all other edges are called *diagonal edges*. For a non-radial edge e , we define e^- to be the open halfplane defined by (the supporting line through) e and not containing v_0 , and similarly e^+ to be the open halfplane containing v_0 .

Additionally, we define a partial order $<_c$ on the set of non-radial edges, where $e <_c f$ if (the relative interior of) e completely lies in f^- (that is, f is “closer” to the center vertex v_0 than e). Two non-radial edges e, f are *incomparable* with respect to $<_c$, if neither $e <_c f$ nor $f <_c e$ holds (we omit “with respect to $<_c$ ” if it is clear from the context). In the following, when speaking of an edge e lying in f^- or in f^+ for another edge f , we always refer to the relative interior of e (that is, an endpoint of e may lie on the line through f – which actually means to coincide with an endpoint of f). A non-radial edge e is *maximal* in some set of edges E , if there is no other edge $e' \in E$ such that $e <_c e'$ (in the following we often consider maximal diagonal edges of plane spanning trees). *Minimal* edges are defined similarly. See Figure 2 for an illustration. Let us emphasize that we never use $<_c$ for radial edges.

² We will consider the index of a group \mathcal{G}_x always modulo k , but tacitly mean $((x-1) \bmod k) + 1$ (since our indexing starts with 1). The same holds for any other objects, e.g., the vertices on the convex hull.



■ **Figure 2** Example of a plane spanning tree on the bumpy wheel set $BW_{5,5}$. The diagonal edges f and f' are maximal. The edges e and e' are boundary edges (they are also the only minimal edges).

2.1 Partition into plane spanning trees

In this section, we prove Theorem 3. We remark that the non-existence direction almost follows from Theorem 4 (not even a partition into plane subgraphs is possible). The only case that is not covered is $BW_{3,5}$, which one can easily verify using computer assistance. However, since the proof of Theorem 3 is more instructive and intuitive, we decided to present it anyway and limit the proof of Theorem 4 to the essentials. We start with the non-existence:

► **Theorem 5.** *For any odd parameters $k \geq 3$ and $\ell \geq 5$, the edges of $BW_{k,\ell}$ cannot be partitioned into $n = \frac{k\ell+1}{2}$ plane spanning trees.*

Towards the proof of Theorem 5, we will first prove several structural results concerning the number and arrangement of radial and diagonal edges in the spanning trees of a potential partition (some of which have a similar flavor as those by Trao et al. [14]). We show that radial edges must lie between maximal diagonal edges and those maximal diagonal edges need to fulfill certain distance constraints. We will show that this cannot be satisfied if $\ell \geq 5$. Due to space constraints, we postpone the proofs of most preliminary results to the full version of this paper [8].

The following observation follows immediately from the construction of bumpy wheel sets and the definition of the partial order $<_c$.

► **Observation 6.** *For two non-radial, non-crossing, incomparable edges e, f the vertices in e^- and f^- are disjoint and neither e^- nor f^- contains an endpoint of the other edge.*

Note that e and f in the above observation may share an endpoint. Furthermore, for any set of edges E , two maximal edges $e, e' \in E$ are always incomparable.

► **Lemma 7.** *Let T be a plane spanning tree of $BW_{k,\ell}$. Then the following properties hold:*

- (i) *for any diagonal edge $e \in E(T)$, T contains at least one boundary edge in e^- ,*
- (ii) *for any pair of incomparable diagonal edges $e, f \in E(T)$, the boundary edges of T in e^- and f^- are distinct, and*
- (iii) *if T contains exactly one maximal diagonal edge, T contains at least $(\frac{k-1}{2}\ell + 1)$ consecutive radial edges (in particular, all radial edges of $\frac{k-1}{2}$ consecutive groups).*

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Note that any spanning tree in a partition of $BW_{k,\ell}$ contains a maximal diagonal edge, since the star around v_0 clearly cannot be used in such a partition.

► **Proposition 8.** *Let T_0, \dots, T_{n-1} be a partition of $BW_{k,\ell}$ into plane spanning trees (if it exists). Then exactly one of those trees, say T_0 , contains a single boundary edge and a single maximal diagonal edge and all other $n - 1$ trees contain exactly two boundary edges and exactly two maximal diagonal edges each. In particular, any diagonal edge $e \in E(T_i)$ contains exactly one boundary edge of T_i in e^- .*

From now on, T_0 always denotes the spanning tree with exactly one boundary edge (when considering a partition into plane spanning trees). Further, we let all radial edges $\{v_0, v_i\}$ for $i \in \{1, 2, \dots, \frac{k-1}{2}\ell + 1\}$ be part of T_0 (which we can assume without loss of generality due to symmetry).

For two non-radial, non-crossing edges e, f , define the *span* of e and f to be the (closed) area between the two edges, that is,

$$\text{span}(e, f) = \begin{cases} \text{cl}(e^+ \cap f^+) & \text{if } e \text{ and } f \text{ are incomparable} \\ \text{cl}(e^+ \cap f^-) & \text{if } e <_c f, \end{cases}$$

where $\text{cl}(\cdot)$ denotes the closure. The shaded area in Figure 3 for instance defines the span of two incomparable edges e and f .

Note, however, that we are more interested in the vertices and edges contained in the span, rather than the area itself. If we want to emphasize this, we may use the notation $V(\text{span}(e, f))$ or $E(\text{span}(e, f))$. In the following we are mostly interested in the span of maximal diagonal edges of some plane spanning tree.

► **Lemma 9.** *Let T_0, \dots, T_{n-1} be a partition of $BW_{k,\ell}$ into plane spanning trees (if it exists) and e, f be the maximal diagonal edges of some T_i ($i \neq 0$). Then, all edges of T_i in the span of e and f are radial (except e and f).*

Define the *distance* $\text{dist}(e)$ of a non-radial edge e to be the number of vertices in e^- plus one (or in other words, the number of boundary edges in $\text{cl}(e^-)$). Clearly, $1 \leq \text{dist}(e) \leq \frac{k+1}{2}\ell - 1$ holds for any non-radial edge e and $\text{dist}(f) < \text{dist}(e)$ holds for any edge $f \subseteq e^-$. It will be convenient to define, for $i \in \{1, \dots, \frac{k+1}{2}\ell - 1\}$:

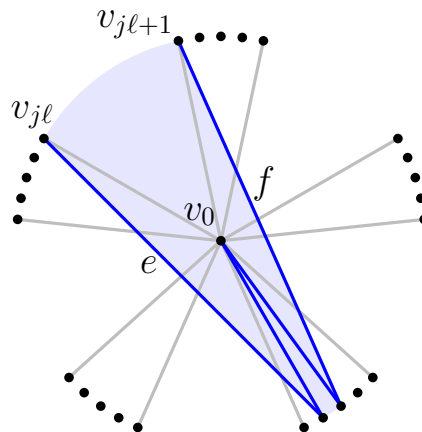
$$d_i = \frac{k+1}{2}\ell - i. \tag{1}$$

We define it in this (slightly counter-intuitive) way, d_1 being the largest distance, since we mostly deal with edges of large distances and thereby aim to improve the readability.

► **Lemma 10.** *Consider a plane spanning tree T of a partition of $BW_{k,\ell}$ and let e be a diagonal edge in T of distance $d = \text{dist}(e) > 1$. Then T also contains exactly one of the edges of distance $d - 1$ in e^- .*

We need a little more preparation towards the proof of Theorem 5. We call the first and last vertex of each group *outmost vertices* (and the corresponding radial edges *outmost radial edges*). Note that there are exactly $2k$ outmost radial edges in $BW_{k,\ell}$. Every hull vertex or radial edge that is not outmost, is called an *inside* vertex or an *inside* radial edge.

Furthermore, define two groups $\mathcal{G}_i, \mathcal{G}_j$ to be *opposite* if $|i - j| = \frac{k-1}{2}$ or $|i - j| = \frac{k+1}{2}$. In particular, each group has two opposite groups and two consecutive groups have exactly one opposite group in common (we call that group *the opposite group* of a pair of consecutive groups).



■ **Figure 3** All outmost radial edges are depicted in gray. The maximal diagonal edges e and f (connecting opposite groups) form a special wedge. Their span is shaded blue.

Let e, f be two maximal (non-crossing) diagonal edges which have an endpoint in a common group. Then the set of vertices of $\text{span}(e, f)$ in the common group is called *apex*. Note that any apex contains at least one vertex (and this lower bound is attained if the endpoints of e and f coincide).

Moreover, two maximal (non-crossing) diagonal edges $e = \{u, v\}$ and $f = \{u', v'\}$ form a *special wedge* if two endpoints (say u and u') are consecutive outmost vertices of different groups (that is, $u = v_{j\ell}$ and $u' = v_{j\ell+1}$ for some j) and v and v' are inside vertices lying in the opposite group of \mathcal{G}_j and \mathcal{G}_{j+1} . See Figure 3 for an illustration of these terms.

► **Proposition 11.** *Let T_0, \dots, T_{n-1} be a partition of $BW_{k,\ell}$ into plane spanning trees (if it exists) and let T_i ($i \neq 0$) be a spanning tree that does not use any outmost radial edge. Then the two maximal diagonal edges e, f of T_i form a special wedge and T_i has to use all radial edges incident to the apex of this wedge.*

Proof. We first argue that all but exactly two radial edges in $\text{span}(e, f)$ must be part of T_i . The subgraph of T_i induced by $V(\text{span}(e, f))$ needs to form a tree. Moreover, $\text{span}(e, f)$ contains $|V(\text{span}(e, f))| - 1$ radial edges. Since T_i uses the two diagonal edges $e, f \in E(\text{span}(e, f))$ and all other edges in the span need to be radial (Lemma 9), it has to use exactly all but two radial edges.

Furthermore, since we cannot have two maximal diagonal edges between the same pair of groups, the span of e and f contains at least two outmost vertices, namely in two different groups which contain an endpoint of e and f , respectively. On the other hand, $\text{span}(e, f)$ can neither contain a third outmost vertex nor an outmost vertex in its interior, since otherwise T_i has to use an outmost radial edge (by Lemma 9 and above argument). In particular, e and f share a common group and the apex does not contain any outmost vertex (hence, e and f form a special wedge, as depicted in Figure 3).

Moreover, since T_i has to use all but two radial edges in the span, it clearly has to use all radial edges incident to the apex. ◀

Note that for two spanning trees T_i, T_j ($i \neq j$) not using an outmost radial edge, their apices must be disjoint.

► **Proposition 12.** *Let T_0, \dots, T_{n-1} be a partition of $BW_{k,\ell}$ into plane spanning trees (if it exists). Then for each pair $\mathcal{G}, \mathcal{G}'$ of opposite groups and each $j \in \{1, \dots, \ell\}$ there is a unique diagonal edge (connecting \mathcal{G} and \mathcal{G}') of distance d_j (recall Equation (1)) that is maximal in its tree.*

Proof. Observe first that for any $j \in \{1, \dots, \ell\}$ there are exactly j edges of distance d_j (between \mathcal{G} and \mathcal{G}') and all edges of the same distance (between \mathcal{G} and \mathcal{G}') pairwise cross. Also note, for any two edges e, e' (between \mathcal{G} and \mathcal{G}') with $\text{dist}(e) > \text{dist}(e')$, either $e' \subseteq e^-$ holds or they cross. In particular, if they do not cross and belong to the same tree, the shorter is not a maximal edge.

Consider now for some $j \in \{2, \dots, \ell\}$ the distance d_j and let c_1, \dots, c_j be the colors³ used for all edges of this distance. By Lemma 10, there are $j - 1$ edges of (the larger) distance d_{j-1} using the same color as an edge of distance d_j , w.l.o.g. c_1, \dots, c_{j-1} . By the above arguments the corresponding edges of distance d_j cannot be maximal.

On the other hand, the color c_j cannot be used by any edge of larger distance, since again by Lemma 10 this color would have to appear in d_{j-1} as well. Hence, indeed the only edge of distance d_j that is maximal in its tree is the one of color c_j .

Lastly, for $j = 1$ observe that the single edge of distance d_1 is clearly maximal. ◀

Finally, we are ready to prove Theorem 5, which we restate here for the ease of readability:

► **Theorem 5.** *For any odd parameters $k \geq 3$ and $\ell \geq 5$, the edges of $BW_{k,\ell}$ cannot be partitioned into $n = \frac{k\ell+1}{2}$ plane spanning trees.*

Proof. Assume to the contrary that there is such a partition T_0, \dots, T_{n-1} . There are $2k$ outmost radial edges and T_0 uses (at least) k of them (see the remark after Proposition 8). Hence, there are at most $k + 1$ spanning trees (including T_0) containing an outmost radial edge.

Next, let us count how many spanning trees *not* containing an outmost radial edge we can have. Since, by Proposition 11, the apex of such a tree cannot use any outmost vertex nor any vertex already incident to a radial edge in T_0 , there remain $\frac{k+1}{2}(\ell - 2)$ possible vertices (to be used by apexes), namely the inside vertices of the last $\frac{k+1}{2}$ groups $\mathcal{G}_{\frac{k+1}{2}}, \dots, \mathcal{G}_k$ (which are not fully connected to v_0 by radial edges in T_0). Also recall that each apex contains at least one vertex.

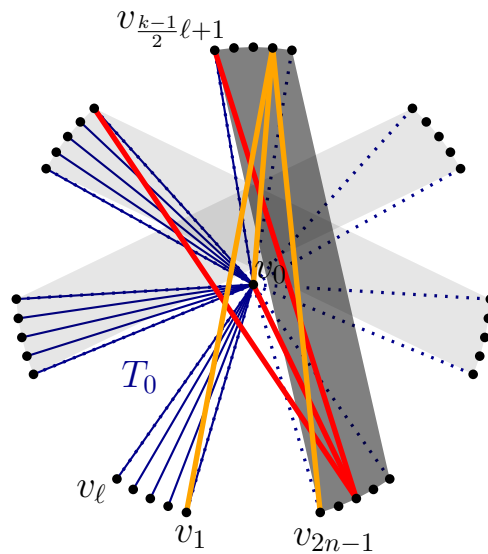
It is crucial to emphasize that among those last $\frac{k+1}{2}$ groups, group $\mathcal{G}_{\frac{k+1}{2}}$ and group \mathcal{G}_k are opposite (the only opposite pair). Therefore, by Proposition 11, two spanning trees with an apex in group $\mathcal{G}_{\frac{k+1}{2}}$ and group \mathcal{G}_k respectively, must each have a maximal diagonal edge between these two groups. Hence, by Proposition 12, we can have at most $(\ell - 2)$ spanning trees with apex in one of these two groups (instead of $2(\ell - 2)$); see Figure 4.

So, in total there can be at most $\frac{k-1}{2}(\ell - 2)$ spanning trees which do not use an outmost radial edge. Hence, whenever

$$k + 1 + \frac{k-1}{2}(\ell - 2) < \frac{k\ell + 1}{2}$$

holds, we cannot find enough spanning trees. Rearranging terms, this inequality is equivalent to $\ell > 3$. ◀

³ Instead of always spelling out that an edge belongs to a plane subgraph, we associate edges with colors.



■ **Figure 4** In the black stripes (the darker one is the crucial one) the maximal diagonal edges (of those trees without outmost radial edge) need to have distinct distances. That allows $\ell - 2$ many for each stripe. Two spanning trees (red and orange) with apex in group $\mathcal{G}_{\frac{k+1}{2}}$ and group \mathcal{G}_k both need to have a maximal diagonal edge in the dark stripe.

Next, we prove the other direction of Theorem 3:

► **Theorem 13.** *For any odd parameter $k \geq 3$, the edges of $BW_{k,3}$ can be partitioned into plane spanning trees.*

We only sketch the construction very briefly (the details can be found in the full version [8]).

Proof sketch. Our construction consists of three steps. In the first step, we give an explicit construction of a *partial partition* that covers all radial edges, each (partial) tree in the partition covers exactly its span, and between any pair of opposite groups exactly one diagonal edge of each distance d_1, d_2, d_3 is covered.

After that we extend this partial partition in two steps (these extensions actually work for arbitrary ℓ , but we stick to $\ell = 3$ for now). First we show that there is a unique way to extend the partial partition to one that covers all diagonal edges of distance d_1, \dots, d_3 . Roughly speaking, whenever we want to include some edge of distance d_i (between a certain pair of groups) we have two choices to which tree we can join it (see Lemma 10). However, since by construction exactly one edge of each distance is already covered, this determines the *orientation* how we can include the other edges of the same distance.

Once we covered all edges down to distance d_3 , there are precisely $2n - 1$ edges of each following distance and no edge of any smaller distance is already covered. Therefore, in each iteration (considering some distance $d_j < d_3$) we have the choice to fix some orientation (“left” or “right”) which determines how we need to extend all edges of distance d_j . Hence, in this second extension step there are $2^{\frac{3(k-1)}{2}-1}$ possible extensions. ◀

2.2 Partition into plane subgraphs

In the previous section, we gave a classification of which bumpy wheels can be partitioned into plane spanning trees and which cannot. Surprisingly it turns out that allowing arbitrary plane subgraphs does not help much. The only bumpy wheel that can be partitioned into plane subgraphs but not into plane spanning trees is $BW_{3,5}$.

Note that before we also heavily exploited the structure enforced by spanning trees. This is not possible anymore for the case of arbitrary plane subgraphs. We cannot make any assumptions on the number of edges, not even about connectedness. The only property we can (and will) exploit is the fact that we still have maximal diagonal edges and radial edges may only be contained in their span.

We split the proof of Theorem 4 into two parts, first focusing on the case $\ell > 5$.

► **Theorem 14.** *For any odd parameters $k \geq 3$ and $\ell > 5$, the edges of $BW_{k,\ell}$ cannot be partitioned into $n = \frac{k\ell+1}{2}$ plane subgraphs.*

The proof is more technical than for spanning trees. We give a detailed overview of the main ideas and postpone the full proof to the full version [8].

Proof sketch. Assume D_0, \dots, D_{n-1} is a partition into plane subgraphs. Then, a crucial insight is that between any pair of opposite groups and any distance $d_i = \frac{k+1}{2}\ell - i$ (for $1 \leq i \leq \ell$) there have to be at least i diagonal edges of distance at least d_i which are maximal in their subgraph. This follows from the fact that all edges of distance d_i between a fixed pair of opposite groups $\mathcal{G}, \mathcal{G}'$ form a crossing family. In particular, all of them get a different color and are either maximal or have another (larger) maximal edge between \mathcal{G} and \mathcal{G}' .

Further, this enables us to define a set E_{forced} of exactly $k \cdot \ell$ forced diagonal edges fulfilling the just mentioned distance constraints. In particular,

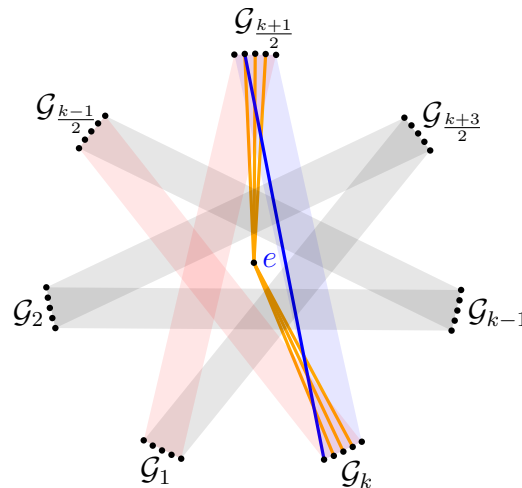
$$\sum_{e \in E_{\text{forced}}} \text{dist}(e) \geq k \sum_{i=1}^{\ell} \left(\frac{k+1}{2}\ell - i \right)$$

holds. Our goal will be to argue that we cannot accommodate all these forced diagonal edges and all radial edges at the same time.

To this end, note that we cannot have too many pairwise incomparable edges in a plane subgraph, more precisely their distance sums to at most $2n - 2$. In fact, it turns out that again we have one subgraph, say D_0 , containing exactly one forced diagonal edge, while all other $n - 1$ subgraphs contain exactly two of them.

Now the pairs of forced diagonal edges in our subgraphs again form a *span* (similar as in the spanning tree setting). Furthermore, radial edges may only be contained in this span (be careful, we are not assuming that there are radial edges in the span, but if the subgraph wants to use a radial edge it has to be in the span). We noted above that the distances of forced diagonal edges in the subgraph D_i sum up to at most $2n - 2$, say they sum up to $2n - 2 - x_i$ (and $\text{dist}(e) = d_1 - x_0$ for the single forced diagonal edge e of D_0). Then these x_i 's allow some additional margin to accommodate radial edges in the spans (or *additional vertices* as we call them). However, and this is the second crucial insight, we show that this additional margin is at most

$$\sum_{i=0}^{n-1} x_i \leq \frac{\ell - 1}{2}.$$



■ **Figure 5** High level overview of the proof of Theorem 15. We have at most $\frac{5-1}{2} = 2$ additional vertices in total and the blue stripe (which contains the single forced diagonal edge e of D_0) has to use both of them. Then, in the grey stripes we must use all forced diagonal edges of distances d_2, d_3, d_4 . Finally, since the two red stripes intersect ($k \geq 5$), there will not be enough forced diagonal edges left to pair all 6 forced diagonal edges of distances d_2, d_3, d_4 from the red stripes.

Finally, we consider only the $2\ell - 4$ inside radial edges of the opposite pair of groups $\mathcal{G}, \mathcal{G}'$ containing the endpoints of e (the single forced diagonal edge of D_0). Any subgraph with an apex in one of the two groups also has a forced diagonal edge between them. Putting everything together, this implies that we can cover at most

$$(\ell - 1) + \frac{\ell - 1}{2} = \frac{3}{2}(\ell - 1)$$

of these $2\ell - 4$ inside radial edges. In other words, whenever $\frac{3}{2}(\ell - 1) < 2\ell - 4$ holds, we cannot cover all edges. This inequality is equivalent to $\ell > 5$. ◀

For the case $\ell = 5$, we need to go even deeper into the structure of our plane subgraphs.

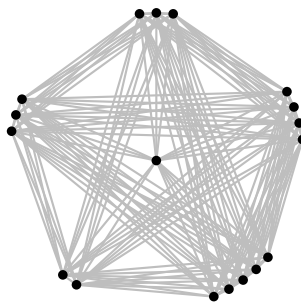
► **Theorem 15.** *For any odd parameter $k \geq 5$, the edges of $BW_{k,5}$ cannot be partitioned into $n = \frac{5k+1}{2}$ plane subgraphs.*

Figure 5 gives a brief sketch of the proof from a high level view. The full proof can also be found in the full version [8].

Finally, using Theorem 3, it only remains to show that there is a partition for $BW_{3,5}$, which is easy to compute (using computer assistance), and can be found in the full version [8].

3 Generalized wheels

In this section we generalize our construction to non-regular wheel sets. We give a sufficient condition in the setting of plane spanning trees and a full characterization for partitioning into plane double stars. For $N = [n_1, \dots, n_k]$ and integers $n_i \geq 1$, GW_N denotes the *generalized wheel* with group sizes n_i (in the given circular order). As before, the arrangement of the k groups resembles a regular k -gon around the center vertex, the vertices within each group are ε -close, and k is odd (see Figure 6). And for our purpose we also require $\sum_i n_i$ to be odd.



■ **Figure 6** Illustration of a generalized wheel ($GW_{[2,3,3,4,5]}$).

Note that the geometric regularity of generalized wheels is not strictly required (but eases the proofs). In fact, one can show that for any point set P (in general position) with exactly one point inside its convex hull, there is a generalized wheel with the exact same set of crossing edge pairs (further details can be found in the full version [8]).

► **Theorem 16.** *Let GW_N be a generalized wheel with k groups and $2n$ vertices. Then GW_N cannot be partitioned into plane spanning trees if each family of $\frac{k-1}{2}$ consecutive groups contains (strictly) less than $n - 2$ vertices.*

The proof, which is similar to the one of Theorem 5, can be found in the full version [8].

Plane double stars. Considering the other side of the story, it turns out that many generalized wheels can already be partitioned into plane double stars⁴:

► **Theorem 17.** *Let GW_N be a generalized wheel with k groups and $2n$ vertices. Then GW_N cannot be partitioned into plane spanning double stars if and only if there are three families of $\frac{k-1}{2}$ consecutive groups, each of which contains at most $n - 2$ vertices, such that each group is in at least one family.*

The proof requires several tools introduced by Schnider [13]. In a first step we identify conditions under which a point set admits a so-called *spine matching* – the collection of spine edges from a partition into double stars. Using these conditions we show that a generalized wheel GW_N cannot be partitioned into plane double stars if and only if GW_N has three *bad halfplanes* whose intersection is empty (for a non-radial *halving* edge e , the closure of e^- defines a *bad halfplane*). All details can be found in the full version [8].

We phrased Theorem 17 this way to make it consistent with Theorem 16; however, let us rephrase it in a way that better indicates the gap between the two theorems. Let F_i denote the family of $\frac{k-1}{2}$ consecutive groups starting at \mathcal{G}_i in clockwise order (whenever speaking of a family without further specification, we refer to such a family of $\frac{k-1}{2}$ groups for the remainder of this section). Two families F_i and F_{i+1} are called *consecutive* and $|F_i|$ denotes the number of vertices in F_i . If $|F_i| \leq n - 2$ holds, we call F_i *small*, and otherwise *large*.

► **Corollary 18.** *Let GW_N be a generalized wheel with k groups and $2n$ vertices. Then GW_N can be partitioned into plane spanning double stars if and only if there are $\frac{k-1}{2}$ consecutive families each containing (strictly) more than $n - 2$ vertices.*

⁴ All double stars in this section are spanning (which we may not always spell out for readability).

Proof. If, for the one direction, there are $\frac{k-1}{2}$ large consecutive families, then there is a group \mathcal{G}^* (namely the one that is contained in all these $\frac{k-1}{2}$ families) such that any family containing \mathcal{G}^* is large. In particular, there cannot be three small families covering all groups. Hence, by Theorem 17, there is a partition into plane double stars.

On the other hand, if there are no $\frac{k-1}{2}$ large consecutive families, we can find three small families as follows. Note first that every group is contained in some small family. Pick a small family F arbitrarily and let \mathcal{G} be the first group after F (in clockwise order). Among all small families containing \mathcal{G} , pick the one that is “furthest” from F , that is, has least overlap with F , and call it F' . Let \mathcal{G}' again be the first group after F' and among all small families containing \mathcal{G}' pick the one furthest from F' and call it F'' . Since F'' cannot contain \mathcal{G} , we conclude that the three small families F, F', F'' cover all groups. ◀

4 Partitions into k -planar subgraphs

In this section, we consider a generalization to partitioning into k -planar subgraphs (for $k = 0$ this amounts to the previous partitioning into plane subgraphs). We focus on the special case where the input point set is in convex position. Our first result fully resolves this problem for $k = 1$. Note that we do not require even sized point sets.

► **Proposition 19.** *For a point set P in convex position with $|P| = n \geq 5$, $K(P)$ can be partitioned into $\lceil \frac{n}{3} \rceil$ 1-planar subgraphs and $\lceil \frac{n}{3} \rceil$ subgraphs are required in every 1-planar partition.*

The proof can be found in the full version [3]. More generally, we show the following bounds:

► **Theorem 20.** *For an n -point set P in convex position and every $k \in \mathbb{N}$, $K(P)$ admits a partition into at most $\frac{n}{\sqrt{2k}}$ k -planar subgraphs. More precisely, for every integer $s \geq 2$, $K(P)$ admits a $\frac{(s-1)(s-2)}{2}$ -planar partition into $\lceil \frac{n}{s} \rceil$ subgraphs.*

Conversely, for every $k \in \mathbb{N}$, at least $\frac{n-1}{4.93\sqrt{k}}$ subgraphs are required in any k -planar partition of $K(P)$.

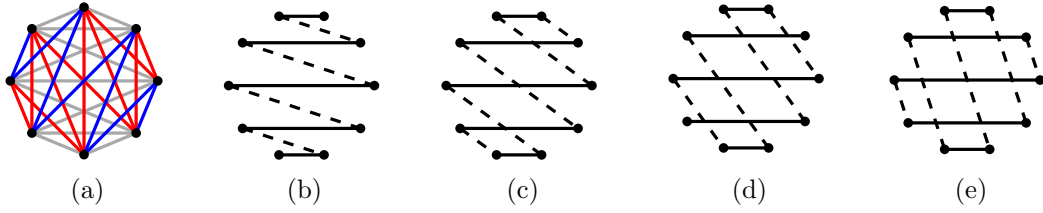
For the proofs of Proposition 19 and Theorem 20 (in particular for the lower bounds) it will be necessary to understand how many edges a single color class, or in other words, how many edges a k -planar subgraph of a convex geometric K_n , can maximally have. Once such bounds are established, we will be able to lower-bound the number of colors required in any k -planar partition of a convex geometric K_n by considering the “largest” color class.

We postpone this analysis, which also includes an improvement of the well-known crossing lemma for convex geometric graphs, to the full version [3] and only state the main ingredient that we need for the proof of Theorem 20:

► **Theorem 21.** *For every $k \geq 5$, every convex k -plane graph G on n vertices has at most $\sqrt{\frac{243}{40}k} \cdot n$ edges.*

Proof of Theorem 20. Let us first prove the upper bound. To this end, suppose that $s \geq 2$ is such that $\frac{(s-1)(s-2)}{2} \leq k$, and let us show that $K(P)$ can be partitioned into $\lceil \frac{n}{s} \rceil$ k -planar subgraphs. W.l.o.g. assume that the points in P form a regular n -gon. Consider all possible n slopes of segments and sort those in circular order. Next, partition this list of slope values into $\lceil \frac{n}{s} \rceil$ (contiguous) intervals of size at most s . Then, define a color class for all edges whose slopes fall into a common interval of this partition, see Figure 7(a).

6:14 Edge Partitions of Complete Geometric Graphs



■ **Figure 7** (a) Partition into 1-planar subgraphs by composing groups of (at most) 3 consecutive slopes each. (b)-(e) Edges with slope distance 1/2/3/4 intersect at most 0/1/2/3 times.

We show that all these subgraphs are $\frac{(s-1)(s-2)}{2}$ -planar. To this end, define the *slope distance* to be the distance between two slope values in the circularly sorted list of slopes. Note that edges cannot be crossed by other edges of the same slope or slope distance 1; by at most one edge of slope distance 2, by at most two edges of slope distance 3, etc. (see Figure 7(b)-(e)). Hence, if an edge e has color i , and if the slope of e is the j -th slope ($j \in \{1, \dots, s\}$) in its circular interval of slopes, then e can cross with at most the following amount of edges of color i :

$$\begin{aligned} \sum_{1 \leq k < j-1} (j-k-1) + \sum_{j+1 < k \leq s} (k-j-1) &= \frac{(j-1)(j-2)}{2} + \frac{(s-j)(s-j-1)}{2} = \\ &= \frac{(s-1)(s-2)}{2} - (s-j)(j-1) \leq \frac{(s-1)(s-2)}{2}. \end{aligned}$$

For the lower bound, note that $K(P)$ has $\frac{n(n-1)}{2}$ edges, and that in every k -planar partition of $K(P)$, every color class induces a convex k -plane subgraph on n vertices. Hence, by Theorem 21, every color class has size at most $\sqrt{\frac{243}{40}k} \cdot n$. So, the number of colors required in any k -planar partition is at least

$$\frac{\binom{n(n-1)}{2}}{\sqrt{\frac{243}{40}k} \cdot n} \geq \frac{n-1}{4.93\sqrt{k}}.$$

This concludes the proof. ◀

The following intriguing question is left open by our study.

► **Question 22.** *Is the upper bound in Theorem 20 tight up to lower-order terms?*

More generally, it would be interesting to shed some more light on the “in-between-cases” coming out of the upper bound in Theorem 20, where the term $\frac{(s-1)(s-2)}{2}$ covers only the values 0, 1, 3, 6, 10, For instance, can we partition convex complete geometric graphs with fewer colors into 2-planar subgraphs than we need for the 1-planar partition? More generally, for $\frac{(s-1)(s-2)}{2} < k < \frac{s(s-1)}{2}$, can we improve upon the $\lceil \frac{n}{s} \rceil$ bound from Theorem 20 for k -planar partitions? This question is surprisingly difficult (even for $k = 2$)⁵ and we do not know of any improvements of the bounds for these “in-between-cases”.

⁵ Using computer assistance, we can show that $\frac{3n}{10}$ colors are required for any 2-planar partition (almost matching the $\frac{n}{3}$ bound from the 1-planar partition). We omit this computer assisted result as it is a very special case and not even answering the question whether or not the bound can be improved for $k = 2$.

5 Partitions into k -quasi-planar subgraphs and spanning trees

In this section, we develop bounds on the number of colors required in a k -quasi-planar partition for point sets in general position (for $k = 2$ this again amounts to the setting of plane subgraphs, hence we assume $k \geq 3$ in the following). The setting of spanning trees is easily resolved by the following lemma (whose proof can be found in the full version [3]).

► **Lemma 23.** *Let P be a point set of size $2n$, then the complete geometric graph $K(P)$ can be partitioned into n 3-quasi-planar spanning trees.*

So, we turn our attention to the subgraph setting. The main ingredient towards the proof of Theorem 25 is the following lemma concerning point sets admitting a perfect crossing-matching, that is, a crossing family of size $|P|/2$. Note that in this case any edge in the crossing family determines a halving line [10].

► **Lemma 24.** *Let P be a point set of size $2n$, with a crossing family of size n , then $\lceil \frac{n}{k-1} \rceil$ colors are required and sufficient to partition $K(P)$ into k -quasi-planar subgraphs.*

Again, due to space constraints, we postpone the proof to the full version [3].

► **Theorem 25.** *Let P be a set of n points in general position and denote the size of a largest crossing family on P by m . Also let $k \geq 3$ s.t. $k \leq m$ (otherwise one color is always sufficient). Then, at least $\lceil \frac{m}{k-1} \rceil$ colors are required and at most $\lceil \frac{m}{k-1} \rceil + \lceil \frac{n-2m}{k-1} \rceil$ colors are needed to partition the complete geometric graph $K(P)$ into k -quasi-planar subgraphs.*

Proof. Let $P' \subseteq P$ be the subset of endpoints induced by a largest crossing family of size m .

Then, the lower bound follows immediately from Lemma 24 applied on P' .

For the upper bound, divide the point set $P \setminus P'$ into disjoint subsets Q_1, \dots, Q_c of size $k-1$, where $c = \lceil \frac{n-2m}{k-1} \rceil$. For each edge with an endpoint in some Q_i assign it the color i (for edges that have two choices, pick one arbitrarily). Certainly, each color class is k -quasi-planar, since it consists of (at most) the union of $k-1$ stars. Together with $K(P')$, which we can clearly partition by using $\lceil \frac{m}{k-1} \rceil$ colors, the upper bound follows. ◀

6 Conclusion

We showed that there are complete geometric graphs that cannot be partitioned into plane spanning trees and gave a full characterization of partitionability for bumpy wheels (even in the much broader setting of partitioning into plane subgraphs). Also, for generalized wheels we gave sufficient and necessary conditions. There are two natural directions for further research in this setting. On the one hand, one could try to further classify which point sets can be partitioned and which cannot (this might also be a useful approach towards the question concerning the complexity of the decision problem whether a given complete geometric graphs admits a partition into plane spanning trees). On the other hand, we initiated the study of partitions into broader classes of subgraphs, namely k -planar and k -quasi-planar.

The intriguing question to determine *how far* we may get from the $\frac{|P|}{2}$ bound is still open:

► **Question 26** ([6]). *Can any complete geometric graph on n vertices be partitioned into $\frac{n}{c}$ plane subgraphs for some constant $c > 1$?*

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