UNIVERSITÀ DEGLI STUDI DI PADOVA

Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica Pura ed Applicata

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE

INDIRIZZO MATEMATICA

XXI CICLO

MONOTONE 2-GROUPS

Direttore della Scuola: Ch.mo Prof. Paolo Dai Pra

Supervisore: Ch.mo Prof. Federico Menegazzo

Dottorando: Eleonora Crestani

Sommario

I problemi di generazione sono problemi estremamente interessanti nella teoria dei gruppi finiti. Tali problemi spesso si riducono a problemi sui generatori di p-gruppi. Questo ha portato ad un sempre maggiore interesse per i problemi di generazione nei p-gruppi e allo studio di classi di p-gruppi finiti in cui i generatori del gruppo e dei sottogruppi soddisfano alcune precise condizioni.

Di particolare interesse è la classe dei p-gruppi finiti G tali che il numero di generatori di ogni sottogruppo H di G è minore o uguale del numero di generatori di G.
Esempi di p-gruppi appartenenti a questa classe sono i p-gruppi abeliani, i p-gruppi modulari e i p-gruppi powerful. Soddisfano tale proprietà anche i p-gruppi monotoni. Per questi ultimi ricordiamo la definizione.

Definizione. *Dato G un gruppo, sia $d(G)$ il numero di generatori di G. Un p-gruppo G si dice monotono se per ogni H e K sottogruppi di G con H contenuto in K, si ha $d(H) \leq d(K)$.*

I p-gruppi monotoni sono stati introdotti da Mann durante una conferenza tenutasi a Saint Andrews nel 1985.
Lo stesso autore, in "*The number of generators of finite p-groups*" (vedi [10]), lavoro pubblicato nel 2005, studia i p-gruppi monotoni e li classifica per p dispari. Del caso $p = 2$, non viene data alcuna classificazione ma vengono date alcune proprietà interessanti. Ad esempio, Mann dimostra che un 2-gruppo G è monotono se e solo se i sottogruppi 2-generati di G sono metaciclici.

In questa tesi vengono studiati e classificati completamente i 2-gruppi monotoni. Per i risultati si rimanda ai Teoremi 1.4, 1.5 e 1.6.

Abstract

The generation problems are very interesting in the theory of finite groups. These problems can often be reduced to problems on the generators of p-groups. This has led to an increasing interest on the problems of generation in p-groups and on the study of classes of p-groups in which generators satisfy some precise conditions.

In particular, it is very interesting the class of finite p-groups G with the property that the rank of G is equal to the number of generators of G (i.e. the number of generators of every subgroup of G is smaller than or equal to the number of generators of G). For instance, the abelian, the modular and the powerful p-groups belong to this class. Also the monotone p-groups lie in this class. We recall here the definition of monotone p-groups.

Definition. *Let G be a group. We denote with $d(G)$ the number of generators of G. A p-group G is monotone if for every H and K subgroups of G with H contained in K, we have that $d(H) \leq d(K)$.*

The class of monotone p-groups was introduced by A. Mann during the 1985 Saint Andrews Conference. In the paper "*The number of generators of finite p-groups*" (see [10]) published in 2005, Mann studies the monotone p-groups and classifies the monotone p-groups for p odd. When $p = 2$, Mann does not classify the monotone 2-groups, but he gives some remarkable properties. For instance, he proves that a 2-group G is monotone if and only if the 2-generated subgroups of G are metacyclic.

In this thesis, the monotone 2-groups are studied and completely determined. For the main results we refer to Theorems 1.4, 1.5 and 1.6.

Contents

Chapter 1

Introduction

Very many problems in finite group theory are concerned with generators. Also, very often problems on the generators of a general group can be reduced to problems on the generators of p-groups.

For instance, we recall that if every Sylow p-subgroup of a group G is d-generated, then G is $d+1$-generated (see [8]). Another result in this direction states that every permutation group of degree $n \geq 3$ is $n/2$-generated (a proof reduces to dealing with p-groups).

These and many other applications lead to investigations on generation problems of p-groups. Since general problems concerning generators of p-groups are quite hard to deal with, authors try to investigate particular classes of p-groups, where generating systems satisfy some conditions.

A very interesting class is the class of those finite p-groups G in which the rank of G is equal to the number of generators of G, i.e. the class of finite p-groups G such that the number of generators of any subgroup H of G is smaller than or equal to the number of generators of G.

Abelian, modular and powerful p-groups are examples of p-groups belonging to this class. Other important p-groups, where the rank equals the number of generators, are the d-maximal p-groups (a p-group G is said to be d-maximal if G is d-generated and every proper subgroup of G is generated by strictly less than d elements). Laffey, in [6], proves that the nilpotency class of a d-maximal p-group for $p \geq 3$ is at most 2. For $p = 2$ not much is known, but it seems to be very difficult to investigate the d-maximal 2-groups. Some results about d-maximal 2-groups are in [5].

Another family of p-groups that belongs to this class is the family of monotone p-groups. We first recall the definition.

Definition 1.1. *Let G be a group. We denote with $d(G)$ the number of generators of G. A p-group G is monotone if for every H and K subgroups of G with H contained in K, we have that $d(H) \leq d(K)$.*

The monotone p-groups were introduced by Avinoam Mann during the 1985 Saint Andrews Conference (see [9]).
More precisely, he defined the classes of p-groups M_s, where $1 \leq s \leq p$.

Definition 1.2. *A p-group G is said to be in M_s, where $1 \leq s \leq p$, if it satisfies the following condition:*
if $H \leq K \leq G$, $|K : H| = p$ and K is not cyclic then $d(H)-1 < s(d(K)-1)$.

The restriction on the range of s is imposed to avoid trivialities. In fact, for $s = 1$, the p-groups in M_1 are the elementary abelian p-groups, the cyclic p-groups and the quaternion group of order 8.
For $s > p$, all p-groups lie in M_s, because of Schreier's inequality (i.e. in every group G, if H is a subgroup of finite index, then $d(H) - 1 \leq |G : H|(d(G) - 1)$).

Among the classes M_s, the class M_2 is strictly related with the class of the monotone p-groups.
In fact, every monotone p-group is in M_2. Moreover, in the paper "*The number of generators of finite p-groups*" (see [10]), Mann studies the classes M_s, and he shows that, when $p = 2$ the class M_2 coincides with the class of the monotone 2-groups.
In the same paper, Mann gives a strong characterization for the groups in M_2.

Proposition 1.1. *A p-group G is in M_2 if and only if every subgroup of a 2-generated subgroup of G is 2-generated.*

Using this property, Mann classifies completely, except for some uncertainty for exponent p^2, the monotone p-groups and the p-groups in M_2, when p is odd.
In particular, he shows that for $p > 3$, the monotone p-groups are (apart from a small number of exceptions) modular, whereas for $p = 3$, there are also some monotone 3-groups of maximal class and some other related groups. See Theorems 8, 9, 10 in [10] for a full account.

For $p = 2$, things are much harder and in literature there is no complete classification of monotone 2-groups.

Nevertheless, in his work, Mann gives some important properties for monotone 2-groups.

In fact for $p = 2$, Proposition 1.1 can be refined into the following:

Proposition 1.2. *A 2-group G is monotone if and only if every 2-generated subgroup of G is metacyclic.*

Another remarkable property of the class M_2 is related with the subgroups $H_q(G)$ of G. We first recall the definition of $H_q(G)$.

Definition 1.3. *Let G be a p-group and let $q = p^e$. Then $H_q(G) = \langle x \in G : x^q \neq 1\rangle$.*

Mann shows the following:

Proposition 1.3. *Let p be equal to 2 or 3. If G is a p-group in M_2, and $exp(G) > q$, then $|G : H_q(G)| \leq p$.*

Moreover, in his work Mann classifies the power-closed 2-groups.
We recall that a p-group G is said to be power-closed if in each section of G a product of p-th powers is again a p-th power.
It turns out that, for p odd, every monotone p-group is power-closed. For $p = 2$ every power-closed 2-group is monotone but the converse is not true (for example the group $\langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1}\rangle$ is monotone but not power-closed).

In this thesis, we classify all the finite monotone 2-groups. In Chapter 3 we completely determine the monotone 2-groups of exponent 4. Chapter 4 and Chapter 5 deal with monotone 2-groups of exponent greater than 4. By Proposition 1.3, we get that, when G is a monotone 2-group of exponent greater than 4, the subgroup $H_4(G)$ is either maximal or the whole group. In particular, in Chapter 4 the monotone 2-groups of exponent greater than 4 and such that $|G : H_4(G)| = 2$ are determined. The last chapter is dedicated to the monotone 2-groups of exponent greater than 4 and such that $G = H_4(G)$.

Given a 2-group G, we write $G = H * K$ to mean that $G = HK$ and $[H, K] = 1$, i.e. G is a central product of the two subgroups H and K. Each time we use the symbol $G = H * K$ we shall specify the intersection $H \cap K$ in G.

We now report the main results of each chapter.

Theorem 1.4. *Let G be a monotone 2-group of exponent 4. Then G is either abelian or isomorphic to one of the groups in the following list:*

- $E * A$, *where E is an extraspecial group and A is either an abelian group of the form $C_4 \times C_2 \times \cdots \times C_2$ and $E^2 = A^2$ or an elementary abelian group and $E \cap A = 1$;*
- $A\langle b\rangle$, *where A is an abelian group of exponent 4, $|b| \leq 4$, and $a^b = a^{-1}$ for all $a \in A$;*
- $\langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b\rangle \times A$, *where A is elementary abelian;*
- $\langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b^3\rangle \times A$, *where A is elementary abelian;*
- $\langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = b^3, c^d = cb^2\rangle \times A$, *where A is elementary abelian;*
- $\langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = bd^2, c^d = c^3\rangle \times A$, *where A is elementary abelian;*
- $\langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b^3, a^d = ad^2, b^d = b, c^d = ca^2\rangle \times A$, *where A is elementary abelian.*

Theorem 1.5. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$. Then G is isomorphic to one of the groups in the following list:*

- $A\langle u\rangle$, *where A is abelian of exponent $2^n \geq 8$, $u^2 \in \Omega_1(A)$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$ for every $a \in A$;*
- $\langle a, b, u\rangle \times A$, *where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 2$, $\langle a, b\rangle$ is abelian, $u^2 = a^{2^{n-1}}$, $b^u = ba^{2^{n-1}}$, $a^u = a^{-1}$;*
- $\langle a, b, u\rangle \times A$, *where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 4$, $\langle a, b\rangle$ is abelian, $u^2 = b^2$ and $a^u = a^{-1}$, $b^u = b^{-1}a^{2^{n-1}}$;*
- $\langle a, u\rangle * E \times A$, *where $|a| = 2^n \geq 8$, E is extraspecial, A is elementary abelian, $u^2 \in \langle a^{2^{n-1}}\rangle$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$ and $E^2 = \langle a^{2^{n-1}}\rangle$;*
- $\langle a, u\rangle * E * A$, *where E is extraspecial, A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$, $|a| = 2^n \geq 8$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$ and $A^2 = E^2 = \langle a^{2^{n-1}}\rangle$;*

- $\langle a,u,b\rangle * E \times A$, *where* $|a| = 2^n \geq 8$, E *is extraspecial,* A *is elementary abelian,* $|b| = 2$, $u^2 \in \langle a^{2^{n-1}}\rangle$, $a^b = a^{1+2^{n-1}}$, $a^u = a^{-1}$, $b^u = ba^{4h}$ *with* $|a^{4h}| \leq 2$ *and* $E^2 = \langle a^{2^{n-1}}\rangle$*;*

- $\langle a,u,b\rangle * E * A$, *where* E *is extraspecial,* A *is abelian of the form* $C_4 \times C_2 \times \cdots \times C_2$, *and* $|a| = 2^n \geq 8$, $|b| = 2$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1}$, $b^u = b$, $a^b = a^{1+2^{n-1}}$, *and* $A^2 = E^2 = \langle a^{2^{n-1}}\rangle$*;*

- $\langle a,b,u\rangle \times A$, *where* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$, $a^b = a^{1+2^{n-1}}$ *and* $u^b = u^{-1}$*;*

- $\langle a,c,b,u\rangle \times A$, *where* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|c| = 2$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1}$, $a^b = a^{1+2^{n-1}}$, $u^b = u^{-1}$, $c^a = c$, $c^b = c$ *and* $c^u = ca^{2^{n-1}}$*;*

- $\langle a,b,u\rangle \times A$, *where* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|b| = 4$, $|u| = 4$, $u^2 = b^2$, $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$, $b^u = b^{-1}a^{2^{n-1}}$, $a^b = a^{1+2^{n-1}}$.

Theorem 1.6. *Let* G *be a non-trivial monotone* 2*-group such that* $G = H_4(G)$*. Then* G *is either a modular group that does not involve* Q_8 *or is isomorphic to one of the groups in the following list:*

- $\langle a,c\rangle * E \times A$, *where* E *is extraspecial,* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|c| = 2$ *and* $a^c = a^{1+4h}$ *with* $|a^{4h}| \leq 2$, $E^2 = \langle a^{2^{n-1}}\rangle$*;*

- $\langle a,b,c\rangle \times A$, *where* A *is elementary abelian,* $|c| = 2^n \geq 8$, $|a| = 4$, $a^2 = b^2$, $c^a = c^{1+4h_1}$, $c^b = c^{-1+4h_2}$ *and* $a^b = a^{-1}c^{4h_3}$, *with* $|c^{4h_i}| \leq 2$ *for* $i = 1,2,3$*;*

- $\langle a,b,c,d\rangle \times A$, *where* A *is elementary abelian,* $|c| = 2^n \geq 8$, $|a| = 4$, $a^2 = b^2$, $|d| = 2$, $c^a = c$, $c^b = c$ *and* $a^b = a^{-1}c^{4h}$ *with* $|c^{4h}| \leq 2$, *and* $c^d = c^{1+2^{n-1}}$, $a^d = a$ *and* $b^d = b$*;*

- $\langle a,b,c,d\rangle * E \times A$, *where* A *is elementary abelian,* E *is extraspecial,* $|a| = 2^n \geq 8$, $b^4 = a^{2^{n-1}}$, $\langle c,d\rangle$ *is elementary abelian,* $a^b = a^{-1+4h}$, *with* $|a^{4h}| \leq 2$, $a^c = a^{1+4h_1}$, $a^d = a$, $b^c = b$, $b^d = b^{1+4h_2}$, *where* $|a^{4h_1}| \leq 2$ *and* $|b^{4h_2}| \leq 2$ *and* $E^2 = \langle a^{2^{n-1}}\rangle$*;*

- $A\langle b\rangle$, *where* A *is an abelian group,* $|b| \geq 8$ *and* $a^b = a^{-1+4h}$, *for every* $a \in A$*;*

- $\langle A, c, b\rangle$, *where* A *is an abelian group of exponent* 2^n, *with* $n \geq 3$, $A^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $|b| \geq 8$, $a^b = a^{-1+4h}$, $a^c = a^{1+2^{n-1}}$ *for every* $a \in A$, $c^b = c^{-1+4h}$ *and* $exp\,(\langle A, c\rangle^{4h}) < |b^2| < 2^n$;

- $\langle A, c, b\rangle$ *where* A *is an abelian group of exponent* 2^n, *with* $n \geq 3$, $A^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $|b| \geq 8$, $a^b = a^{-1+4h+2^{n-1}}$, $a^c = a^{1+2^{n-1}}$ *for every* $a \in A$, $c^b = c^{-1+4h}$, $|c| > 2^n$, $|c^{4h}| = |b^2|$, $|b^2| < 2^n$, *and* $\langle b\rangle \cap \langle c\rangle = 1$.

We note that all the groups in the above lists are in fact monotone and so Theorem 1.4, Theorem 1.5 and Theorem 1.6 comprise the classification of monotone 2-groups.

All the groups considered in this thesis are finite.

Chapter 2

Preliminaries

2.1 Definitions and Notations

In this first section, we give some definitions and state some well-known facts that will be often used throughout the thesis.

A very basic fact, that is often used in this thesis is the following:

Remark 2.1. *If a group H is generated by X, then the derived subgroup of H is the normal closure in H of the subgroup $\langle [x_1, x_2] : x_1, x_2 \in X \rangle$.*
If H_1 and H_2 are subgroups of H generated respectively by X_1 and by X_2, then $[H_1, H_2]$ is the normal closure in $\langle X_1, X_2 \rangle$ of the subgroup generated by $\langle [x_1, x_2] : x_1 \in X_1, x_2 \in X_2 \rangle$.

Let us now recall the definition of powerful p-groups.
As usual, if G is a p-group we denote with $G^n = \langle x^n : x \in G \rangle$.

Definition 2.1. *A p-group G is powerful if $[G, G] \leq G^p$ for $p \neq 2$, or $[G, G] \leq G^4$ for $p = 2$.*
A subgroup N of a p-group G is said to be powerfully embedded in G, if $[N, G] \leq N^p$ for $p \neq 2$, or $[N, G] \leq N^4$ for $p = 2$.
A subgroup N of a p-group G is said to be almost-powerfully-embedded if $[N, G] \leq N^p$. In particular, for $p \neq 2$, the definition of powerfully embedded and of almost-powerfully-embedded coincide.

In the following proposition we recall some properties of powerful p-groups.

Proposition 2.2. *Let G be a powerful p-group. Then the followings hold:*

- *for any $i \geq 1$, the subgroup G^{p^i} is equal to $\{x^{p^i} : x \in G\}$;*
- *if $X = \{x_1, \ldots, x_s\}$ is a set of generators of G, then G^{p^i} is generated by $\{x_1^{p^i}, \ldots, x_s^{p^i}\}$;*
- *a 2-generated powerful group is metacyclic.*

Standard references for the theory of powerful p-groups are [1] and [7]. We refer the reader to [5] for the theory of almost-powerfully-embedded groups.

We now recall the definition and some properties of the modular p-groups. First of all the definition of permutable subgroup:

Definition 2.2. *Let G be a group. Let H and K be subgroups of G. The subgroup H permutes with K if $HK = KH$. A subgroup H of G is called permutable if H permutes with K, for all subgroups K of G.*

By (3.11) and (3.12) on page 24 of [14], we get the following characterization for permutable subgroups:

Remark 2.3. *Let G be a group and let H and K be subgroups of G. The subgroups H and K permute if and only if $|H \cap K||HK| = |H||K|$.*

We report the definition of modular group:

Definition 2.3. *A group G is said to be modular if its lattice of subgroups is modular, i.e. $\langle X, Y \cap Z\rangle = \langle X, Y\rangle \cap Z$ for all subgroups X, Y, Z of G such that $X \leq Z$.*

A finite p-group G is modular if and only if all subgroups of G are permutable.

The modular p-groups are well-known and are classified in the following theorem.

Theorem 2.4. *A finite p-group G is modular if and only if*

(a) *G is a direct product of a quaternion group of order 8 with an elementary abelian 2-group, or*

(b) *G contains an abelian normal subgroup A with cyclic factor G/A; further there exists an element $b \in G$ with $G = A\langle b\rangle$ and a positive integer s such that $a^b = a^{1+p^s}$, for all $a \in A$, with $s \geq 2$ in case $p = 2$.*

We refer the reader to Chapter 2 of [13] for more details on modular groups.

2.2 General results

In this section, we state some results about monotone 2-groups that will be often used in our work.

The class of monotone 2-groups is closed for taking quotients and subgroups, but, in general, it is not closed for taking direct products. For example, the direct product of a dihedral group of order 8 with a cyclic group of order 4 is not monotone. Nevertheless, in some special cases the direct product of monotone 2-groups is monotone. In fact, we have the following lemma.

Lemma 2.5. *If G is a monotone 2-group and A is an elementary abelian 2-group, then the direct product $G \times A$ is monotone.*

Proof. By Proposition 1.2, we have to show that all the 2-generated subgroups of G are metacyclic. Pick $x_1, x_2 \in G \times A$. Then for $i = 1, 2$ there exist $g_i \in G$ and $a_i \in A$ such that $x_i = g_i a_i$. The subgroup $\langle g_1, g_2 \rangle$ is a 2-generated subgroup of G and so it is metacyclic. Hence there exists $g_3 = g_1^{h_1} g_2^{k_1}$ and $g_4 = g_1^{h_2} g_2^{k_2}$ such that $\langle g_1, g_2 \rangle = \langle g_3, g_4 \rangle$ and $\langle g_3 \rangle \trianglelefteq \langle g_3, g_4 \rangle$. We have $h_1 k_2 - h_2 k_1 \equiv 1 \mod 2$. Put $x_3 = x_1^{h_1} x_2^{k_1} = g_3 a_1^{h_1} a_2^{k_2}$, and $x_4 = x_1^{h_2} x_2^{k_2} = g_4 a_1^{h_2} a_2^{k_2}$. We get $\langle x_3, x_4 \rangle \leq \langle x_1, x_2 \rangle$ and the condition $h_1 k_2 - h_2 k_1 \equiv 1 \mod 2$ guarantees that $\langle x_1, x_2 \rangle = \langle x_3, x_4 \rangle$. Moreover $\langle [x_3, x_4] \rangle = \langle [g_3, g_4] \rangle \leq \langle g_3^2 \rangle = \langle x_3^2 \rangle$. Hence $\langle x_1, x_2 \rangle$ is a metacyclic subgroup. Therefore $G \times A$ is a monotone 2-group. $\square$

Therefore, given a 2-group $G = H \times A$, where H is a 2-group and A is elementary abelian, in order to check that G is monotone, it is sufficient to check that H is monotone.

In the following lemma, we introduce a very important family of monotone 2-groups.

Lemma 2.6. *Let G be a group isomorphic to $A\langle b \rangle$ where A is abelian and b is such that $a^b = a^r$, for all a in A. Then G is monotone.*

Proof. We have to prove that each 2-generated subgroup of G is metacyclic (see Proposition 1.2). Since A is abelian, and so monotone, it is enough to check that the subgroups of the form $\langle a_1 b^i, a_2 \rangle$ are metacyclic, where a_1 and a_2 are in A. So let $H = \langle a_1 b^i, a_2 \rangle$, with a_1 and a_2 in A. Since $a_2^{a_1 b^i} = a_2^{b^i} = a_2^{r^i}$, we have that $\langle a_2 \rangle \trianglelefteq H$ and so H is metacyclic. Therefore, our lemma is proved. $\square$

The next part is related with the power structure of a monotone 2-group. In Lemma 2.7 we show that the Frattini subgroup of a monotone 2-group is powerful. This implies that any element in G^4 is a square in G. Moreover, we show in Lemma 2.9 that the cyclic group generated by a square of an element of G is permutable in G, and so we get (Corollary 2.10) that G^4 is modular.

Lemma 2.7. *Let G be a monotone 2-group. The subgroup G^2 is powerful. Moreover, G^2 is almost-powerfully-embedded in G.*

Proof. We want to show that $[G^2, G^2] \leq (G^2)^4$. By Remark 2.1, since G^2 is generated by the set $\{a^2 : a \in G\}$, it is enough to show that for all a and b in G the commutator $[a^2, b^2]$ is contained in $(G^2)^4$.

Pick a and b in G. Since G is a monotone 2-group, all the 2-generated subgroups are metacyclic. Hence, there exist $x, y \in \langle a, b\rangle$ such that $\langle x, y\rangle = \langle a, b\rangle$, $\langle x\rangle \trianglelefteq \langle x, y\rangle$ and $x^y = x^r$, where $r \equiv \pm 1 \mod 4$. So $r = \pm 1 + 4h$. Since $\langle x, y\rangle$ is metacyclic and $\langle x\rangle \trianglelefteq \langle x, y\rangle$, we have that $\langle x, y\rangle^2 = \langle x^2, y^2\rangle$ and $[\langle x, y\rangle^2, \langle x, y\rangle^2] = \langle [x^2, y^2]\rangle$. Now, we have that $(x)^{y^2} = x^{(\pm 1+4h)^2} = x^{1\pm 8h+16h^2}$ and so $(x^2)^{y^2} = x^{2\pm 16h+32h^2}$. Hence, we obtain that $[x^2, y^2] = x^{\pm 16h+32h^2}$. In particular, we have $[x^2, y^2] \in \langle x^8\rangle$ and $[\langle x, y\rangle^2, \langle x, y\rangle^2] \leq \langle x^8\rangle$. Now, being $\langle x^8\rangle = \langle x^2\rangle^4$, we have $[\langle x, y\rangle^2, \langle x, y\rangle^2] \leq (G^2)^4$. Since a^2, b^2 are in $\langle x, y\rangle^2$, it follows that $[a^2, b^2] \in (G^2)^4$, and the first part of the lemma is proved.

In order to prove that G^2 is almost-powerfully-embedded, we have to show that $[G^2, G] \leq (G^2)^2$. Arguments similar to the previous ones show that if a and b are in G, then $[a^2, b] \in (G^2)^2$ and the result follows from Remark 2.1. $\square$

Corollary 2.8. *Let G be a monotone 2-group. Then $\Phi(G^2) = G^4$ and $G^{2^{i+1}} = (G^2)^{2^i}$, for $i \geq 1$.*

Proof. If P is a powerful group and X is a generating set for P, then P^{2^i} is generated by the set $\{x^{2^i} : x \in X\}$ (see Theorem 2.7 on page 40 of [1]).

Since G^2 is powerful with generating set $\{a^2 : a \in G\}$, we get that $(G^2)^{2^i}$ is generated by the set $\{a^{2^{i+1}} : a \in G\}$. In particular, we obtain that $(G^2)^{2^i} = G^{2^{i+1}}$ for $i \geq 1$. Moreover, $G^4 = (G^2)^2 = \Phi(G^2)$. $\square$

Lemma 2.9. *Let G be a monotone 2-group. If a is an element of G, then the subgroup $\langle a^2\rangle$ is permutable in G.*

Proof. We first show that, if H is a metacyclic 2-group, then, for all $a \in H$, the subgroup $\langle a^2 \rangle$ is permutable in H.
Let $H = \langle x, y \rangle$, where $\langle x \rangle$ is normal in $\langle x, y \rangle$ and $x^y = x^r$. If $r \equiv 1 \mod 4$, then, by Lemma 2.3.4 on page 56 of [13], H is a modular subgroup, and, by Lemma 2.3.2 on page 55 of [13], each subgroup of H is permutable.

We assume now $r \equiv -1 \mod 4$, i.e. $r = -1 + 4h$. Let a be an element of H. Since a^2 is in $\langle x, y^2 \rangle$, which is a modular subgroup, it is sufficient to check that $\langle a^2 \rangle$ permutes with the subgroup $\langle y^{k_1} x^{k_2} \rangle$, where k_1 is odd. Replacing $y^{k_1} x^{k_2}$ with a suitable power, we may assume $k_1 = 1$. Moreover, we have that $\langle x, y \rangle = \langle x, yx^{k_2} \rangle$, and yx^{k_2} acts as y on $\langle x \rangle$. Therefore, it is enough to prove that, for all the elements a in H, the subgroups $\langle y \rangle$ and $\langle a^2 \rangle$ permute. Since $H = \{y^i x^j : i, j \in \mathbb{N}\}$, we assume $a = y^i x^j$. We distinguish two cases, depending on the parity of i.

- Suppose i odd. Replacing eventually a with a suitable power, we may assume $a = yx^j$. We get $a^2 = y^2 x^{4hj}$. The subgroup $\langle y^2 x^{4hj}, y \rangle$ is equal to $\langle x^{4hj}, y \rangle$. Since the subgroups $\langle x^{4hj} \rangle$ and $\langle y \rangle$ permute, in order to prove that the subgroups $\langle y \rangle$ and $\langle a^2 \rangle$ permute, we have to show that

$$|\langle y^2 x^{4hj}, y \rangle| = |\langle x^{4hj}, y \rangle| = \frac{|x^{4hr}||y|}{|\langle x^{4hj} \rangle \cap \langle y \rangle|}.$$

Since $\langle x^2, y^2 \rangle$ is a modular subgroup we get that $(y^2 x^{4hj})^l \equiv y^{2l} x^{4hjl} \mod \langle x^{8hjl} \rangle$. In particular $(y^2 x^{4hj})^l \in \langle y \rangle$ if and only if $(x^{4hj})^l \in \langle y \rangle$. It follows that $[\langle y^2 x^{4hj} \rangle : \langle y^2 x^{4hj} \rangle \cap \langle y \rangle] = [\langle x^{4hj} \rangle : \langle x^{4hj} \rangle \cap \langle y \rangle]$. Then $\frac{|x^{4hr}||y|}{|\langle x^{4hj} \rangle \cap \langle y \rangle|} = \frac{|y^2 x^{4hr}||y|}{|\langle y^2 x^{4hj} \rangle \cap \langle y \rangle|} = |\langle y^2 x^{4hj}, y \rangle|$.

- Suppose now i even. Then, we have $i = 2k$ and $a = y^{2k} x^j$. Now, $a^2 = y^{4k} x^{2j+4s}$, for some $s \in \mathbb{N}$. We get that $\langle y^{4k} x^{2j+4s}, y \rangle = \langle x^{2j+4s}, y \rangle$. Since the subgroups $\langle x^{2j+4s} \rangle$ and $\langle y \rangle$ permute, in order to prove that the subgroups $\langle y \rangle$ and $\langle a^2 \rangle$ permute, we have to show that

$$|\langle y^{4k} x^{2j+4s}, y \rangle| = |\langle x^{2j+4s}, y \rangle| = \frac{|x^{2j+4s}||y|}{|\langle x^{2j+4s} \rangle \cap \langle y \rangle|}.$$

Since $\langle x^2, y^2 \rangle$ is a modular subgroup, we get that $(y^{4k} x^{2j+4s})^l \in \langle y \rangle$ if and only if $(x^{2j+4s})^l \in \langle y \rangle$. Hence, we get $[\langle y^{4k} x^{2j+4s} \rangle : \langle y^{4k} x^{2j+4s} \rangle \cap \langle y \rangle] = [\langle x^{2j+4s} \rangle : \langle x^{2j+4s} \rangle \cap \langle y \rangle]$ and so $\frac{|x^{2j+4s}||y|}{|\langle x^{2j+4s} \rangle \cap \langle y \rangle|} = \frac{|y^{4k} x^{2j+4s}||y|}{|\langle y^{4k} x^{2j+4s} \rangle \cap \langle y \rangle|} = |\langle y^{4k} x^{2j+4s}, y \rangle|$.

Therefore, in both cases, we get that $\langle a^2 \rangle$ and $\langle y \rangle$ permute and it follows that the subgroup $\langle a^2 \rangle$ is permutable in H.
This concludes our preliminary claim.

To conclude, let a be an element of G. In order to show that $\langle a^2 \rangle$ is a permutable subgroup of G, it suffices to prove that it permutes with $\langle b \rangle$ for all $b \in G$. Hence, it is enough to prove that $\langle a^2 \rangle$ is a permutable subgroup in $H = \langle a, b \rangle$. Being a subgroup of a monotone 2-group, H is metacyclic, and so the result follows from the first part of the proof. $\square$

Corollary 2.10. *Let G be a monotone 2-group. The subgroups of G^4 are permutable in G. In particular, the subgroup G^4 is modular.*

Proof. Since G^2 is a powerful subgroup, by Proposition 2.6 of page 40 in [1], the elements of $(G^2)^2$ are squares of elements in G^2. By Corollary 2.8, we have that $(G^2)^2 = G^4$. By Lemma 2.9, the cyclic subgroups of G^4 are permutable in G. Hence all the subgroups of G^4 are permutable in G^4 and we get, by Lemma 2.3.2 on page 55 of [13], that G^4 is modular. $\square$

The following lemma deals with metacyclic 2-groups that have a generator of order 2.

Lemma 2.11. *Let $\langle a, b \rangle$ be a metacyclic group with $|b| = 2$.
Then $\langle a, b \rangle$ is either semidihedral or $\Omega_1(\langle a, b \rangle)$ is contained in the normalizer of $\langle c \rangle$ for all $c \in \langle a, b \rangle$ of order greater than or equal to 4.*

Proof. Suppose that $\langle a, b \rangle = \langle x, y \rangle$ with $\langle x \rangle \trianglelefteq \langle x, y \rangle$.
If $x^y = x^{1+4h}$, then $\langle x, y \rangle$ is modular and so $\Omega_1(\langle x, y \rangle)$ is contained in the normalizer of $\langle c \rangle$ for all $c \in \langle a, b \rangle$ (see Lemma 2.3.6 on page 57 of [13]).
Hence, we may now assume $x^y = x^{-1+4h}$. The proof is now a case-by-case analysis depending on the order of x.

- Suppose that $|x| = 2$. The subgroup $\langle x, y \rangle$ is abelian and so the statement is true.

- Suppose that $|x| = 4$. Then, $x^y = x^{-1}$.
  If $|y| = 2$, then $\langle x, y \rangle$ is isomorphic to D_8 and the statement holds.
  If $|y| \geq 4$, then either there are no elements in $\langle x, y \rangle \backslash \langle x, y \rangle^2$ of order 2 (and so we contradict our assumptions) or $\langle x, y \rangle$ is modular (and the lemma is true).
  In fact, the generators of $\langle x, y \rangle$ are of the form $y^i x^j$ with i odd or

$y^{2i}x^j$ with j odd. In the first case, $(y^ix^j)^2 = y^{2i}$, and so, y^ix^j, with i odd has order 4. In the second case, $(y^{2i}x^j)^2 = y^{4i}x^2$. Hence, $|y^{2i}x^j| = 2$ if and only if $y^{4i} = x^2$ for some i. Therefore, $|y| = 2^r \geq 8$ and $\langle y \rangle \cap \langle x \rangle = \langle x^2 \rangle = \langle y^{2^{r-1}} \rangle$. Now, we get $y^x = y^{1+2^{r-1}}$, i.e. the subgroup $\langle x, y \rangle$ is modular. By Lemma 2.3.6 on page 57 of [13], the statement holds.

- Suppose that $|x| = 2^n \geq 8$. If $|y| = 2$, then $x = x^{y^2} = x^{1-8h+16h^2}$ and so we get $x^y = x^{-1+2^{n-1}h}$ where $h \in \{0, 1\}$. So, either $\langle x, y \rangle$ is semidihedral or $x^y = x^{-1}$.
  In the latter case, since an element of order greater than or equal to 4 is contained in $\langle x \rangle$, the statement is true. So, we may assume that $|y| \geq 4$.
  Now, the generators of $\langle x, y \rangle$ are of the form y^ix^j with i odd or $y^{2i}x^j$ with j odd.
  We now study under which conditions there exists a generator of $\langle x, y \rangle$ of order 2.
  Consider now, $y^{2i}x^j$, with j odd. Since $\langle y^2, x \rangle$ is a modular subgroup and $x^{y^2} = x^{1-8h+16h^2}$, we have that $(y^{2i}x^j)^2 = y^{4j}x^{2i}x^{8s}$, for some s. Therefore, since $|x| \geq 8$, and $\langle y \rangle \cap \langle x \rangle \leq \langle x^{2^{n-1}} \rangle$, we get that $|y^{2i}x^j| \geq 4$.
  So, if there exists a generator of $\langle x, y \rangle$ of order 2, then it is of the form y^ix^j with i odd. Now, up to replacing y^ix^j with a suitable power, we may assume yx^j. Now, we get that $(yx^j)^2 = y^2x^{4jh}$. Hence, if $y^2x^{4jh} = 1$, then $|y| = 4$ and $y^2 \in \langle x^{2^{n-1}} \rangle$. The automorphism induced by y on $\langle x \rangle$ has order at most 2 and so, $x^y = x^{-1+2^{n-1}h}$ with $h \in \{0, 1\}$ and $y^2 = x^{2^{n-1}}$. Now, all the elements of order greater than or equal to 8 are in $\langle x \rangle$, and so the statement is true.

□

We conclude the section with the Lemma 2.12, Lemma 2.13 and Proposition 2.14, where we give properties of subgroups H of a monotone 2-group G for which HG^4/G^4 satisfy some particular condition.

Lemma 2.12. *Let G be a monotone 2-group.*
Let a and b in G such that $\langle a, b \rangle G^4/G^4$ is isomorphic to $C_4 \times C_4$.
Then, the group $\langle a, b \rangle$ is modular.
Moreover, $\langle a, b \rangle^2 = \langle a^2, b^2 \rangle$ and $G^4 \cap \langle a, b \rangle = \langle a^4, b^4 \rangle$.

Proof. We consider a non-modular metacyclic group $\langle x, y\rangle$ with $\langle x\rangle \trianglelefteq \langle x, y\rangle$, and $x^y = x^{-1+4h}$. If $\langle x, y\rangle/N$ is an abelian quotient of $\langle x, y\rangle$, then the derived subgroup of $\langle x, y\rangle$ is contained in N. This means that $x^2 \in N$. In particular, $\langle x, y\rangle/\langle x^2\rangle$ is isomorphic to $C_2 \times C_{2^n}$. Therefore, a non-modular metacyclic group has no abelian quotient isomorphic to $C_4 \times C_4$. This implies that $\langle a, b\rangle$ is a modular metacyclic subgroup that does not involve Q_8. Then, by Proposition 2.5.9 on page 94 of [13], we have that $\langle a, b\rangle$ is lattice isomorphic to an abelian group. Then $\langle a^2, b^2\rangle = \langle a, b\rangle^2$, $\langle a^2, b^2\rangle^2 = \langle a^4, b^4\rangle$. Now, since $\langle a^4, b^4\rangle \leq G^4 \cap \langle a, b\rangle$ and $|\langle a, b\rangle : G^4 \cap \langle a, b\rangle| = 16$, we have the equality $\langle a^4, b^4\rangle = G^4 \cap \langle a, b\rangle$. □

Lemma 2.13. *Let G be a monotone 2-group.*
Let a and b be in G such that $|aG^4| = 4$, $|bG^4| = 4$, $\langle aG^4\rangle \cap \langle bG^4\rangle = G^4$ and $a^bG^4 = a^{-1}G^4$.
Then the group $\langle a, b\rangle$ is a metacyclic non-modular group.
Moreover, $\Phi(\langle a, b\rangle) = \langle a^2, b^2\rangle$, $G^4 \cap \langle a, b\rangle = \Phi(\Phi(\langle a, b\rangle))$.

Proof. The group $\langle a, b\rangle G^4/G^4$ is non-modular and isomorphic to $\langle a, b\rangle/(G^4 \cap \langle a, b\rangle)$.
In particular, $\langle a, b\rangle$ is non-modular.

The subgroup $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle)$ is normal in $\langle a, b\rangle$ with elementary abelian quotient of order 4. Since $\langle a, b\rangle$ is metacyclic, we get $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle) = \Phi(\langle a, b\rangle)$. Moreover, $G^4 \cap \langle a, b\rangle$ is normal in $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle)$ with elementary abelian quotient of order 4. Being a subgroup of a metacyclic group, the subgroup $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle)$ is metacyclic. Hence, we get that $G^4 \cap \langle a, b\rangle$ is the Frattini subgroup of $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle)$ and $\langle a^2, b^2\rangle(G^4 \cap \langle a, b\rangle) = \langle a^2, b^2\rangle = \Phi(\langle a, b\rangle)$. Now, since the Frattini subgroup of a metacyclic group is powerful we get that $\Phi(\langle a^2, b^2\rangle) = \langle a^4, b^4\rangle = G^4 \cap \langle a, b\rangle$. □

Proposition 2.14. *Let G be a monotone 2-group.*
Let H be a subgroup of G such that HG^4/G^4 is isomorphic to a direct product of C_4s, $|H^2G^4/G^4| \geq 4$, and $H \cap G^4 \leq \Phi(H)$.
Then, the subgroup H is modular and it does not involve Q_8, the quaternion group of order 8.

Proof. Suppose that $H = \langle a_1, \cdots, a_n\rangle$, where $H/(G^4 \cap H) = \langle a_1(G^4 \cap H)\rangle \times \cdots \times \langle a_n(G^4 \cap H)\rangle$, with $n \geq 2$.

We first show that the subgroup H is powerful.
In order to prove that H is powerful, it is sufficient, by Remark 2.1, to show that $[a_i, a_j] \leq H^4$, for all i and j.
Now, since $\langle a_i, a_j \rangle/(G^4 \cap H)$ is isomorphic to $C_4 \times C_4$, by Lemma 2.12, we get that $\langle a_i, a_j \rangle$ is modular, hence powerful. In particular, $[a_i, a_j] \in \langle a_i, a_j \rangle^4$ and, since $\langle a_i, a_j \rangle \leq H$, we get that $[a_i, a_j] \in H^4$. Hence, the subgroup H is powerful.

Since H is powerful and generated by $\{a_1, \cdots, a_n\}$, we have that $H^2 = \langle a_1^2, \cdots, a_n^2 \rangle$. Now, $H^2(G^4 \cap H)$ contains H^2 and $[H : H^2(G^4 \cap H)] = [H : H^2]$. Then, we have that $H^2(G^4 \cap H) = H^2$, i.e. $(G^4 \cap H) \leq \Phi(H^2) \leq H^4$. Since $H^2/(G^4 \cap H)$ is elementary abelian, we have that $H^4 \leq (G^4 \cap H)$. Therefore, we have $H^4 \leq (G^4 \cap H) \leq H^4$, which implies $H^4 = (G^4 \cap H)$. This proves that if H is a subgroup of G such that HG^4/G^4 is isomorphic to a direct product of C_4 and $(H \cap G^4) \leq \Phi(H)$, then H is powerful with $H^4 = (G^4 \cap H)$ and H/H^4 is isomorphic to a direct product of C_4.

We now show that, for all a and b in $H \backslash H^2$, the subgroup $\langle a, b \rangle$ is modular.
We may assume that $\langle a, b \rangle$ is maximal among the subgroups of this form. We distinguish three cases depending on the form of the quotient $\langle a, b \rangle H^4/H^4$

- Suppose that $\langle a, b \rangle H^4/H^4$ is isomorphic to $C_4 \times C_4$. By Lemma 2.12, we have that $\langle a, b \rangle$ is a modular metacyclic subgroup.

- Suppose that $\langle a, b \rangle H^4/H^4$ is isomorphic to $C_4 \times C_2$. Then, there exists $z \in H^2 \backslash H^4$ such that $b = az$. Since H is powerful, there exists a $c \in H \setminus H^2$ such that $c^2 = z$. Now, we have that $\langle a, b \rangle < \langle a, c \rangle$, so that $\langle a, b \rangle$ is not maximal, a contradiction.

- $\langle a, b \rangle H^4/H^4$ is isomorphic to C_4. Since $\langle a, b \rangle H^4/H^4 \simeq C_4$, we have that $a = bz$, where $z \in H^4$. Since H is powerful, there exists $c \in H$ such that $c^4 = z$. The maximality of $\langle a, b \rangle$ forces $\langle a, b \rangle = \langle a, c \rangle$, but $\langle a, b \rangle = \langle a, z \rangle = \langle a, c^4 \rangle$. Since $c^4 \in \Phi(\langle a, c \rangle)$, this implies that $\langle a, b \rangle = \langle a \rangle$ and so the subgroup is modular.

Then, we have that for all $a, b \in H \backslash H^2$, the subgroup $\langle a, b \rangle$ is modular.

In order to show that H is modular, we have to prove that, for every x and y in H, the subgroups $\langle x \rangle$ and $\langle y \rangle$ permute. Then, let x and y be elements in H. Since H is powerful, there exist a and b in $H \backslash H^2$ such that $a^{2^i} = x$ and $b^{2^j} = y$. So, we have that $\langle x, y \rangle \leq \langle a, b \rangle$, where a and b are in

$H \backslash H^2$. By the previous paragraph, the subgroup $\langle a, b \rangle$ is modular. Hence, the subgroups $\langle x \rangle$ and $\langle y \rangle$ permute and H is modular.

A modular group that involves Q_8 is isomorphic to $Q_8 \times A$ with A elementary abelian, and so it does not have quotients isomorphic to a direct product of C_4. Therefore, H is a modular group that does not involve a Q_8. $\square$

Chapter 3

Monotone 2-groups of exponent 4

Throughout all this chapter G will be a monotone 2-group of exponent 4.

Definition 3.1. *We introduce the following families of* 2*-groups:*

- $\mathscr{A}_1$ *is the family of* 2*-groups of the form* $K_2 \times A$*, where* A *is elementary abelian and* $K_2 = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^3 \rangle$*;*

- $\mathscr{A}_2$ *is the family of* 2*-groups of the form* $E * A$*, where* E *is an extraspecial group and* A *is either an abelian group of the form* $C_4 \times C_2 \times \cdots \times C_2$ *and* $E^2 = A^2$ *or an elementary abelian group and* $E \cap A = 1$*;*

- $\mathscr{A}_3$ *is the family of* 2*-groups of the form* $A \rtimes \langle b \rangle$*, where* A *is an abelian group of exponent* 4*,* b *has order* 2 *and* $a^b = a^{-1}$ *for all* $a \in A$*;*

- $\mathscr{A}_4$ *is the family of* 2*-groups of the form* $A \rtimes \langle b \rangle$*, where* A *is an abelian group of exponent* 4 *with* $|A^2| \geq 4$*,* b *has order* 4 *and* $a^b = a^{-1}$ *for all* $a \in A$*;*

- $\mathscr{A}_5$ *is the family of* 2*-groups of the form* $A\langle b \rangle$*, where* A *is an abelian group of exponent* 4 *with* $|A^2| \geq 8$*,* b *has order* 4*,* $b^2 \in A^2$ *and* $a^b = a^{-1}$ *for all* $a \in A$*;*

- $\mathscr{A}_6$ *is the family of* 2*-groups of the form* $K_6 \times A$*, where* $K_6 = \langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b \rangle$ *and* A *is elementary abelian;*

- $\mathscr{A}_7$ *is the family of* 2*-groups of the form* $K_7 \times A$*, where* $K_7 = \langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b^3\rangle$ *and* A *is elementary abelian;*

- $\mathscr{A}_8$ *is the family of* 2*-groups of the form* $K_8 \times A$*, where* $K_8 = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b^3a^2\rangle$*, and* A *is an elementary abelian group;*

- $\mathscr{A}_9$ *is the family of* 2*-groups of the form* $K_9 \times A$*, where* A *is elementary abelian and* $K_9 = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = b^3, c^d = cb^2\rangle$*;*

- $\mathscr{A}_{10}$ *is the family of* 2*-groups of the form* $K_{10} \times A$*, where* A *is elementary abelian and* $K_{10} = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = bd^2, c^d = c^3\rangle$*;*

- $\mathscr{A}_{11}$ *is the family of* 2*-groups of the form* $K_{11} \times A$*, where* A *is elementary abelian and* $K_{11} = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b^3, a^d = ad^2, b^d = b, c^d = ca^2\rangle$.

We start by proving that the groups in $\mathscr{A}_i$, for $i \in \{1, \ldots, 11\}$, introduced in Definition 4.1, are actually monotone.
The main tools are Proposition 1.2 and Lemma 2.5.

Proposition 3.1. *The groups in the families* $\mathscr{A}_i$*, for* $i \in \{1, \ldots, 11\}$ *are monotone.*

Proof. We want to show that if G is a group in $\mathscr{A}_i$, for $i \in \{1, \ldots, 11\}$, then G is monotone. Now, the proof is a case-by-case analysis depending on the family in which G lies.

- Let G be a group in $\mathscr{A}_1$. Then $G = K_2 \times A$, where $K_2 = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1}\rangle$ and A is elementary abelian. By Lemma 2.5, in order to check that G is monotone, it is sufficient to prove that K_2 is monotone. Since K_2 is metacyclic, K_2 is monotone and so is G.

- Let G be a group in $\mathscr{A}_2$. Then $G = E * A$, where E is an extraspecial group and A is either an abelian group of the form $C_4 \times C_2 \times \cdots \times C_2$ or an elementary abelian group. By Lemma 2.5, in order to check that G is monotone, it is sufficient to prove that $E * C_4$ and E are monotone.

Since E is a subgroup of $E * C_4$, it is enough to check that $E * C_4$ is a monotone group.

Each cyclic subgroup of order 4 in $E*C_4$ contains the derived subgroup of $E * C_4$. Therefore, for every a and b in $E * C_4$, if either $|a| = 4$ or $|b| = 4$, then the subgroup $\langle a, b\rangle$ is metacyclic.

If $|a| = 2$ and $|b| = 2$, then $\langle a, b\rangle$ is abelian or a dihedral group.

This proves that for every a and b in $E * C_4$, the subgroup $\langle a, b\rangle$ is metacyclic. Hence, $E * C_4$ is monotone, and so is G.

- Let G be a group in $\mathscr{A}_3$. Then $G = A \rtimes \langle b\rangle$, where A is an abelian group of exponent 4, b has order 2 and $a^b = a^{-1}$ for all $a \in A$. Lemma 2.6 proves that G is monotone.

- Let G be a group in $\mathscr{A}_4$. Then $G = A \rtimes \langle b\rangle$, where A is an abelian group of exponent 4 with $|A^2| \geq 4$, b has order 4 and $a^b = a^{-1}$ for all A. Lemma 2.6 proves that G is monotone.

- Let G be a group in $\mathscr{A}_5$. Then $G = A\langle b\rangle$, where A is an abelian group of exponent 4 with $|A^2| \geq 8$, b has order 4, $b^2 \in A^2$ and $a^b = a^{-1}$ for all $a \in A$. Lemma 2.6 proves that G is monotone.

- Let G be a group in $\mathscr{A}_6$. Then $G = K_6 \times A$, where A is elementary abelian and $K_6 = \langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b\rangle$. By Lemma 2.5, it is sufficient to check that K_6 is monotone.

  Since $\langle a, c\rangle$ is abelian, it is enough to check that the subgroups of the form $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ are metacyclic. Now, $(a^{i_1}c^{i_2})^2 = a^{2i_1}c^{2i_2}$, $(ba^{j_1}c^{j_2})^2 = b^2a^{2j_2}$, and $[a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}] = a^{2i_1}b^{2i_2}$.

  If $i_2 \equiv 1 \mod 2$ and $i_1 \equiv j_2 \mod 2$, then $(ba^{j_1}c^{j_2})^2 = [a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}]$, and so $\langle ba^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$, i.e. the subgroup $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

  Suppose that $i_2 \equiv 0 \mod 2$. Then $(a^{i_1}c^{i_2})^2 = [a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}]$, and so $\langle a^{i_1}c^{i_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$, i.e. the subgroup $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

  Suppose now that $i_2 \equiv 1 \mod 2$ and $i_1 \equiv j_2 + 1 \mod 2$. Then, $(a^{i_1}c^{i_2})^2 = (ba^{j_1}c^{j_2})^2$, and so the element $a^{i_1}c^{i_2}ba^{j_1}c^{j_2}$ is such that $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is equal to $\langle a^{i_1}c^{i_2}ba^{j_1}c^{j_2}, ba^{j_1}c^{j_2}\rangle$ and $\langle a^{i_1}c^{i_2}ba^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$. Then, the subgroup $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

This shows that each 2-generated subgroup of K_6 is metacyclic. Hence, K_6 is monotone, and so is G.

- Let G be a group in $\mathscr{A}_7$. Then $G = K_7 \times A$, where A is elementary abelian and $K_7 = \langle a, b, c : a^4 = 1, b^4 = 1, a^2b^2 = c^2, a^b = a^3, a^c = a, b^c = b^3\rangle$. By Lemma 2.5, it is sufficient to check that K_7 is monotone.

  The subgroup $\langle a, c\rangle$ is abelian. Hence, it is enough to check that the subgroups of the form $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ are metacyclic.

  Now, we have that $(a^{i_1}c^{i_2})^2 = a^{2i_1}c^{2i_2}$, $(ba^{j_1}c^{j_2})^2 = b^2c^{2j_2}b^{2j_2} = b^2a^{2j_2}$, $[a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}] = a^{2i_1}b^{2i_2}$.

  If $i_2 \equiv 0 \mod 2$, then $(a^{i_1}c^{i_2})^2 = [a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}]$, and so the subgroup $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

  Suppose that $i_2 \equiv 1 \mod 2$. If $i_1 \equiv j_2 \mod 2$, then $(ba^{j_1}c^{j_2})^2 = [a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}]$, and so the subgroup $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

  Suppose now that $i_2 \equiv 1 \mod 2$ and $i_1 \equiv j_2 + 1 \mod 2$. Then $(a^{i_1}c^{i_2})^2 = (ba^{j_1}c^{j_2})^2$. Therefore, the element $a^{i_1}c^{i_2}ba^{j_1}c^{j_2}$ is such that $\langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$ is equal to $\langle a^{i_1}c^{i_2}ba^{j_1}c^{j_2}, ba^{j_1}c^{j_2}\rangle$ and $\langle a^{i_1}c^{i_2}ba^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, ba^{j_1}c^{j_2}\rangle$. Then, the subgroup is metacyclic.

  This shows that each 2- generated subgroup of K_7 is metacyclic. Hence, K_7 is monotone, and so is G.

- Let G be a group in $\mathscr{A}_8$. Then $G = K_8 \times A$, where $K_8 = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b^3a^2\rangle$, and A is an elementary abelian group. By Lemma 2.5, it is sufficient to check that K_8 is monotone. Now, the subgroup $\langle a, c\rangle$ is abelian and b acts as inversion on $\langle a, c\rangle$. Therefore, K_8 is monotone by Lemma 2.6.

- Let G be a group in $\mathscr{A}_9$. Then $G = K_9 \times A$, where A is elementary abelian and $K_9 = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = b^3, c^d = cb^2\rangle$. By Lemma 2.5, it is sufficient to check that K_9 is monotone.

  The subgroup $\langle a, b, c\rangle$ is isomorphic to K_6. Since K_6 is monotone, every 2-generated subgroup of $\langle a, b, c\rangle$ is metacyclic. Therefore, it is enough to check that the subgroups of the form $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ are metacyclic. If i_1 and j_1 are both even, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle b, c, d\rangle$ which is isomorphic to K_6 (the isomorphism

is given by setting $\overline{a} = b$, $\overline{b} = d$, $\overline{c} = bc$). Hence, the subgroup $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.
If i_2 and j_2 are both even, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle a, b, d\rangle$ which is isomorphic to K_7 (the isomorphism is given by setting $\overline{a} = a$, $\overline{b} = b$, $\overline{c} = d$). Hence, the subgroup $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.
If i_3 and j_3 are both even, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle a, c, d\rangle$ which is isomorphic to K_6 (the isomorphism is given by setting $\overline{a} = ac$, $\overline{b} = d$, $\overline{c} = c$). Hence, the subgroup $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic. So, for every $k = 1, 2, 3$, we may assume that i_k and j_k are not both even.

We now distinguish two cases depending on the parity of i_3.

Suppose firstly that i_3 is odd and j_3 is even. Since $b^2 \in \langle a, c\rangle$, we may assume $i_3 = 1$ and $j_3 = 0$. The subgroup has the form $\langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$. Now, $(a^{i_1}c^{i_2}b)^{da^{j_1}c^{j_2}} = a^{i_1}c^{i_2}bb^{2i_2}b^2$. Now, if i_2 is odd, then the subgroup is abelian. If i_2 is even, then $(a^{i_1}c^{i_2}b)^2 = b^2$, and so $\langle a^{i_1}c^{i_2}b\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}b^{j_3}\rangle$ and the subgroup is metacyclic.

Suppose now that i_3 is even and j_3 is odd. Since $b^2 = a^2c^2$, we may assume that $i_3 = 0$ and $j_3 = 1$. Hence, the subgroup has the form $\langle a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}\rangle$. Now, $(a^{i_1}c^{i_2})^2 = a^{2i_1}c^{2i_2}$, $(dba^{j_1}c^{j_2})^2 = d^2a^{2j_1}c^{2j_2}b^{2j_2}a^{2j_1} = d^2a^{2j_2}$, and $[a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}] = b^{2i_2}a^{2i_1}$. In particular, we have that $|dba^{j_1}c^{j_2}| = 4$.

Moreover, if $i_1 \equiv j_2 \mod 2$ and $i_2 \equiv 1 \mod 2$, then $(a^{i_1}c^{i_2})^2 = (dba^{j_1}c^{j_2})^2$. It follows that $a^{i_1}c^{i_2}dba^{j_1}c^{j_2}$ is a non-Frattini element such that $\langle a^{i_1}c^{i_2}dba^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}\rangle$. Hence, the subgroup is metacyclic.

Suppose that $i_2 \equiv 0 \mod 2$. Then we may assume that j_2 is odd. Hence, $(a^{i_1}c^{i_2})^2 = a^{2i_1}$, $(dba^{j_1}c^{j_2})^2 = d^2a^2$, and $[a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}] = a^{2i_1}$. Then, we have that $[a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}] = (a^{i_1}c^{i_2})^2$, and so $\langle a^{i_1}c^{i_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}\rangle$. It follows that the subgroup is metacyclic.

Finally it remains to consider the case $i_2 \equiv 1 \mod 2$ and $i_1 \equiv j_2 + 1 \mod 2$. Then, we get that $(a^{i_1}c^{i_2})^2 = a^{2i_1}c^2$, $(dba^{j_1}c^{j_2})^2 = b^2a^{2i_1}$, and $[a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}] = b^2a^{2i_1}$. Then, we have that $[a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}] = (dba^{j_1}c^{j_2})^2$, and so $\langle a^{i_1}c^{i_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, dba^{j_1}c^{j_2}\rangle$. It follows that the subgroup is metacyclic.

This shows that every 2-generated subgroup of $\langle a, b, c, d\rangle$ is metacyclic. Therefore, the group K_9 is monotone, and so is G.

- Let G be a group in $\mathscr{A}_{10}$. Then, $G = K_{10} \times A$, where A is elementary abelian and $K_{10} = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = bd^2, c^d = c^3\rangle$. By Lemma 2.5, it is sufficient to check that K_{10} is monotone.

  The subgroup $\langle a, b, c\rangle$ is isomorphic to K_6 (the isomorphism is given by setting $\overline{a} = c$, $\overline{b} = ad$, $\overline{c} = a$). Since K_6 is monotone, every 2-generated subgroup of $\langle a, b, c\rangle$ is metacyclic.

  Therefore, in order to show that K_{10} is monotone, it is enough to check that the subgroups of the form $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ are metacyclic.

  If i_1 and j_1 are both even, then $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is contained in $\langle b, c, d\rangle$ which is isomorphic to K_6 (the isomorphism is given by setting $\overline{a} = d$, $\overline{b} = b$, $\overline{c} = bc$). Hence the subgroup $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is metacyclic.
  If i_2 and j_2 are both even, then $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is contained in $\langle a, b, d\rangle$ which is isomorphic to K_8 (the isomorphism is given by setting $\overline{a} = a$, $\overline{b} = b$, $\overline{c} = d$). Hence the subgroup $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is metacyclic.
  If i_3 and j_3 are both even, then $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is contained in $\langle a, c, d\rangle$ which is isomorphic to K_6 (the isomorphism is given by setting $\overline{a} = c$, $\overline{b} = ad$, $\overline{c} = a$). Hence the subgroup $\langle a^{i_1}c^{i_2}d^{i_3}, ba^{j_1}c^{j_2}d^{j_3}\rangle$ is metacyclic.
  Therefore, we may assume that i_k and j_k are not both even, for $k = 1, 2, 3$.

  Suppose now that i_3 is even and j_3 is odd. Since $d^2 \in \langle a, c\rangle$, we may assume that $i_3 = 0$ and, being j_3 odd we may assume that $j_3 = 1$. So, the subgroup has the form $\langle a^{i_1}c^{1_2}, ba^{j_1}c^{j_2}d\rangle$ and it is metacyclic, because $(a^{i_1}c^{i_2})^{ba^{j_1}c^{j_2}d} = (a^{i_1}c^{i_2})^{-1}$.

  So we suppose now that i_3 is odd and so we may assume $i_3 = 1$ and that $j_3 = 0$. The subgroup is of the form $\langle a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}\rangle$.
  Now, we have $(a^{i_1}c^{i_2}d)^2 = a^{2i_1}c^{2i_2}d^2c^{2i_2} = a^{2^{i_1}}d^2$, and so $|a^{i_1}c^{i_2}d| = 4$. Moreover, $(ba^{j_1}c^{j_2})^2 = b^2a^{2j_1}c^{2j_2}a^{2j_1} = b^2c^{2^{j_2}}$, and so also $|ba^{j_1}c^{j_2}| = 4$. The commutator $[a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}] = a^{2i_1}d^2c^{2j_2}$. So, if j_2 is even,

then $(a^{i_1}c^{i_2}d)^2 = [a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}]$, and so $\langle a^{i_1}c^{i_2}d\rangle \trianglelefteq \langle a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}\rangle$, i.e. the subgroup $\langle a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}\rangle$ is metacyclic.
If j_2 is odd, then $[a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}] = a^{2i_1}$, and $(ba^{j_1}c^{j_2})^2 = a^2$. This means that $[a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}] \leq \langle (ba^{j_1}c^{j_2})^2\rangle$, and so we get that $\langle ba^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}\rangle$, i.e. the subgroup $\langle a^{i_1}c^{i_2}d, ba^{j_1}c^{j_2}\rangle$ is metacyclic.

This shows that every 2-generated subgroup of $\langle a, b, c, d\rangle$ is metacyclic. Hence the subgroup $\langle a, d, c, d\rangle$ is monotone and so is G.

- $\mathscr{A}_{11}$ is the family of 2-groups of the form $K_{11} \times A$, where A is elementary abelian and $K_{11} = \langle a, b, c, d : a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b^3, a^d = ad^2, b^d = b, c^d = ca^2\rangle$.

  By Lemma 2.5, it is sufficient to check that K_{11} is monotone.
  The subgroup $\langle a, b, c\rangle$ is isomorphic to K_7 (the isomorphism is given by $\overline{a} = a$, $\overline{b} = b$, $\overline{c} = c$). Since K_7 is monotone, every 2-generated subgroup of $\langle a, b, c\rangle$ is metacyclic.
  Therefore, it is sufficient to check that the subgroups of the form $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ are metacyclic.

  If i_1 and j_1 are even, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle b, c, d\rangle$ which is monotone, being isomorphic to K_7 (the isomorphism is given by setting $\overline{a} = b$, $\overline{b} = c$, $\overline{c} = bd$). It follows that $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.
  If i_2 and j_2 are even, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle a, b, d\rangle$ which is monotone, being isomorphic to K_7 (the isomorphism is given by setting $\overline{a} = d$, $\overline{b} = a$, $\overline{c} = b$). It follows that $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.
  If i_3 and j_3 are even, then $\langle ai_1c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is contained in $\langle a, c, d\rangle$ which is monotone, being isomorphic to K_7 (the isomorphism is given by setting $\overline{a} = ac$, $\overline{b} = d$, $\overline{c} = a$). It follows that $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.

  Now, suppose that i_3 is even. Then, we may assume that j_3 is odd. Since $b^2 \in \langle a, c\rangle$ we may assume that $i_3 = 0$ and $j_3 = 1$. The subgroup has the form $\langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$. We have that $(a^{i_1}c^{i_2})^2 = a^{2i_1}c^{2i_2}$, $(da^{j_1}c^{j_2}b)^2 = d^2a^{2j_1}c^{2j_2}b^2a^{2j_1}b^{2j_2}d^{2j_1}a^{2j_2} = d^2b^2d^{2j_1} = a^2d^{2j_1}$ and $[a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b] = d^{2i_1}a^{2i_2}a^{2i_1}b^{2i_2} = d^{2i_1+2i_2}a^{2i_1}$.

  In particular, if i_1 is odd and $i_2 \equiv j_1 \mod 2$, then $(a^{i_1}c^{i_2})^2 = (da^{j_1}c^{j_2}b)^2$, and so the subgroup is metacyclic. Namely, $(a^{i_1}c^{i_2})(da^{j_1}c^{j_2}b)$ is a gen-

erator and $\langle (a^{i_1}c^{i_2})(da^{j_1}c^{j_2}b)\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$.
If i_1 is even, then we may suppose that j_1 is odd and so $(a^{i_1}c^{i_2})^2 = c^{2i_2}$, $(da^{j_1}c^{j_2}b)^2 = a^2d^2$, and $[a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b] = d^{2i_2}$. Then we have that $[a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b] = (a^{i_1}c^{i_2})^2$, and so $\langle a^{i_1}c^{i_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$, and so the subgroup $\langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$ is metacyclic.

Suppose now i_1 odd and $i_2 \not\equiv j_1 \mod 2$. This means that $j_1 \equiv i_2+1 \mod 2$, and so we get $(a^{i_1}c^{i_2})^2 = a^2c^{2i_2}$, $(da^{j_1}c^{j_2}b)^2 = a^2d^{2+2i_2}$ and $[a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b] = d^{2+2i_2}a^2$. Then, we have that $[a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b] = (da^{j_1}c^{j_2}b)^2$, and so $\langle da^{j_1}c^{j_2}b\rangle \trianglelefteq \langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$, and the subgroup $\langle a^{i_1}c^{i_2}, da^{j_1}c^{j_2}b\rangle$ is metacyclic.

So, we now may assume that i_3 is odd. In particular, since $b^2 \in \langle a, c\rangle$, we may assume that $i_3 = 1$ and that $j_3 = 0$. Hence, the subgroup has the form $\langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$.
Now, we have that $(a^{i_1}c^{i_2}b)^2 = a^{2i_1}c^{2i_2}b^2a^{2i_1}b^{2i_2} = a^{2i_2+2}c^2$, $(da^{j_1}c^{j_2})^2 = d^2a^{2j_1}c^{2j_2}d^{2j_1}a^{2j_2} = d^2b^{2j_1+2j_2}$, and $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] = d^{2i_1}a^{2i_2}a^{2j_1}b^{2j_2} = a^{2i_1+2i_2+2j_1}b^{2i_1+2j_2}$.
In particular, we get that $|a^{i_1}c^{i_2}b| = 4$, and also $|da^{j_1}c^{j_2}| = 4$.

If $i_2 \equiv 1 \mod 2$ and $j_1 \equiv j_2 \mod 2$, then $(a^{i_1}c^{i_2}b)^2 = (da^{j_1}c^{j_2})^2$, and so we have that $a^{i_1}c^{i_2}bda^{j_1}c^{j_2}$ is a generator such that $\langle a^{i_1}c^{i_2}bda^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$, and so the subgroup is metacyclic.

Suppose that $i_2 \equiv 0 \mod 2$. Then, we may assume that j_2 is odd and we get $(a^{i_1}c^{i_2}b)^2 = b^2$, $(da^{j_1}c^{j_2})^2 = a^2b^{2j_1}$, and $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] = a^{2i_1+2j_1}b^{2i_1+2} = b^2c^{2i_1}a^{2j_1}$.
If $i_1 \equiv j_1 \mod 2$, then $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] \in \langle (a^{i_1}c^{i_2}b)^2\rangle$, and so we obtain that $\langle a^{i_1}c^{i_2}b\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$, and the subgroup $\langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$ is metacyclic.
If $i_1 \equiv j_1 \mod 2$, then $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] = (da^{j_1}c^{j_2})^2$, and so we get that $\langle da^{j_1}c^{j_2}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$, and the subgroup $\langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$ is metacyclic.

To conclude, suppose that $i_2 \equiv 1 \mod 2$ and $j_1 \equiv j_2+1 \mod 2$. Then, we have $(a^{i_1}c^{i_2}b)^2 = c^2$, $(da^{j_1}c^{j_2})^2 = d^2b^2$, and $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] = d^{2i_1}a^{2i_2}a^{2j_1}b^{2j_2} = a^{2i_1+2i_2+2j_1}b^{2i_1+2j_2} = c^{2+2i_1+2i_2}$, and so we get that $[a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}] \in \langle (a^{i_1}c^{i_2}b)^2\rangle$. Hence, $\langle a^{i_1}c^{i_2}b\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$, and the subgroup $\langle a^{i_1}c^{i_2}b, da^{j_1}c^{j_2}\rangle$ is metacyclic.

This shows that every 2-generated subgroup of $\langle a, b, c, d\rangle$ is metacyclic.

Therefore, the group K_{11} is monotone, and so is G.

$\square$

Remark 3.2. *It is worth mentioning that K_6, K_9 and K_{10} admit a more intuitive presentation. Indeed, K_6 is isomorphic to $Q_8 \times C_4$, K_{10} is isomorphic to $Q_8 \times Q_8$ and K_9 is isomorphic to $Q_8 * K_2$, where, if $K_2 = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1} \rangle$, then $Q_8^2 = \langle a^2 b^2 \rangle$.*

The presentation given in Definition 3.1 is more convenient for the results we need to prove.

The aim of this chapter is to prove the following :

Theorem 3.3. *Let G be a monotone 2-group of exponent 4. Then G is either abelian or in $\mathscr{A}_i$, for some $i \in \{1, \dots, 11\}$.*

3.1 General Results

In this section we prove some preliminary results about monotone 2-groups of exponent 4.
More precisely, in Lemma 3.4, we describe the metacyclic 2-groups of exponent 4, whereas in Lemma 3.5 and Lemma 3.6, we give some properties of the normalizers of the cyclic subgroups of order 4.

Lemma 3.4. *Let G be a metacyclic 2-group of exponent 4.
Then G is either abelian or isomorphic to a group in the following list:*

1. *D_8, the dihedral group of order 8;*
2. *Q_8, the quaternion group of order 8;*
3. *$K_2 = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^3 \rangle$, a metacyclic group of order 16.*

Proof. Suppose $G = \langle a, b \rangle$ with $\langle a \rangle \trianglelefteq G$ and assume it is non-abelian. Since $exp(G) = 4$, the elements a and b have order ≤ 4.
We distinguish the possible cases:

1. if either $|a| = 2$ and $|b| = 2$ or $|a| = 4$ and $|b| = 2$, then G is the dihedral group of order 8.
2. if $|a| = 4$, $|b| = 4$ and $\langle a \rangle \cap \langle b \rangle \neq 1$, then G is the quaternion group of order 8.

3. if $|a| = 4$, $|b| = 4$ and $\langle a \rangle \cap \langle b \rangle = 1$, then G is isomorphic to K_2.

$\square$

Lemma 3.5. *Let G be a monotone 2-group of exponent 4.*
If a and b are elements of order 4 such that $\langle a \rangle \cap \langle b \rangle = 1$, then either $\langle a \rangle \trianglelefteq \langle a, b \rangle$ or $\langle b \rangle \trianglelefteq \langle a, b \rangle$.

Proof. Since G is monotone, $H = \langle a, b \rangle$ is metacyclic and, by Lemma 3.4, H is either abelian or isomorphic to K_2.

If H is abelian, then the statement is true.

If H is isomorphic to K_2, then there exist c and d in H such that H is equal to $\langle c, d : c^4 = 1, d^4 = 1, c^d = c^3 \rangle$. The possible pairs $\{a, b\}$ such that $\langle a, b \rangle = H$, and $\langle a \rangle \cap \langle b \rangle = 1$ are $\{c^h, c^i d^k\}$,$\{c^i d^k, c^h d^2\}$, with $h \in \{1, 3\}$, $k \in \{1, 3\}$ and $i \in \{0, 1, 2, 3\}$. Since $\langle c \rangle$ and $\langle c^h d^2 \rangle$ are normal subgroups in H, the lemma is proved. $\square$

Lemma 3.6. *Let G be a monotone 2-group of exponent 4. If a is an element of order 4, then $\Omega_1(G)$ is contained in the normalizer of $\langle a \rangle$.*

Proof. Take $a, b \in G$ such that $|a| = 4$ and $|b| = 2$. By Lemma 3.4, we have that $\langle a, b \rangle$ is either abelian or dihedral of order 8. In particular, each element of order 2 normalizes every cyclic subgroup of order 4, and the statement is proved. $\square$

3.2 Monotone 2-Groups Of Exponent 4 not Involving K_2

In this section we study monotone 2-groups of exponent 4 that do not involve a subgroup isomorphic to K_2 (see Definition 3.1). The main results are in Proposition 3.9 and in Proposition 3.12, where we describe the monotone 2-groups of exponent 4 with the property that K_2 is not involved.

When a monotone 2-group of exponent 4 does not involve a subgroup isomorphic to K_2, we can refine Lemma 3.4 into the following lemma.

Lemma 3.7. *Let G be a monotone 2-group of exponent 4 that does not involve a subgroup isomorphic to K_2. The followings hold:*

1. *$G^2 \leq Z(G)$;*

2. *if a and b are elements of G of order 4 such that $\langle a\rangle \cap \langle b\rangle = 1$, then $\langle a, b\rangle$ is abelian;*

3. *if X is a cyclic subgroup of G of order 4, then X is normal in G and $[G : C_G(X)] \leq 2$.*

Proof. The lemma is an easy consequence of Lemma 3.4. □

Lemma 3.8 and Proposition 3.9 deal with non-abelian monotone 2-groups G of exponent 4, that do not involve subgroups isomorphic to K_2 and with $|G^2| \geq 4$.
We claim that such a non-abelian group G does not contain a subgroup isomorphic to Q_8. Arguing by contradiction, let Q be a subgroup of G isomorphic to Q_8. Since, by Lemma 3.7(1), G^2 is abelian and generated by $\{x^2 : x \in G\}$, we obtain that there exists a cyclic subgroup X of order 4 such that $Q \cap X = 1$. By Lemma 3.7(2), we get that X centralizes Q, and so G contains a subgroup isomorphic to $Q_8 \times C_4$. Since $Q_8 \times C_4$ involves a K_2, we have a contradiction, and our claim is proved.

This means, by Lemma 3.4, that the group G contains a subgroup D isomorphic to D_8 and a cyclic group X of order 4 such that $D \cap X = 1$. In particular, for studying the structure of a monotone 2-group G of exponent 4, that does not involve subgroups isomorphic to K_2 and with $|G^2| \geq 4$, we may assume that G contains a subgroup D isomorphic to D_8 and a cyclic subgroup X of order 4 such that $D \cap X = 1$.

Lemma 3.8. *Let G be a monotone 2-group of exponent 4 such that*

(i) G does not involve a subgroup isomorphic to K_2;

(ii) G contains a subgroup D isomorphic to D_8 and there exists X a cyclic subgroup of order 4 of G such that $X \cap D = 1$.

Then G contains a subgroup isomorphic to $K_1 = A \rtimes \langle b\rangle$, where A is abelian of the form $C_4 \times C_4$, b has order 2 and $a^b = a^{-1}$ for all $a \in A$.
More precisely, if $D = \langle a, b : a^4 = 1, b^2 = 1, a^b = a^3\rangle$ and $X = \langle c\rangle$, then $\langle D, X\rangle = \langle a, b, c : a^4 = 1, c^4 = 1, b^2 = 1, a^b = a^3, a^c = a, c^b = c^3\rangle$.

Proof. Let D be generated by a and b, with a of order 4 and let X be $\langle c\rangle$.

By Lemma 3.7, $\langle a, c\rangle$ is abelian. Since $\langle b, c\rangle$ is either abelian or dihedral, we get the following cases:

- $a^c = a$, $b^c = b$: since $\langle bc, a\rangle$ is isomorphic to K_2, we contradict (i).
- $a^c = a$, $c^b = c^3$: the group $\langle a, c, b\rangle$ is isomorphic to K_1.

□

Proposition 3.9. *Let G be a monotone 2-group of exponent 4 such that*

(i) *G does not involve a subgroup isomorphic to K_2;*

(ii) *G properly contains a subgroup K isomorphic to K_1 (see Lemma 3.8).*

Then G is isomorphic to a semidirect product $A \rtimes \langle b\rangle$, where A is abelian of exponent 4, $|A^2| \geq 4$, b has order 2 and $a^b = a^{-1}$ for all $a \in A$.

Proof. Let $K = \langle a, b, c : a^4 = 1, c^4 = 1, b^2 = 1, a^b = a^3, c^b = c^3, a^c = a\rangle$.

Let A be the centralizer of a. We want to prove that A is abelian.

First of all we prove that, if d is an element of order 4 in G, then it commutes with a and c and is inverted by b.
In fact, if $\langle d\rangle \cap \langle a, b\rangle \neq 1$, then, by Lemma 3.7, we have $\langle d\rangle \cap \langle a, b\rangle = \langle a^2\rangle$. Hence, we get that $\langle d\rangle \cap \langle c, b\rangle = 1$ and so, by Lemma 3.8, we have that $\langle c, d\rangle$ is abelian and $d^b = d^3$. Moreover, since $\langle cd\rangle \cap \langle a, b\rangle = 1$, by Lemma 3.8, we have that $\langle a, cd\rangle$ is abelian. Since $[a, d] = [a, cd] = 1$, our preliminary claim is proved.

Now, let d and e be elements of order 4 of A. If $\langle d\rangle \cap \langle e\rangle = 1$, then, by Lemma 3.7(2), the subgroup $\langle d, e\rangle$ is abelian.
Suppose $\langle d\rangle \cap \langle e\rangle \neq 1$. Then either $\langle d\rangle \cap \langle a\rangle = \langle e\rangle \cap \langle a\rangle = 1$ or $\langle d\rangle \cap \langle c\rangle = \langle e\rangle \cap \langle c\rangle = 1$. In the first case, since $\langle ad\rangle \cap \langle e\rangle = 1$, using Lemma 3.7(2), we have that $\langle ad, e\rangle$ is abelian. Since e is in A, we have that also $\langle d, e\rangle$ is abelian. In the second case, since $\langle cd\rangle \cap \langle e\rangle = 1$, using Lemma 3.7(2), we have that $\langle cd, e\rangle$ is abelian. By our preliminary claim, the element e is in the centralizer of $\langle c\rangle$, and so the subgroup $\langle d, e\rangle$ is abelian. Hence, the elements of order 4 of A commute.

Let u be an element of A of order 2. By the previous paragraph, being au an element of order 4 in A, we have that au centralizes all the elements of order 4 of A. Since A is the centralizer of a, we have that u commutes with all the elements of order 4 of A.

Let u and v be elements of order 2 in A. By the previous paragraph, the element u centralizes av, and since u centralizes a, we get that u and v commute.

Therefore, the subgroup A is abelian, and so A is generated by its elements of order 4. Since, by our preliminary claim, each element of A of order 4 is inverted by b, we get that b acts by inversion on A. Since by Lemma 3.7(3), A is a maximal subgroup of G, the statement is proved. □

Complementary to Lemma 3.8 and Proposition 3.9, in Lemma 3.10, Lemma 3.11 and Proposition 3.12, we deal with monotone 2-groups G of exponent 4, that do not involve subgroups isomorphic to K_2 and with $|G^2| = 2$. In particular, if G is as above and contains a subgroup D isomorphic to D_8, then every cyclic subgroup of G of order 4 intersects non-trivially D.
Hence, for studying a monotone 2-group G of exponent 4, that does not involve a subgroup isomorphic to K_2 and with $|G^2| = 2$, we may assume that any dihedral subgroup D of G and any cyclic subgroup of order 4 intersect non-trivially.
In Lemma 3.10, we treat monotone 2-groups G of exponent 4, that do not involve a subgroup isomorphic to K_2 with $|G^2| = 2$ and that contain a subgroup isomorphic to Q_8.
In Lemma 3.11, we determine the non-abelian monotone 2-groups G of exponent 4, that do not involve a subgroup isomorphic neither to K_2 nor to Q_8 and with $|G^2| = 2$. Proposition 3.12 concludes the description of monotone 2-groups G of exponent 4 not involving a subgroup isomorphic to K_2 and with $|G^2| = 2$.

Lemma 3.10. *Let G be a monotone 2-group of exponent 4 such that*

(i) G does not involve a subgroup isomorphic to K_2;

(ii) if G contains a subgroup D isomorphic to D_8, then there are no cyclic subgroups X of order 4 such that $X \cap D = 1$;

(iii) G contains a subgroup Q isomorphic to Q_8.

Then $G = E * C$, where E is an extraspecial group of the form $Q_8 * \cdots * Q_8$ and C is a subgroup that does not involve Q_8.

Proof. We prove the lemma by induction on the order of G.
Let $Q = \langle a, b : a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle$. If X is a cyclic subgroup of order 4 not contained in Q, then $X \cap Q = \langle a^2 \rangle$ (otherwise G would contain a subgroup isomorphic to K_2). Moreover, by Lemma 3.7(3), the group $C_G(Q)$ has index 4 in G and $G = Q * C_G(Q)$. Now, $C_G(Q)$ is a monotone 2-group

of exponent 4 that satisfies (i) and (ii). Moreover $|C_G(Q)| < |G|$. Now, if $C_G(Q)$ does not involve a subgroup isomorphic to Q_8, then the lemma is proved. Otherwise we conclude by induction. $\square$

Lemma 3.11. *Let G be a monotone 2-group of exponent 4 such that*

(i) *G does not involve a subgroup isomorphic to K_2;*

(ii) *G contains a subgroup D isomorphic to D_8 and there are no cyclic subgroups X of order 4 such that $X \cap D = 1$;*

(iii) *G does not involve a subgroup isomorphic to Q_8.*

Then $G = D \times A$, where A is an elementary abelian subgroup.

Proof. We prove the lemma by induction on the order of G.
Let $D = \langle a, b : a^4 = 1, b^2 = 1, a^b = a^{-1}\rangle$ be a subgroup of G isomorphic to D_8. By Lemma 3.7(3), the subgroup $C_G(\langle a\rangle)$ is maximal in G. We now show that $C_G(\langle a\rangle)$ is abelian.

Let $c, d \in C_G(\langle a\rangle)$. If c has order 4, then $c^2 = a^2$ and ac has order 2. The same holds for d. Then we may assume that c and d have order 2.
Assume $\langle c, d\rangle$ is non-abelian. Then $\langle c, d\rangle$ is dihedral. In particular $(cd)^2 = a^2$, $(cd)^c = (ad)^{-1}$ and G contains the subgroup $\langle ac, cd\rangle \simeq Q_8$, against (iii). Therefore $C_G(\langle a\rangle)$ is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$.

We now prove that $\Omega_1(C_G(\langle a\rangle)) \leq C_G(\langle b\rangle)$. It is sufficient to prove that if c is an element of order 2 in $C_G(\langle a\rangle)$, then $c \in C_G(\langle b\rangle)$.
Now $\langle b, c\rangle$ is either abelian or isomorphic to D_8. Suppose that $\langle b, c\rangle$ is isomorphic to D_8. The element cb has order 4 and so $(cb)^2 = a^2$ and $(bc)^b = (bc)^{-1}$. Now $\langle bc, a\rangle$ is isomorphic to Q_8, against (iii). It follows that the subgroup $\langle b, c\rangle$ is abelian, for all $c \in C_G(\langle a\rangle)$ of order 2. Therefore $C_G(D)$ is an elementary abelian subgroup of G such that $[G : C_G(D)] = 4$ and so $G = D * C_G(D) = D \times A$, where A is elementary abelian. $\square$

Proposition 3.12. *Let G be a monotone 2-group of exponent 4 such that*

(i) *G does not involve a subgroup isomorphic to K_2;*

(ii) *if G contains a subgroup D isomorphic to D_8, then there are no cyclic subgroups X of order 4 such that $X \cap D = 1$.*

Then G is isomorphic to one of the following groups:

1. *$E \times A$, where E is extraspecial and A is elementary abelian;*

2. *$E * A$, where E is extraspecial and A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$ and $E^2 = A^2$.*

Proof. If G does not involve a subgroup isomorphic to Q_8, then by Lemma 3.11 the group G is in *1*.

Suppose now that G involves a subgroup isomorphic to Q_8. By Lemma 3.10, we have $G = E * C$, where E is extraspecial of the form $Q_8 * \cdots * Q_8$ and C is a subgroup of G that does not involve Q_8. By (i), it is easy to check that, if C is abelian, then C is either of the form $C_4 \times C_2 \times \ldots \times C_2$ (and so G is in *2*) or elementary abelian (and so G is in *1*).
If C is non-abelian, then C satisfies the hypothesis of Lemma 3.11. So $C = D \times A$, where D is isomorphic to D_8 and A is elementary abelian.
Therefore $G = E \times A$ with A elementary abelian and G is in *1*. □

Summing up, in this section, we determined the monotone 2-groups of exponent 4 not containing a subgroup isomorphic to K_2. Namely, any such a group is in the class $\mathscr{A}_2$ or in the class $\mathscr{A}_3$.

3.3 Monotone 2-Groups Of Exponent 4 Involving K_2

In this section we study monotone 2-groups G of exponent 4 containing a subgroup K isomorphic to K_2.

First of all we state a preliminary result in which we give some properties of the centralizer of K.

Lemma 3.13. *Let G be a monotone 2-group of exponent 4 containing a subgroup K isomorphic to K_2.
Then $\Omega_1(G) \leq C_G(K)$.*

Proof. Let $K = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1} \rangle$.
Let c be an element of order 2 in G. If c is in K, then we get $c \in C_G(K)$ (because $\Omega_1(K) = Z(K)$).

Suppose now that c is not in K.
By Lemma 3.6, the element c normalizes $\langle a \rangle$ and $\langle b \rangle$. So, we have the following cases:

1. $a^c = a$, $b^c = b$: the element c is in $C_G(K)$.

2. $a^c = a^{-1}$, $b^c = b$: the element ab has order 4 and $(ab)^2 = b^2$. Since $(ab)^c = aba^2$, we have $c \notin N_G(\langle ab \rangle)$ and we contradict Lemma 3.6.

3. $a^c = a$, $b^c = b^{-1}$: the elements ac and b have order 4 and $\langle ac \rangle \cap \langle b \rangle = 1$. Since $(ac)^b = a^3cb^2 = aca^2b^2$, neither $\langle ac \rangle$ nor $\langle b \rangle$ is normal in G, and we contradict Lemma 3.5.

4. $a^c = a^{-1}$, $b^c = b^{-1}$: the element ab has order 4 and $(ab)^2 = b^2$. Since $(ab)^c = aba^2b^2$, we have $c \notin N_G(\langle ab \rangle)$ and we contradict Lemma 3.6.

□

In the first part of this section, we describe the structure of the monotone 2-groups G of exponent 4 containing a subgroup K isomorphic to K_2 and with the property that $|G^2| \geq 8$. In particular, this implies that there exists a cyclic subgroup X of order 4 such that $X \cap K = 1$.
More precisely, in Lemma 3.14, we determine the structure of $\langle K, X \rangle$, and in Proposition 3.15 we conclude the description of the monotone 2-groups G of exponent 4 containing a subgroup isomorphic to K_2 and such that $|G^2| \geq 8$.

Lemma 3.14. *Let G be a monotone 2-group of exponent 4 such that*

(i) G contains a subgroup K isomorphic to K_2;

(ii) there exists X, a cyclic subgroup of order 4 of G, such that $X \cap K = 1$.

Then $\langle K, X \rangle$ is isomorphic to $K_3 = A \rtimes \langle b \rangle$, where A is a 2-generated abelian group, $|A^2| = 4$, b has order 4 and $a^b = a^{-1}$ for all $a \in A$.
More precisely, if $K = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^3 \rangle$ and $X = \langle c \rangle$, then $\langle K, X \rangle = \langle a, b, c : a^4 = 1, b^4 = 1, c^4 = 1, a^c = a, a^b = a^3, c^b = c^3 \rangle$.

Proof. Let $K = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1} \rangle$ and let $X = \langle c \rangle$. Since c and a have order 4 and $\langle a \rangle \cap \langle c \rangle = 1$, by Lemma 3.5, either $\langle a \rangle$ or $\langle c \rangle$ is normal in $\langle a, c \rangle$. The same holds for $\langle b, c \rangle$. Hence $[a, c] \in \{1, a^2, c^2\}$, and $[b, c] \in \{1, b^2, c^2\}$. The possibilities are:

1. $a^c = ac^2$: now, either $c^b = c^h$, with $h \in \{1, 3\}$, or $c^b = cb^2$. In every case, the elements a and cb have order 4 and $\langle a \rangle \cap \langle cb \rangle = 1$. As $\langle a \rangle$ and $\langle cb \rangle$ are not normal in $\langle a, bc \rangle$, we contradict Lemma 3.5.

2. $a^c = a^3$: now, either $c^b = c^h$, with $h \in \{1,3\}$, or $c^b = cb^2$. In every case, the subgroup $\langle ac, b\rangle$ is non-metacyclic. In fact, $(ac)^2 = c^2$ and $(ac)^b = aca^2[c,b]$. Since $[c,b] \in \{1, b^2, c^2\}$, we have that $\langle ac, b\rangle$ contains the 3-generated elementary abelian subgroup $\langle c^2, b^2, a^2\rangle$. Therefore $\langle ac, b\rangle$ is not monotone and this case does not arise.

3. $a^c = a$: now, either $c^b = c^h$, with $h \in \{1,3\}$, or $c^b = cb^2$.
   If $h = 1$ or $c^b = cb^2$, then ac and b are elements of order 4 and $\langle ac\rangle \cap \langle b\rangle = 1$. As $\langle ac\rangle$ and $\langle b\rangle$ are not normal in $\langle ac, b\rangle$, we contradict Lemma 3.5.
   If $h = 3$, then we get the group $\langle a, b, c : a^4 = 1, b^4 = 1, c^4 = 1, a^c = a, a^b = a^3, c^b = c^3\rangle$ which is isomorphic to K_3.

$\square$

Proposition 3.15. *Let G be a monotone 2-group of exponent 4.*
Suppose that G properly contains a subgroup K isomorphic to K_3 (see Lemma 3.14).
Then G is isomorphic to $A\langle b\rangle$, where A is an abelian group of exponent 4, $|A^2| \geq 4$, b has order 4, $a^b = a^{-1}$ for all $a \in A$, and either the extension is splitting or $b^2 \in A^2$.

Proof. Let K be $\langle a, c\rangle \rtimes \langle b\rangle$ where $\langle a, c\rangle$ is a 2-generated abelian group of order 16, b has order 4 and acts as inversion on $\langle a, c\rangle$.

We firstly prove that if $\langle d\rangle$ is a cyclic subgroup such that $\langle d\rangle \cap \langle b\rangle = 1$, then the subgroup $\langle a, c, d\rangle$ is abelian and $d^b = d^3$.
Let d be an element of order 4 such that $\langle d\rangle \cap \langle b\rangle = 1$.
Up to renaming the generators of $\langle a, c\rangle$, we may assume that $\langle d\rangle \cap \langle a, b\rangle = 1$. Using Lemma 3.14, we have that $\langle a, d\rangle$ is abelian and $d^b = d^3$. Now, if $\langle d\rangle \cap \langle c, b\rangle = 1$, then, by Lemma 3.14, the subgroup $\langle c, d\rangle$ is also abelian. If $\langle d\rangle \cap \langle c, b\rangle \neq 1$, then $d^2 \in \langle c^2, b^2\rangle$. Since $(ad)^2 = a^2d^2$ is not in $\langle c, b\rangle$, we have that $\langle ad\rangle \cap \langle c, b\rangle = 1$ and, by Lemma 3.14, the subgroup $\langle c, ad\rangle$ is abelian. Since a and c commute, we get that $\langle c, d\rangle$ is abelian. This concludes the proof of the claim and, in the sequel, we refer to $(*)$ to recall this fact.

Let A be the centralizer of a.

We now show that $G = \langle A, b\rangle$.
If d is an element of order 2 of G then, by Lemma 3.13, the element d lies in A.

If d is an element of order 4 of G such that $\langle d\rangle \cap \langle b\rangle = 1$, then, by $(*)$, we have that d is in A.
Suppose now that d is an element of order 4 of G such that $\langle d\rangle \cap \langle b\rangle \neq 1$. If d does not lie in A, then $\langle a, d\rangle$ is non-abelian. Since $\langle a\rangle \cap \langle d\rangle = 1$, by Lemma 3.5, we get that either $\langle a\rangle$ or $\langle d\rangle$ is normal in $\langle a, d\rangle$. If $\langle d\rangle \trianglelefteq \langle a, d\rangle$, then we have $d^a = d^3$. The element ad has order 4 and $\langle ad\rangle \cap \langle b\rangle = 1$. Hence, by $(*)$, we get that ad lies in A, i.e. d lies in A, a contradiction. Then, we get $\langle a\rangle \trianglelefteq \langle a, d\rangle$ and so $a^d = a^3$. Then, the element db lies in A. Therefore, our claim is proved.

We prove that b acts as inversion on A. This also implies that A is abelian.
We show that, if d is in A, then $d^b = d^3$.
If d is an element of A of order 2, then, by Lemma 3.13, we get that $d \in C_G(\langle a, b\rangle)$.
If d is an element of A of order 4 and $\langle d\rangle \cap \langle b\rangle = 1$, then, by $(*)$, we get that $d^b = d^3$.
If d is an element of A of order 4 and $\langle d\rangle \cap \langle b\rangle \neq 1$, then ad is an element of order 4 lying in A and $\langle ad\rangle \cap \langle b\rangle = 1$. Therefore, by $(*)$, we get that $(ad)^b = (ad)^3$, and since d is in A and $a^b = a^3$, we get that $d^b = d^3$.

Summing up, the group $G = \langle A, b\rangle$ is such that A is abelian, $|A^2| \geq 4$ and $d^b = d^3$ for all $d \in A$. The proposition is proved.

□

The previous proposition concludes the first part of this section and the classification of the monotone 2-groups G that contain a subgroup K isomorphic to K_2 and a cyclic subgroup X of order 4, such that $K \cap X = 1$ (i.e. $|G^2| \geq 8$).

In the rest of this section, we study the monotone 2-groups of exponent 4 that contain a subgroup K isomorphic to K_2 and such that there exist no cyclic subgroups X of order 4 such that $K \cap X = 1$ (i.e. $|G^2| = 4$).
Lemma 3.16 and Lemma 3.17 gives some properties of these groups.

Lemma 3.16. *Let G be a monotone 2-group of exponent 4 such that*

(i) G contains a subgroup K isomorphic to K_2;

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then $G^2 = K^2$.

Proof. The subgroup G^2 is generated by $\{a^2 : a \in G\}$. Now, if a is an element of order 4, then $\langle a \rangle \cap K \neq 1$ and so $a^2 \in \Omega_1(K) = K^2$. Hence $G^2 = \{x^2 : x \in G\} \leq \Omega_1(K) = K^2$. □

Lemma 3.17. *Let G be a monotone 2-group of exponent 4 such that*

(i) G contains a subgroup K isomorphic to K_2;

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then $\Omega_1(G)$ is an abelian subgroup.

Proof. Let $K = \langle a, b : a^4 = 1, b^4 = 1, a^b = a^{-1} \rangle$ and let c, d be elements of order 2 in G.
If $c, d \in K$, then $\langle c, d \rangle$ is abelian because $\Omega_1(K) = Z(K)$.

If $c \in K$ and $d \notin K$, then $d \in C_G(K)$ because of Lemma 3.13. Therefore the subgroup $\langle c, d \rangle$ is abelian.

Suppose now $c, d \notin K$. Assume that $\langle c, d \rangle$ is not abelian. By Lemma 3.4, the subgroup $\langle c, d \rangle$ is a dihedral group of order 8. Also by Lemma 3.13, we get $\Omega_1(G) \leq C_G(K)$ and we have that $\langle c, d \rangle \leq C_G(K)$. Moreover cd is an element of order 4 and (ii) yields $(cd)^2 \in \langle a^2, b^2 \rangle$. Furthermore, either $\langle cd \rangle \cap \langle a \rangle = 1$ or $\langle cd \rangle \cap \langle b \rangle = 1$. If $\langle cd \rangle \cap \langle a \rangle = 1$, then the subgroup $\langle c, d, a \rangle$ is isomorphic to $D_8 \times C_4$. If $\langle cd \rangle \cap \langle b \rangle = 1$, then the subgroup $\langle c, d, b \rangle$ is isomorphic to $D_8 \times C_4$. In both cases, G contains $D_8 \times C_4$, that is not a monotone group, a contradiction. Therefore $\langle c, d \rangle$ is abelian. □

Let G be a monotone 2-group of exponent 4 containing a subgroup K isomorphic to K_2. We point out that, by Lemma 3.16, G has no cyclic subgroups X of order 4 with $K \cap X = 1$ if and only if $|G^2| = 4$.
Lemma 3.18, Lemma 3.21, Lemma 3.22, Lemma 3.23 and Lemma 3.24 describe the structure of some subgroups of a monotone 2-group that involves a subgroup isomorphic to K_2 and such that $|G^2| = 4$.

Lemma 3.18. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_2;

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then G contains a subgroup isomorphic to a group in the following list:

$K_2 \times C_2$;

$K_6 = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b\rangle$;

$K_7 = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b^3\rangle$;

$K_8 = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b^3a^2\rangle$.

Proof. Let K be $\langle a, b : a^4 = b^4 = 1, a^b = a^{-1}\rangle$ and let $c \in G \setminus G^2$ with $c \notin K$.

Suppose that c has order 2. Since an element of order 2 centralizes K (see Lemma 3.13), the subgroup $\langle K, c\rangle$ is isomorphic to $K \times \langle c\rangle$. Also, if there exists $k \in K$ such that kc has order 2, then, replacing c with kc, we get $\langle K, c\rangle = K \times \langle kc\rangle \simeq K_2 \times C_2$.

Therefore, from now on, we may assume that all the elements of Kc have order 4. In the sequel, we refer to $(*)$ to recall the previous assumption. Since $G^2 = K^2 = \langle a^2, b^2\rangle$ (see Lemma 3.16), we have that $[c, K] \leq K^2$ and $c^2 \in K^2$. The rest of the proof is a case-by-case analysis depending on where c^2 lies in K^2.

1. Suppose that $c^2 = a^2b^2$. Since $\langle c\rangle \cap \langle a\rangle = 1$, by Lemma 3.5 either $\langle a\rangle$ or $\langle c\rangle$ is normal in $\langle a, c\rangle$.
   Likewise, being $\langle c\rangle \cap \langle b\rangle = 1$, either $\langle b\rangle$ or $\langle c\rangle$ is normal in $\langle c, b\rangle$.
   Therefore $[a, c] \in \{1, a^2, c^2\}$ and $[b, c] \in \{1, b^2, c^2\}$. Now, we analyze all the possibilities:

   - $a^c = a$: now $b^c = b^h$, where $h \in \{1, 3\}$ or $b^c = bc^2$.
     If $h = 1$, then the group $\langle a, b, c\rangle$ is isomorphic to K_6.
     If $h = 3$, then the group $\langle a, b, c\rangle$ is isomorphic to K_7.
     If $b^c = bc^2$, then the group $\langle a, b, c\rangle$ is isomorphic to K_8.
   - $a^c = a^3$: now $b^c = b^h$, where $h \in \{1, 3\}$ or $b^c = bc^2$.
     If $h = 1$, then the subgroup $\langle ab, c\rangle$ is neither abelian nor isomorphic to K_2, and this case does not arise by Lemma 3.5.
     In the other cases, the group $\langle a, b, c\rangle$ is isomorphic to K_7. More precisely, if $h = 3$, then we get an isomorphism by setting $\overline{a} = a, \overline{b} = ab, \overline{c} = bc$.
     If $b^c = bc^2$, then we get an isomorphism by setting $\overline{a} = a, \overline{b} = c, \overline{c} = bc$.

- $a^c = a^3b^2$: now $b^c = b^h$, where $h \in \{1,3\}$ or $b^c = bc^2$.
  If $h = 3$, then the subgroup $\langle ab, c\rangle$ is neither abelian nor isomorphic to K_2, and this case does not arise by Lemma 3.5.
  In the other cases, the group $\langle a, b, c\rangle$ is isomorphic to K_7.
  More precisely, if $h = 1$, then we get the isomorphism by setting $\overline{a} = c, \overline{b} = a, \overline{c} = b$.
  Also if $b^c = bc^2$, then we get the isomorphism by setting $\overline{a} = c, \overline{b} = a, \overline{c} = abc$.

2. Suppose now $c^2 = a^2$. Since $\langle c\rangle \cap \langle b\rangle = 1$, by Lemma 3.5, either $\langle b\rangle$ or $\langle c\rangle$ is normal in $\langle b, c\rangle$.
   Likewise, being $\langle c\rangle \cap \langle ab\rangle = 1$, either $\langle ab\rangle$ or $\langle c\rangle$ is normal in $\langle c, ab\rangle$.
   Therefore $[b, c] \in \{1, b^2, c^2\}$ and $[ab, c] \in \{1, b^2, c^2\}$. Now, we analyze all the possibilities:

   - $b^c = b$: replacing c with bc, we are in the case 1;
   - $b^c = b^3$: now $(ab)^c = abb^{2h}$ (i.e. $a^c = ab^{2(h+1)}$), where $h \in \{0,1\}$, or $(ab)^c = (ab)c^2$ (i.e. $a^c = a^3b^2$). If $h = 0$, then, replacing c with abc, we are in case 1 ($(abc)^2 = b^2c^2 = b^2a^2$). Likewise if $a^c = a^3b^2$, then, replacing c with ac, we are in case 1 ($(ac)^2 = a^2b^2$). To conclude, if $h = 1$, then ac has order 2 and we contradict $(*)$.
   - $b^c = ba^2$: now $(ab)^c = abb^{2h}$ (i.e. $a^c = aa^2b^{2h}$), where $h \in \{0,1\}$, or $(ab)^c = (ab)c^2$ (i.e. $a^c = a$). If $h = 0$, then, replacing c with abc, we are in case 1 ($(abc)^2 = a^2b^2$). Likewise, if $h = 1$, then replacing c with ac, we are in case 1 ($(ac)^2 = a^2b^2$). To conclude, if $a^c = a$, then ac has order 2 and we contradict $(*)$.

3. Suppose $c^2 = b^2$. Since $\langle c\rangle \cap \langle a\rangle = 1$, by Lemma 3.5, we get that either $\langle a\rangle$ or $\langle c\rangle$ is normal in $\langle a, c\rangle$. Moreover $[c, b] \in \langle a^2, b^2\rangle$. Therefore, we analyze all the possibilities:

   - $a^c = a$: replacing c with ac, we are in case 1 ($(ac)^2 = a^2b^2$).
   - $a^c = a^3$: now $b^c = ba^{2h}b^{2k}$, where $h, k \in \{0,1\}$.
     Consider the element bc: $(bc)^2 = a^{2h}b^{2k}$. Hence, if $h = 0$ and $k = 0$, then bc has order 2 and we contradict $(*)$. If $h = 1$ and $k = 1$, then, replacing c with bc, we are in case 1.
     Consider the element abc: $(abc)^2 = a^{2+2h}b^{2k}$. Hence, if $h = 1$ and

$k = 0$, then abc has order 2 and we contradict $(*)$. If $h = 0$ and $k = 1$, then, replacing c with abc, we are in case 1.

- $a^c = ac^2$: now $b^c = ba^{2h}b^{2k}$, where $h, k \in \{0, 1\}$.
  Consider the element bc: $(bc)^2 = a^{2h}b^{2k}$. Hence, if $h = 0$ and $k = 0$, then bc has order 2 and we contradict $(*)$. If $h = 1$ and $k = 1$, then, replacing c with bc, we are in case 1.
  Consider the element abc: $(abc)^2 = a^{2h}c^{2k+2}$. Hence, if $h = 0$ and $k = 1$, then abc has order 2 and we contradict $(*)$. If $h = 1$ and $k = 0$, then, replacing c with abc, we are in case 1.

Hence the statement is proved.

□

Remark 3.19. *Let G and K be as in Lemma 3.16. Let $c \in G \setminus G^2$ but $c \notin K$. From the proof of Lemma 3.18, we have that there exists an element $k \in K$ such that $|ck| = 2$ or such that $(ck)^2 = a^2b^2$.*

Remark 3.20. *Let G and K be as in Lemma 3.16.*
If T is a subgroup of G such that $T^2 = G^2 = K^2$, then there are no cyclic subgroups X of order 4 of G such that $X \cap T = 1$ (otherwise there exists X cyclic subgroup of G such that $X \cap K = 1$, against the assumption).

In the following three lemmas, we study the structure of a monotone 2-group G such that $|G^2| = 4$ and containing K_6 or K_7 and K_8.

Lemma 3.21. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_6 (see Lemma 3.18);

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then K is contained in a subgroup isomorphic to a group in the following list:

$K_6 \times C_2$;

$$K_9 = \langle a, b, c, d \ : \ a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = b^3, c^d = cb^2 \rangle;$$

$$K_{10} = \langle a, b, c, d \ : \ a^4 = 1, b^4 = 1, c^2 = a^2b^2, c^2 = d^2, a^b = a^3, a^c = a, b^c = b, a^d = a, b^d = bd^2, c^d = c^3 \rangle.$$

Moreover, the subgroup $\Omega_1(G)$ is in the centralizer of K.

Proof. Suppose $K = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b\rangle$ and let d be in $G\backslash K$.

Suppose d has order 2. By Lemma 3.13, we get $d \in C_G(\langle a, b\rangle)$ and $d \in C_G(\langle ab, bc\rangle)$. Hence $\langle K, d\rangle$ is isomorphic to $K \times C_2$. In particular we get that $\Omega_1(G) \leq C_G(K)$. Furthermore, from now on, we may assume that all the elements in Kc have order 4. In the sequel, we refer to $(*)$ to recall the previous assumption.

Suppose d has order 4. By (ii), it is easy to check that $[d, K] \leq K^2$ and $d^2 \in K^2$. We have $\langle a, b\rangle \simeq K_2$ and $\langle a, b\rangle^2 = G^2 = K^2$. By Remark 3.19, we can assume $d^2 = a^2b^2$. Since $\langle d\rangle \cap \langle a\rangle = 1$, $\langle d\rangle \cap \langle b\rangle = 1$ and $\langle d\rangle \cap \langle bc\rangle = 1$, we get that $\langle a, d\rangle$, $\langle d, b\rangle$, $\langle d, bc\rangle$ are either abelian or isomorphic to K_2.
Hence $[a, d] \in \{1, a^2, a^2b^2\}$, $[b, d] \in \{1, b^2, a^2b^2\}$, $[bc, d] \in \{1, a^2, a^2b^2\}$. Now, the rest of the proof is a case-by-case analysis depending on the possible values of $[a, d]$,$[b, d]$ and $[bc, d]$. By $(*)$, we may assume that none of the followings happen:

- $[d, c] = 1$ (otherwise $(cd)^2 = 1$);
- $[a, d][b, d] = a^2$ (otherwise $(abd)^2 = 1$);
- $[a, d][c, d] = a^2$ (otherwise $(acd)^2 = 1$);
- $[b, d][c, d] = b^2$ (otherwise $(bcd)^2 = 1$);
- $[a, d][b, d][c, d] = b^2$ (otherwise $(abcd)^2 = 1$).

Excluding the cases in which one of the previous is satisfied, it remains to study the following possibilities:

- $a^d = a$, $b^d = b$, $(bc)^d = bca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = c, \overline{b} = abcd, \overline{c} = ab, \overline{d} = b$.
- $a^d = a$, $b^d = bb^2$, $(bc)^d = bc$, i.e. $a^d = a$, $b^d = bb^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9.
- $a^d = a$, $b^d = bb^2$, $(bc)^d = (bc)a^2$, i.e. $a^d = a$, $b^d = bb^2$, $c^d = ca^2b^2$: the group $\langle a, b, c, d\rangle$ so obtained is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = a, \overline{b} = bcd, \overline{c} = ad, \overline{d} = acd$.

- $a^d = a$, $b^d = ba^2b^2$, $(bc)^d = bc$, i.e. $a^d = a$, $b^d = ba^2b^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is K_{10}.

- $a^d = a$, $b^d = ba^2b^2$, $(bc)^d = (bc)a^2$, i.e. $a^d = a$, $b^d = ba^2b^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = bc, \overline{b} = cd, \overline{c} = adc, \overline{d} = c$.

- $a^d = aa^2$, $b^d = bb^2$, $(bc)^d = bc$, i.e. $a^d = aa^2$, $b^d = bb^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = a, \overline{b} = acd, \overline{c} = bd, \overline{d} = c$.

- $a^d = aa^2$, $b^d = bb^2$, $(bc)^d = (bc)a^2$, i.e. $a^d = aa^2$, $b^d = bb^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = abc, \overline{b} = acd, \overline{c} = abd, \overline{d} = abcd$.

- $a^d = aa^2$, $b^d = ba^2b^2$, $(bc)^d = bc$, i.e. $a^d = aa^2$, $b^d = ba^2b^2$, $c^d = ca^2b^2$. The group $\langle a, gb, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = bc, \overline{b} = abcd, \overline{c} = bcd, \overline{d} = bd$.

- $a^d = aa^2$, $b^d = ba^2b^2$, $(bc)^d = (bc)a^2$, i.e. $a^d = aa^2$, $b^d = ba^2b^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = bc, \overline{b} = ab, \overline{c} = c, \overline{d} = abd$.

- $a^d = aa^2b^2$, $b^d = b$, $(bc)^d = bca^2b^2$, i.e. $a^d = aa^2b^2$, $b^d = b$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = a, \overline{b} = acd, \overline{c} = bd, \overline{d} = bcd$.

- $a^d = aa^2b^2$, $b^d = ba^2b^2$, $(bc)^d = bc$, i.e. $a^d = aa^2b^2$, $b^d = ba^2b^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = d, \overline{b} = abcd, \overline{c} = ab, \overline{d} = ac$.

□

Lemma 3.22. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_7 (see Lemma 3.18);

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then K is contained in a subgroup isomorphic to

$K_7 \times C_2$;

K_9 *(see Lemma 3.21);*

$$K_{11} = \langle a, b, c, d \ : \ a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, \\ a^c = a, b^c = bb^2a^d = ad^2, b^d = b, c^d = ca^2\rangle.$$

Moreover, the subgroup $\Omega_1(G)$ is in the centralizer of K.

Proof. Suppose $K = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = b^3\rangle$ and let d be in $G\backslash K$.

Suppose d has order 2: then, by Lemma 3.13, $d \in C_G(\langle a, b\rangle)$ and $d \in C_G(\langle b, c\rangle)$ that means that $\langle K, d\rangle$ is isomorphic to $K \times C_2$.
In particular, the subgroup $\Omega_1(G)$ is contained in $C_G(K)$. Furthermore, from now on, we may assume that all the elements in Kc have order 4. In the sequel, we refer to $(*)$ to recall the previous assumption.

Suppose d has order 4. By (ii), it is easy to check that $[d, K] \in K^2$ and $d^2 \in K^2$. We have $\langle a, b\rangle \simeq K_2$ and $\langle a, b\rangle^2 = G^2 = K^2$. By Remark 3.19, we can assume $d^2 = a^2b^2$. Since $\langle d\rangle \cap \langle a\rangle = 1$, $\langle d\rangle \cap \langle b\rangle = 1$ and $\langle d\rangle \cap \langle ac\rangle = 1$, we get that $\langle a, d\rangle$, $\langle d, b\rangle$ and $\langle d, ac\rangle$ are either abelian or isomorphic to K_2. Hence $[a, d] \in \{1, a^2, a^2b^2\}$, $[b, d] \in \{1, b^2, a^2b^2\}$, $[ac, d] \in \{1, b^2, a^2b^2\}$. Now, the rest of the proof is a case-by-case analysis depending on the possible values of $[a, d]$,$[b, d]$ and $[ac, d]$. By $(*)$, we may assume that none of the followings happen:

- $[d, c] = 1$ (otherwise $(cd)^2 = 1$);
- $[a, d][b, d] = a^2$ (otherwise $(abd)^2 = 1$);
- $[a, d][c, d] = a^2$ (otherwise $(acd)^2 = 1$);
- $[b, d][c, d] = 1$ (otherwise $(bcd)^2 = 1$);
- $[a, d][b, d][c, d] = 1$ (otherwise $(abcd)^2 = 1$).

Excluding the cases in which one of the previous is satisfied, it remains to study the following possibilities:

- $a^d = a$, $b^d = b$, $(ac)^d = acb^2$, i.e. $a^d = a$, $b^d = b$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = abd, \overline{b} = bcd, \overline{c} = c, \overline{d} = d$.
- $a^d = a$, $b^d = b$, $(ac)^d = aca^2b^2$, i.e. $a^d = a$, $b^d = b$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = abd, \overline{b} = abcd, \overline{c} = ac, \overline{d} = acd$.

- $a^d = a$, $b^d = bb^2$, $(ac)^d = aca^2b^2$, i.e. $a^d = a$, $b^d = bb^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = a, \overline{b} = bd, \overline{c} = ac, \overline{d} = ad$.

- $a^d = a$, $b^d = ba^2b^2$, $(ac)^d = acb^2$, i.e. $a^d = a$, $b^d = ba^2b^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9.An isomorphism is given by setting: $\overline{a} = abd, \overline{b} = bcd, \overline{c} = c, \overline{d} = d$.

- $a^d = aa^2$, $b^d = bb^2$, $(ac)^d = ac$, i.e. $a^d = aa^2$, $b^d = bb^2$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = cd, \overline{b} = bc, \overline{c} = abd, \overline{d} = ac$.

- $a^d = aa^2$, $b^d = ba^2b^2$, $(ac)^d = ac$, i.e. $a^d = aa^2$, $b^d = ba^2b^2$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = acd, \overline{b} = c, \overline{c} = ac, \overline{d} = bd$.

- $a^d = aa^2b^2$, $b^d = b$, $(ac)^d = acb^2$, i.e. $a^d = aa^2b^2$, $b^d = b$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_{11}.

- $a^d = aa^2b^2$, $b^d = ba^2b^2$, $(ac)^d = acb^2$, i.e. $a^d = aa^2b^2$, $b^d = ba^2b^2$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_{11}. An isomorphism is given by setting: $\overline{a} = abc, \overline{b} = cd, \overline{c} = bcd, \overline{d} = bd$.

□

Lemma 3.23. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_8 (see Lemma 3.18);

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then K is contained in a subgroup isomorphic to

$K_8 \times C_2$*;*

K_9 *(see Lemma 3.21);*

K_{10}*(see Lemma 3.21).*

Moreover, the subgroup $\Omega_1(G)$ is in the centralizer of K.

Proof. Suppose $K = \langle a, b, c : a^4 = 1, b^4 = 1, c^2 = a^2b^2, a^b = a^3, a^c = a, b^c = ba^2b^2\rangle$ and let d be in $G \backslash K$.

Suppose d has order 2: then, by Lemma 3.13, $d \in C_G(\langle a, b\rangle)$ and $d \in C_G(\langle ab, bc\rangle)$. Hence $\langle K, d\rangle$ is isomorphic to $K \times C_2$. In particular, the subgroup $\Omega_1(G)$ is contained in $C_G(K)$. Furthermore, from now on, we may assume that all the elements in Kc have order 4. In the sequel, we refer to $(*)$ to recall the previous assumption.

Suppose d has order 4. By (ii), it is easy to check that $[d, K] \leq K^2$ and $d^2 \in K^2$. We have $\langle a, b\rangle \simeq K_2$ and $\langle a, b\rangle^2 = G^2 = K^2$. By Remark 3.19, we can assume $d^2 = a^2b^2$. Since $\langle d\rangle \cap \langle a\rangle = 1$, $\langle d\rangle \cap \langle b\rangle = 1$ and $\langle d\rangle \cap \langle ac\rangle = 1$, we get that $\langle a, d\rangle$, $\langle b, d\rangle$ and $\langle ac, d\rangle$ are either abelian or isomorphic to K_2. Hence $[a, d] \in \{1, a^2, a^2b^2\}$, $[b, d] \in \{1, b^2, a^2b^2\}$, $[ac, d] \in \{1, b^2, a^2b^2\}$.
Now, the rest of the proof is a case-by-case analysis depending on the possible values of $[a, d]$,$[b, d]$ and $[ac, d]$.
By $(*)$, we assume that none of the followings happen:

- $[d, c] = 1$ (otherwise $(cd)^2 = 1$);
- $[a, d][b, d] = a^2$ (otherwise $(abd)^2 = 1$);
- $[a, d][c, d] = a^2$ (otherwise $(acd)^2 = 1$);
- $[b, d][c, d] = a^2$ (otherwise $(bcd)^2 = 1$);
- $[a, d][b, d][c, d] = a^2$ (otherwise $(abcd)^2 = 1$).

Excluding the cases in which one of the previous is satisfied, it remains to study the following possibilities:

- $a^d = a$, $b^d = b$, $(ac)^d = acb^2$, i.e. $a^d = a$, $b^d = b$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = bd, \overline{b} = cd, \overline{c} = acd, \overline{d} = d$.
- $a^d = a$, $b^d = b$, $(ac)^d = aca^2b^2$, i.e. $a^d = a$, $b^d = b$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_{10}. An isomorphism is given by setting: $\overline{a} = cd, \overline{b} = ad, \overline{c} = a, \overline{d} = abd$.
- $a^d = a$, $b^d = bb^2$, $(ac)^d = acb^2$, i.e. $a^d = a$, $b^d = bb^2$, $c^d = cb^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = abcd, \overline{b} = ab, \overline{c} = acd, \overline{d} = d$.
- $a^d = a$, $b^d = ba^2b^2$, $(ac)^d = aca^2b^2$, i.e. $a^d = a$, $b^d = ba^2b^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_{10}. An isomorphism is given by setting: $\overline{a} = cd, \overline{b} = ad, \overline{c} = a, \overline{d} = abd$.

- $a^d = aa^2$, $b^d = bb^2$, $(ac)^d = ac$, i.e. $a^d = aa^2$, $b^d = bb^2$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = acd, \overline{b} = abd, \overline{c} = bd, \overline{d} = abcd$.

- $a^d = aa^2$, $b^d = ba^2b^2$, $(ac)^d = ac$, i.e. $a^d = aa^2$, $b^d = ba^2b^2$, $c^d = ca^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = acd, \overline{b} = bd, \overline{c} = abd, \overline{d} = bcd$.

- $a^d = aa^2b^2$, $b^d = b$, $(ac)^d = ac$, i.e. $a^d = aa^2b^2$, $b^d = b$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_{10}. An isomorphism is given by setting: $\overline{a} = a, \overline{b} = b, \overline{c} = bcd, \overline{d} = abcd$.

- $a^d = aa^2b^2$, $b^d = ba^2b^2$, $(ac)^d = ac$, i.e. $a^d = aa^2b^2$, $b^d = ba^2b^2$, $c^d = ca^2b^2$. The group $\langle a, b, c, d\rangle$ is isomorphic to K_9. An isomorphism is given by setting: $\overline{a} = d, \overline{b} = abcd, \overline{c} = abd, \overline{d} = bcd$.

□

The next lemma shows that the structure of a monotone 2-group G such that $|G^2| = 4$ and containing K_9 or K_{10} or K_{11} is very restricted.

Lemma 3.24. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_i, $i = 9, 10, 11$;

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then G is isomorphic to $K \times A$ where A is elementary abelian.

Proof. Let

$$\begin{aligned} K = \langle a, b, c, d \ &: \ a^4 = 1, b^4 = 1, c^2 = a^2b^2, d^2 = c^2, a^b = a^3, \\ &\quad a^c = a, b^c = b, a^d = a, b^d = b^3, c^d = cb^2\rangle \end{aligned}$$

be isomorphic to K_9.

Let $f \in G \backslash K$.

Suppose f has order 2. By Lemma 3.13, f centralizes all the subgroups isomorphic to K_2. Since $\langle a, b\rangle$, $\langle d, b\rangle$ and $\langle cd, d\rangle$ are isomorphic to K_2, we get that $f \in C_G(K)$. Therefore, $\Omega_1(G)$ is elementary abelian (see Lemma 3.17), $\Omega_1(G) \leq C_G(K)$ and so $\langle K, \Omega_1(G)\rangle = K \times A$, where A is elementary abelian. Then we may assume that all the elements in Kf have order 4. In the sequel we refer to $(*)$ to recall this assumption. Moreover, since $\langle a, b\rangle \simeq K_2$ and $f^2 \in \langle a^2, b^2\rangle$, by Remark 3.19, we may assume $f^2 = a^2b^2$. Now, we have $\langle f\rangle \cap \langle a\rangle = 1$, $\langle f\rangle \cap \langle b\rangle = 1$, $\langle f\rangle \cap \langle ac\rangle = 1$ and $\langle f\rangle \cap \langle ad\rangle = 1$. Hence the subgroups $\langle a, f\rangle$, $\langle f, b\rangle$, $\langle f, ac\rangle$ and $\langle f, ad\rangle$ are abelian or isomorphic to K_2. Hence $[a, f] \in \{1, a^2, a^2b^2\}$, $[b, f] \in \{1, b^2, a^2b^2\}$, $[ac, f] \in \{1, b^2, a^2b^2\}$, $[ad, f] \in \{1, b^2, a^2b^2\}$. Because of $(*)$, we assume that none of the followings happen:

- $[c, f] = 1$ (otherwise $(cf)^2 = 1$);
- $[a, f][b, f] = a^2$ (otherwise $(abf)^2 = 1$);
- $[a, f][c, f] = a^2$ (otherwise $(acf)^2 = 1$);
- $[b, f][c, f] = b^2$ (otherwise $(bcf)^2 = 1$);
- $[a, f][b, f][c, f] = b^2$ (otherwise $(abcf)^2 = 1$);
- $[d, f] = 1$ (otherwise $(df)^2 = 1$);
- $[a, f][d, f] = a^2$ (otherwise $(adf)^2 = 1$);
- $[b, f][d, f] = 1$ (otherwise $(bdf)^2 = 1$);
- $[c, f][d, f] = a^2$ (otherwise $(cdf)^2 = 1$);
- $[a, f][b, f][d, f] = 1$ (otherwise $(abdf)^2 = 1$);
- $[a, f][c, f][d, f] = 1$ (otherwise $(acdf)^2 = 1$);
- $[b, f][c, f][d, f] = a^2$ (otherwise $(bcdf)^2 = 1$);
- $[a, f][b, f][c, f][d, f] = a^2$ (otherwise $(abcdf)^2 = 1$);

It is not difficult to see that for all the possible choices of the commutators $[a, f], [b, f], [c, f], [d, f]$ one of the previous conditions is satisfied. Therefore if $i = 9$, then the statement is true.

The arguments above hold also for K_i, where $i = 10, 11$.

□

The next proposition, that concludes the section, completes the classification of the monotone 2-groups of exponent 4 involving a subgroup K isomorphic to K_2 and such that there is no cyclic subgroup X of order 4 with $X \cap K = 1$, i.e. $|G^2| = 4$.

Proposition 3.25. *Let G be a monotone 2-group of exponent 4 such that*

(i) G properly contains a subgroup K isomorphic to K_2;

(ii) there are no cyclic subgroups X of order 4 of G such that $X \cap K = 1$.

Then G is isomorphic to $H \times A$, where A is elementary abelian and H is isomorphic to K_i, where $i \in \{2, 6, 7, 8, 9, 10, 11\}$ (see Lemma 3.18, Lemma 3.21 and Lemma 3.22).

Proof. Suppose K is a subgroup of G isomorphic to K_2.
If for all the elements c of $G \backslash K$ there exists $k \in K$ such that kc has order 2, then $G = \langle K, \Omega_1(G) \rangle$. By Lemma 3.17, $\Omega_1(G)$ is elementary abelian. Since by Lemma 3.13, the subgroup $\Omega_1(G)$ is in $C_G(K)$, we get that $G \simeq K \times A$, where A is elementary abelian.
Suppose now that there exists an element c in $G \backslash K$ such that there is no $k \in K$ with kc of order 2. Then, by Lemma 3.18, the subgroup K is contained in a subgroup isomorphic to K_6 or to K_7 or to K_8.
Then we may assume that G contains a subgroup T isomorphic to K_i, where i is in $\{6, 7, 8\}$.
Suppose that for all the elements c of $G \backslash T$, there exists an element $k \in T$ such that kc has order 2. Then $G = \langle T, \Omega_1(G) \rangle$. By Lemma 3.17, $\Omega_1(G)$ is elementary abelian. Moreover, by Lemma 3.21 if $i = 6$, by Lemma 3.22 if $i = 7$ and by Lemma 3.23 if $i = 8$, the subgroup $\Omega_1(G)$ is in $C_G(T)$. So, we get that $G \simeq T \times A$, where $T \simeq K_i$ with i in $\{6, 7, 8\}$ and A is elementary abelian.
Suppose now that there exists an element c in $G \backslash T$ such that there is no $k \in K$ with kc of order 2. Then, by Lemma 3.21 if $i = 6$, by Lemma 3.22 if $i = 7$ and by Lemma 3.23 if $i = 8$, the subgroup T is contained in a subgroup of G isomorphic to K_9 or to K_{10} or to K_{11}.
So, we may assume that that G contains a subgroup S isomorphic to K_i, where i is in $\{9, 10, 11\}$.
By Lemma 3.24, the group G is isomorphic to $S \times A$, where $S \simeq K_i$ with i is in $\{9, 10, 11\}$ and A is elementary abelian. Hence the statement is proved. $\square$

Summing up, in this section we determined the monotone 2-groups of exponent 4 containing a subgroup isomorphic to K_2. Namely, any such a group is in the class $\mathscr{A}_i$, with $i \in \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

In particular, combining the previous two sections Theorem 3.3 is proved.

Chapter 4

Monotone 2-Groups of exponent greater than 4 in which $|G : H_4(G)| = 2$

Since the 2-groups of exponent at most 2 are elementary abelian and the monotone 2-groups of exponent 4 were fully classified in Chapter 3, in the rest of this thesis we study monotone 2-groups of exponent greater than 4. Let G be such a group. By Proposition 1.3, we have that $|G : H_4(G)| \leq 2$. In this chapter we investigate the monotone 2-groups of exponent greater than or equal to 8 and such that $|G : H_4(G)| = 2$.

Definition 4.1. *We introduce the following families of 2-groups:*

- $\mathscr{B}_1$ *is the family of 2-groups of the form* $A\langle u\rangle$, *where* A *is abelian of exponent* $2^n \geq 8$, $u^2 \in \Omega_1(A)$, $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$ *for every* $a \in A$;

- $\mathscr{B}_2$ *is the family of 2-groups of the form* $\langle a, b, u\rangle \times A$, *where* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|b| = 2$, $\langle a, b\rangle$ *is abelian,* $u^2 = a^{2^{n-1}}$, $b^u = ba^{2^{n-1}}$, $a^u = a^{-1}$;

- $\mathscr{B}_3$ *is the family of 2-groups of the form* $\langle a, b, u\rangle \times A$, *where* A *is elementary abelian,* $|a| = 2^n \geq 8$, $|b| = 4$, $\langle a, b\rangle$ *is abelian,* $u^2 = b^2$ *and* $a^u = a^{-1}$, $b^u = b^{-1}a^{2^{n-1}}$;

- $\mathscr{B}_4$ *is the family of 2-groups of the form* $\langle a, u\rangle * E \times A$, *where* $|a| = 2^n \geq 8$, E *is extraspecial,* A *is elementary abelian,* $u^2 \in \langle a^{2^{n-1}}\rangle$, $a^u = a^{-1+4h}$

with $|a^{4h}| \leq 2$ *and* $E^2 = \langle a^{2^{n-1}} \rangle$*;*

- $\mathscr{B}_5$ *is the family of* 2*-groups of the form* $\langle a, u \rangle * E * A$*, where* E *is extraspecial,* A *is abelian of the form* $C_4 \times C_2 \times \cdots \times C_2$*,* $|a| = 2^n \geq 8$*,* $u^2 = a^{2^{n-1}}$*,* $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$ *and* $A^2 = E^2 = \langle a^{2^{n-1}} \rangle$ *;*

- $\mathscr{B}_6$ *is the family of* 2*-groups of the form* $\langle a, u, b \rangle * E \times A$*, where* $|a| = 2^n \geq 8$*,* E *is extraspecial,* A *is elementary abelian,* $|b| = 2$*,* $u^2 \in \langle a^{2^{n-1}} \rangle$*,* $a^b = a^{1+2^{n-1}}$*,* $a^u = a^{-1}$*,* $b^u = ba^{4h}$ *with* $|a^{4h}| \leq 2$ *and* $E^2 = \langle a^{2^{n-1}} \rangle$*;*

- $\mathscr{B}_7$ *is the family of* 2*-groups of the form* $\langle a, u, b \rangle * E * A$*, where* E *is extraspecial,* A *is abelian of the form* $C_4 \times C_2 \times \cdots \times C_2$*, and* $|a| = 2^n \geq 8$*,* $|b| = 2$*,* $u^2 = a^{2^{n-1}}$*,* $a^u = a^{-1}$*,* $b^u = b$*,* $a^b = a^{1+2^{n-1}}$*, and* $A^2 = E^2 = \langle a^{2^{n-1}} \rangle$*;*

- $\mathscr{B}_8$ *is the family of* 2*-groups of the form* $\langle a, b, u \rangle \times A$*, where* A *is elementary abelian,* $|a| = 2^n \geq 8$*,* $|b| = 2$*,* $|u| = 4$*,* $u^2 = a^{2^{n-1}}$*,* $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$*,* $a^b = a^{1+2^{n-1}}$ *and* $u^b = u^{-1}$*;*

- $\mathscr{B}_9$ *is the family of* 2*-groups of the form* $\langle a, c, b, u \rangle \times A$*, where* A *is elementary abelian,* $|a| = 2^n \geq 8$*,* $|c| = 2$*,* $|b| = 2$*,* $|u| = 4$*,* $u^2 = a^{2^{n-1}}$*,* $a^u = a^{-1}$*,* $a^b = a^{1+2^{n-1}}$*,* $u^b = u^{-1}$*,* $c^a = c$*,* $c^b = c$ *and* $c^u = ca^{2^{n-1}}$*;*

- $\mathscr{B}_{10}$ *is the family of* 2*-groups of the form* $\langle a, b, u \rangle \times A$*, where* A *is elementary abelian,* $|a| = 2^n \geq 8$*,* $|b| = 4$*,* $|u| = 4$*,* $u^2 = b^2$*,* $a^u = a^{-1+4h}$ *with* $|a^{4h}| \leq 2$*,* $b^u = b^{-1}a^{2^{n-1}}$*,* $a^b = a^{1+2^{n-1}}$*.*

We start by proving that the groups in $\mathscr{B}_i$, for $i \in \{1, \ldots, 10\}$, defined in Definition 4.1, are actually monotone.

Proposition 4.1. *The groups in the families* $\mathscr{B}_i$*, for* $i \in \{1, \ldots, 10\}$ *are monotone.*

Proof. We want to show that if G is a group in $\mathscr{B}_i$, for $i \in \{1, \ldots, 10\}$, then G is monotone. Now, the proof is a case-by-case analysis depending on the family in which G lies.

- Suppose that G is in $\mathscr{B}_1$. Lemma 2.6 proves that G is monotone.

- Suppose that G is in $\mathscr{B}_2$. Then, we have $G = \langle a, b, u \rangle \times A$, where A is elementary abelian, $|a| = 2^n$ with $n \geq 3$, $|b| = 2$,$\langle a, b \rangle$ is abelian, $u^2 = a^{2^{n-1}}$, $b^u = ba^{2^{n-1}}$, $a^u = a^{-1}$. We have to prove that the 2-generated subgroups are metacyclic. By Lemma 2.5, it is sufficient to prove that the group $\langle a, b, u\rangle$ is monotone. Moreover, since $\langle a, b\rangle$ is abelian, it is enough to check that the subgroups of the form $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ are metacyclic.

We distinguish two cases depending on the order of a^{j_1}.

If $|a^{j_1}| \geq 8$, then $\Omega_1(\langle a^{j_1}b^{j_2}\rangle) = \langle a^{2^{n-1}}\rangle$. Since $(a^{j_1}b^{j_2})^{a^{i_1}b^{i_2}u} = a^{-j_1}b^{j_2}a^{2^{n-1}j_2}$, we have that $\langle a^{j_1}b^{j_2}\rangle \trianglelefteq \langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ and so the subgroup $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ is metacyclic.

Suppose now that $|a^{j_1}| \leq 4$. The subgroup $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ is contained in $\langle a^{2^{n-2}}, b, a^{i_1}u\rangle$. Now, the group $\langle a^{2^{n-2}}, b, a^{i_1}u\rangle = \langle a^{2^{n-2}}, a^{i_1}u\rangle * \langle a^{2^{n-2}}b\rangle$ is isomorphic to $Q_8 * C_4$, where $Q_8^2 = C_4^2$. Since $Q_8 * C_4$, where $Q_8^2 = C_4^2$, is monotone, the subgroup $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ is metacyclic.

Therefore, the group G is monotone.

- Suppose that G is in $\mathscr{B}_3$. Then, we have $G = \langle a, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n$, $|b| = 4$, $\langle a, b\rangle$ is abelian, $u^2 = b^2$ and $a^u = a^{-1}$, $b^u = b^{-1}a^{2^{n-1}}$. By Lemma 2.5, it is sufficient to prove that the group $\langle a, b, u\rangle$ is monotone. Moreover, since $\langle a, b\rangle$ is abelian it is enough to check that the subgroups of the form $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ are metacyclic.

  Now, we have that $(a^{j_1}b^{j_2})^{a^{i_1}b^{i_2}u} = (a^{j_1}b^{j_2})^u = a^{-j_1}b^{-j_2}a^{2^{n-1}j_2} = (a^{j_1}b^{j_2})^{-1}a^{2^{n-1}j_2}$. We distinguish two cases depending on the order of a^{j_1}.

  If $|a^{j_1}| \geq 8$, then $|a^{j_1}b^{j_2}| = |a^{j_1}|$ and, in particular, $\Omega_1(\langle a^{j_1}b^{j_2}\rangle) = \langle a^{2^{n-1}}\rangle$. Then, we have $\langle a^{j_1}b^{j_2}\rangle \trianglelefteq \langle a^{j_1}b^{j_2}, a^{i_1}b^{i_2}u\rangle$, and so the group $\langle a^{j_1}b^{j_2}, a^{i_1}b^{i_2}u\rangle$ is metacyclic.

  Suppose now that $|a^{j_2}| \leq 4$. The subgroup $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ is contained in the subgroup $\langle a^{2^{n-2}}, b, a^{i_1}u\rangle$, which is isomorphic to K_7 (see Lemma 3.18) by setting $\overline{a} = a^{2^{n-2}}, \overline{b} = a^{i_1}ub$ and $\overline{c} = b$. Since K_7 is monotone, the subgroup $\langle a^{i_1}b^{i_2}u, a^{j_1}b^{j_2}\rangle$ is metacyclic.

  Therefore, the group $\langle a, b, u\rangle$ is monotone and so G is monotone.

- Suppose that G is in $\mathscr{B}_4$. Then, we have $G = \langle a, u\rangle * E \times A$, where a has order $2^n \geq 8$, E is extraspecial, A is elementary abelian, u is such that $u^2 \in \langle a^{2^{n-1}}\rangle$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$, $E^2 = \langle a^{2^{n-1}}\rangle$. By Lemma3.6, it is sufficient to show that $\langle a, u\rangle * E$ is monotone.

We first prove that the subgroup $\langle a, E\rangle$ is monotone. It is enough to check that the subgroups $\langle a^i t_1, t_2\rangle$, where t_1 and t_2 are in E, are metacyclic. Now, $(a^i t_1)^{t_2} = a^i t_1[t_1, t_2] = a^i t_1 a^{2^{n-1}s}$, for some $s \in \mathbb{N}$. If $|a^i| \geq 8$, then $\Omega_1(\langle a^i t_1\rangle) = \langle a^{2^{n-1}}\rangle$, and so $\langle a^i t_1\rangle \trianglelefteq \langle a^i t_1, t_2\rangle$. If $|a^i| \leq 4$, then $\langle a^i t_1, t_2\rangle \leq \langle a^{2^{n-2}}\rangle * E$, which is a monotone group (see Theorem 3.3).
This proves that the subgroup $\langle a\rangle * E$ is monotone.

Now, in order to conclude that the group $\langle a, u\rangle * E$ is monotone, it is enough to prove that the subgroups $\langle a^i t_1 u, a^j t_2\rangle$ are metacyclic, where $t_1, t_2 \in E$. We have that $(a^j t_2)^{a^i t_1 u} = (a^j t_2)^{-1} a^{4hj} t_2^2[t_1, t_2]$. We now distinguish two cases depending on the order of a^j.

If $|a^j| \geq 8$, then $\Omega_1(\langle a^j t_2\rangle) = \langle a^{2^{n-1}}\rangle$. Since $t_2^2[t_1, t_2] \in \langle a^{2^{n-1}}\rangle$, we have that $\langle a^j t_2\rangle \trianglelefteq \langle a^i t_1 u, a^j t_2\rangle$.

If $|a^j| \leq 4$, then $\langle a^i t_1 u, a^j t_2\rangle$ is contained in $\langle a^{2^{n-2}}, a^i u\rangle * E$ which is extraspecial and so monotone. In particular, the subgroup $\langle a^j t_2, a^i t_1 u\rangle$ is metacyclic.

Therefore, the group G is monotone.

- Suppose that G is in $\mathscr{B}_5$. Then, we have $G = \langle a, u\rangle * E * A$, where E is extraspecial, A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$, and $|a| = 2^n$, with $n \geq 3$, $u^2 = a^{2^{n-1}}$ and $a^u = a^{-1+4h}$, with $|a^{4h}| \leq 2$ and $A^2 = E^2 = \langle a^{2^{n-1}}\rangle$. By Lemma 3.6, it is sufficient to show that $\langle a, u\rangle * E * C_4$ is monotone.

  We first prove that the subgroup $\langle a\rangle * E * C_4$ is monotone. Since $\langle a\rangle * E * C_4 = \langle a\rangle * E \times C_2$, by Lemma 3.6, it suffices to check that the subgroups $\langle a^i t_1, t_2\rangle$, where t_1 and t_2 are in E, are metacyclic. Now, $(a^i t_1)^{t_2} = a^i t_1[t_1 t_2] = a^i t_1 a^{2^{n-1}s}$, for some $s \in \mathbb{N}$. If $|a^i| \geq 8$, then $\Omega_1(\langle a^i t_1\rangle) = \langle a^{2^{n-1}}\rangle$, and so $\langle a^i t_1\rangle \trianglelefteq \langle a^i t_1, t_2\rangle$. If $|a^i| \leq 4$, then $\langle a^i t_1, t_2\rangle \leq \langle a^{2^{n-2}}\rangle * E$, which is a monotone group (see Theorem 3.3). This proves that the subgroup $\langle a\rangle * E * C_4$ is monotone.

  In order to conclude that the group $\langle a, u\rangle * E * C_4$ is monotone, we prove that the subgroups $\langle a^i t_1 u, a^j t_2\rangle$ are metacyclic, where $t_1, t_2 \in E * C_4$. We have that $(a^j t_2)^{a^i t_1 u} = (a^j t_2)^{-1} a^{4hj} t_2^2[t_1, t_2]$. If $|a^j| \geq 8$, then $\Omega_1(\langle a^j t_2\rangle) = \langle a^{2^{n-1}}\rangle$. Since $t_2^2[t_1, t_2] \in \langle a^{2^{n-1}}\rangle$, we have that $\langle a^j t_2\rangle \trianglelefteq \langle a^i t_1 u, a^j t_2\rangle$. If $|a^j| \leq 4$, then $\langle a^i t_1 u, a^j t_2\rangle$ is contained in

$\langle a^{n-2}, a^i u\rangle * E * C_4$ which is monotone. In particular, the subgroup $\langle a^j t_2, a^i t_1 u\rangle$ is metacyclic.

Therefore, the group G is monotone.

- Suppose that G is in $\mathscr{B}_6$. Then, we have $G = \langle a, u, b\rangle * E \times A$, where a has order $2^n \geq 8$, E is extraspecial, A is elementary abelian, $|b| = 2$, $u^2 \in \langle a^{2^{n-1}}\rangle$, $a^b = a^{1+2^{n-1}}$, $a^u = a^{-1}$, $b^u = ba^{4h}$, where $|a^{4h}| \leq 2$ and $E^2 = \langle a^{2^{n-1}}\rangle$. By Lemma 2.5, it is enough to check that $\langle a, u, b\rangle * E$ is monotone. Therefore, we have to control that all the 2-generated subgroups of $\langle a, u, b\rangle * E$ are monotone.

  We first prove that $\langle a, b\rangle * E$ is monotone. We control that the subgroups of the form $\langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$ are metacyclic.
  We have $(a^{j_1} t_2)^{a^{i_1} b^{i_2} t_1} = a^{j_1} t_2 [a^{j_1}, b^{i_2}][t_2, t_1]$. Therefore, we obtain $[a^{i_1} b^{i_2} t_1, a^{j_1} t_2] = [a^{j_1}, b^{i_2}][t_2, t_1] \leq \langle a^{2^{n-1}}\rangle$. We distinguish two cases depending on the orders of a^{j_1} and a^{i_1}.

  Suppose that $|a^{j_1}| \geq 8$ or $|a^{i_1}| \geq 8$. In the first case, we get $\Omega_1(\langle a^{j_1} t_2\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle a^{j_1} t_2\rangle \trianglelefteq \langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$. In the second case, we have that $\Omega_1(\langle a^{i_1} b^{i_2} t_1\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle a^{i_1} b^{i_2} t_1\rangle \trianglelefteq \langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$.
  Hence, if $|a^{j_1}| \geq 8$ or $|a^{i_1}| \geq 8$, then the subgroup $\langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$ is metacyclic.

  Suppose now that both $|a^{j_1}| \leq 4$ and $|a^{i_1}| \leq 4$. Then $\langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$ is contained in $\langle a^{2^{n-2}}\rangle * E \times \langle b\rangle$, which is monotone (see Theorem 3.3) . Therefore, if both $|a^{j_1}| \leq 4$ and $|a^{i_1}| \leq 4$, then the subgroup $\langle a^{i_1} b^{i_2} t_1, a^{j_1} t_2\rangle$ is metacyclic.

  This shows that the group $\langle a, b\rangle * E$ is monotone and, in order to conclude that $\langle a, u, b\rangle * E$ is monotone, we check that the subgroups of the form $\langle a^{i_1} b^{i_2} t_1, a^{j_1} b^{i_3} t_2 u\rangle$ are metacyclic.
  We have $(a^{i_1} b^{i_2} t_1)^{a^{j_1} b^{i_3} t_2 u} = (a^{i_1} b^{i_2} t_1)^{-1} a^{2^{n-1} s}$, for some $s \in \mathbb{N}$.
  We distinguish two cases depending on the order of a^{i_1}.

  Suppose that $|a^{i_1}| \geq 8$. Then, we have $\Omega_1(\langle a^{i_1} b^{i_2} t_1\rangle) = \langle a^{2^{n-1}}\rangle$ It follows that $\langle a^{i_1} b^{i_2} t_1\rangle \trianglelefteq \langle a^{i_1} b^{i_2} t_1, a^{j_1} b^{i_3} t_2 u\rangle$. Hence, if $|a^{i_1}| \geq 8$, the subgroup $\langle a^{i_1} b^{i_2} t_1, a^{j_1} b^{i_3} t_2 u\rangle$ is metacyclic.

  Suppose now $|a^{i_1}| \leq 4$. We have that $\langle a^{i_1} b^{i_2} t_1, a^{j_1} b^{i_3} t_2 u\rangle$ is a subgroup of $\langle a^{2^{n-2}}, b, u, E\rangle = \langle a^{2^{n-2}}, u\rangle * E * \langle a^{2^{n-2}} b\rangle$, which is isomorphic to a monotone group of the form $F * C_4$, where F is extraspecial. Therefore, if $|a^{i_1}| \leq 4$, then $\langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$ is metacyclic.

This shows that all the 2-generated subgroups of $\langle a, b, u, E\rangle$ are metacyclic. It follows that the group G is monotone.

- Suppose that G is in $\mathscr{B}_7$. Then, we have $G = \langle a, u, b\rangle * E * A$, where E is extraspecial, A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$, and $|a| = 2^n$, with $n \geq 3$, $|b| = 2$, $a^b = a^{1+2^{n-1}}$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1}$, $b^u = b$ and $A^2 = E^2 = \langle a^{2^{n-1}}\rangle$.

  By Lemma 2.5, it is enough to check that $\langle a, u, b\rangle * E * C_4$ is monotone. Therefore, we have to control that all the 2-generated subgroups of $\langle a, u, b\rangle * E * C_4$ are metacylcic.

  We first prove that $\langle a, b\rangle * E * C_4$ is monotone. Since $\langle a, b\rangle * E * C_4 = \langle a, b\rangle * E \times C_2$, by Lemma 2.5, it is sufficient to check that the subgroups of the form $\langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$, where t_1 and t_2 are in E, are metacyclic. Now, we have $(a^{j_1}t_2)^{a^{i_1}b^{i_2}t_1} = a^{j_1}t_2[a^{j_1}, b^{i_2}][t_2, t_1]$. Therefore $[a^{i_1}b^{i_2}t_1, a^{j_1}t_2] = [a^{j_1}, b^{i_2}][t_2, t_1] \in \langle a^{2^{n-1}}\rangle$.

  Suppose that $|a^{j_1}| \geq 8$ or $|a^{i_1}| \geq 8$. In the first case, we have that $\Omega_1(\langle a^{j_1}t_2\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle a^{j_1}t_2\rangle \trianglelefteq \langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$. In the second case, we get $\Omega_1(\langle a^{i_1}b^{i_2}t_1\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle a^{i_1}b^{i_2}t_1\rangle \trianglelefteq \langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$. Hence, if $|a^{j_1}| \geq 8$ or $|a^{i_1}| \geq 8$, then the subgroup $\langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$ is metacyclic.

  Suppose now that both $|a^{j_1}| \leq 4$ and $|a^{i_1}| \leq 4$. Then $\langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$ is contained in $\langle a^{2^{n-2}}\rangle * E \times \langle b\rangle$, which is monotone (see Theorem 3.3) . Therefore, if both $|a^{j_1}| \leq 4$ and $|a^{i_1}| \leq 4$, then the subgroup $\langle a^{i_1}b^{i_2}t_1, a^{j_1}t_2\rangle$ is metacyclic, being a subgroup of a monotone group.

  In order to conclude that $\langle a, u, b\rangle * E * C_4$ is monotone, we check that the subgroups of the form $\langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$, where t_1 and t_2 are in $E * C_4$, are metacyclic. We have that $(a^{i_1}b^{i_2}t_1)^{a^{j_1}b^{i_3}t_2u} = (a^{i_1}b^{i_2}t_1)^{-1}a^{2^{n-1}s}$ for some $s \in \mathbb{N}$.

  Suppose that $|a^{i_1}| \geq 8$. Then, we have $\Omega_1(\langle a^{i_1}b^{i_2}t_1\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle a^{i_1}b^{i_2}t_1\rangle \trianglelefteq \langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$. Hence if $|a^{i_1}| \geq 8$, the subgroup $\langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$ is metacyclic.

  Suppose now $|a^{i_1}| \leq 4$. Then the subgroup $\langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$ is a subgroup of $\langle a^{2^{n-2}}, b, u, E\rangle = \langle a^{2^{n-2}}, u\rangle * E * \langle a^{2^{n-2}}b\rangle$, which is isomorphic to a monotone group of the form $F * C_4$, where F is extraspecial.

Therefore, if $|a^{i_1}| \leq 4$, then $\langle a^{i_1}b^{i_2}t_1, a^{j_1}b^{i_3}t_2u\rangle$ is metacyclic (being a subgroup of a monotone group). This shows that all the 2-generated subgroups of $\langle a, b, u\rangle * E * C_4$ are metacyclic.

It follows that the group G is monotone.

- Suppose that G is in $\mathscr{B}_8$. Then, we have $G = \langle a, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1+2^{n-1}h}$, with h in $\{0, 1\}$, $a^b = a^{1+2^{n-1}}$ and $u^b = u^{-1}$. By Lemma 2.5, it is enough to check that $\langle a, b, u\rangle$ is a monotone group. Therefore, we have to check that all the 2-generated subgroups of $\langle a, b, u\rangle$ are metacyclic. Since $\langle a, b\rangle$ is a modular metacyclic subgroup, it is monotone and so it is sufficient to check the subgroups of the form $\langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$. If $i_1 = 0$, then the subgroup $\langle a^{i_2}, ua^{j_2}b^{j_1}\rangle$ is metacyclic since $\langle a^{i_2}\rangle \trianglelefteq \langle a^{i_2}, ua^{j_2}b^{j_1}\rangle$. If $i_2 = 1$ then, replacing if necessary $ua^{j_2}b^{j_1}$ with $ua^{j_2}b^{j_1}(ba^{i_2})^{j_1}$, we may assume that the subgroup is of the form $\langle ba^{i_2}, ua^{j_2}\rangle$. Now $(ba^{i_2})^{ua^{j_2}} = b^{a^{j_2}}a^{(-1+2^{n-1})i_2} = (ba^{i_2})^{-1}a^{2^{n-1}i_2+2^{n-1}j_2}$. Now, we distinguish two cases depending on the order of a^{i_2}.

  If $|a^{i_2}| \geq 8$, then $\Omega_1(\langle ba^{i_2}\rangle) = \langle a^{2^{n-1}}\rangle$ and so $\langle ba^{i_2}\rangle \trianglelefteq \langle ba^{i_2}, ua^{j_2}\rangle$.

  If $|a^{i_2}| \leq 4$, then $\langle ba^{i_2}, ua^{j_2}\rangle$ is contained in the group $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle$. We have that $(ua^{j_2})^2 = u^2a^{2^{n-1}hj_2} = a^{2^{n_1}hj_2+1}$, and so $(ua^{j_2})^2 \in \langle a^{2^{n-1}}\rangle$. Therefore, $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle$ is equal to $\langle a^{2^{n-2}}, ua^{j_2}\rangle * \langle a^{2^{n-2}}b\rangle$. If $|ua^{j_2}| = 2$, then $\langle a^{2^{n-2}}, ua^{j_2}\rangle$ is isomorphic to D_8. Therefore, the subgroup $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle = \langle a^{2^{n-2}}, ua^{j_2}\rangle * \langle a^{2^{n-2}}b\rangle$ is isomorphic to the monotone group $D_8 * C_4$, where $D_8^2 = C_4^2$.
  If $|ua^{j_2}| = 4$, then $\langle a^{2^{n-2}}, ua^{j_2}\rangle$ is isomorphic to Q_8. Therefore, the subgroup $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle = \langle a^{2^{n-2}}, ua^{j_2}\rangle * \langle a^{2^{n-2}}b\rangle$ is isomorphic to the monotone group $Q_8 * C_4$, where $Q_8^2 = C_4^2$.
  In both cases, the subgroup $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle$ is monotone and in particular, the subgroup $\langle ba^{i_2}, ua^{j_2}\rangle$ is metacyclic.

  Therefore, the group G is monotone.

- Suppose that G is in $\mathscr{B}_9$. Then, we have $G = \langle a, c, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|c| = 2$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1}$, $a^b = a^{1+2^{n-1}}$, $u^b = u^{-1}$, $c^a = c$, $c^b = c$ and $c^u = ca^{2^{n-1}}$.
  By Lemma 2.5, it is enough to check that $\langle a, c, b, u\rangle$ is a monotone

group. Therefore, we have to check that all the 2-generated subgroups of $\langle a, c, b, u\rangle$ are metacyclic. Since the subgroup $\langle a, b, c\rangle$ is modular (and so monotone), it is enough to check the subgroups of the form $\langle a^{i_1}b^{i_2}c^{i_3}, ua^{j_1}b^{j_2}c^{j_3}\rangle$. We distinguish two cases depending on the order of a^{i_1}.

If $|a^{i_1}| \geq 8$, then $\langle a^{i_1}b^{i_2}c^{i_3}\rangle \trianglelefteq \langle a^{i_1}b^{i_2}c^{i_3}, ua^{j_1}b^{j_2}c^{j_3}\rangle$. In fact, we have that

$$\begin{aligned}(a^{i_1}b^{i_2}c^{i_3})^{ua^{j_1}b^{j_2}c^{j_3}} &= (a^{-i_1}b^{i_2}c^{i_3}a^{2^{n-1}(i_2+i_3)})^{b^{j_2}c^{j_3}}\\ &= a^{-i_1}b^{i_2}c^{i_3}a^{2^{n-1}(i_2+i_3+i_1j_2)}\\ &= (a^{i_1}b^{i_2}c^{i_3})^{-1}a^{2^{n-1}(i_2+i_3+i_1j_2)}.\end{aligned}$$

Since $\langle (a^{i_1}b^{i_2}c^{i_3})^2\rangle = \langle a^{2i_1}\rangle$, we have that $\langle a^{2^{n-1}(i_2+i_3+i_1j_2)}\rangle \leq \langle a^{2^{n-1}}\rangle \leq \Omega_1(\langle a^{i_1}b^{i_2}c^{i_3}\rangle)$

Suppose now that $|a^{i_1}| \leq 4$, then the subgroup $\langle a^{i_1}b^{i_2}c^{i_3}, ua^{j_1}b^{j_2}c^{j_3}\rangle$ is contained in $\langle a^{2^{n-2}}, b, c, ua^{j_1}\rangle$. Therefore, in order to prove that $\langle a^{i_1}b^{i_2}c^{i_3}, ua^{j_1}b^{j_2}c^{j_3}\rangle$ is metacyclic, it is sufficient to prove that the group $\langle a^{2^{n-2}}, b, c, ua^{j_1}\rangle$ is monotone. Now, this group has exponent 4 and $\langle a^{2^{n-2}}, b, c, ua^{j_1}\rangle = \langle a^{2^{n-2}}, ua^{j_1}\rangle * \langle a^{2^{n-2}}b\rangle \times \langle bc\rangle$. Then, the group $\langle a^{2^{n-2}}, b, c, ua^{j_1}\rangle$ is isomorphic to $Q_8 * C_4 \times C_2$, which is monotone.

Therefore, the group G is monotone.

- Suppose that G is in $\mathscr{B}_{10}$. Then, we have $G = \langle a, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 4$, $|u| = 4$, $u^2 = b^2$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$, $b^u = b^{-1}a^{2^{n-1}}$, $a^b = a^{1+2^{n-1}}$.
  By Lemma 2.5, it is enough to check that $\langle a, b, u\rangle$ is a monotone group. Therefore, we have to check that all the 2-generated subgroups of $\langle a, b, u\rangle$ are metacyclic.
  Since $\langle a, b\rangle$ is a modular metacyclic subgroup, it is monotone and so it is sufficient to check the subgroups of the form $\langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$. We distinguish two cases depending on the order of a^{i_1}.

  Suppose that $|a^{i_2}| \geq 8$. Then $\Omega_1(\langle b^{i_1}a^{i_2}\rangle) = \langle a^{2^{n-1}}\rangle$.
  Since $(b^{i_1}a^{i_2})^{ua^{j_2}b^{j_1}} = (b^{-i_1}a^{2^{n-1}}a^{-i_2(1+4h)})^{a^{j_2}b^{j_1}} = (b^{i_1}a^{i_2})^{-1}a^{2^{n-1}s}$, for some $s \in \mathbb{N}$, we have that $\langle b^{i_1}a^{i_2}\rangle \trianglelefteq \langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$ and so the subgroup $\langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$ is metacyclic.

  If $|a^{i_2}| \leq 4$, then we have that the subgroup $\langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$ is contained in $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle$.
  If j_2 is even, then $\langle a^{2^{n-2}}, ua^{j_2}, ba^{2^{n-2}}\rangle$ is isomorphic to K_7 (see Lemma

3.18).
If j_2 is odd, $\langle a^{2^{n-2}}, ua^{j_2}, ba^{2^{n-2}}\rangle$ is isomorphic to K_8 (see Lemma 3.18).
Hence, we obtain that the group $\langle a^{2^{n-2}}, b, ua^{j_2}\rangle$ is monotone and so $\langle b^{i_1}a^{i_2}, ua^{j_2}b^{j_1}\rangle$ is metacyclic.

Therefore, the group G is monotone.

□

The aim of this chapter is to prove the following :

Theorem 4.2. *Let G be a monotone 2-group of exponent greater than or equal to 8 and such that $|G : H_4(G)| = 2$. Then G is in one of the families $\mathscr{B}_i$, for some $i \in \{1, \ldots, 10\}$, defined in Definition 4.1.*

4.1 The subgroup $H_4(G)$

In this section we study the structure of $H_4(G)$, where G is a monotone 2-group of exponent at least 8 and $H_4(G)$ is maximal in G.
In the following two lemmas we give some informations on the normalizer of the cyclic subgroups of order greater than 4 in G and on the metacyclic subgroups of $H_4(G)$ having exponent greater than 4.

Lemma 4.3. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Any cyclic subgroup X of order greater than or equal to 8 is normal in G. Moreover, if u is an element of G not in $H_4(G)$ and $X = \langle a\rangle$, then $a^u = a^{-1+4h}$, where $|a^{4h}| \leq 2$.

Proof. Let a be an element of $H_4(G)$, with $|a| \geq 8$, and let u be an element lying in $G\backslash H_4(G)$.

We first prove the following claim: each cyclic subgroup of $\langle a, u\rangle$ of order greater than 4 is normal in $\langle a, u\rangle$.
Since u is not in $H_4(G)$, the order of u is smaller than or equal to 4. Moreover, the subgroup $\langle a, u\rangle$ is not contained in $H_4(G)$ and, since $|a| \geq 8$, the exponent of $\langle a, u\rangle$ is greater than 4. Now a modular metacyclic group is generated by its elements of maximal order, and so the group $\langle a, u\rangle$ is non-modular metacyclic. Therefore, there exist $x, y \in \langle a, u\rangle$ such that $\langle a, u\rangle = \langle x, y\rangle$, $\langle x\rangle \trianglelefteq \langle x, y\rangle$ and $x^y = x^{-1+4h}$.

We want to show that each cyclic subgroup of $\langle x, y\rangle$ of order greater than or equal to 8 is normal in $\langle x, y\rangle$.

Assume $|x| \leq 4$. In this case we have $x^y = x^{-1}$. If $|y| \leq 4$, then we get $exp(\langle x, y\rangle) \leq 4$, a contradiction. Hence, we have $|y| \geq 8$. Since $|yx| = |y|$, the subgroup $\langle x, y\rangle = \langle yx, y\rangle$ is contained in $H_4(G)$, a contradiction.
Therefore, this case does not arise.

So, we have that the order of x is 2^n, where $n \geq 3$. If $|y| \geq |8|$, then the subgroup $\langle x, y\rangle$ is contained in $H_4(G)$, a contradiction.
Therefore, we get that $|x| = 2^n \geq 8$ and $|y| \leq 4$. We now analyze the cases depending on the order of y and on the action of y on $\langle x\rangle$.

If y has order 2, then the automorphism induced by y on $\langle x\rangle$ has order 2, i.e. $x^y = x^{-1+2^{n-1}i}$, where $i \in \{0, 1\}$. Now, an element t of $\langle x, y\rangle$ of order greater than or equal to 8 lies in $\langle x\rangle$. In particular, we get $\langle t\rangle \trianglelefteq \langle x, y\rangle$. Therefore, the subgroup $\langle x, y\rangle$ is contained in the normalizer of every cyclic subgroup of $\langle x, y\rangle$ of order at least 8. In this case the claim is proved.

Suppose now $|y| = 4$ and $|\langle y\rangle \cap \langle x\rangle| = 2$. Then, we have $y^2 = x^{2^{n-1}}$. The automorphism induced by y on $\langle x\rangle$ has order 2, and so $x^y = x^{-1+2^{n-1}i}$, where $i \in \{0, 1\}$. Now, an element t of $\langle x, y\rangle$, with $|t| \geq 8$, lies in $\langle x\rangle$. In particular, we have $\langle t\rangle \trianglelefteq \langle x, y\rangle$. So, the subgroup $\langle x, y\rangle$ is contained in the normalizer of every cyclic subgroup of $\langle x, y\rangle$ of order at least 8 and, also in this case, the claim is proved.

Suppose now $|y| = 4$ and $|\langle y\rangle \cap \langle x\rangle| = 1$. If yx has order greater than 4, then $\langle x, y\rangle = \langle x, yx\rangle$ and $\langle x, yx\rangle$ is contained in $H_4(G)$, a contradiction. Hence, we have that $|yx| \leq 4$.
Since $x^y = x^{-1+4h}$, we have that $(yx)^4 = x^{8h+64h^3-32h^2}$. Therefore, the element yx is such that $|yx| \leq 4$ if and only if $8h \equiv 0 \bmod 2^n$, i.e. $x^y = x^{-1+4h}$, with $|x^{4h}| \leq 2$.
The elements t in $\langle x, y\rangle$, with $|t| \geq 8$, are of the form $x^i y^{2k}$, where $|x^i| \geq 8$ and $k \in \{0, 1\}$. Now, we get that x centralizes $x^i y^{2k}$. Also, since $(x^i y^{2k})^y = (x^i y^{2k})^{-1} x^{4hi}$ and $(x^i y^{2k})^2 = x^{2i}$, we have that y is in the normalizer of $\langle x^i y^{2k}\rangle$. It follows that $\langle x, y\rangle$ is contained in the normalizer of every cyclic subgroup of $\langle x, y\rangle$ of order at least 8. Therefore, in this case, the claim is proved.

In particular, we have that, if X is a cyclic subgroup of G having order greater than or equal to 8 and u is an element in $G \backslash H_4(G)$, then u is in $N_G(X)$.

Now, since $\{u \in G : u \notin H_4(G)\}$ is a set of generators of G, we have that X is a normal subgroup of G, and the first part of the statement is proved.

We now prove the second part of the lemma.
Let $X = \langle a \rangle$, where a element of G of order 2^n, $n \geq 3$. Let u be an element lying in $G \backslash H_4(G)$. By the previous part, we get that $\langle a \rangle \trianglelefteq \langle a, u \rangle$. Since $\langle a, u \rangle$ is not contained $H_4(G)$, the subgroup $\langle a, u \rangle$ is non-modular metacyclic, i.e. $a^u = a^{-1+4h}$. If the order of ua is greater than 4, then $\langle a, u \rangle = \langle a, ua \rangle$ and $\langle a, ua \rangle$ is contained in $H_4(G)$, a contradiction. Hence, we have that $|ua| \leq 4$. Now, we have $(ua)^2 = u^2 a^{4h}$ and so $(ua)^4 = a^{8h+64h^3-32h^2}$. Then, we have $(ua)^4 = 1$ if and only if $|a^{8h}| = 1$, i.e. $|a^{4h}| \leq 2$. Therefore, also the second part of the statement is proved. $\square$

Lemma 4.4. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Let a and b be elements of $H_4(G)$ with $|a| \geq 8$.
Then, the metacyclic subgroup $\langle a, b \rangle$ is modular.
Moreover, the subgroup $H_4(G)$ is powerful and is generated by its elements of maximal order.

Proof. By Lemma 4.3, if c is an element of G and $|c| \geq 8$, then $\langle c \rangle \trianglelefteq G$. Let a and b be elements of $H_4(G)$, with $|a| = 2^n \geq 8$ and $|b| = 2^m$.
Since $\langle a, b \rangle$ is a 2-generated subgroup of a monotone group, the subgroup $\langle a, b \rangle$ is metacyclic and $a^b = a^r$.

Suppose $r \equiv -1 \mod 4$, i.e. $a^b = a^{-1+4r}$. Let u be in $G \backslash H_4(G)$. By Lemma 4.3, we get $a^u = a^{-1+2^{n-1}h}$. Hence $a^{ub} = (a^{-1+2^{n-1}h})^b = a^{(-1+4r)(-1+2^{n-1}h)}$. So, the subgroup $\langle a, ub \rangle$ is modular. Since a modular group is generated by its elements of maximal order and a has order at least 8, we get that $\langle a, ub \rangle$ is contained in $H_4(G)$. In particular, the element ub lies in $H_4(G)$ and so, since $b \in H_4(G)$, we have $u \in H_4(G)$, a contradiction. Hence, this case does not arise.

It follows that $a^b = a^{1+4h}$ and so the subgroup $\langle a, b \rangle$ is modular (see Lemma 2.3.4 on page 56 of [13]).
This proves the first part of the statement.

The subgroup $H_4(G)$ is generated by the set $T = \{c \in H_4(G) : |c| \geq 8\}$. The first part of the proof shows that if c_1 and c_2 are in T, then $\langle c_1, c_2 \rangle$ is a modular metacyclic subgroup and so $[c_1, c_2] \in \langle c_1, c_2 \rangle^4$. Then, if c_1 and c_2 are in T, then $[c_1, c_2] \in H_4(G)^4$ and, by Remark 2.1, we conclude that $H_4(G)$ is powerful.

Furthermore, let a be an element of $H_4(G)$ of maximal order and let $c \in T$. Since the subgroup $\langle a, c\rangle$ is modular, there exists a $\overline{c} \in \langle a, c\rangle$ such that $|\overline{c}| = |a|$ and $\langle a, \overline{c}\rangle = \langle a, c\rangle$. In particular, $c \in \langle a, \overline{c}\rangle$ and so the set $\overline{T} = \{x : x \text{ is of maximal order in } H_4(G)\}$ is a set of generators of maximal order of $H_4(G)$. This concludes the proof of the lemma. □

The following lemmas deal with the elements of order 4 in $H_4(G)$. In fact, in Lemma 4.5 and in Lemma 4.6 we give some properties of $\Omega_2(H_4(G))$. More precisely, we show in Remark 4.7 that $\Omega_2(H_4(G))$ is a monotone 2-group of exponent 4 that does not involve a subgroup isomorphic to K_2 (i.e. a group well studied in Section 3.2).

Lemma 4.5. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
If a and b are in $H_4(G)$ with $|a| \leq 4$ and $|b| \leq 4$, then $exp(\langle a, b\rangle) \leq 4$. Moreover, if a and b have order 4 and $\langle a\rangle \cap \langle b\rangle = 1$, then the subgroup $\langle a, b\rangle$ is abelian.

Proof. Let a and b be elements of order at most 4 in $H_4(G)$.
If $exp(\langle a, b\rangle) \geq 8$, then, by Lemma 4.4, the subgroup $\langle a, b\rangle$ is modular. In particular, the subgroups $\langle a\rangle$ and $\langle b\rangle$ permute, a contradiction, because if H and K permute, then the exponent of HK equals $max\{exp(H), exp(K)\}$. Therefore, the exponent of $\langle a, b\rangle$ is at most 4.

Suppose now that a and b have order 4 and $\langle a\rangle \cap \langle b\rangle = 1$. Since the subgroup $\langle a, b\rangle$ has exponent 4, by Lemma 3.4, the subgroup $\langle a, b\rangle$ is either abelian or isomorphic to K_2.

If $\langle a, b\rangle$ is abelian, then the lemma holds.

Suppose that $\langle a, b\rangle$ is isomorphic to K_2. Up to renaming the generator of $\langle a, b\rangle$, we may assume that $a^b = a^{-1}$. Since $H_4(G)$ is powerful (see Lemma 4.4) and $a^2 \in H_4(G)'$, we have that $a^2 \in H_4(G)^4$ and there exists a $c \in H_4(G)$ such that $c^4 = a^2$. In particular, the element c has order 8 and, by Lemma 4.3, the subgroup $\langle c\rangle$ is normal in G. Moreover, by Lemma 4.4, the subgroups $\langle c, a\rangle$ and $\langle c, b\rangle$ are modular. Then, we have that $c^a = c^{1+4h_1}$ and $c^b = c^{1+4h_2}$. Now c^2 has order 4 and lies in the centralizer of $\langle a, b\rangle$. In particular, we get $c^2 \notin \langle a, b\rangle$. Now, the subgroup $\langle c^2a, c^2b\rangle$ is non-metacyclic, a contradiction. In fact, we have that $[c^2a, c^2b] = a^2$ and $(c^2ac^2b)^2 = (ab)^2 = b^2$. Therefore, the subgroup $\langle c^2a, c^2b\rangle$ contains the 3-generated elementary abelian subgroup $\langle c^2a, a^2, b^2\rangle$.

Therefore, this case does not arise and the second part of the statement is proved. $\square$

Lemma 4.6. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
If a is an element of $H_4(G)$ of order at most 4 and u is an element in $G \backslash H_4(G)$, then the exponent of $\langle a, u \rangle$ is at most 4.

Proof. Let a be in $H_4(G)$ of order at most 4 and u be in $G \backslash H_4(G)$.
Suppose that the exponent of $\langle a, u \rangle$ is 2^n, with $n \geq 3$, and let c be an element of maximal order in $\langle a, u \rangle$. Then, $\langle c \rangle \trianglelefteq G$ and $c \in H_4(G)$. Therefore, there exists $d \notin H_4(G)$ with $|d| \leq 4$ such that $\langle c, d \rangle = \langle a, u \rangle$. By Lemma 4.3, $c^d = c^{-1+4j}$, with $|c^{4j}| \leq 2$. Now, any element x of $\langle c, d \rangle$ such that $|x| = 4$ and $\langle x, d \rangle = \langle c, d \rangle$ is of the form dc^i, with i odd. All the elements of this form are not in $H_4(G)$. Since by hypothesis $\langle c, d \rangle = \langle a, u \rangle$, where u is not in $H_4(G)$ and a is in $H_4(G)$ and has order 4, we get a contradiction. $\square$

Remark 4.7. *By Lemma 4.5 and 4.6, we have that $\Omega_2(H_4(G))$ is a monotone 2-group of exponent 4.*
Moreover, Lemma 4.5 shows that $\Omega_2(H_4(G))$ is a monotone 2-group of exponent 4 that does not involve a subgroup isomorphic to K_2.

The following lemmas conclude the first part of the investigation of $H_4(G)$ and show that $H_4(G)$ is either abelian or there exists an element of maximal order a in $H_4(G)$ such that $H_4(G) = \langle a, \Omega_2(H_4(G)) \rangle$, see Lemma 4.9

Lemma 4.8. *Let H be a non-abelian metacyclic modular group of exponent greater than or equal to 8. Suppose that every cyclic subgroup of order greater than 4 is normal in H. Then, H is isomorphic to the split extension $C_{2^n} \rtimes X$, where $2^n = exp(H)$ and X is a cyclic subgroup of order either 2 or 4.*
More precisely, let $\langle a \rangle$ be a normal cyclic subgroup of H of maximal order. Then, there exists $d \in H$ such that the order of d is at most 4 and $H = \langle a \rangle \rtimes \langle d \rangle$.

Proof. Let a be an element in H of maximum order. Then $|a| \geq 8$ and $\langle a \rangle \trianglelefteq H$. Since H is not a quaternion group, H is lattice isomorphic to an abelian group (see Lemma 2.5.9 on page 94 of [13]). In particular, $\langle a \rangle$ has a complement $\langle d \rangle$ in H. If the order of d is greater than 4, then $\langle d \rangle$ is a normal subgroup. Since $\langle a\rangle$ is also normal in H, we get that $[a, d] \in \langle a\rangle \cap \langle b\rangle = 1$, i.e. H is abelian, a contradiction. Then, the element d has order at most 4, and the statement is proved. □

Lemma 4.9. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Then, either $H_4(G)$ is abelian or there exists an element a in $H_4(G)$ of maximal order such that $H_4(G) = \langle a, \Omega_2(H_4(G))\rangle$.

Proof. By Lemma 4.4, the subgroup $H_4(G)$ is generated by the set $T = \{a \in H_4(G) : |a| = 2^n\}$, where $2^n = exp(G)$. If for all $a_1, a_2 \in T$ the subgroup $\langle a_1, a_2\rangle$ is abelian, then, by Remark 2.1, the group $H_4(G)$ is abelian.
Suppose there exist $a_1, a_2 \in T$, such that the subgroup $\langle a_1, a_2\rangle$ is non-abelian. Then, by Lemma 4.4, the subgroup $\langle a_1, a_2\rangle$ is non-abelian modular and, by Lemma 4.3, every cyclic subgroup of order greater than 4 is normal in $\langle a_1, a_2\rangle$. Therefore, by Lemma 4.8, there exists an element d in $\Omega_2(H_4(G))$ such that $\langle a_1, a_2\rangle = \langle a_1\rangle \rtimes \langle d\rangle$. Hence, $a_2 \in \langle a_1, \Omega_2(H_4(G))\rangle$. Now, let x be in $H_4(G)$. We want to show that $x \in \langle a_1, \Omega_2(H_4(G))\rangle$. Since $a_2 \in \langle a_1, \Omega_2(H_4(G))\rangle$, we may replace in case x with a_2x and assume that x is an element in $H_4(G)$ such that $\langle a_1, x\rangle$ is non-abelian. By Lemma 4.4, the subgroup $\langle a_1, x\rangle$ is modular non-abelian and, by Lemma 4.3, every cyclic subgroup of order greater than 4 is normal in $\langle a_1, x\rangle$. Therefore, by Lemma 4.8, there exists an element $d \in \Omega_2(H_4(G))$ such that $\langle a_1, x\rangle = \langle a_1\rangle \rtimes \langle d\rangle$. Hence, $x \in \langle a_1, \Omega_2(H_4(G))\rangle$. In particular, T is contained in $\langle a_1, \Omega_2(H_4(G))\rangle$ and the statement is proved. □

In the rest of this section, we assume that $H_4(G)$ is not abelian. By Lemma 4.8, there exists an element a in $H_4(G)$ of maximal order such that $H_4(G) = \langle a, \Omega_2(H_4(G))\rangle$.

If $\Omega_2(H_4(G))$ is abelian, then, by Lemma 4.4 and by Lemma 4.5, we have that $H_4(G)$ is a modular group that does not involve a quaternion group. In particular, the structure of $H_4(G)$ is fully understood by Theorem 2.4.

If $\Omega_2(H_4(G))$ is not abelian, then, by Remark 4.7, we get that $\Omega_2(H_4(G))$ involves a subgroup isomorphic to D_8 or to Q_8, but not a subgroup isomorphic to K_2.
We determine in Lemma 4.10, in Lemma 4.12 and in Lemma 4.13 the structure of $H_4(G)$ in this latter case.

Lemma 4.10. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $H_4(G)$ contains a subgroup D isomorphic to D_8. Then, it contains a subgroup isomorphic to Q_8.

Proof. Let D be $\langle a, b\rangle$ with a of order 4. Since $H_4(G)$ is powerful and $a^2 \in H_4(G)'$, the element a^2 is in $H_4(G)^4$ and there exists $c \in H_4(G)$ such that $c^4 = a^2$. In particular, the element c has order 8, and, by Lemma 4.3, $\langle c\rangle \trianglelefteq G$. By lemma 4.4, the subgroups $\langle c, a\rangle$ and $\langle c, b\rangle$ are modular. Therefore, we have that $c^a = c^{1+4h_1}$ and $c^b = c^{1+4h_2}$. Now, the element c^2 has order 4 and lies in the centralizer of $\langle a, b\rangle$. Now, the subgroup $\langle a, c^2 b\rangle$ is isomorphic to a quaternion group of order 8 and the lemma is proved. $\square$

Remark 4.11. *By Lemma 4.5, the subgroup $\Omega_2(H_4(G))$ has exponent 4 and does not involve a subgroup isomorphic to K_2. Moreover, by Lemma 4.10, if $\Omega_2(H_4(G))$ is non-abelian then it contains a subgroup isomorphic to Q_8. Hence, by Theorem 3.3, we have that $\Omega_2(H_4(G))$ is either abelian or isomorphic to $E * A$, where E is an extraspecial group and A is either elementary abelian with $E \cap A = 1$ or abelian of the form $C_4 \times C_2 \times \cdots \times C_2$ with $E^2 = A^2$.*

Lemma 4.12. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $\Omega_2(H_4(G))$ contains a subgroup Q isomorphic to Q_8. Then, there are no cyclic subgroups X of order greater than 4 in $H_4(G)$ such that $X \cap Q = 1$. More precisely, if X is not contained in Q, then we have $X \cap Q = Q^2$.

Proof. Let Q be $\langle a, b\rangle$ and let X be $\langle c\rangle$. Suppose $\langle c\rangle \cap \langle a, b\rangle = 1$. Then, by Lemma 4.5, the subgroups $\langle a, c\rangle$ and $\langle b, c\rangle$ are abelian. The elements ca and b are such that $\langle ca\rangle \cap \langle b\rangle = 1$ but, since $\langle ac, b\rangle$ is non-abelian, we contradict Lemma 4.5. Hence, $\langle c\rangle \cap \langle a, b\rangle \neq 1$. Moreover, by Lemma 4.3, the subgroup $\langle c\rangle$ is normal in G and, by Lemma 4.4, the subgroups $\langle c, a\rangle$ and $\langle c, b\rangle$ are modular. At least one of $\langle c, a\rangle$ and $\langle c, b\rangle$ is not cyclic. Up to relabeling the generators of Q, we may assume that $\langle c, a\rangle$ is a non-cyclic modular subgroup. Since $\langle c\rangle \cap \langle a, b\rangle \neq 1$, we have $a^2 \in \langle c\rangle$. So the automorphism induced by a on $\langle c\rangle$ has order at most 2, i.e. $c^a = c^{1+4i}$, where $|c^{4i}| \leq 2$. Since a inverts all the element of order 4 of Q, we get that $\langle c\rangle \cap \langle a, b\rangle \leq \Omega_1(\langle a, b\rangle) = \langle a, b\rangle^2$. $\square$

Lemma 4.13. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$. Suppose that $H_4(G)$ contains a subgroup Q isomorphic to Q_8.*
Then, $H_4(G)$ is isomorphic to $\langle a, b\rangle * E \times A$, where $|a| = 2^n$, $n \geq 3$, E is extraspecial, A is elementary abelian, $|b| = 2$ and $a^b = a^{1+4i}$, where $|a^{4i}| \leq 2$ and $E^2 = \langle a^{2^{n-1}}\rangle$.

Proof. By Lemma 4.9, since $H_4(G)$ is non-abelian, we have that there exists an element a in $H_4(G)$ of maximal order 2^n (and so $n \geq 3$) such that $H_4(G) = \langle a, \Omega_2(H_4(G))\rangle$. Let x be an element in $H_4(G)$ such that $|x| \leq 4$. Then, the subgroup $\langle a, x\rangle$ is modular. Then $a^x = a^{1+4h}$, where $|a^{4h}| \leq 4$.
Thence, the subgroup $\Omega_2(\langle a\rangle)$ is contained in the centralizer of x. Therefore, since $H_4(G) = \langle a, \Omega_2(H_4(G))\rangle$, we have that $\Omega_2(\langle a\rangle) = \langle a^{2^{n-2}}\rangle$ is in the center of $H_4(G)$.
By Remark 4.11, we have that $\Omega_2(H_4(G)) = E * A$, where A is either elementary abelian with $E \cap A = 1$ or abelian of the form $C_4 \times C_2 \times \cdots \times C_2$ with $E^2 = A^2$.
Since $\Omega_2(H_4(G))$ contains the central element $a^{2^{n-2}}$ of order 4, we have that $\Omega_2(H_4(G)) = E * A$, where A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$. In particular, since $a^{2^{n-2}}$ is central of order 4, we may assume that $\Omega_2(H_4(G)) = \langle a^{2^{n-2}}\rangle * E \times A$, where E is extraspecial and A is elementary abelian. Moreover, since $C_4 * D_8$ is isomorphic to $C_4 * Q_8$, we may assume that $E = \langle x_1, y_1\rangle * \cdots * \langle x_m, y_m\rangle$, where $\langle x_i, y_i\rangle \simeq Q_8$. Since $a^{2^{n-1}} = x_i^2 = y_i^2$, using Lemma 4.4, we have that $a^{x_i} = a^{1+4h_i}$ and $a^{y_i} = a^{1+4k_i}$, where $|a^{4h_i}| \leq 2$ and $|a^{4k_i}| \leq 2$.
Replacing in case a with $a\prod_{i=1}^{n} x_i^{k_i} y_i^{h_i}$, we may assume that $[a, E] = 1$.

Suppose now that $A = \langle a_1\rangle \times \cdots \times \langle a_r\rangle$. If $[a, A] = 1$, then $H_4(G) = (C_{2^n} * E \times A)$, where E is extraspecial and A is elementary abelian and so $H_4(G)$ is isomorphic to one of the groups in the statement.

Suppose now that $[a, A] \neq 1$. Up to renaming the generators, we may assume that $a^{a_1} = a^{1+2^{n-1}}$, and $a^{a_i} = a$, for all $i \geq 2$.
Let $\bar{A}$ be the subgroup $\langle a_2, \ldots, a_r\rangle$.
Then, the group $H_4(G)$ is equal to $(\langle a\rangle * E \times \bar{A}) \rtimes \langle a_1\rangle$, where $x^{a_1} = x^{1+2^{n-1}}$, for all $x \in \langle a\rangle * E \times \bar{A}$.
Therefore, the group $H_4(G)$ is isomorphic to one of the groups in the statement. □

4.2 The Classification of Monotone 2-Groups G with $exp(G) \geq 8$ and $|G : H_4(G)| = 2$.

Lemma 4.9 shows that if G is a monotone 2-group of exponent at least 8 and with $H_4(G)$ maximal, then $H_4(G)$ is either abelian or there exists an element of maximal order a in $H_4(G)$ such that $H_4(G) = \langle a, \Omega_2(H_4(G))\rangle$.
In this second case, if $H_4(G)$ contains a subgroup isomorphic to Q_8, then $H_4(G)$ is isomorphic to a group of the form $\langle a, b\rangle * E \times A$, where $|a| = 2^n$, $n \geq 3$, E is extraspecial, A is elementary abelian, $|b| = 2$ and $a^b = a^{1+4i}$, where $|a^{4i}| \leq 2$ and $E^2 = \langle a^{2^{n-1}}\rangle$ (see Lemma 4.13).
If $H_4(G)$ does not contain a subgroup isomorphic to Q_8, then $H_4(G)$ is modular with no quaternion subgroups (note that these groups are fully classified by Theorem 2.4).

The following proposition classifies completely the monotone 2-groups of exponent at least 8 and with $H_4(G)$ maximal and abelian.

Proposition 4.14. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $H_4(G)$ is abelian. Then G is in $\mathscr{B}_1$ or in $\mathscr{B}_2$ or in $\mathscr{B}_3$ (see Definition 4.1).

Proof. Since $H_4(G)$ is an abelian group of exponent greater than 4, we have that $H_4(G) = \prod_{i=1}^{m} \langle a_i \rangle$, where $|a_1| = 2^n \geq 8$ and $|a_i| \geq |a_j|$, for all $i \leq j$.
Let u be in $G \backslash H_4(G)$. By Lemma 4.3, we have that $a_1^u = a_1^{-1+4h_1}$, where $|a_1^{4h_1}| \leq 2$.

Suppose that $|a_2| \geq 8$. Then, the subgroup $H_4(G)$ is generated by $\{a_1\} \cup T$, where the set T is given by $\{a : |a| \geq 8, \langle a\rangle \cap \langle a_1\rangle = 1\}$.
Let a be an element in T. Then, by Lemma 4.3, we have that $a^u = a^{-1+4k}$, where $|a^{4k}| \leq 2$. Then $(a_1 a)^u = (a_1 a)^{-1+4h_1} a^{4h_1+4k}$. Therefore, if $a^{4h_1+4k} \neq 1$, we have that the subgroup $\langle a_1 a\rangle$ has order greater than 4. But $\langle a_1 a\rangle$ is not normal, contradicting Lemma 4.3. Hence, we get that $a^u = a^{-1+4h_1}$. Since $H_4(G)$ is abelian and $\{a_1\} \cup T$ is a generating set, we get that G is isomorphic to $A\langle u\rangle$, where A is abelian of exponent 2^n, A^4 is not cyclic, $a^u = a^{-1+4h}$, with $exp(H_4(G)^{4h}) \leq 2$ and $u^2 \in \Omega_1(H_4(G))$. This shows that G is isomorphic to a group in $\mathscr{B}_1$.

Suppose now that $|a_i| \leq 4$, for all $i \geq 2$. We have that $H_4(G) = \prod_{i=1}^{m} \langle a_i\rangle = \langle a_1\rangle \prod_{i=2}^{m_1-1} \langle a_i\rangle \prod_{i=m_1}^{m} \langle a_i\rangle$, where $|a_1| = 2^n \geq 8$, $|a_i| = 4$ for $2 \leq i \leq m_1 - 1$, $|a_i| = 2$ for $m_1 \leq i \leq m$.

Let a be an element of $H_4(G)$ of order smaller than or equal to 4. The element a_1a has order $2^n \geq 8$, and so, by Lemma 4.3, $(a_1a)^u = (a_1a)^{-1+4k}$, where $|(a_1a)^{4k}| \leq 2$. Then $a^u = a^{-1}a_1^{4(h_1+k)}$. In particular, if a has order 4, then a^2 is central in G, i.e. $\Omega_1(H_4(G)) \cap \Phi(H_4(G)) = \langle a_1^{2^{n-1}}, a_2^2, \cdots, a_{m_1-1}^2\rangle$ is a central subgroup.

If for all $a \in H_4(G)$ of order smaller than or equal to 4 we have that $a^u = a^{-1}$, then G is isomorphic to $(C_{2^n} \times A)\langle u\rangle$, where A is abelian of exponent 4, $u^2 \in \Omega_1(C_{2^n} \times A)$ and for every $a \in C_{2^n} \times A$ we have $a^u = a^{-1+4h}$ with $exp((C_{2^n} \times A)^{4h}) \leq 2$. This shows that G is $\mathscr{B}_1$.

Now, we may assume that there exists an element a such that $|a| \leq 4$ and $a^u = a^{-1}a_1^{2^{n-1}}$.

We first assume that there exists a with $|a| = 2$ and $a^u = aa_1^{2^{n-1}}$. Since all the elements of order 2 of the Frattini subgroup of $H_4(G)$ are centralized by u, up to relabeling the generators of $\langle a_{m_1}, \ldots, a_m\rangle$, we may assume that $a_{m_1}^u = a_{m_1}a_1^{2^{n-1}}$. Up to replacing perhaps for some i, the element a_i with $a_ia_{m_1}$, we may assume $a_i^u = a_i^{-1}$ for every $i \in \{m_1+1, \ldots, m\}$. In particular, by Lemma 4.5 and Lemma 4.6, the subgroup $\langle a_{m_1}, u\rangle$ is metacyclic non-abelian of exponent 4. Then, $\langle a_{m_1}, u\rangle$ is isomorphic to a D_8. Replacing in case u with $a_{m_1}u$, we may assume $u^2 = a_1^{2^{n-1}}$. Consider now a_2. Replacing perhaps a_2 with $a_2a_{m_1}$, we may assume that $a_2^u = a_2^{-1}a_1^{2^{n-1}}$.
The subgroup $\langle a^{2^{n-2}}a_2, u\rangle$ is not metacyclic. In fact, we have $(a^{2^{n-2}}a_2)^2 = a^{2^{n-1}}a_2^2$, and $(a^{2^{n-2}}a_2)^u = (a^{-2^{n-2}}a_2^{-1})a_1^{2^{2^{n-1}}} = (a_1^{2^{n-2}}a_2)^{-1}a_1^{2^{n-1}}$. In particular, we have that $(ua_1^{2^{n-2}}a_2)^2 = u^2a_1^{-2^{n-2}}a_2^{-1}a_1^{2^{n-1}}a_1^{2^{n-2}}a_2 = 1$. Then, the group $\langle(a^{2^{n-2}}a_2), u\rangle$ contains the 3-generated elementary abelian subgroup $\langle a^{2^{n-2}}a_2a_{m_1}u, a_2^2, a^{2^{n-1}}\rangle$.
Therefore, we obtain that $m_1 = 2$.
Then, replacing perhaps for some $i \neq 2$ the generators a_i with $a_ia_{m_1}$, we get that G is isomorphic to the group $\langle a, a_{m_1}, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $a^{a_{m_1}} = a$ and $a^u = a^{-1}$, $|a_{m_1}| = 2$, $u^a = u^{-1}$ and $u^2 = a^{2^{n-1}}$. This shows that the group G is in $\mathscr{B}_2$.

We may now assume that $u \in C(\langle a_i\rangle)$, for all $i \geq m_1$, and that there exists an element $a \in H_4(G)$ of order 4 such that $a^u = a^{-1}a_1^{2^{n-1}}$. Up to renaming the generators, we may assume that $a_2^u = a_2^{-1}a_1^{2^{n-1}}$. Up to replacing a_1 with a_1a_2, we may assume $a_1^u = a_1^{-1}$.
If u has order 2, then the subgroup $\langle a_2, u\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle u, a_2^2, a_1^{2^{n-1}}\rangle$. Then,

the element u has order 4. If u^2 does not lie in $\langle a_2^2, a_1^{2^{n-1}}\rangle$, then the subgroup $\langle a_2, u\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle u^2, a_2^2, a_1^{2^{n-1}}\rangle$. Then, we have that $u^2 \in \langle a_1^{2^{n-1}}, a_2^2\rangle$. Suppose that $u^2 = a_1^{2^{n-1}}$. The subgroup $\langle a_2, u\rangle$ is non-metacyclic, because it contains the 3-generated abelian subgroup $\langle a_2^2, u^2, a_2 u\rangle$. Therefore $u^2 = a^2 a_1^{2^{n-1}t}$. Replacing in case u with ua_2, we get that we may assume $u^2 = a_2^2$. Suppose $m_1 - 1 \geq 3$, and consider a_3. Since a_3 has order 4, we have that $a_3^u = a_3^{-1} a_1^{2^{n-1}h_3}$. Replacing in case a_3 with $a_2 a_3$, we may assume that $a_3^u = a_3^{-1} a_1^{2^{n-1}}$. The subgroup $\langle a_3, u\rangle$ is not metacyclic since it contains the 3-generated abelian subgroup $\langle a_1^{2^{n-1}}, a_3^2, u^2\rangle$. Therefore, this case does not arise and $m_1 = 3$. This proves that G is isomorphic to $\langle a_1, a_2, u\rangle \times A$, where A is elementary abelian, $|a_1| = 2^n$, $|a_2| = 4$, $\langle a_1, a_2\rangle$ is abelian, $u^2 = a_2^2$ and $a_1^u = a_1^{-1}$, and $a_2^u = a_2^{-1} a_1^{2^{n-1}}$. Therefore, the group G is in $\mathscr{B}_3$. $\square$

Now, we study the cases in which $H_4(G)$ is a non-abelian modular group not involving Q_8. Indeed, the following proposition classifies the monotone 2-groups of exponent greater than 4 with $H_4(G)$ maximal and modular without subgroups isomorphic to Q_8.

Proposition 4.15. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $H_4(G)$ is non-abelian and does not involve a subgroup isomorphic to Q_8. Then G is in one of the families $\mathscr{B}_8, \mathscr{B}_9, \mathscr{B}_{10}$ (see Definition 4.1).

Proof. By Remark 4.11, we have that $\Omega_2(H_4(G))$ is an abelian group of exponent 4 and $H_4(G) = \langle a\rangle\Omega_2(H_4(G))$.

Suppose that the order of a is 8. By Lemma 4.4, if x is an element of $\Omega_1(H_4(G))$, then $\langle a, x\rangle$ is a modular subgroup and the automorphism induced on $\langle a\rangle$ has order at most 2.
Suppose that a has order greater than 8. By Lemma 4.3 and Lemma 4.4, if x is an element of A, then $\langle a, x\rangle$ is a modular subgroup, and $\langle a\rangle$ is a normal subgroup. Suppose that there exists an x of order 4 such that $a^x = a^{1+4h}$, with $|a^{4h}| = 4$. Let u be in $G\backslash H_4(G)$. Then, by Lemma 4.3, we have $a^u = a^{-1+4k}$, with $|a^{4k}| \leq 2$. We have that $a^{xu} = a^{-1-4h+4k}$. Therefore, the element axu has order 8. Since a and x are in $H_4(G)$, we have that $u \in H_4(G)$, a contradiction.

This proves that if a has order greater than or equal to 8 and x is an element of $\Omega_2(H_4(G))$, then $\langle a, x\rangle$ is a modular subgroup and the automorphism induced by x on $\langle a\rangle$ has order at most 2.

Since $H_4(G)$ is non-abelian, we have that $H_4(G) = (\langle a\rangle \times \langle a_1\rangle \times \cdots \times \langle a_m\rangle \times \langle c_1\rangle \times \cdots \times \langle c_r\rangle) \rtimes \langle b\rangle$, where $|a| = 2^n$, $|a_i| = 4$, $|c_i| = 2$, $|b| \leq 4$ and $x^b = x^{1+2^{n-1}}$, for all $x \in \langle a, a_1, \ldots, a_m, c_1, \ldots, c_r\rangle$.

Let u be in $G\backslash H_4(G)$. Then $u^2 \in \Omega_1(H_4(G)) \cap Z(H_4(G))$.

We now distinguish two cases depending on the order of b.

Suppose first $|b| = 2$. Then, we have $H_4(G) = (\langle a\rangle \times \langle a_1\rangle \times \cdots \times \langle a_m\rangle \times \langle c_1\rangle \times \cdots \times \langle c_r\rangle) \rtimes \langle b\rangle$, where $|a| = 2^n$, $|a_i| = 4$, $|c_i| = 2$, $|b| = 2$, $x^b = x^{1+2^{n-1}}$ for all $x \in \langle a, a_1, \ldots, a_m, c_1, \ldots, c_r\rangle$.

By Lemma 4.4, we have that $a^u = a^{-1+4h}$, with $|a^{4h}| \leq 2$. Since ab has order 2^n, by Lemma 4.4, we have that $(ab)^u = (ab)^{-1+4k}$ with $|(ab)^{4k}| \leq 2$, i.e. $b^u = ba^{4(k+h+1)}$. Replacing perhaps u with au, we may assume that $b^u = ba^{2^{n-1}}$. In particular, if u has order 4 and $u^2 \neq a^{2^{n-1}}$, then $\langle b, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle b, u^2, a^{2^{n-1}}\rangle$. Then, up to replacing if necessary u with ub, we may assume that u has order 4 and $u^2 = a^{2^{n-1}}$.
So we now have $a^u = a^{-1+4h}$, with $|a^{4h}| \leq 2$, $u^2 = a^{2^{n-1}}$ and $u^b = u^{-1}$.

We now show that $m = 0$.
Suppose that $m > 0$. Since aa_1 has order 2^n, by Lemma 4.4, we have that $(aa_1)^u = (aa_1)^{-1}a^{4h_1}$ with $|a^{4h_i}| \leq 2$, i.e. $a_1^u = a_1^{-1}a^{4(h+h_1)}$. If $a_1^u = a_1^{-1}a^{2^{n-1}}$, then the subgroup $\langle a_1, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle ua_1, a_1^2, u^2\rangle$. If $a_1^u = a_1^{-1}$, then $(a_1b)^u = (a_1b)^{-1}a^{2^{n-1}}$ and the group $\langle a_1b, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle ua_1b, u^2, a_1^2\rangle$. Then, we have that $m = 0$.

Now, we consider the action of u on $\langle c_1, \cdots, c_r\rangle$.
Since $|ac_i| = 2^n$, by Lemma 4.4, we have that $(ac_i)^u = (ac_i)^{-1}a^{4h_1}$ with $|a^{4h_1}| \leq 2$, i.e. $c_i^u = c_i^{-1}a^{4k_i}$ with $|a^{4k_i}| \leq 2$.
Then, we have two possibilities, depending on the orders of a^{4k_i}.

If $c_i^u = c_i$ for all i, then G is isomorphic to $\langle a, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1+4h}$, $a^b = a^{1+2^{n-1}}$ and $u^b = u^{-1}$ and so G is in the family $\mathscr{B}_8$.

Suppose that there exists an i such that $c_i^u = c_ia^{2^{n-1}}$. Up to reordering the generators, we may assume that $c_1^u = c_1a^{2^{n-1}}$. Moreover, up to replacing in case c_i with c_ic_1 for $i \geq 2$ and a with ac_1, we obtain that G is isomorphic to $\langle a, c_1, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|c_1| = 2$, $|b| = 2$, $|u| = 4$, $u^2 = a^{2^{n-1}}$, $a^u = a^{-1}$, $a^b = a^{1+2^{n-1}}$, $u^b = u^{-1}$, $c_1^a = c_1$, $c_1^b = c_1$ and $c_1^u = c_1a^{2^{n-1}}$. So, the group G is in the family $\mathscr{B}_9$.

Suppose now that $|b| = 4$. Then, we have $H_4(G) = (\langle a\rangle \times \langle a_1\rangle \times \cdots \times \langle a_m\rangle \times \langle c_1\rangle \times \cdots \times \langle c_r\rangle) \rtimes \langle b\rangle$, where $|a| = 2^n$, $|a_i| = 4$, $|c_i| = 2$, $|b| = 4$, $x^b = x^{1+2^{n-1}}$ for all $x \in \langle a, a_1, \dots, a_m, c_1, \dots, c_r\rangle$. By Lemma 4.4, we have that $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$.
Since ab has order 2^n, by Lemma 4.4, we have that $(ab)^u = (ab)^{-1+4k}$ with $|(ab)^{4k}| \leq 2$, i.e. $b^u = b^{-1}a^{4(k+h+1)}$. Replacing perhaps u with au, we may assume that $b^u = b^{-1}a^{2^{n-1}}$. So we get $a^u = a^{-1+4h}$ and $b^u = b^{-1}a^{2^{n-1}}$.

In particular, if $|u| = 2$, then the subgroup $\langle b, u\rangle$ is not metacyclic since it contains the 3-generated elementary abelian subgroup $\langle b^2, a^{2^{n-1}}, u\rangle$. Then, u has order 4.
Moreover, if $u^2 \notin \langle a^{2^{n-1}}, b^2\rangle$, then $\langle b, u\rangle$ is not metacyclic since it contains the 3-generated elementary abelian subgroup $\langle b^2, a^{2^{n-1}}, u^2\rangle$. Then, the element u has order 4 and $u^2 \in \langle a^{2^{n-1}}, b^2\rangle$.
If $u^2 = a^{2^{n-1}}$, then the subgroup $\langle b, u\rangle$ is not metacyclic since it contains the 3-generated elementary abelian subgroup $\langle b^2, a^{2^{n-1}}, ub\rangle$. Then, we have $u^2 = b^2a^{2^{n-1}s}$, and, replacing in case u with ub, we may assume $u^2 = b^2$.
Then, we have that $a^u = a^{-1+4h}$, $b^u = b^{-1}a^{2^{n-1}}$ and $u^2 = b^2$.

We now prove that $m = 0$.
Suppose $m \neq 0$. Then, the element aa_1 has order 2^n, and so, by Lemma 4.4, we have that $(aa_1)^u = (aa_1)^{-1}a^{4(h_1+h)}$ with $|a^{4h_1}| \leq 2$. So we have two possibilities: either $a_1^u = a_1^{-1}$ or $a_1^u = a_1^{-1}a^{2^{n-1}}$. In both cases we reach a contradiction. In fact, if $a_1^u = a_1^{-1}a^{2^{n-1}}$, then the subgroup $\langle a_1, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle a^{2^{n-1}}, a_1^2, u^2\rangle$. If $a_1^u = a_1^{-1}$, then $(a_1b)^u = (a_1b)^{-1}a^{2^{n-1}}$ and the group $\langle a_1b, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle u^2, b^2a_1^2, a^{2^{n-1}}\rangle$.
Therefore this case does not arise, and $m = 0$.

We now consider the action of u on $\langle c_1, \cdots, c_r\rangle$.
The element ac_i has order 2^n and by Lemma 4.4, we have that $(ac_i)^u = (ac_i)^{-1}a^{4(h_1+h)}$ with $|a^{4h_1}| \leq 2$, i.e. $c_i^u = c_ia^{4k_i}$ with $|a^{4k_i}| \leq 2$.
If $c_i^u = c_ia^{2^{n-1}}$, then the subgroup $\langle c_i, u\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle c_i, u^2, a^{2^{n-1}}\rangle$.

Then, we have that $c_i^u = c_i$ for all i, and G is isomorphic to $\langle a, b, u\rangle \times A$, where A is elementary abelian, $|a| = 2^n \geq 8$, $|b| = 4$, $|u| = 4$, $u^2 = b^2$, $a^u = a^{-1+4h}$, with $|a^{4h}| \leq 2$, $b^u = b^{-1}a^{2^{n-1}}$, $a^b = a^{1+2^{n-1}}$. So G is in the family $\mathscr{B}_{10}$. This concludes the proof of the statement. □

To conclude the classification of monotone 2-groups of exponent greater than 4 and where $H_4(G)$ is maximal, it remains to consider the case where $H_4(G)$ contains a subgroup isomorphic to Q_8. We do that in the next two propositions.
We first consider the case $H_4(G) = C_{2^n} * E \times A$, where E is extraspecial and A is elementary abelian.

Proposition 4.16. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $H_4(G)$ is isomorphic to $C_{2^n} * E \times A$, where E is extraspecial and A is elementary abelian. Then, the group G belongs either to $\mathscr{B}_4$ or to $\mathscr{B}_5$ (see Definition 4.1).

Proof. Let $C_{2^n} = \langle a\rangle$. Since $a^{2^{n-2}}$ is central of order 4 in $H_4(G)$ and $D_8 * C_4 \simeq Q_8 * C_4$, we may assume that $E = \langle x_1, y_1\rangle * \cdots * \langle x_m, y_m\rangle$, where $\langle x_i, y_i\rangle \simeq Q_8$. Let $A = \langle a_1\rangle \times \cdots \times \langle a_r\rangle$.

Let u be in $G \backslash H_4(G)$.
We have that $u^2 \in \Omega_1(H_4(G)) = \langle a^{2^{n-1}}\rangle \times \langle a_1\rangle \times \cdots \times \langle a_r\rangle$.
By Lemma 4.3, we get that $a^u = a^{-1+4h}$, where $|a^{4h}| \leq 2$.

We now investigate the action of u on E.
Since ax_i is an element of order 2^n, by Lemma 4.3, we have that $(ax_i)^u = (ax_i)^{-1+4h_i}$, with $|(ax_i)^{4h_i}| \leq 2$, i.e. $x_i^u = x_i^{-1}a^{4(h+h_i)}$. Since $x_i^2 = a^{2^{n-1}}$, we have that $x_i^u = x_i x_i^{2s_i}$, where $s_i \in \{0, 1\}$. Using the same argument with y_i instead of x_i, we also get that $y_i^u = y_i y_i^{2r_i}$, where $r_i \in \{0, 1\}$. Now, replacing perhaps u with $u\prod_{i=1}^m x_i^{r_i} y_i^{s_1}$, we may assume that $a^u = a^{-1+4h}$, where $|a^{4h}| \leq 2$, and $[E, u] = 1$.

We now consider the action of u on A.
Since aa_i is an element of order 2^n, by Lemma 4.3, we have that $(aa_i)^u = (aa_i)^{-1+4k_i}$, with $|(aa_i)^{4k_i}| \leq 2$, i.e. $a_i^u = a_i a^{4h_i}$, where $|a^{4h_i}| \leq 2$. We distinguish two cases depending on the values of the orders of a^{4h_i}.

Suppose first that $a^{4h_i} = 1$ for all i. Then we get that $[A, u] = 1$. Therefore, we get $a^u = a^{-1+4h}$, $[E, u] = 1$, $[A, u] = 1$. The element $u^2 \in \Omega_1(H_4(G)) \cap Z(H_4(G))$.

Suppose that $u^2 \notin \langle a^{2^{n-1}} \rangle$. Then, $\langle a^{2^{n-2}} x_1, u \rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle a^{2^{n-2}} x_1, u^2, a^{2^{n-1}} \rangle$. Hence, this case does not arise and u^2 lies in $\langle a^{2^{n-1}} \rangle$. Then, we obtain that G is isomorphic to $\langle a, u \rangle * E \times A$, where E is extraspecial, A is elementary abelian, $|a| = 2^n \geq 8$, and u is such that $u^2 \in \langle a^{2^{n-1}} \rangle$, $a^u = a^{-1+4h}$ with $|a^{4h}| \leq 2$ and $x^u = x$ for all $x \in E \times A$, i.e. G is in $\mathscr{B}_4$.

Suppose now that there exists an i with $|a^{4h_i}| = 2$. Up to renaming the generators of A, we may suppose that $a_1^u = a_1 a^{2^{n-1}}$, and, replacing in case a_i with $a_1 a_i$, we may assume that $a_i^u = a_i$, for $i \geq 2$. Replacing perhaps u with $a_1 u$, we may assume that u has order 4. If $u^2 \neq a^{2^{n-1}}$, then the subgroup $\langle a_1, u \rangle$ is not metacyclic, since it contains the 3-generated elementary abelian subgroup $\langle a_1, a^{2^{n-1}}, u^2 \rangle$. Hence, $u^2 = a^{2^{n-1}}$.
Replacing perhaps a with aa_1, we may assume $a^u = a^{-1}$.
Replacing a_1 with $a_1 a^{2^{n-2}}$, we obtain that the group G is isomorphic to $\langle a, u \rangle * E * A$, where E is extraspecial, A is abelian of the form $C_4 \times C_2 \times \cdots \times C_2$, and $|a| = 2^n$ with $n \geq 3$, $u^2 = a^{2^{n-1}}$ and $a^u = a^{-1}$, i.e. G is in $\mathscr{B}_5$. $\square$

Finally we are ready to study the last case, i.e. $H_4(G) = \langle a, b \rangle * E \times A$, where $|a| = 2^n \geq 8$, E is extraspecial, A is elementary abelian, $|b| = 2$, $a^b = a^{1+2^{n-1}}$ and $E^2 = \langle a^{2^{n-1}} \rangle$.

Proposition 4.17. *Let G be a monotone 2-group of exponent greater than 4 and such that $|G : H_4(G)| = 2$.*
Suppose that $H_4(G)$ is isomorphic to $\langle a, b \rangle * E \times A$, where $|a| = 2^n \geq 8$, E is extraspecial, A is elementary abelian, $|b| = 2$, $a^b = a^{1+2^{n-1}}$ and $E^2 = \langle a^{2^{n-1}} \rangle$. Then G belongs either to $\mathscr{B}_6$ or to $\mathscr{B}_7$ (see Definition 4.1).

Proof. Since $a^{2^{n-2}}$ is central of order 4 in $H_4(G)$ and $D_8 * C_4 \simeq Q_8 * C_4$, we may assume that $E = \langle x_1, y_1 \rangle * \cdots * \langle x_m, y_m \rangle$, where $\langle x_i, y_i \rangle \simeq Q_8$. Let $A = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$.
Let u be in $G \backslash H_4(G)$.
By Lemma 4.4, we have that $a^u = a^{-1+2^{n-1}h}$. In particular, replacing in case u with ub we may assume that $a^u = a^{-1}$. The element u^2 lies in $\Omega_1(H_4(G)) \cap C_G(\langle a \rangle)$. Hence, since $\Omega_1(H_4(G)) \cap C_G(\langle a \rangle) = \langle a^{2^{n-1}}, A \rangle$, we get that $u^2 \in \langle a^{2^{n-1}}, A \rangle$.

We consider now the action of u on E. Since ax_i is an element of order 2^n, by Lemma 4.3, we have that $(ax_i)^u = (ax_i)^{-1+2^{n-1}h_i}$, with $h_i \in \{0, 1\}$, i.e. $x_i^u = x_i^{-1}a^{2^{n-1}(h_i)}$. Since $x_i^2 = a^{2^{n-1}}$, we have that $x_i^u = x_i x_i^{2s_i}$, where $s_i \in \{0,1\}$. Replacing x_i with y_i and using the same argument, we also get that $y_i^u = y_i y_i^{2r_i}$, where $r_i \in \{0,1\}$. Now, replacing perhaps u with $u\prod_{i=1}^m x_i^{r_i} y_i^{s_1}$, we may assume that $a^u = a^{-1}$, and $[E,u] = 1$.

Now, we consider the action of u on A. Since aa_1 is an element of order 2^n, by Lemma 4.3, we have that $(ax_i)^u = (ax_i)^{-1+2^{n-1}k_i}$, with $k_i \in \{0,1\}$, i.e. $a_i^u = a_i a^{2^{n-1}h_i}$, where $h_i \in \{0,1\}$. We distinguish two cases depending on the values of the h_i's.

Suppose first that $h_i = 0$ for all i. Then we get that $[A,u] = 1$.
If u has order 4 and $u^2 \neq a^{2^{n-1}}$, then the subgroup $\langle a^{2^{n-2}}x_1, u\rangle$ is not metacyclic, because $\langle a^{2^{n-2}}x_1, u\rangle$ contains the 3-generated elementary abelian subgroup $\langle a^{2^{n-2}}x_1, u^2, a^{2^{n-1}}\rangle$. Hence, we have that u^2 lies in $\langle a^{2^{n-1}}\rangle$. Therefore, we get $a^u = a^{-1}$, $[E,u] = 1$, $[A,u] = 1$ and $u^2 \in \langle a^{2^{n-1}}\rangle$. Consider now ab. Since $|ab| = 2^n$, we have that $(ab)^u = (ab)^{-1+2^{n-1}h}$, i.e. $b^u = ba^{2^{n-1}h}$, and G is in $\mathscr{B}_6$.

Suppose now that there exists an i with $h_i \neq 1$. Up to renaming the generators of A, we may suppose that $a_1^u = a_1 a^{2^{n-1}}$, and, up to replacing perhaps a_i with $a_1 a_i$, we may assume $a_i^u = a_i$, for all $i \geq 2$.
Replacing perhaps u with $a_1 u$, we may assume that u has order 4. Moreover, if $u^2 \neq a^{2^{n-1}}$, then $\langle a_1, u\rangle$ is non-metacyclic because it contains the 3-generated elementary abelian subgroup $\langle a_1, a^{2^{n-1}}, u^2\rangle$. Therefore, we have that $u^2 = a^{2^{n-1}}$.
Consider now ab. Since $|ab| = 2^n$, we have that $(ab)^u = (ab)^{-1+2^{n-1}h}$, i.e. $b^u = ba^{2^{n-1}k}$ with $k \in \{0,1\}$. Up to replacing b with $a_1 b$, we may assume that $b^u = b$. Moreover, replacing a_1 with $a_1 a^{2^{n-2}}$, we obtain that the group G is in $\mathscr{B}_7$. $\square$

Summing up, in this section we determined the monotone 2-groups of exponent at least 8 such that $|G : H_4(G)| = 2$. Namely, combining the previous four propositions, we obtain that any such a group is in the class $\mathscr{B}_i$, with $i \in \{1,\ldots,10\}$ and Theorem 4.2 is proved.

Chapter 5

Monotone 2-Groups of exponent greater than 4 in which $G = H_4(G)$

In Chapter 3, the monotone 2-groups of exponent 4 were fully classified. In Chapter 4, we classified completely the monotone 2-groups of exponent greater than or equal to 8 and such that $|G : H_4(G)| = 2$. Therefore, in order to complete the classification of the monotone 2-groups, by Proposition 1.3, it remains to study the monotone 2-groups of exponent greater than or equal to 8 such that $G = H_4(G)$. In this chapter, such groups are determined.

Definition 5.1. *We introduce the following families of* 2*-groups:*

- $\mathscr{C}_1$ *is the family of* 2*-groups of the form* $\langle a, c\rangle * E \times A$*, where* E *is extraspecial,* A *is elementary abelian,* $|a| = 2^n \geq 8$*,* $|c| = 2$*,* $a^c = a^{1+4h}$ *with* $|a^{4h}| \leq 2$ *and* $\langle a^{2^{n-1}}\rangle = E^2$*;*

- $\mathscr{C}_2$ *is the family of* 2*-groups of the form* $\langle a, b, c\rangle \times A$*, where* A *is elementary abelian,* $|c| = 2^n \geq 8$*,* $|a| = 4$*,* $a^2 = b^2$*,* $c^a = c^{1+4h_1}$*,* $c^b = c^{-1+4h_2}$ *and* $a^b = a^{-1}c^{4h_3}$*, with* $|c^{4h_i}| \leq 2$ *for* $i = 1, 2, 3$*;*

- $\mathscr{C}_3$ *is the family of* 2*-groups of the form* $\langle a, b, c, d\rangle \times A$*, where* A *is elementary abelian,* $|c| = 2^n \geq 8$*,* $|a| = 4$*,* $a^2 = b^2$*,* $|d| = 2$*,* $c^a = c$*,* $c^b = c$ *and* $a^b = a^{-1}c^{4h}$ *with* $|c^{4h}| \leq 2$*, and* $c^d = c^{1+2^{n-1}}$*,* $a^d = a$ *and* $b^d = b$*;*

- $\mathscr{C}_4$ *is the family of* 2*-groups of the form* $\langle a, b, c, d\rangle * E \times A$*, where* A *is elementary abelian,* E *is extraspecial,* $|a| = 2^n \geq 8$*,* $b^4 = a^{2^{n-1}}$*,* $\langle c, d\rangle$ *is* *elementary abelian,* $a^b = a^{-1+4h}$*, with* $|a^{4h}| \leq 2$*,* $a^c = a^{1+4h_1}$*,* $a^d = a$*,* $b^c = b$*,* $b^d = b^{1+4h_2}$*, where* $|a^{4h_1}| \leq 2$*,* $|b^{4h_2}| \leq 2$ *and* $\langle a^{2^{n-1}} \rangle = E^2$*;*

- $\mathscr{C}_5$ *is the family of* 2*-groups of the form* $A\langle b \rangle$*, where* A *is an abelian group,* $|b| \geq 8$ *and* $a^b = a^{-1+4h}$*, for every* $a \in A$*;*

- $\mathscr{C}_6$ *is the family* $\langle A, c, b \rangle$*, where* A *is an abelian group of exponent* 2^n*, with* $n \geq 3$*,* $A^{2^{n-1}} = \Omega_1(\langle b \rangle)$*,* $|b| \geq 8$*,* $a^b = a^{-1+4h}$*,* $a^c = a^{1+2^{n-1}}$ *for every* $a \in A$*,* $c^b = c^{-1+4h}$ *and* $exp(\langle A, c \rangle^{4h}) < |b^2| < 2^n$*;*

- $\mathscr{C}_7$ *is the family of* 2*-groups of the form* $\langle A, c, b \rangle$ *where* A *is an abelian group of exponent* 2^n*, with* $n \geq 3$*,* $A^{2^{n-1}} = \Omega_1(\langle b \rangle)$*,* $|b| \geq 8$*,* $a^b = a^{-1+4h+2^{n-1}}$*,* $a^c = a^{1+2^{n-1}}$ *for every* $a \in A$*,* $c^b = c^{-1+4h}$*,* $|c| > 2^n$*,* $|c^{4h}| = |b^2|$*,* $|b^2| < 2^n$*, and* $\langle b \rangle \cap \langle c \rangle = 1$*.*

We start by proving that all the groups in the families just defined are monotone.

Proposition 5.1. *The groups in the families* $\mathscr{C}_i$*, for* $i \in \{1, \ldots, 7\}$ *are monotone.*

Proof. We want to show that if G is a group in $\mathscr{C}_i$, for $i \in \{1, \ldots, 7\}$, then G is monotone. Now, the proof is a case-by-case analysis depending on the family in which G lies.

- Suppose that G is in $\mathscr{C}_1$. Then, we have $G = \langle a, c \rangle * E \times A$, where E is extraspecial, A is elementary abelian, $|a| = 2^n \geq 8$, $|c| = 2$, $a^c = a^{1+4h}$ with $|a^{4h}| \leq 2$ and $\langle a^{2^{n-1}} \rangle = E^2$. We have to prove that every 2-generated subgroup is metacyclic. By Lemma 2.5, it is sufficient to prove that the group $\langle a, c \rangle * E$, where $\langle a^{2^{n-1}} \rangle = E^2$, is monotone. We treat separately the cases $a^{4h} = 1$ and $|a^{4h}| = 2$.

  Suppose at first that $a^{4h} = 1$, i.e. $\langle a, c \rangle$ is abelian. Since c is a central element of order 2 in $\langle a, c \rangle * E$, where $\langle a^{2^{n-1}} \rangle = E^2$, by Lemma 2.12, it is sufficient to check that $\langle a \rangle * E$, where $\langle a^{2^{n-1}} \rangle = E^2$, is monotone. The extraspecial groups are monotone, and so we check that the subgroups of the form $\langle a^i t_1, t_2 \rangle$, with t_1 and t_2 in E, are metacyclic.
  If $|a^i| \leq 4$, then $\langle a^i t_1, t_2 \rangle$ is contained in $\langle a^{2^{n-2}} \rangle * E$, where $\langle a^{2^{n-1}} \rangle = E^2$. The group $\langle a^{2^{n-2}} \rangle * E$, where $\langle a^{2^{n-1}} \rangle = E^2$, is monotone because it is isomorphic to a group in $\mathscr{A}_2$ (see Definition 3.1 and Proposition 3.1).

So, we now assume that $|a^i| > 4$. In particular, $\Omega_1(\langle a^i t_1\rangle) = \Omega_1(\langle a\rangle) = [\langle a\rangle * E, \langle a\rangle * E]$, and so, $\langle a^i t_1\rangle$ is normal in the group $\langle a\rangle * E$. It follows that $\langle a^i t_1, t_2\rangle$ is metacyclic. So, if $a^{4h} = 1$, the group $\langle a\rangle * E$, where $\langle a^{2^{n-1}}\rangle = E^2$, is monotone, and so is G.

Suppose now that $|a^{4h}| = 2$. The group $\langle a^2, c, E\rangle$ is isomorphic to $C_{2^{n-1}} * E \times \langle c\rangle$ where $\Omega_1(C_{2^{n-1}}) = E^2$, and so, it is either in $\mathscr{A}_2$ or isomorphic to a group studied in the previous paragraph. In both cases, $\langle a^2, c, E\rangle$ is monotone. So, it is sufficient to check that the subgroups of the form $\langle a^i t_1, t_2\rangle$, with t_1 and t_2 in $E \times \langle c\rangle$, are metacyclic. Now, $\Omega_1(\langle at_2\rangle) = \Omega_1(\langle a\rangle) = [\langle a, c, E\rangle, \langle a, c, E\rangle]$, and so, $\langle at_1\rangle \trianglelefteq \langle a, c, E\rangle$. It follows that $\langle at_1, t_2\rangle$ is metacyclic, and so the group $\langle a, c, E\rangle$ is monotone. It follows that if $|a^{4h}| = 2$, then G is monotone.

- Suppose that G is in $\mathscr{C}_2$. Then, we have $G = \langle a, b, c\rangle \times A$, where A is elementary abelian, $|c| = 2^n \geq 8$, $|a| = 4$, $a^2 = b^2$, $c^a = c^{1+4h_1}$, $c^b = c^{1+4h_2}$ and $a^b = a^{-1}c^{4h_3}$, with $|c^{4h_i}| \leq 2$ for $i = 1, 2, 3$.
  We have to prove that every 2-generated subgroup is metacyclic. By Lemma 2.5, it is sufficient to prove that the group $\langle a, b, c\rangle$ is monotone. The subgroup $\langle c^{2^{n-2}}, a, b\rangle$ is isomorphic to K_6 and so $\langle c^{2^{n-2}}, a, b\rangle$ is monotone (see Definition 3.1 and Proposition 3.1). Therefore, it is sufficient to check that every subgroup of the form $\langle c^i a^{i_1} b^{i_2}, a^{j_1} b^{j_2}\rangle$, with $|c^i| \geq 8$ is metacyclic. Now, we have $\Omega_1(\langle c^i a^{i_1} b^{i_2}\rangle) = \langle c^{2^{n-1}}\rangle$, and $[c^i a^{i_1} b^{i_2}, a^{j_1} b^{j_2}] = c^{4h_1 i j_1 + 4h_2 j_2 + 4h_3(j_1 i_2 + j_2 i_1)} b^{2i_2 j_1 + 2i_1 j_2}$. We distinguish a few cases depending on the parity of j_1 and of j_2.
  If j_1 and j_2 are even, then the subgroup $\langle c^i a^{i_1} b^{i_2}, a^{j_1} b^{j_2}\rangle$ is abelian, and so it is metacyclic.

  If j_1 is odd and j_2 is even, then, since $a^2 = b^2$, we may assume that $j_1 = 1$ and $j_2 = 0$. Up to replacing $c^i a^{i_1} b^{i_2}$ with $c^i a^{i_1} b^{i_2} a^{-i_1}$, we may assume that $i_1 = 0$ and that the subgroup is of the form $\langle c^i b^{i_2}, a\rangle$. Now, $[c^i b^{i_2}, a] = c^{4h_1 i} b^{2i_2} c^{4h_3 i_2}$.
  If i_2 is even, we have that $[c^i b^{i_2}, a] \in \Omega_1(\langle c^i b^{i_2}\rangle)$, i.e. $\langle c^i b^{i_2}\rangle \trianglelefteq \langle c^i b^{i_2}, a\rangle$ and so the subgroup is metacyclic.
  If i_2 is odd and $c^{h_1 i + 4h_3 i_2} = 1$, then, $\langle a\rangle \trianglelefteq \langle c^i b^{i_2}, a\rangle$ and so the subgroup is metacyclic.
  Suppose i_2 odd, and $|c^{h_1 i + 4h_3 i_2}| = 2$. We have $\Omega_2(\langle c^i b^{j_2}\rangle) = \langle c^{2^{n-2}} b^{2j_3}\rangle$, where if $|c^i| = 8$, then $j_3 = 1$ and if $|c^i| > 8$, then $j_3 = 0$. Therefore,

$ac^{2^{n-2}}b^{2j_3}$ is such that $\langle ac^{2^{n-2}}b^{2j_3}\rangle \trianglelefteq \langle c^ib^{i_2}, a\rangle$ and $\langle ac^{2^{n-2}}b^{2j_3}, c^ib^{i_2}\rangle = \langle a, c^ib^{i_2}\rangle$, i.e. the subgroup is metacyclic.

If j_2 is odd and j_1 is even, then, since $a^2 = b^2$, we may assume that $j_2 = 1$ and $j_1 = 0$. We may swap a with b and, using the same argument of the previous paragraph, we get that $\langle c^ia^{i_1}, b\rangle$ is metacyclic.

Suppose now that j_1 and j_2 are even. Up to replacing in case $c^ia^{i_1}b^{i_2}$ with $c^ia^{i_1}b^{i_2}a^{j_1}b^{j_2}$, we may assume that i_1 is even and since $a^2 = b^2$, we may assume that $i_1 = 0$. Then, the subgroup is of the form $\langle c^ib^{i_2}, a^{j_1}b^{j_2}\rangle$. Now, $[c^ib^{i_2}, a^{j_1}b^{j_2}] = c^{4h_1ij_1+4h_2ij_2}b^{2i_2}c^{4h_3i_2}$.

If i_2 is even, we have that $[c^ib^{i_2}, a^{j_1}b^{j_2}] \in \Omega_1(\langle c^ib^{i_2}\rangle)$, i.e. $\langle c^ib^{i_2}\rangle \trianglelefteq \langle c^ib^{i_2}, a^{j_1}b^{j_2}\rangle$ and so the subgroup is metacyclic.

Suppose that i_2 is odd. Then, $[c^ib^{i_2}, a^{j_1}b^{j_2}] = c^{4h_1ij_1+4h_2ij_2}b^2c^{4h_3i_2}$.
Suppose that $|c^{4h_1ij_1+4h_2ij_2+4h_3i_2}| = 2$. If $|c^{4h_3}| = 2$, then $\langle ab\rangle \trianglelefteq \langle c^ib^{i_2}, a\rangle$ and so the subgroup is metacyclic. If $c^{4h_3} = 1$, then we get $\Omega_2(\langle c^ib^{j_2}\rangle) = \langle c^{2^{n-2}}b^{2i_3}\rangle$, where if $|c^i| = 8$, then $j_3 = 1$ and if $|c^i| > 8$, then $j_3 = 0$. Therefore, $a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}$ is such that $\langle a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}\rangle \trianglelefteq \langle c^ib^{i_2}, a^{j_1}b^{j_2}\rangle$ and $\langle a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}, c^ib^{i_2}\rangle = \langle a^{j_1}b^{j_2}, c^ib^{i_2}\rangle$, i.e. the subgroup is metacyclic.
Suppose i_2 odd, and $c^{4h_1ij_1+4h_2ij_2+4h_3i_2} = 1$. If $|c^{4h_3}| = 2$, then $\Omega_2(\langle c^ib^{j_2}\rangle) = \langle c^{2^{n-2}}b^{2i_3}\rangle$, where if $|c^i| = 8$, then $j_3 = 1$ and if $|c^i| > 8$, then $j_3 = 0$. Therefore, $a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}$ is such that $\langle a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}\rangle \trianglelefteq \langle c^ib^{i_2}, a^{j_1}b^{j_2}\rangle$ and $\langle a^{j_1}b^{j_2}c^{2^{n-2}}b^{2j_3}, c^ib^{i_2}\rangle = \langle a^{j_1}b^{j_2}, c^ib^{i_2}\rangle$, i.e. the subgroup is metacyclic.
If $c^{4h_3} = 1$, then $\langle ab\rangle \trianglelefteq \langle cb^{i_2}, ab\rangle$ and so the subgroup is metacyclic.

This concludes the proof that every 2-generated subgroup of $\langle a, b, c\rangle$ is metacyclic, and so G is monotone.

- Suppose that G is in $\mathscr{C}_3$. Then, we have that $G = \langle a, b, c, d\rangle \times A$, where A is elementary abelian, $|c| = 2^n \geq 8$, $|a| = 4$, $a^2 = b^2$, $|d| = 2$, $c^a = c$, $c^b = c$ and $a^b = a^{-1}c^{4h}$ with $|c^{4h}| \leq 2$, and $c^d = c^{1+2^{n-1}}$, $a^d = a$ and $b^d = b$.

  By Lemma 2.5, it is sufficient to prove that the group $\langle a, b, c, d\rangle$ is monotone. Since $\langle a, b, c\rangle$ is a group in $\mathscr{C}_2$, it is sufficient to check the subgroups of the form $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$.
  If j_1 is odd, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle \leq \langle b, c, ad\rangle$. Since $\langle b, c, ad\rangle$ lies in $\mathscr{C}_2$, the group $\langle b, c, ad\rangle$ is monotone and so $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is

metacyclic.

If j_3 is odd, then $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle \leq \langle bd, c, a\rangle$. Since $\langle bd, c, a\rangle$ lies in $\mathscr{C}_2$, the group $\langle bd, c, a\rangle$ is monotone and so $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic.

Suppose that both j_1 and j_3 are even. Now, $[a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}] = c^{2^{n-1}i_2}$. So, if $|c^{i_2}| \geq 8$, then $\Omega_1(\langle a^{i_1}c^{i_2}b^{i_3}\rangle) = \langle c^{2^{n-1}}\rangle$ and so $\langle a^{i_1}c^{i_2}b^{i_3}\rangle \trianglelefteq \langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$. This means that $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is metacyclic. If $|c^{i_2}| \leq 4$, then i_2 is even, and so $\langle a^{i_1}c^{i_2}b^{i_3}, da^{j_1}c^{j_2}b^{j_3}\rangle$ is abelian.

Hence, every 2-generated subgroup of $\langle a, b, d, c\rangle$ is metacyclic and so $\langle a, b, c, d\rangle$ is monotone. It follows that G is a monotone group.

- Suppose that G is in $\mathscr{C}_4$. Then, we have that $G = \langle a, b, c, d\rangle * E \times A$, where A is elementary abelian, E is extraspecial, $|a| = 2^n \geq 8$, $b^4 = a^{2^{n-1}}$, $\langle c, d\rangle$ is elementary abelian, $a^b = a^{-1+4h}$, with $|a^{4h}| \leq 2$, and $a^c = a^{1+4h_1}$, $a^d = a$, $b^c = b$, $b^d = b^{1+4h_2}$, where $|a^{4h_1}| \leq 2$, $|b^{4h_2}| \leq 2$, and $\langle a^{2^{n-1}}\rangle = E^2$.

  The subgroup $\langle a, b^2, c, d, E, A\rangle$ is in $\mathscr{C}_1$, and so it is monotone. So, it is sufficient to check that the subgroups of the form $\langle ba^{i_1}t_1, a^{i_2}t_2\rangle$ with t_1 and t_2 in $\langle E, c, d\rangle$ are metacyclic. We have that $[b, \langle E, c, d\rangle] \leq \langle b^4\rangle$, $[a, \langle E, c, d\rangle] \leq \langle b^4\rangle$, and $\langle E, c, d\rangle^2 = \langle b^4\rangle$.

  We have $(ba^{i_1}t_1)^2 = b^2a^{-i_1+4hi_1}t_1a^{i_1}t_1[t_1, b] = b^2a^{4hi_1}t_1^2[t_1, b][t_1, a^{i_1}] = b^2b^{4v}$, for some v.

  Moreover, we have that
  $$\begin{aligned}(a^{i_2}t_2)^{ba^{i_1}t_1} &= (a^{i_2(-1+4h)}t_2[t_2, b])^{a^{i_1}t_1}\\ &= a^{i_2(-1+4h)}[a^{i_2(-1+4h)}, t_2]t_2[t_2, a^{i_1}][t_2, t_1]\\ &= (a^{i_2}t_2)^{-1+4h}b^{4u} \text{ for some } u.\end{aligned}$$
  Since $(a^{i_2}t_2(ba^{i_1}t_1)^2)^{ba^{i_1}t_1} = (a^{i_2}t_2(ba^{i_1}t_1)^2)^{-1+4h}$, we have that the subgroup $\langle ba^{i_1}t_1, a^{i_2}t_2\rangle$ is metacyclic. Therefore, the group $\langle a, b, c, d\rangle * E$, where $\langle a^{2^{n-1}}\rangle = E^2$, is monotone, and so is G.

- Suppose that G is in $\mathscr{C}_5$. Then, Lemma 2.6 proves that G is a monotone group.

- Suppose that G is in $\mathscr{C}_6$. Then, $\langle A, c, b\rangle$, where A is an abelian group of exponent 2^n, with $n \geq 3$, $A^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $|b| \geq 8$ and $a^b = a^{-1+4h}$, $a^c = a^{1+2^{n-1}}$ for every $a \in A$, $c^b = c^{-1+4h}$ and $exp(\langle A, c\rangle^{4h}) < |b^2| < 2^n$.

We first prove that the subgroup $\langle A, c, b^2\rangle$ is monotone. Since $\langle A, c\rangle$ is modular, it is enough to check that the subgroups of the form $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ with a_1 and a_2 in A, are metacyclic. Since $a^b = a^{-1+4h}$ and $c^b = c^{-1+4h}$, we have that $a^{b^{2s}} = a^{1+4k}$ and $c^{b^{2s}} = c^{1+4k}$, where $1+4k = (-1+4h)^{2s}$. Now, $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}a_2^{2^{n-1}i_1}a_1^{2^{n-1}i_2}$. Now, if $|a_1| < 2^n$ and $|a_2| < 2^n$, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.
If i_1 and i_2 are even, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.

If $|a_1| = 2^n$, $|a_2| = 2^n$, i_2 and i_1 are odd, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.

Suppose that $|a_1| = 2^n$ and i_2 is odd. Then, we may assume that $|a_2| < 2^n$ or i_1 is even.

Suppose first that i_1 is odd and $|a_2| < 2^n$. We may assume that $i_1 = 1$. Now, $b^{2s}a_2c^{i_2}(a_1c)^{-i_2} = b^{2s}a_2a_1^{1+2^{n-1}}$. Since $\langle A, b^2\rangle$ is modular, $|b^{2s}| < |a_1|$ and $|a_2| < |a_1|$, we have that $|b^{2s}a_2a_1^{1+2^{n-1}}| = 2^n$, and $\Omega_1(\langle b^{2s}a_2a_1^{1+2^{n-1}}\rangle) = \Omega_1(\langle a_1\rangle)$. Therefore, we have that $a_1^{2^{n-1}} \in \langle b^{2s}a_2a_1^{1+2^{n-1}}\rangle^4$. This means that $[a_1c^{i_1}, b^{2s}a_2c^{i_2}]$ lies in the subgroup $\langle (a_1c^{i_1})^4, (b^{2s}a_2a_1^{1+2^{n-1}})^4\rangle$ which is contained in $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

Suppose now that i_1 is even. We have $(b^{2s}a_2c)^2 = b^{4s}a_2^{2+4k+2^{n-1}}c^{2+4k}$. Now, the subgroup $\langle A, c^2, b^2\rangle$ is modular, and so we get that $(b^{2s}a_2c)^{i_1} = b^{2si_1}a_2^{j_1}c^{j_2}$, for some j_1 and j_2 with $|c^{j_2}| = |c^{i_1}|$ and j_1 is even. Moreover, there exists i_3 odd such that $(a_1c^{i_1})^{i_3} = a_1^{i_3}c^{-j_2}$ and so we have that $(b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3} = b^{2si_1}a_2^{j_1}a_1^{i_3}$. Hence, we obtain that $|(b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}| = |a_1|$ and $\Omega_1(\langle (b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}\rangle) = \Omega_1(\langle a_1\rangle)$. Therefore, we have that $a_1^{2^{n-1}} \in \langle (b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}\rangle^4$. This means that $[a_1c^{i_1}, b^{2s}a_2c^{i_2}] \in \langle (a_1c^{i_1})^4, ((b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3})^4\rangle \leq \langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

Suppose now that $|a_2| = 2^n$ and i_1 is odd. We may assume that $i_1 = 1$ and up to replacing $ba_2c^{i_2}$ with $(a_1c)^{-i_2}$, we may assume that $i_2 = 0$. Therefore, the subgroup has the form $\langle a_1c^{i_1}, b^{2s}a_2\rangle$. Now, $|b^{2s}a_2| = 2^n \geq 8$, and moreover, $\Omega_1(\langle b^{2s}a_2\rangle) = \Omega_1(\langle a_2\rangle)$. This means that

$[a_1c^{i_1}, b^{2s}a_2c^{i_2}] \in \langle (a_1c^{i_1})^4, ((b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3})^4\rangle \leq \langle a_1c^{i_1}, b^{2s}a_2\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

This shows that $\langle A, c, b^2\rangle$ is monotone. Hence, in order to prove that $\langle A, c, b\rangle$ is monotone, we have to check that the subgroups of the form $\langle bx_1, x_2\rangle$, where x_1 and x_2 are in $\langle A, c\rangle$, are metacyclic. Put $|b| = 2^k$. We have that $[\langle A, c\rangle, \langle A, c\rangle] = \langle b^{2^{k-1}}\rangle$.
We have that $x_2^b = x_2^{-1+4h}b^{2^{k-1}u}$, for some u. In fact we have that $x_2 = ac^i$ with $a \in A$, and $(ac^i)^b = a^{-1+4h}c^{i(-1+4h)} = (ac^i)^{-1+4h}[a, c^i] = (ac^i)^{-1+4h}b^{2^{k-1}u}$. Moreover, $[\langle A, c\rangle, \langle A, c\rangle] = A^{2^{n-1}} = \Omega_1(\langle b\rangle)$.
So we have that $x_2^{bx_1} = x_2^{-1+4h}b^{2^{k-1}v}$, for some $v \in \{0, 1\}$. Now, $(bx_2)^2 = b^2x_2^{4h}b^{2^{k-1}}$ and, since $|b^2| > exp(\langle A, c\rangle)^{4h}$, we have that $\Omega_1(\langle bx_2\rangle) = \Omega_1(\langle b\rangle)$ and $|bx_2| = |b|$. Therefore, $\langle x_2(bx_1)^{2^{k-2}v}\rangle$ is normal in $\langle x_1, bx_2\rangle$ and so the subgroup is metacyclic.

- Suppose that G is in $\mathscr{C}_7$. Then $G = \langle A, c, b\rangle$ where A is an abelian group of exponent 2^n, with $n \geq 3$, $A^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $|b| \geq 8$ and $a^b = a^{-1+4h+2^{n-1}}$, $a^c = a^{1+2^{n-1}}$ for every $a \in A$, $|c| > 2^m$, $c^b = c^{-1+4h}$ and $|c^{4h}| = |b^2|$, $|b^2| < 2^m$ and $\langle b\rangle \cap \langle c\rangle = 1$.

  We first prove that the subgroup $\langle A, c, b^2\rangle$ is monotone. Since $\langle A, c\rangle$ is modular, it is enough to check that the subgroups of the form $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ with a_1 and a_2 in A, are metacyclic. Since for every $a \in A$ we have that $a^b = a^{-1+4h+2^{n-1}}$ and $c^b = c^{-1+4h}$ and $2^n = exp(A)$, we have that $a^{b^{2s}} = a^{1+4k}$ and $c^{b^{2s}} = c^{1+4k}$. Now, $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}a_2^{2^{n-1}i_1}a_1^{2^{n-1}i_2}$. Now, if $|a_1| < 2^n$ and $|a_2| < 2^n$, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.

  If i_1 and i_2 is even, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.

  If $|a_1| = 2^n$, $|a_2| = 2^n$, i_2 and i_1 are odd, then $(a_1c^{i_1})^{b^{2s}a_2c^{i_2}} = (a_1c^{i_1})^{1+4k}$, and so the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is metacyclic.

  Suppose that $|a_1| = 2^n$ and i_2 is odd. Then we may assume that $|a_2| < 2^n$ or i_1 is even.

  Suppose first that i_1 is odd and $|a_2| < 2^n$. We may assume that $i_1 = 1$. Now, $b^{2s}a_2c^{i_2}(a_1c)^{-i_2} = b^{2s}a_2a_1^{1+2^{n-1}}$. Since $\langle A, b^2\rangle$ is modular, $|b^{2s}| < |a_1|$ and $|a_2| < |a_1|$, we have that $|b^{2s}a_2a_1^{1+2^{n-1}}| = 2^n$, and $\Omega_1(\langle b^{2s}a_2a_1^{1+2^{n-1}}\rangle) = \Omega_1(\langle a_1\rangle)$. Therefore, we have that $a_1^{2^{n-1}} \in \langle b^{2s}a_2a_1^{1+2^{n-1}}\rangle^4$. This means that $[a_1c^{i_1}, b^{2s}a_2c^{i_2}]$ lyes in the subgroup $\langle (a_1c^{i_1})^4, (b^{2s}a_2a_1^{1+2^{n-1}})^4\rangle$ which is contained in $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

Suppose now that i_1 is even. We have $(b^{2s}a_2c)^2 = b^{4s}a_2^{2+4k+2^{n-1}}c^{2+4k}$. Now, $\langle b^2, A, c^2\rangle$ is modular, and so $(b^{2s}a_2c)^{i_1} = b^{2si_1}a_2^{j_1}c^{j_2}$, for some j_1 and j_2 where $|c^{j_2}| = |c^{i_1}|$ and j_1 is even . Moreover, there exists i_3 odd such that $(a_1c^{i_1})^{i_3} = a_1^{i_3}c^{-j_2}$ and so we have that $(b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3} = b^{2si_1}a_2^{j_1}a_1^{i_3}$. Hence, we obtain that $|(b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}| = |a_1|$. Moreover, $\Omega_1(\langle (b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}\rangle) = \Omega_1(\langle a_1\rangle)$. Therefore, we have that $a_1^{2^{n-1}} \in \langle (b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3}\rangle^4$. This means that $[a_1c^{i_1}, b^{2s}a_2c^{i_2}]$ lies in the subgroup $\langle (a_1c^{i_1})^4, ((b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3})^4\rangle$ which is contained in $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

Suppose now that $|a_2| = 2^n$ and i_1 is odd. We may assume that $i_1 = 1$ and up to replacing $ba_2c^{i_2}$ with $(a_1c)^{-i_2}$, we may assume that $i_2 = 0$. Therefore, the subgroup has the form $\langle a_1c^{i_1}, b^{2s}a_2\rangle$. Now, $|b^{2s}a_2| = 2^n \geq 8$, and moreover, $\Omega_1(\langle b^{2s}a_2\rangle) = \Omega_1(\langle a_2\rangle)$. This means that $[a_1c^{i_1}, b^{2s}a_2c^{i_2}] \in \langle (a_1c^{i_1})^4, ((b^{2s}a_2c)^{i_1}(a_1c^{i_1})^{i_3})^4\rangle \leq \langle a_1c^{i_1}, b^{2s}a_2\rangle^4$. Therefore, the subgroup $\langle a_1c^{i_1}, b^{2s}a_2c^{i_2}\rangle$ is powerful and 2-generated. Hence, it is metacyclic.

This shows that $\langle A, c, b^2\rangle$ is monotone. Hence, in order to prove that $\langle A, c, b\rangle$ is monotone, we have to check that the subgroups of the form $\langle bc^{i_1}a_1, c^{i_2}a_2\rangle$, where a_1 and a_2 are in A, are metacyclic. Put $|b| = 2^k$. We distinguish various cases.
Suppose that i_2 is odd. Then, we may assume $i_2 = 1$, and, up to replacing $bc^{i_1}a_1$ with $(bc^{i_1}a_1)(ca_2)^{-i_1}$, we may assume that $i_1 = 0$. Therefore, the subgroup is of the form $\langle ba_1, ca_2\rangle$. We have $(ca_2)^{ba_1} = (ca_2)^{-1+4r}a_1^{2^{n-1}}$.
So, if $|a_1| < 2^n$, then $\langle ca_2\rangle \trianglelefteq \langle ba_1, ca_2\rangle$ and we are done.
If $|a_1| = 2^n$, then, since $\Omega_1(\langle a_1\rangle) = \Omega_1(\langle b\rangle)$ and $|b^2| < |a_1^{4h+2^{n-1}}|$, we have that $\Omega_2(\langle ba_1\rangle) = \langle b^{2^{k-2}}\rangle$ and $\langle ca_2b^{2^{k-2}}\rangle \trianglelefteq \langle ba_1, ca_2\rangle$. Therefore, in this case the subgroup is metacyclic.
Suppose that i_2 is even. If also i_1 is even, then we have that $\Omega_1(\langle bc^{i_2}a_2\rangle) = \Omega_1(\langle b\rangle)$ and $(c^{i_2}a_2)^{bc^{i_1}a_1} = (c^{i_2}a_2)^{-1+4r}a_2^{2^{n-2}}$.

If $|a_2| < 2^n$, then $\langle c^{i_2}a_2\rangle \trianglelefteq \langle bc^{i_1}a_1, c^{i_2}a_2\rangle$ and we are done.
If $|a_2| = 2^n$, then $\langle ca_2(bc^{i_1}a_1)^{2^{k-2}}\rangle \trianglelefteq \langle ba_1, ca_2\rangle$ and so the subgroup $\langle bc^{i_1}a_1, c^{i_2}a_2\rangle = \langle bc^{i_1}a_1, ca_2(bc^{i_1}a_1)^{2^{k-2}}\rangle$ is metacyclic.
Suppose that i_2 is even, and i_1 is odd. Then, we may assume that $i_1 = 1$. If $|a_1| = 2^n$, we have that $(c^{i_2}a_1)^{bca_2} = (c^{i_2})^{-1+4h}a_2^{-1+4h+2^{n-1}}a_2^{2^{n-1}} = (c^{i_2}a_2)^{-1+4h}$. If $|a_1| < 2^n$, then we have that $(c^{i_2}a_1)^{bca_2} = (c^{i_2})^{-1+4h}a_1^{-1+4h} = (c^{i_2}a_2)(-1+4h)$. In both cases, we get that $\langle c^{i_2}a_2\rangle \trianglelefteq \langle bc^{i_1}a_1, c^{i_2}a_2\rangle$.
This shows that G is monotone.

□

The aim of this chapter is to prove the following theorem.

Theorem 5.2. *Let G be a monotone 2-groups of exponent greater than or equal to 4 such that $G = H_4(G)$. Then G is either a modular group that does not involve Q_8 or is in $\mathscr{C}_i$, for some $i \in \{1, \dots, 7\}$.*

First of all, we want to stress the following fact.

Remark 5.3. *Let G be a non-trivial monotone 2-group Then, the quotient G/G^4 is a monotone 2-group of exponent 4. Therefore, the group G/G^4 is isomorphic to one of the groups listed in Theorem 3.3.*

5.1 Monotone 2-Groups with $H_4(G) = G$ and G/G^4 abelian

Let G be a non-trivial monotone 2-group, such that $H_4(G) = G$ and such that G/G^4 is abelian. In the following proposition, we determine such groups when $|G^2/G^4| = 2$.

Proposition 5.4. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$.*
Suppose that the quotient G/G^4 is abelian and $|G^2/G^4| = 2$. Then G is either abelian or isomorphic to a group in $\mathscr{C}_1$.

Proof. Let $G = \langle a, c_1, \dots, c_s\rangle$, where $|aG^4| = 4$, and $|c_iG^4| = 2$ for $i \in \{1, \dots, s\}$ and G/G^4 abelian.

First of all we prove that we may assume that c_i has order 2, and that the derived subgroup of G is contained in $\Omega_1(\langle a\rangle)$.

The group G is powerful, because G/G^4 is abelian. In particular, we have $G^2 = \Phi(G^2)$. Since $G^2 = \langle a^2 \rangle G^4$, we have that G^2 is cyclic and generated by $\langle a^2 \rangle$. Since G is powerful and $G^2 = \langle a^2 \rangle$, we have $G^{2^i} = \langle a^{2^i} \rangle$ and so $|a| = 2^n$, where $exp(G) = 2^n$.

Now, from $c_j^2 \in G^4$ and $[a, c_i] \in G^4$, it follows that $c_i^2 = a^{4r_i}$ and $a^{c_i} = a^{1+4k_i}$. Since c_i^2 is in the centralizer of a, we get that $|a^{4k_i}| \leq 2$. In particular, a^2 commutes with c_i for every i, and so $\langle a^2 \rangle \leq Z(G)$. Replacing c_i with $c_i a^{-2r_i}$, we may assume that c_i has order 2, for every i.

Since $[c_i, c_j] \in \langle a^4 \rangle$ and $a^4 \in Z(G)$, we have that $c_i^{c_j} = c_i a^{4k_{ij}}$, with $|a^{4k_{ij}}| \leq 2$. So, the subgroup $\langle c_i, c_j \rangle$ is either abelian or isomorphic to D_8.

Therefore, we may assume $|c_i| = 2$, $a^{c_i} = a^{1+4k_i}$ with $|a^{4k_i}| \leq 2$, $\langle a \rangle \trianglelefteq G$ and $c_i^{c_j} = c_i a^{4k_{ij}}$ with $|a^{4k_{ij}}| \leq 2$. By Remark 2.1, we get that $[G, G] \leq \Omega_1(\langle a \rangle)$.
Moreover, the previous argument also shows that $|G : C_G(c_i)| \leq 2$.

We now prove the proposition by induction on the order of G.

Suppose that $\langle c_1, \ldots, c_s \rangle$ is abelian. If $[a, c_i] = 1$ for every i, then the group G is abelian. Suppose there exists i such that $[a, c_i] \neq 1$. Then, up to reordering the indices, we may assume that $a^{c_1} = a^{1+2^{n-1}}$ and, up to perhaps replacing c_i with $c_i c_1$, we have that G is isomorphic to $\langle a, c_1 \rangle \times \langle c_2, \ldots, c_s \rangle$ with $|a| = 2^n$, $|c_i| = 2$, $a^{c_1} = a^{1+2^{n-1}}$ and $\langle c_1, \ldots, c_r \rangle$ elementary abelian. Hence G is a modular 2-group in $\mathscr{C}_1$.

Suppose now that $\langle c_1, \ldots, c_s \rangle$ is not abelian. Then, up to reordering the indices, we may assume that $\langle c_1, c_2 \rangle$ is non-abelian. Since $[c_1, c_2] \in \langle a^4 \rangle$, and $a^4 \in C_G(\langle c_i \rangle)$ for every i, we get that $c_1^{c_2} = c_1 a^{2^{n-1}}$, i.e. $\langle c_1, c_2 \rangle \simeq D_8$. Since $C = C_G(\langle c_1, c_2 \rangle)$ has index 4 in G, we have that $G = \langle c_1, c_2 \rangle C$. Since $a^{c_1} = a^{1+4h_1}$ and $a^{c_2} = a^{1+4h_2}$ with $|a^{4h_1}| \leq 2$ and $|a^{4h_2}| \leq 2$, we may assume, up to replacing a with $a c_1^{h_2} c_2^{h_1}$, that $a \in C$. Therefore, C is a proper subgroup of G, $G = \langle c_1, c_2 \rangle * C$, where $C^{2^{n-1}} = \langle c_1, c_2 \rangle^2$, and $G^4 = C^4$. Therefore, C is a proper subgroup of G such that C/C^4 is isomorphic to $C_4 \times C_2 \times \cdots \times C_2$. Hence, we can conclude by induction. $\square$

The previous proposition determines the non-trivial monotone 2-groups G such that $H_4(G) = G$, G/G^4 is abelian and $|G^2/G^4| = 2$.
In the next proposition, we deal with the complementary case and we classify the non-trivial monotone 2-groups G such that $H_4(G) = G$, G/G^4 is abelian and $|G^2/G^4| \geq 4$.

Proposition 5.5. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. Suppose that the quotient G/G^4 is abelian with $|G^2/G^4| \geq 4$. Then G is a modular 2-group not involving Q_8.*

Proof. Since, by hypothesis, G/G^4 is abelian, the group G is powerful.

Let G be such that $G/G^4 = \langle a_1G^4\rangle \times \cdots \times \langle a_rG^4\rangle \times \langle c_1G^4\rangle \times \cdots \times \langle c_sG^4\rangle$, where $|a_iG^4| = 4$, and $|c_jG^4| = 2$. We have that $G^2 = \langle a_1^2, \ldots, a_r^2\rangle G^4$. Since G^2 is powerful with $\Phi(G^2) = G^4$ (see Lemma 2.7 and Corollary 2.8), we get that $G^{2^i} = \langle a_1^{2^i}, \cdots, a_r^{2^i}\rangle$. Let H be the group $\langle a_1, \cdots, a_r\rangle$. By Proposition 2.14, H is modular and does not involve Q_8. Moreover, $H^{2^i} = G^{2^i}$ for all $i \geq 1$, and, in particular, we have that $exp(H) = exp(G) = 2^n \geq 8$.

We now prove that we may assume $|c_i| = 2$ for every i. Since $c_i^2 \in H^4$ and H is powerful with $H^2 = G^2$ and $H^4 = G^4$, there exists $b_i \in H\backslash H^2$ with $b_i^{2^r} = c_i^2$ with $r \geq 2$. If $|b_i| = 4$, then $c_i^2 = 1$. Hence, we may assume that $|b_i| = 2^{n_i} \geq 8$. Now, $\langle b_i^{2^r}\rangle$ is a central subgroup of $\langle b_i, c_i\rangle$, and the quotient $\langle b_i, c_i\rangle/\langle b_i^{2^r}\rangle$ is a metacyclic group with c_i generator of order 2 and b_i generator of order greater than or equal to 4. Moreover, since $\langle b_i, c_i\rangle$ has a quotient isomorphic to $C_4 \times C_2$, we have that $\langle b_i, c_i\rangle/\langle b_i^{2^r}\rangle$ is not semidihedral. By Lemma 2.11, we get that $c_i\langle b_i^{2^r}\rangle$ is in the normalizer of $\langle b_i\rangle/\langle b_i^{2^r}\rangle$. This implies that $\langle b_i\rangle$ is normalized by c_i. If $b_i^{c_i} = b_i^{-1+4h_i}$, then b_i^2 is in the derived subgroup of G, a contradiction, because $b_i \in H\backslash H^2$ and so $b_i^2 \in H^2\backslash H^4$, but $[G, G] \leq H^4$. So, we have that $b_i^{c_i} = b_i^{1+4h_i}$, and since $c_i^2 \in \langle b_i\rangle$, we get that $|b_i^{4h_i}| \leq 2$. In particular, since $[b_i^2, c_i] = 1$ and $c_i^2 = b_i^{2^r}$ with $r \geq 2$, replacing c_i with $c_ib_i^{-2^{r-1}}$, we may assume that $|c_i| = 2$, and the claim is proved.

We now prove that the subgroup $\langle c_i, c_j\rangle$ is abelian.
Suppose that $\langle c_i, c_j\rangle$ is non-abelian. Then, $\langle c_i, c_j\rangle$ is dihedral. Moreover $[c_i, c_j]^{c_i} = [c_i, c_j]^{c_j} = [c_i, c_j]^{-1}$. Since $[c_i, c_j] \in H^4$, and H is powerful, there exists $a \in H\backslash H^2$, such that $a^{2^r} = [c_i, c_j]$, with $r \geq 2$. Since $|[c_i, c_j]| \geq 2$, we have that $|a| = 2^n \geq 8$.
Since $\langle a, c_i\rangle H^4/H^4$ is isomorphic to $C_4 \times C_2$, the subgroup $\langle a, c_i\rangle$ cannot be semi-dihedral. Hence, by Lemma 2.12, we get that $\langle a\rangle$ is normalized by c_i. If $a^{c_i} = a^{-1+4h}$, then $a^2 \in [G, G]$, a contradiction because $a^2 \in H^2\backslash H^4$ and $[G, G] \leq G^4 = H^4$. Then, we get that $a^{c_i} = a^{1+4h}$, and since ${c_i}^2 = 1$, we get that $|a^{4h}| \leq 2$. Now, since $[c_i, c_j] \in \langle a\rangle$ and $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$, we get that $\langle c_i, c_j\rangle$ is isomorphic to a D_8 with $\langle c_i, c_j\rangle \cap H \leq \langle a\rangle$. Now, since H is modular and does not involve Q_8, by Proposition 2.5.9 on page 94 of

[13], H is lattice isomorphic to an abelian group. In particular, we have $[H^2 : H^4] = |\Omega_1(H^2)|$. Since $|\Omega_1(H^2)| \geq 4$, there exists an involution c with $c \in H^2$ and $c \notin \langle c_i, c_j \rangle$. Since $c \in H^2$, there exists $d \in H \backslash H^2$, with $d^{2^l} = c$. Since $\langle d, c_i \rangle H^4/H^4$ is isomorphic to $C_4 \times C_2$, the subgroup $\langle d, c_i \rangle$ cannot be semi-dihedral. Hence, by Lemma 2.12, we get that $\langle d \rangle$ is normalized by c_i. If $d^{c_i} = d^{-1+4k}$, then $d^2 \in [G, G]$, a contradiction because $d^2 \in H^2 \backslash H^4$ and $[G, G] \leq G^4 = H^4$. Then, we get that $d^{c_i} = d^{1+4k_i}$, and since ${c_i}^2 = 1$, we get that $|d^{4k_i}| \leq 2$. Using the same argument with c_j instead of c_i, we get that $d^{c_j} = d^{1+4k_j}$ with $|d^{4k_j}| \leq 2$. In particular, c_i and c_j are in the $C_G(\Omega_2(\langle d \rangle))$, and so G contains the subgroup $\Omega_2(\langle d \rangle) \times \langle c_i, c_j \rangle$. Since $\Omega_2(\langle d \rangle) \times \langle c_i, c_j \rangle$ is isomorphic to $C_4 \times D_8$, which is not monotone, we have a contradiction. Therefore $\langle c_i, c_j \rangle$ is abelian for every i and j and so $\langle c_1, \cdots, c_s \rangle$ is abelian.

We now show that c_i is in the normalizer of every element of H and in the centralizer of $\Omega_2(H)$.
Let $b \in H$. Since H is powerful, there exists $a \in H \backslash H^2$ such that $a^{2^r} = b$ for some r. In order to show that $c \in N_G(\langle b \rangle)$, it is sufficient to show that c_i normalizes $\langle a \rangle$.
Now, $\langle a, c_i \rangle$ is a metacyclic subgroup with a generator of order 2. Moreover since $\langle a, c_i \rangle G^4/G^4 \simeq C_4 \times C_2$, we get that $\langle a, c_i \rangle$ is not semidihedral. By Lemma 2.12, we have that c_i is in the normalizer of every cyclic subgroup having order at least 4 of $\langle a, c_i \rangle$. In particular, $c \in N_G(\langle a \rangle)$.
Moreover, since $a^2 \in H^2 \backslash H^4$ and $[G, G] \leq H^4$, we have that $\langle a, c_i \rangle$ is modular metacyclic and $a^{c_i} = a^{1+4t}$, with $|a^{4t}| \leq 2$. So, we have that $[a^2, c_i] = 1$, for each $a \in H \backslash H^2$ and since $\{a^2 : a \in H \backslash H^2\}$ is a generating set for H^2, we have that c_i centralizes H^2. In particular, if $b \in H^2$ and has order 4, then it is centralized by c_i. Moreover, if b has order 4 and is in $H \backslash H^2$, then the argument just used shows that $b^c = b$. Hence, the element c_i is in $C_G(\Omega_2(H))$, for all i.

We now study the action of c_i on H.
If $[H, c_i] = 1$ for every i, then $G = H \times \langle c_1 \rangle \times \cdots \times \langle c_s \rangle$. Since H is modular without subgroups isomorphic to Q_8, then G is modular without Q_8.
Suppose that there exists a c_i such that $[c_i, H] \neq 1$. Since c_i is in the normalizer of every element of H and $c_i^2 = 1$, we have that c_i acts as a power automorphism of order 2 on H. The group H is modular and does not involve a subgroup isomorphic to Q_8. Moreover $[c_i, \Omega_2(H)] = 1$. Hence, by Theorem 2.3.24 on page 68 of [13], we have that c_i acts as an universal automorphism,

i.e. $h^{c_i} = h^{1+2^{n-1}}$ for every $h \in H$ with $2^n = exp(G) = exp(H)$. Up to reordering the indices, we may assume that $h^{c_1} = h^{1+2^{n-1}}$ and, up to replacing perhaps c_i with $c_i c_1$ for $i \geq 2$, we may assume that $[H, c_i] = 1$ for $i \geq 2$.
If H is abelian, then $G = (H \times \langle c_2 \rangle \times \cdots \times \langle c_s \rangle) \rtimes \langle c_1 \rangle$, with $a^{c_1} = a^{1+2^{n-1}}$ for every $a \in H \times \langle c_2 \rangle \times \cdots \langle c_s \rangle$.
If H is modular non-abelian, then $H\langle c_1 \rangle$ satisfies the hypothesis of Lemma 1.3 of [12], and so $H\langle c_1 \rangle$ is a modular subgroup without Q_8. Since $\langle H, c_1 \rangle$ is centralized by the elementary abelian subgroup $\langle c_2, \ldots, c_s \rangle$, we obtain that $G = \langle H, c_1 \rangle \times \langle c_2, \ldots, c_s \rangle$ is modular without subgroup isomorphic to Q_8.

We have proved that if G is a group such that the quotient G/G^4 is abelian and $|G^2/G^4| \geq 4$, then G is a modular group without subgroups isomorphic to Q_8. Hence the proof is complete. □

Summing up, this section shows that a non-trivial monotone 2-group G with $H_4(G) = G$ and G/G^4 abelian is either modular without subgroups isomorphic to Q_8 or is in the class $\mathscr{C}_1$.

5.2 Monotone 2-Groups with $H_4(G) = G$ and G/G^4 non-abelian

In the first seven propositions, we show that if G is a monotone non-trivial 2-group such that $H_4(G) = G$, then G/G^4 can not be isomorphic to a group in $\mathscr{A}_i$ where $i \in \{2, 3, 5, 7, 8, 9, 10, 11\}$ (see Theorem 3.3).

Proposition 5.6. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$.*
The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_2$.

Proof. Let G be $\langle x_1, y_1, \ldots, x_n, y_n, a_1, \ldots, a_m \rangle$, where $\langle x_1, y_1 \rangle G^4/G^4 * \cdots * \langle x_n, y_n \rangle G^4/G^4$ is extraspecial, $\langle x_i, y_i \rangle G^4/G^4$ is isomorphic to D_8 or to Q_8 with $|x_i G^4| = 4$ and $A = \langle a_1 \rangle G^4/G^4 \times \cdots \times \langle a_m \rangle G^4/G^4$ is abelian, with $|a_1 G^4| \leq 4$ and $|a_i G^4| = 2$ for every $i \geq 2$.
We have that $G^2/G^4 = \langle x_1^2 \rangle G^4/G^4$. By Corollary 2.8, we have that $G^4 = \Phi(G^2)$ and so we obtain $G^2 = \langle x_1^2 \rangle$. Moreover, by Corollary 2.8, we also have that $\langle x_1^{2^i} \rangle = G^{2^i}$. Hence, since $exp(G) \geq 8$, we get that $G^4 = \langle x_1^4 \rangle \neq 1$, i.e. $|x_1| \geq 8$.

We first show that G/G^4 cannot involve Q_8.
Suppose that G/G^4 involves a Q_8. Then, up to relabeling the generators of G, we may assume that $\langle x_1, y_1\rangle G^4/G^4$ is isomorphic to Q_8. We have that $x_1^2G^4 = y_1^2G^4$, and $x_1^{y_1} = x_1^{-1}G^4$. This means that $x_1^{2+4h} = y^2$ and $x_1^{y_1} = x_1^{-1+4k}$. In particular, the element y_1 centralizes $\langle x_1^{2+4h}\rangle = \langle x_1^2\rangle$. From $x_1^{y_1} = x_1^{-1+4k}$, we have that $(x_1^2)^{y_1} = x_1^{-2+8k}$ and so $x_1^{-2+8k} = x_1^2$. This means that $x_1^4 = 1$, and so $G^4 = \langle x_1^4\rangle = 1$. It follows that $H_4(G) = 1$, a contradiction. This proves the first claim.

Since $D_8 * D_8 \simeq Q_8 * Q_8$, $D_8 * C_4 \simeq Q_8 * C_4$ and G/G^4 can not involve Q_8, we obtain that $G/G^4 = \langle x, y\rangle G^4/G^4 \times \langle a_1G^4\rangle \times \cdots \times \langle a_mG^4\rangle$, where $\langle x, y\rangle G^4/G^4 \simeq D_8$, with $\langle x\rangle G^4/G^4 \trianglelefteq \langle x, y\rangle G^4/G^4$, and $|a_iG^4| = 2$.
By Corollary 2.8, since $G^2 = \langle x^2\rangle$, we have that $G^{2^i} = \langle x^{2^i}\rangle$ and so $|x| = 2^n$, where $exp(G) = 2^n \geq 8$. Moreover, we have $G^4 = \langle x^4\rangle$.

Since $y^2 \in \langle x^4\rangle$ and $x^y = x^{-1+4h}$, we get that $|x^{4h}| \leq 2$. In particular, since $y^2 \in \langle x^4\rangle$, we have that $|y| \leq 4$.

Let now u be an element of G not in $\langle x, a_1, \ldots, a_m\rangle$. Then, $u \notin G^2 = \langle x^2\rangle$, uG^4 has order 2 and acts as inversion on $\langle x, a_1, \ldots, a_m\rangle G^4/G^4$. Therefore, repeating the argument of the previous paragraph with u instead of y, we get that $|u| \leq 4$. This shows that each element of $G\backslash\langle x, a_1, \cdots, a_m\rangle$ has order at most 4 and this means that $H_4(G)$ is a proper subgroup of G, a contradiction.
In particular, this shows that G/G^4 cannot be in $\mathscr{A}_2$.

□

Proposition 5.7. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_3$.*

Proof. Let G be $\langle a_1, \ldots, a_m, c_1, \ldots, c_s, b\rangle$ with $(\langle a_1G^4\rangle \times \cdots \times \langle a_mG^4\rangle \times \langle c_1G^4\rangle \times \cdots \times \langle c_sG^4\rangle)\langle bG^4\rangle$, where $m \geq 2$, $|a_iG^4| = 4$, $|bG^4| = 2$, $|c_iG^4| = 2$, $(a_iG^4)^{bG^4} = a_i^{-1}G^4$ and $(c_jG^4)^{bG^4} = c_jG^4$, for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, s\}$.

We get that $G^2 = \langle a_1^2, \ldots, a_m^2\rangle G^4$. By Lemma 2.7 and Corollary 2.8, we have that $G^4 = \Phi(G^2)$. So $G^2 = \langle a_1^2, \cdots, a_m^2\rangle$ and $G^{2^i} = \langle a_1^{2^i}, \cdots, a_m^{2^i}\rangle$. So, if $H = \langle a_1, \cdots, a_m, c_1, \ldots, c_s\rangle$, then, we have that H is a monotone 2-group such that H/H^4 is abelian of exponent 4 and $|H^2/H^4| \geq 4$. Therefore, by Proposition 5.5, the group H is modular and does not involve Q_8.

We now show that b is an element of order at most 4.
Since $b^2 \in G^4$, $G^4 = \langle a_1^4, \cdots, a_m^4\rangle$ and $\langle a_1, \cdots, a_m\rangle$ is powerful, there exists

$a \in \langle a_1, \cdots, a_m \rangle \setminus \langle a_1^2, \ldots, a_m^2 \rangle$ such that $a^{2^r} = b^2$. Since $\langle a, b \rangle G^4/G^4$ is isomorphic to D_8 with $\langle a \rangle G^4$ of order 4, we have that $[a, b]G^4 = a^2G^4$. Since $\langle a, b \rangle$ is 2-generated and $(\langle a^2 \rangle G^4) \cap \langle a, b \rangle$ is normal of index 4 in $\langle a, b \rangle$ with elementary abelian quotient, we get that $\langle a^2 \rangle G^4 \cap \langle a, b \rangle = \Phi(\langle a, b \rangle)$. Since $\langle a, b \rangle$ is metacyclic, there exist $c, d \in \langle a, b \rangle$ such that $\langle c, d \rangle = \langle a, b \rangle$ with $\langle c \rangle \trianglelefteq \langle a, b \rangle$. In particular, $\langle c \rangle (G^4 \cap \langle a, b \rangle) \trianglelefteq \langle a, b \rangle (G^4 \cap \langle a, b \rangle)$, and $c \notin \Phi(\langle a, b \rangle) = \langle a^2 \rangle (G^4 \cap \langle a, b \rangle)$. Therefore, $\langle c \rangle (G^4 \cap \langle a, b \rangle) = \langle a \rangle (G^4 \cap \langle a, b \rangle)$. Since $\langle c, b \rangle (G^4 \cap \langle a, b \rangle) = \langle a, b \rangle (G^4 \cap \langle a, b \rangle)$ and $G^4 \cap \langle a, b \rangle \leq \Phi(\langle a, b \rangle)$, we have that $\langle a, b \rangle = \langle c, b \rangle$ with $\langle c \rangle \trianglelefteq \langle c, b \rangle$.
Since $[a, b](G^4 \cap \langle a, b \rangle) = [a, c](G^4 \cap \langle a, b \rangle) = a^2(G^4 \cap \langle a, b \rangle)$, we have that $c^b = c^{-1+4h}$. Now, $a = c^i b^j$, for some i and j. If j is odd then we get that $a^2 \in (G^4 \cap \langle a, b \rangle)$, a contradiction. Then j is even and so, being $b^2 \in \langle a \rangle$, we have $\langle a \rangle = \langle c \rangle$. Then, $c^b = c^{-1+4h}$, with $|c^{4h}| \leq 2$. Moreover, $b^2 \in \Omega_1(\langle c \rangle)$ and so $|b| \leq 4$.

Let d be an element not in H. Then dG^4 has order 2 and acts on H/H^4 as inversion. Therefore, using the same argument with d instead of b, we get that $|d| \leq 4$ for all $d \notin H$. Therefore, $H_4(G)$ is a proper subgroup of G. This final contradiction proves the statement. □

Proposition 5.8. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_5$ or in $\mathscr{A}_8$.*

Proof. Suppose that $G = \langle a_1, \ldots, a_m, c_1, \ldots, c_s, b \rangle$, with $(\langle a_1G^4 \rangle \times \cdots \times \langle a_mG^4 \rangle \times \langle c_1G^4 \rangle \times \cdots \times \langle c_sG^4 \rangle)\langle bG^4 \rangle$, where $m \geq 2$, $|a_iG^4| = 4$, $|bG^4| = 4$ and $b^2G^4 \in \langle a_1, \ldots, a_m, c_1, \ldots, c_s \rangle^2 G^4/G^4$, $|c_iG^4| = 2$, $(a_iG^4)^{bG^4} = a_i^{-1}G^4$ and $(c_jG^4)^{bG^4} = c_jG^4$, for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, s\}$.

Now, $G^2 = \langle a_1^2, \ldots, a_m^2 \rangle G^4$.
By Lemma 2.7 and Corollary 2.8 we have that $G^4 = \Phi(G^2)$ and so $G^2 = \langle a_1^2, \ldots, a_m^2 \rangle$ and $G^{2^i} = \langle a_1^{2^i}, \ldots, a_m^{2^i} \rangle$. So, if $H = \langle a_1, \ldots, a_m, c_1, \ldots, c_s \rangle$, then, we have that H is a monotone 2-group such that H/H^4 is abelian and $|H^2/H^4| \geq 4$.
Therefore, by Proposition 5.5, the group H is modular and does not involve Q_8. Moreover, the subgroup $K = \langle a_1, \ldots, a_m \rangle$ is such that K/K^4 is isomorphic to a direct product of m copies of C_4. Hence, K is powerful with $K^{2^i} = H^{2^i} = G^{2^i}$.
Since $b^2 \in K^2 \backslash K^4$, and K is powerful, there exists $a \in K \setminus K^2$ such that $a^2 = b^2$.

We now show that $|b| \leq 8$.
Since the subgroup K is modular and does not involve Q_8, we have that $[K, K^2] \leq K^8$. Then, for all $h \in K$ we have that $K^8 = [h, a^2]K^8 = [h, b^2]K^8$. Hence $[h, b]^b K^8 = [h, b]^{-1}K^8$, for all $h \in K$. Since K^2 is generated by $\{[h, b] : h \in K\}$ and K^2/K^8 is abelian, we have that b acts as inversion on K^2/K^8. Therefore, since $b^2 \in K^2 \backslash K^4$, we have that $|bK^8| \leq 4$, i.e. $b^4 \in K^8$. Since K is powerful, there exists $c \in K\backslash K^2$ such that $b^4 = a^4 \in \langle c^8 \rangle$. Let c be of maximal order with this property. In particular, we have that $a^4 = b^4 = c^{2^r}$, with $r \geq 3$, and $c^4 \notin K^8$ (otherwise there exists $\overline{c}$ such that $c^4 \in \langle \overline{c}^8 \rangle$, and so $\overline{c}$ contradicts the maximality of c).
Since c and a are in K, the subgroup $\langle a, c \rangle$ is modular and it is not isomorphic to Q_8, and so $\langle a, c \rangle^{2^i} = \langle a^{2^i}, c^{2^i} \rangle$. In particular, the subgroup $\langle a, c \rangle^4 = \langle c^4 \rangle$. Moreover, since $\langle a, c \rangle / \langle a, c \rangle^4$ is abelian (because $\langle a, c \rangle$ is modular), we get that $\langle c \rangle \trianglelefteq \langle a, c \rangle$. Since $a^4 \in \langle c \rangle$, we have that $c^a = c^{1+4h}$, with $|c^{4h}| \leq 4$.
In particular, this implies that $\langle a^2, c^2 \rangle$ is abelian.
We claim that the subgroup $\langle a^2, c^2 \rangle K^4/K^4$ is elementary abelian of order 4. Suppose that $a^2K^4 = c^2K^4$. Then, there exists $q \in K^4$, such that $a^2 = c^2q$. Since $[a^2, c^2] = 1$, we get that $[c^2, q] = 1$. In particular, $c^4q^2 = a^4 = c^{2^r}$, with $r \geq 3$. Since $q \in K^4$ and K is powerful, we have that $q^2 \in K^8$, and so $c^4 = c^{2^r}q^{-2} \in K^8$, which contradicts the choice of c. Therefore, we have that $\langle a^2, c^2 \rangle K^4/K^4$ is elementary abelian of order 4. In particular, since $\langle a^2, c^2 \rangle$ is the Frattini subgroup of $\langle a, c \rangle$, we get that $\langle a^2, c^2 \rangle$ is 2-generated and moreover, $K^4 \cap \langle a, c \rangle = \Phi(\langle a, c \rangle) = \langle c^4 \rangle$.

It follows that $\langle b, c \rangle K^4/K^4$ is as in Lemma 2.13, and $\langle b, c \rangle^4 = \langle c^4 \rangle$. Since $c^b = c^{-1}(K^4 \cap \langle b, c \rangle)$ and $K^4 \cap \langle b, c \rangle = \langle c^4 \rangle$, we have that $\langle c \rangle \trianglelefteq \langle c, b \rangle$ and $c^b = c^{-1+4k}$, with $|c^{4k}| \leq 4$. In particular, being $|\langle b \rangle \cap \langle c \rangle| \leq 2$, we obtain that the order of b is smaller than or equal to 8.

We now prove that if $|b| = 8$, then we reach a contradiction.
Suppose that $|b| = 8$. Since $a^2 = b^2$, we get that $|a| = 8$. Now, since the element ac is such that $(ac)^2 = a^2c^{2+4h}$,we have that $(ac)^2K^2 = a^2c^2K^4$ and $\langle (ac)^4 \rangle = \langle c^4 \rangle$.
In particular, $\langle ac, b \rangle$ satisfies the hypothesis of Lemma 2.13, and $\langle ac \rangle \trianglelefteq \langle ac, b \rangle$. Since $b^4 \in \langle ac \rangle$, we have that $(ac)^b = (ac)^{-1+4t}$, with $|(ac)^{4t}| \leq 4$ i.e. $a^b = a^{-1}c^{4s}$, for some s with $|c^{4s}| \leq 4$.
Now, since $a^2 = b^2$, we have $a = a^{b^2} = (a^{-1}c^{4s})^{-1}(c^{4s})^b = c^{-4s}ac^{-4s}$. Since $\Omega_2(\langle c \rangle) \leq C_G(a)$, we have that $a = ac^{-8s}$, and so $|c^{4s}| \leq 2$.

Being $\Omega_1(\langle c\rangle) = \langle a^4\rangle$, it follows that $\langle a\rangle \trianglelefteq \langle a, b\rangle$ with $a^b = a^{-1+4r}$. Now, from $a^2 = b^2$, we get that $a^2 = (a^2)^b = a^{-2}$ and so $|a| = 4$, a contradiction. Therefore, this case does not arise and $|b| \leq 4$.

If d is an element not in H, then $dH = bH$. Moreover, we have that $d^2G^4 = b^2G^4$, and dG^4 acts as inversion on H/H^4. The argument above shows that $|d| \leq 4$. Hence, we get a contradiction because we obtain that for any d such that $d \notin H$, then $|d| \leq 4$ and so $d \notin H_4(G)$, i.e. $G \neq H_4(G)$ a contradiction. $\square$

Proposition 5.9. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a subgroup in $\mathscr{A}_7$.*

Proof. Suppose that $G = \langle a, b, c, A\rangle$ with AG^4/G^4 elementary abelian and central in G/G^4, $|aG^4| = 4$, $|cG^4| = 4$, $\langle aG^4\rangle \cap \langle cG^4\rangle = G^4$, $b^2G^4 = a^2c^2G^4$ and $a^bG^4 = a^{-1}G^4$, $a^cG^4 = aG^4$ and $b^cG^4 = b^{-1}G^4$.

The subgroup $\langle a, c, A\rangle$ is monotone, being a subgroup of G which is monotone. Moreover, $\langle a, c, A\rangle$ satisfies the hypothesis of Proposition 5.5. Then, $\langle a, c, A\rangle$ is a powerful modular group not involving Q_8. In particular, $\langle a, c\rangle$ is monotone, powerful and $\langle a^2, c^2\rangle G^4 = G^2$. By Corollary 2.8, $G^4 = \Phi(G^2)$ and so $\langle a^2, c^2\rangle = G^2$. Since $b^2G^4 = a^2c^2G^4$, we have that $b^2 = a^{2+4s}c^{2+4r}$, for some r and s. Therefore, up to replacing a and c with suitable powers, we may assume that $b^2 = a^2c^2$.
Since $\langle a, c\rangle$ is modular metacyclic, we have that $[\langle a, c\rangle, \langle a^2, c^2\rangle] \leq \langle a, c\rangle^8$. In particular, we have that $[\langle a, c\rangle, b^2] \leq \langle a^8, b^8\rangle$. Hence, $[a, b^2]G^8 = G^8$ and $[c, b^2]G^8 = G^8$. It follows that $[a, b]^bG^8 = [a, b]^{-1}G^8$ and $[c, b]^bG^8 = [c, b]^{-1}G^8$. Since $\langle [a, b], [c, b]\rangle G^8 = \langle a^2, c^2\rangle G^8$, and $\langle a^2, c^2\rangle G^8/G^8$ is abelian, we have that b acts as inversion on $\langle a^2, c^2\rangle G^8/G^8$. Since $b^2 \in \langle a^2, c^2\rangle$, we get that $|b^2G^8| \leq 2$, i.e. $b^4 \in G^8$. Hence $(a^2c^2)^2 \in G^8$. This implies that $a^4c^4 \in G^8$. Now, the subgroup $G^8 = \Phi(G^4)$ and $G^4 = \langle a^4, c^4\rangle$. Hence, the condition $a^4c^4 \in G^8$ implies that $a^4G^8 = c^{-4}G^8$ and so $G^4 = \langle a^4\rangle = \langle c^4\rangle$.
Therefore, since $a^b \in a^{-1}G^4$, we have that $a^b = a^{-1+4h}$, for some h. Since $b^c = b^{-1}G^4$, we have that $b^c = b^{-1}a^{4k}$, for some k. Now, $(a^2)^b = (a^2)^{-1+4h} = a^{-2+8h}$ and $b^{c^2} = (b^{-1}a^{4k})^{-1}a^{4k} = a^{-4k}ba^{4k} = ba^{-4k(-1+4h)}a^{4k} = ba^{-8k+16hk}$. Since $a^2c^2 = b^2$, we have that $a^2c^2 = (a^2c^2)^b = a^{-2+8h}c^2a^{-8k+16hk} = a^{-2}c^2a^{8h-8k+16hk}$ and so $a^2c^2 = a^{-2}c^2a^{8h-8k+16hk}$. Hence, we obtain that $a^4 \in \langle a^8\rangle$ but, since $G^4 = \langle a^4\rangle$ and $G^8 = \langle a^8\rangle = \Phi(G^4)$, we obtain that $G^4 = 1$, and so also $H_4(G) = 1$, a contradiction. $\square$

Proposition 5.10. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_9$.*

Proof. Suppose that $G = \langle a, b, c, d, A\rangle$ with $|aG^4| = 4$, $|cG^4| = 4$, $\langle aG^4\rangle \cap \langle cG^4\rangle = G^4$, $b^2G^4 = a^2c^2G^4$, $d^2G^4 = c^2G^4$, $a^bG^4 = a^{-1}G^4$, $a^cG^4 = aG^4$, $b^cG^4 = bG^4$, $a^dG^4 = aG^4$, $b^dG^4 = b^{-1}G^4$, $c^dG^4 = cb^2G^4$, AG^4/G^4 is elementary abelian and central in G/G^4.

We have that $\langle a^2, d^2\rangle G^4 = G^2$. Since by Corollary 2.8, we have that $G^4 = \Phi(G^2)$, we get $\langle a^2, d^2\rangle = G^2$. Since $b^2G^4 = a^2d^2G^4$, we have $b^2 = a^{2+4s}d^{2+4r}$, for some r and s. Therefore, up to replacing a and d with suitable powers, we may assume that $b^2 = a^2d^2$.
Since $\langle a, d\rangle$ satisfies the hypothesis of Lemma 2.12, the subgroup $\langle a, d\rangle$ is modular metacyclic not isomorphic to Q_8, and so $[\langle a, d\rangle, \langle a^2, d^2\rangle] \leq \langle a, d\rangle^8$. Since $b^2 \in \langle a^2, d^2\rangle$, we have that $[\langle a, d\rangle, b^2] \leq \langle a^8, d^8\rangle$. In particular, from $[a, b^2]G^8 = G^8$ and $[c, b^2]G^8 = G^8$, it follows that $[a, b]^bG^8 = [a, b]^{-1}G^8$ and $[d, b]^bG^8 = [d, b]^{-1}G^8$. Since $\langle [a, b], [d, b]\rangle G^8 = \langle a^2, d^2\rangle G^8$ and $\langle a^2, d^2\rangle G^8/G^8$ is abelian, we have that b acts as inversion on $\langle a^2, d^2\rangle G^8/G^8$. Since $b^2 \in \langle a^2, d^2\rangle$, we get that $|b^2G^8| \leq 2$, i.e. $b^4 \in G^8$. Hence $(a^2d^2)^2 \in G^8$. This implies that $a^4d^4 \in G^8$. Now, the subgroup $G^8 = \Phi(G^4)$ and $G^4 = \langle a^4, d^4\rangle$. Hence, the condition $a^4d^4 \in G^8$ implies that $a^4G^8 = d^{-4}G^8$ and so $G^4 = \langle a^4\rangle = \langle d^4\rangle$. Therefore, since $a^b \in a^{-1}G^4$, we have that $a^b = a^{-1+4h}$, for some h. Since $b^d \in b^{-1}G^4$, we have that $b^d = b^{-1}a^{4k}$, for some k. Now, $(a^2)^b = (a^2)^{-1+4h} = a^{-2+8h}$ and $b^{d^2} = (b^{-1}d^{4k})^{-1}a^{4k} = a^{-4k}ba^{4k} = ba^{-4k(-1+4h)}a^{4k} = ba^{-8k+16hk}$. Since $a^2d^2 = b^2$, we have that $a^2d^2 = (a^2d^2)^b = a^{-2+8h}d^2a^{-8k+16hk} = a^{-2}d^2a^{8h-8k+16hk}$ and so $a^2d^2 = a^{-2}d^2a^{8h-8k+16hk}$. Then, we obtain that $a^4 \in \langle a^8\rangle$. Since $G^4 = \langle a^4\rangle$ and $\Phi(G^4) = G^8 = \langle a^8\rangle$, we obtain that $G^4 = 1$, and so also $H_4(G) = 1$, a contradiction. □

Proposition 5.11. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_{10}$.*

Proof. Suppose that $G = \langle a, b, c, d, A\rangle$ with $|aG^4| = 4$, $|cG^4| = 4$, $\langle aG^4\rangle \cap \langle cG^4\rangle = G^4$, $b^2G^4 = a^2c^2G^4$, $d^2G^4 = c^2G^4$, $a^bG^4 = a^{-1}G^4$, $a^cG^4 = aG^4$, $b^cG^4 = bG^4$, $a^dG^4 = aG^4$, $d^bG^4 = d^{-1}G^4$, $c^dG^4 = c^{-1}G^4$, AG^4/G^4 is elementary abelian and central in G/G^4.

We have that $\langle a^2, d^2\rangle G^4 = G^2$. By Corollary 2.8, we have that $G^4 = \Phi(G^2)$ and so $\langle a^2, d^2\rangle = G^2$. Since $b^2G^4 = a^2d^2G^4$, we have $b^2 = a^{2+4s}d^{2+4r}$,

for some r and s. Therefore, up to replacing a and d with suitable powers, we may assume that $b^2 = a^2d^2$.
Since $\langle a, d\rangle$ satisfies the hypothesis of Lemma 2.12, the subgroup $\langle a, d\rangle$ is modular metacyclic not isomorphic to Q_8, and so $[\langle a, d\rangle, \langle a^2, d^2\rangle] \leq \langle a, d\rangle^8$. This implies that $[\langle a, d\rangle, b^2] \leq \langle a^8, d^8\rangle$. In particular, from $[a, b^2]G^8 = G^8$ and $[d, b^2]G^8 = G^8$, it follows $[a, b]^bG^8 = [a, b]^{-1}G^8$ and $[d, b]^bG^8 = [d, b]^{-1}G^8$. Since $\langle [a, b], [d, b]\rangle G^8 = \langle a^2, d^2\rangle G^8$ and $\langle a^2, d^2\rangle G^8/G^8$ is abelian, we have that b acts as inversion on $\langle a^2, d^2\rangle G^8/G^8$. Since $b^2 \in \langle a^2, d^2\rangle$, we get that $|b^2G^8| \leq 2$, i.e. $b^4 \in G^8$. Hence, $(a^2d^2)^2 \in G^8$. This implies that $a^4d^4 \in G^8$. Now, the subgroup $G^8 = \Phi(G^4)$ and $G^4 = \langle a^4, d^4\rangle$. Hence, the condition $a^4d^4 \in G^8$ implies that $a^4G^8 = d^{-4}G^8$ and so $G^4 = \langle a^4\rangle = \langle d^4\rangle$. Now c is an element such that $c^2G^4 = d^2G^4$, and $d^cG^4 = d^{-1}G^4$. Since $G^4 = \langle d^4\rangle$, we have that $c^2 = d^{2+4h_1}$ and $d^c = d^{-1+4h_2}$. Since $\langle d^2\rangle = \langle c^2\rangle$, we have that $d^2 = (d^2)^c = d^{-2+8h_1}$, i.e. $d^4 = 1$. Since $G^4 = \langle d^4\rangle$, we get that $G^4 = 1$, a contradiction because we are assuming that $exp(G) \geq 8$. $\square$

Proposition 5.12. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. The quotient G/G^4 cannot be isomorphic to a group in $\mathscr{A}_{11}$.*

Proof. Suppose that $G = \langle a, b, c, d, A\rangle$, where $|aG^4| = 4$, $|cG^4| = 4$, $\langle aG^4\rangle \cap \langle cG^4\rangle = G^4$, $b^2G^4 = a^2c^2G^4$, $d^2G^4 = c^2G^4$, $a^bG^4 = a^{-1}G^4$, $a^cG^4 = aG^4$, $b^cG^4 = b^{-1}G^4$, $d^aG^4 = d^{-1}G^4$, $d^bG^4 = dG^4$, $c^dG^4 = ca^2G^4$ and AG^4/G^4 is elementary abelian and central in G/G^4.

We have that $\langle a^2, c^2\rangle G^4 = G^2$. By Corollary 2.8, we have that $G^4 = \Phi(G^2)$ and so $\langle a^2, c^2\rangle = G^2$. Since $b^2G^4 = a^2c^2G^4$, we have $b^2 = a^{2+4s}c^{2+4r}$, for some r and s. Therefore, up to replacing a and c with suitable powers, we may assume that $b^2 = a^2c^2$.
Since $\langle a, c\rangle$ satisfies the hypothesis of Lemma 2.12, the subgroup $\langle a, c\rangle$ is modular metacyclic not isomorphic to Q_8, and so $[\langle a, c\rangle, \langle a^2, c^2\rangle] \leq \langle a, d\rangle^8$. This implies that $[\langle a, c\rangle, b^2] \leq \langle a^8, c^8\rangle$. In particular, from $[a, b^2]G^8 = G^8$ and $[c, b^2]G^8 = G^8$, it follows $[a, b]^bG^8 = [a, b]^{-1}G^8$ and $[c, b]^bG^8 = [d, b]^{-1}G^8$. Since $\langle [a, b], [c, b]\rangle G^8 = \langle a^2, c^2\rangle G^8$, and $\langle a^2, c^2\rangle G^8/G^8$ is abelian, we have that b induces the inversion on $\langle a^2, c^2\rangle G^8/G^8$. Since $b^2 \in \langle a^2, c^2\rangle$, we get that $|b^2G^8| \leq 2$, i.e. $b^4 \in G^8$. Hence, $(a^2c^2)^2 \in G^8$. This implies that $a^4c^4 \in G^8$. Now, the subgroup $G^8 = \Phi(G^4)$ and $G^4 = \langle a^4, c^4\rangle$. Hence, the condition $a^4c^4 \in G^8$ implies that $a^4G^8 = c^{-4}G^8$ and so $G^4 = \langle a^4\rangle = \langle c^4\rangle$. Therefore, since $a^b \in a^{-1}G^4$, we have that $a^b = a^{-1+4h}$, for some h. Since $b^c \in b^{-1}G^4$, we have that $b^c = b^{-1}a^{4k}$, for some k. Now, $(a^2)^b = (a^2)^{-1+4h} = a^{-2+8h}$

and $b^{c^2} = (b^{-1}c^{4k})^{-1}a^{4k} = a^{-4k}ba^{4k} = ba^{-4k(-1+4h)}a^{4k} = ba^{-8k+16hk}$. Since $a^2c^2 = b^2$, we have that $a^2c^2 = (a^2c^2)^b = a^{-2+8h}c^2a^{-8k+16hk} = a^{-2}c^2a^{8h-8k+16hk}$ and so $a^2c^2 = a^{-2}c^2a^{8h-8k+16hk}$. Then, we obtain that $a^4 \in \langle a^8 \rangle$. Since $\Phi(G^4) = G^8 = \langle a^8 \rangle$, we obtain that $G^4 = 1$, a contradiction because we are assuming $exp(G) \geq 8$. $\square$

This concludes the first part of this section.
From now on, we deal with non-trivial monotone 2-groups such that $G = H_4(G)$ and G/G^4 is isomorphic to a group in $\mathscr{A}_1$, in $\mathscr{A}_4$ or in $\mathscr{A}_6$.

The next proposition determines the non-trivial monotone 2-groups such that $G = H_4(G)$ and G/G^4 is isomorphic to a group in $\mathscr{A}_6$.

Proposition 5.13. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. Let G/G^4 be isomorphic to a group in $\mathscr{A}_6$. Then G is either in $\mathscr{C}_2$ or in $\mathscr{C}_3$.*

Proof. Let G be $\langle a, c, b, A \rangle$ where $|aG^4| = 4$, $|cG^4| = 4$, $\langle aG^4 \rangle \cap \langle cG^4 \rangle = G^4$, $b^2G^4 = a^2G^4$, $a^cG^4 = aG^4$, $a^bG^4 = a^{-1}G^4$, $c^bG^4 = cG^4$, and AG^4/G^4 is elementary abelian and central in G/G^4.

We have that $\langle a^2, c^2 \rangle G^4 = G^2$. By Corollary 2.8, we have that $\Phi(G^2) = G^4$. Therefore, $\langle a^2, c^2 \rangle = G^2$. Moreover, since by Lemma 2.7, G^2 is powerful, we have that $\langle a^{2^i}, c^{2^i} \rangle = G^{2^i}$. From $b^2G^4 = a^2G^4$, we have $b^2 = a^{2+4s}c^{4r}$, for some r and s. Therefore, up to replacing a with a suitable power, we may assume that $b^2 = a^2c^{4r}$.
By Lemma 2.12, the subgroup $\langle a, c \rangle$ is modular and so $[\langle a, c \rangle, \langle a^2, c^2 \rangle] \leq \langle a, c \rangle^8$. Since $a^b = a^{-1}q_a$ and $c^b = cq_c$, where q_a and q_c are in G^4, we have that $(a^2)^b = (a^{-1}q_a)^2 = a^{-2}q_a^2[q_a, a^{-1}]^{q_a}$ and $(c^2)^b = c^2q_c^2[q_c, c]^{q_c}$. Then $(a^2)^bG^8 = a^{-2}G^8$ and $(c^2)^bG^8 = c^2G^8$. Since $b^2 = a^2c^{4r}$, we have that $(a^2c^{4r})^b = a^2c^{4r}$ and so, in particular, $a^2c^{4r}G^8 = (a^2c^{4r})^bG^8 = a^{-2}c^{4r}G^8$. This means that $a^4G^8 \in G^8$. Since $G^4 = \langle a^4, c^4 \rangle G^8$ and $G^8 = \Phi(G^4)$, we get that $G^4 = \langle c^4 \rangle$.

Since $[c, G] \leq G^4$, we have that $\langle c \rangle$ is normal in G. From $c^aG^4 = cG^4$ and $c^bG^4 = cG^4$, we have that $c^a = c^{1+4s_1}$ and $c^b = c^{1+4s_2}$. Now, a^4 and b^4 lie in $\langle c^4 \rangle$, and so $|c^{4s_1}| \leq 4$ and $|c^{4s_2}| \leq 4$.

Suppose that $|c^{4s_1}| = 4$. Then $c^{a^2} = c^{1+8s_1}$, with $|c^{8s_1}| = 2$. Since $b^2 = a^2c^{4r}$, we have that $c^{b^2} = c^{1+8s_1}$ with $|c^{8s_1}| = 2$, and so $c^b = c^{1+4s_2}$ with $|c^{4s_2}| = 4$. Now $c^{ab} = c^{1+4s_1+4s_2}$ and $|c^{4s_1+4s_2}| \leq 2$. This is a contradiction, because $(ab)^2G^4 = a^2G^4$, i.e. $(ab)^2 = a^2c^{4t}$, for some t, and so $c^{1+8s_1} =$

$c^{a^2} = c^{(ab)^2} = c$.
This shows that $|c^{4s_1}| \leq 2$ and, since $c^{a^2} = c^{b^2}$, we have that also $|c^{4s_2}| \leq 2$.

In particular, the subgroup $\langle a, c^2 \rangle$ is abelian and so, if $a^4 = c^{8l}$, then we have that $\tilde{a} = ac^{-2l}$ has order 4. In the same way, since the subgroup $\langle b, c^2 \rangle$ is abelian and $b^2 = a^2c^{4r}$ we have that $b^4 = c^{8l+4r}$. So we get that $\tilde{b} = bc^{-2(r+l)}$ has order 4. Moreover, $\tilde{b}^2 = b^2c^{-4r-4l} = a^2c^{4r}c^{-4r-4l} = a^2c^{-4l} = \tilde{a}^2$, $c^{\tilde{a}} = c^{1+4s_1}$ and $c^{\tilde{b}} = c^{1+4s_2}$.
Since $\tilde{a}^{\tilde{b}}G^4 = \tilde{a}^{-1}G^4$, we have also that $\tilde{a}^{\tilde{b}} = \tilde{a}^{-1}c^{4s_3}$. Since $\tilde{a^2} = \tilde{b^2}$, we get $\tilde{a}^{\tilde{b}} = \tilde{a}^{-1}c^{4s_3}$, with $|c^{4s_3}| \leq 2$. Therefore, we have $\langle a, b, c \rangle = \langle \tilde{a}, \tilde{b}, c \rangle$ with $|c| = 2^n$, $|\tilde{a}| = 4$, $\tilde{a}^2 = \tilde{b}^2$, $c^{\tilde{a}} = c^{1+4s_1}$, $c^{\tilde{b}} = c^{1+4s_2}$ and $\tilde{a}^{\tilde{b}} = \tilde{a}^{-1}c^{4s_3}$, with $|c^{4s_i}| \leq 2$ for $i = 1, 2, 3$.

Let A be $\langle c_1, \ldots, c_m \rangle$ with $|c_iG^4| = 2$. Since $c_i^2 \in G^4$ and $c^{c_i}G^4 = cG^4$, we get $c_i^2 = c^{4r_i}$ and $c^{c_i} = c^{1+4k_i}$ with $|c^{4k_i}| \leq 2$. In particular, replacing c_i with $c_ic^{-r_i}$, we may assume that $|c_i| = 2$. Moreover, since $\tilde{a}^{c_i}G^4 = \tilde{a}G^4$ and $\tilde{b}^{c_i}G^4 = \tilde{b}G^4$, we get $\tilde{a}^{c_i} = \tilde{a}c^{4j_i}$ and $\tilde{b}^{c_i} = \tilde{b}c^{4l_i}$. If $|c^{4j_i}| \geq 2$, then the subgroup $\langle \tilde{a}, c_i \rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle \tilde{a}^2, c_i, c^{2^{n-1}} \rangle$. If $|c^{4l_i}| \geq 2$, then the subgroup $\langle \tilde{b}, c_i \rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle \tilde{b}^2, c_i, c^{2^{n-1}} \rangle$.
Therefore, we have $c^{c_i} = c^{1+4k_i}$, $\tilde{a}^{c_i} = \tilde{a}$, $\tilde{b}^{c_i} = \tilde{b}$.
In particular, up to reordering the indices and replacing perhaps c_i with c_ic_1, we have that G is $\langle \tilde{a}, \tilde{b}, c, c_1 \rangle \times A$, where A is elementary abelian, $|c| = 2^n$, $|\tilde{a}| = 4$, $\tilde{a}^2 = \tilde{b}^2$, $c^{\tilde{a}} = c^{1+4s_1}$, $c^{\tilde{b}} = c^{1+4s_2}$ and $\tilde{a}^{\tilde{b}} = \tilde{a}^{-1}c^{4s_3}$, $|c_1| = 2$ and $c^{c_1} = c^{1+4k_1}$, $a^{c_1} = a$ and $b^{c_1} = b$, where $|c^{4s_1}| \leq 2$, $|c^{4s_2}| \leq 2$, $|c^{4s_3}| \leq 2$ and $|c^{4k_1}| \leq 2$.
Now if $c^{4k_1} = 1$, then G is in $\mathscr{C}_2$. If $|c^{4k_1}| = 2$, then, up to replacing perhaps a with ac_1 and b with bc_1, we have that G is in $\mathscr{C}_3$. $\square$

The next lemma states a preliminary result useful in order to classify the non-trivial monotone 2-groups such that $G = H_4(G)$ and G/G^4 is in $\mathscr{A}_1$ or in $\mathscr{A}_4$.

Lemma 5.14. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$.*
Suppose that $G = \langle a, b, c_1, \ldots, c_r \rangle$, with $|aG^4| = 4$, $|bG^4| = 4$, $\langle aG^4 \rangle \cap \langle bG^4 \rangle = G^4$, $a^bG^4 = a^{-1}G^4$ and $\langle c_1, \ldots, c_r \rangle G^4/G^4$ is elementary abelian and central in G/G^4.
Then, we may assume that $|c_i| = 2$, for every $i \in \{1, \ldots, r\}$.

Proof. $G = \langle a, b, c_1, \dots, c_r\rangle$, with $|aG^4| = 4$, $|bG^4| = 4$, $\langle aG^4\rangle \cap \langle bG^4\rangle = G^4$, $a^b G^4 = a^{-1}G^4$ and $\langle c_1, \dots, c_r\rangle G^4/G^4$ is elementary abelian and central in G/G^4.

Since $\langle a, b\rangle$ satisfies the hypothesis of Lemma 2.13, we have that $\langle a, b\rangle$ is non-modular metacyclic. Hence, up to renaming the generators, we may assume that $\langle a\rangle \trianglelefteq \langle a, b\rangle$, and $a^b = a^{-1+4h}$. By Lemma 2.13, we also have that $\langle a^2, b^2\rangle = \Phi(\langle a, b\rangle)$. Since the Frattini subgroup of a metacyclic group is powerful, we obtain $\langle a^4, b^4\rangle = G^4 \cap \langle a, b\rangle = \Phi(\langle a^2, b^2\rangle)$.

In particular, $c_i^2 \in \langle a^4, b^4\rangle$ and so $c_i^2 = a^{4r}b^{4s}$.
We now prove that we may assume that c_i has of order 2. We distinguish three cases depending on the values of r and s.

- Suppose first that $c_i^2 = b^{4s}$. The subgroup $\langle c_i^2\rangle$ is normal in $\langle c_i, b\rangle$ and the quotient $\langle b, c_i\rangle/\langle c_i^2\rangle$ is metacyclic with a generator of order 2. Since $\langle c_i, b\rangle G^4/G^4$ is isomorphic to $C_4 \times C_2$, we have that $\langle b, c_i\rangle$ is not semidihedral. Therefore, by Lemma 2.11, we have that c_i is in the normalizer of b and, since $\langle c_i^2\rangle \leq \langle b^4\rangle$, we have that $b^c = b^{1+4h}$, with $|b^{4h}| \leq 2$. In particular, the subgroup $\langle b^2, c_i\rangle$ is abelian and so, up to replacing c_i with $c_i b^{-2s}$, we may assume that c_i has order 2. Hence, in this case the claim is true.

- Suppose now $c_i^2 = a^{4r}b^{4s}$ with $|b^{4s}| \geq 2$, $|a^{4r}| \geq 2$ and $a^{4r} \notin \langle b^4\rangle$.

  We first show that $\langle c_i, b^2\rangle$ is modular. Since $\langle c_i, b\rangle$ is metacyclic, there exists $c_i^{j_1}b^{j_2} \notin \Phi(\langle c_i, b\rangle)$ such that $\langle c_i^{j_1}b^{j_2}\rangle \trianglelefteq \langle c_i, b\rangle$.
  We distinguish two cases depending on the parity of j_2.

  Suppose that j_2 is even. Since $c_i^{j_1}b^{j_2}$ is a generator we have that j_1 is odd and so we have that $\langle c_i^{j_1}b^{j_2}, b\rangle = \langle c_i, b\rangle$ with $(c_i^{j_1}b^{j_2})^b = (c_i^{j_1}b^{j_2})^r$, for some r. It follows that $(c_i^{j_1}b^{j_2})^{b^2} = (c_i^{j_1}b^{j_2})^{r^2}$, and, being $r^2 \equiv 1 \mod 4$, we have that $\langle c_i^{j_1}b^{j_2}, b^2\rangle = \langle c_i, b^2\rangle$ is modular metacyclic.

  Suppose that j_2 is odd. Since $\langle bG^4, c_iG^4\rangle$ is isomorphic to $C_4 \times C_2$, we have that $[c_i, b] \in G^4 \cap \langle b, c_i\rangle$. Since $(c_i^{j_1}b^{j_2})^2G^4 = b^2G^4$, we have that $(c_i^{j_1}b^{j_2})^{c_i} = (c_i^{j_1}b^{j_2})^{1+4r}$. If follows that $\langle c_i^{j_1}b^{j_2}, c_i\rangle$ is modular, and so $\langle b^2, c_i\rangle$ is modular.

  Therefore, we get that $\langle b^2, c_i\rangle$ is modular.

  We now show that we may assume $c_i^2 \in \langle a^4\rangle$.
  Since $\langle b^2, c_i\rangle$ is modular, we have that $\langle b^2, c_i\rangle^2 = \langle b^4, a^{4r}\rangle$ and there

exists an element $c_i^{l_i}b^{2l_2}$ in $\langle b^2, c_i\rangle$ such that $(c_i^{l_i}b^{2l_2})^2 = a^{4r}$.
If l_i is even, then $c_i^{l_i}b^{2l_2} \in \langle c_i^2, b^2\rangle = \langle a^{4r}, b^2\rangle$. Since $\langle a, b^2\rangle$ is modular we have that $\langle a^{4r}, b^2\rangle^2 = \langle a^{8r}, b^4\rangle$ and so $(c_i^{l_i}b^{2l_2})^2 \in \langle a^{8r}, b^4\rangle$. Now, $(c_i^{l_i}b^{2l_2})^2 = a^{4r}$, and so we have that $a^{4r} \in \langle a^{8r}, b^4\rangle$. This means that $\langle a^{4r}, b^2\rangle = \langle b^2\rangle$, i.e. $c_i^2 \in \langle b\rangle$, against our assumption. Therefore, we have that l_i is odd and, so up to replacing c_i with $c_i^{l_i}b^{2l_2}$, we may assume that $c_i^2 \in \langle a^{4r}\rangle$. In particular, we reduce the proof of this case to the following.

- Suppose that $c_i^2 = a^{4r}$. Then, $\langle c_i^2\rangle$ is a normal subgroup in $\langle c_i, a\rangle$ and the quotient $\langle a, c_i\rangle/\langle c_i^2\rangle$ is a metacyclic group with a generator of order 2. Since $\langle c_i, a\rangle G^4/G^4$ is isomorphic to $C_4 \times C_2$, we have that $\langle a, c_i\rangle$ is not semidihedral. Therefore, by Lemma 2.11, we have that c_i is in the normalizer of a and, since $\langle c_i^2\rangle \leq \langle a^4\rangle$, we have that $a^c = a^{1+4k}$ with $|a^{4k}| \leq 2$. In particular, the subgroup $\langle a^2, c_i\rangle$ is abelian and, up to replacing c_i with $c_i a^{-2r}$, we may assume that c_i has order 2.

$\square$

In the next proposition, we determine completely the non-trivial monotone 2-groups such that $G = H_4(G)$ and G/G^4 is isomorphic to a group in $\mathscr{A}_1$.

Proposition 5.15. *Let G be a non-trivial monotone 2-group such that $G = H_4(G)$. Suppose that G/G^4 is in $\mathscr{A}_1$.*
Then G is isomorphic to a group in $\mathscr{C}_4$, or in $\mathscr{C}_5$, or in $\mathscr{C}_6$.

Proof. Let $G = \langle a, b, c_1, \dots, c_r\rangle$, with $|aG^4| = 4$, $|bG^4| = 4$, $\langle aG^4\rangle \cap \langle bG^4\rangle = G^4$, $a^bG^4 = a^{-1}G^4$ and $\langle c_1, \dots, c_r\rangle G^4/G^4$ is elementary abelian and central in G/G^4.

Since $\langle a, b\rangle$ satisfies the hypothesis of Lemma 2.13, we have that $\langle a, b\rangle$ is non-modular metacyclic. Hence, up to renaming the generators, we may assume that $\langle a\rangle \trianglelefteq \langle a, b\rangle$, and $a^b = a^{-1+4h}$. By Lemma 2.13, we also have that $\langle a^2, b^2\rangle = \Phi(\langle a, b\rangle)$ and, since the Frattini subgroup of a metacyclic group is powerful, we have also $\langle a^4, b^4\rangle = G^4 \cap \langle a, b\rangle = \Phi(\langle a^2, b^2\rangle)$.
Moreover, by Lemma 2.7 and Corollary 2.8, since $G^2/G^4 = \langle a^2, b^2\rangle G^4/G^4$, we get that $G^2 = \langle a^2, b^2\rangle$ and also $G^{2^i} = \langle a^{2^i}, b^{2^i}\rangle$ for every $i \geq 1$.
By Lemma 5.14, we may assume that c_i has order 2 and so we have that $G = \langle a, b\rangle\langle c_1, \cdots, c_r\rangle$, with $a^b = a^{-1+4h}$ and $|c_i| = 2$, for all i.

Since $\langle a, c_i\rangle$ and $\langle b, c_i\rangle$ are metacyclic groups with a generator of order 2, and they are not semidihedral (because they both have a quotient isomorphic to $C_4 \times C_2$), we have that c_i lies in the normalizer of $\langle a\rangle$ and $\langle b\rangle$. Moreover, since c_i has order 2, and both $\langle a, c_i\rangle$ and $\langle b, c_i\rangle$ have a quotient isomorphic to $C_4 \times C_2$, we get $a^{c_i} = a^{1+4h_i}$ with $|a^{4h_i}| \le 2$ and $b^{c_i} = b^{1+4k_i}$ with $|b^{4k_i}| \le 2$. In particular, we get that c_i in the centralizer of $\langle a^2, b^2\rangle$, for all i.

We distinguish various cases depending on the structure of $\langle a, b\rangle$.

- Suppose that $|b| = 4$. Since $\langle a, b\rangle$ is such that $G^4 = \langle a^4, b^4\rangle$, and $b^4 = 1$, we have that $G^4 = \langle a^4\rangle$. In particular, we have that $a^b = a^{-1+4h}$, with $|a^{4h}| \le 4$, $\langle a\rangle \cap \langle b\rangle = 1$ and $|a| = 2^n = exp(G)$.

  We now prove that $\langle c_i, c_j\rangle$ is abelian.
  Suppose that c_i and c_j do not commute. Then, $[c_i, c_j] \in G^4$ and, being $c_i^2 = 1$, we get that $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$. Since $G^4 = \langle a^4\rangle$ and $\langle a^2\rangle$ is contained in the centralizer of c_i, we get that $[c_i, c_j] = a^{2^{n-1}}$. Moreover, since b has order 4, we have that $b \in C_G(\langle c_i, c_j\rangle)$. This means that G contains $\langle b\rangle \times \langle c_i, c_j\rangle$ which is isomorphic to $C_4 \times D_8$, a contradiction, because $C_4 \times D_8$ is not monotone. Then, $\langle c_i, c_j\rangle$ is abelian.

  Now, suppose $a^b = a^{-1+4h}$, with $|a^{4h}| \le 2$. We have that $(ba^{i_1} \prod {c_i}^{j_i})^2 = b^2 a^{2^{n-1}s}$, for some s. Therefore, for every $g \in \langle a, c_1, \ldots, c_r\rangle$, we have that $|bg| \le 4$. This means that $b \notin H_4(G)$, a contradiction.
  This implies that $a^b = a^{-1+4h}$, with $|a^{4h}| = 4$.

  So we have that $G = \langle a, b, c_1, \ldots, c_r\rangle$ where $|a| = 2^n$, $|b| = 4$, $|c_i| = 2$, $a^b = a^{-1+4h}$ with $|a^{4h}| = 4$, $a^{c_i} = a^{1+4h_i}$ with $|a^{4h_i}| \le 2$, $b^{c_i} = b$ and $c_i^{c_j} = c_i$. Suppose that there exists c_i such that $a^{c_i} = a^{1+4h_i}$, with $|a^{4h_i}| = 2$. Up to replacing c_i with $c_i b^2$, we may assume that c_i is in the centralizer of a.
  So we have that $G = \langle a, b, c_1, \ldots, c_r\rangle$, with $|a| = 2^n$, $|b| = 4$, $|c_i| = 2$, $\langle c_1, \ldots, c_r\rangle$ is elementary abelian and central in G and $a^b = a^{-1+4h}$ with $|a^{4h}| = 4$. Up to replacing b with ba, we obtain that G is in $\mathscr{C}_5$.

- Suppose that $|a| = 4$. Since $\langle a, b\rangle$ is such that $G^4 = \langle a^4, b^4\rangle$, and $a^4 = 1$, we have that $G^4 = \langle b^4\rangle$. In particular, since $exp(G) \ge 8$, we have that $a^b = a^{-1}$, $|b| = 2^k \ge 8$ and, since $\langle a, b\rangle G^4/G^4$ is isomorphic to K_2 we have that $\langle a\rangle \cap \langle b\rangle = 1$.

We now prove that $\langle c_i, c_j\rangle$ is abelian. Suppose that c_i and c_j do not commute. Then, $[c_i, c_j] \in G^4 = \langle b^4\rangle$. Since $c_i^2 = 1$, we have that $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$. Now, $\langle b^2\rangle$ is contained in the centralizer of c_i. Then, we get that $[c_i, c_j] = b^{2^{k-1}}$. Moreover, since a has order 4, we have that $a \in C_G(\langle c_i, c_j\rangle)$. This means that G contains $\langle a\rangle \times \langle c_i, c_j\rangle$ which is isomorphic to $C_4 \times D_8$, a contradiction, because $C_4 \times D_8$ is not monotone. Then, $\langle c_i, c_j\rangle$ is abelian.

So we have that $G = \langle a, b, c_1, \dots, c_r\rangle$ where $|a| = 4$, $|b| = 2^k$, $|c_i| = 2$, $a^b = a^{-1}$, $a^{c_i} = a$, $b^{c_i} = b^{1+4k_i}$ with $|b^{4k_i}| \leq 2$ and $c_i^{c_j} = c_i$. If $b^{4k_i} = 1$ for every i, then G is in $\mathscr{C}_5$. Suppose that $|b^{4k_i}| = 2$ for some i. Up to reordering the indices and up to replacing in case c_i with c_ic_1, we may assume that $b^{c_i} = b$ for every $i \geq 2$ and $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| = 2$. Let $c = c_1 b^{2^{k-2}}$. Then $a^c = a$, $c^b = c^{-1}$, $a^b = a^{-1}$ and so $G = \langle a, b, c\rangle \times \langle c_2, \dots, c_r\rangle$ is in $\mathscr{C}_5$.

- Suppose that $|a| = 2^n \geq 8$, $|b| = 8$ and $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{n-1}}\rangle$. Since $b^4 \in \langle a\rangle$, we have that $a^b = a^{-1+4h}$, with $|a^{4h}| \leq 4$. If $|a^{4h}| = 4$, then we have that $|ba| = 4$, and so, up to relacing b with ba we are in the first case studied.
 So, we may assume that $a^b = a^{-1+4h}$ with $|a^{4h}| \leq 2$.

 We now prove by induction that G is isomorphic to $\langle a, b, c, d\rangle * E \times A$, with A elementary abelian, E extraspecial, $|a| = 2^n \geq 8$, $|b| = 8$ and $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{n-1}}\rangle$ and $a^b = a^{-1+4h}$ with $|a^{4h}| \leq 2$, $a^c = a^{1+4h_1}$ with $|a^{4h_1}| \leq 2$, $b^c = b$ and $a^d = a$, $b^d = b^{1+4h_2}$ with $|b^{4h_2}| \leq 2$.
 This means that if $E \neq 1$, then G is a group in $\mathscr{C}_4$. Otherwise G is in $\mathscr{C}_5$ or in $\mathscr{C}_6$.

 We now show that, for all i and j, the commutator $[c_i, c_j]$ is contained in $\langle a^{2^{n-1}}\rangle$, and so the subgroup $\langle c_i, c_j\rangle$ is either abelian or dihedral. Suppose that $\langle c_i, c_j\rangle$ is not abelian. Since $c_i^2 = 1$, we have that $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$ and since $[c_i, c_j] \in G^4 = \langle a^4\rangle \leq C_G(\langle c_i, c_j\rangle)$, we have that $[c_i, c_j] = a^{2^{n-1}}$. So our claim is proved.

 Suppose firstly that $\langle c_1, \cdots, c_r\rangle$ is abelian. Then, since we have that $a^{c_i} = a^{1+4h_i}$, with $|a^{4h_i}| \leq 2$ and $b^{c_i} = b^{1+4k_i}$ with $|b^{4k_i}| \leq 2$. Up to reordering the indices and up to replacing in case c_i with c_ic_1, we may assume that $a^{c_1} = a^{1+4h_1}$ with $|a^{4h_1}| \leq 2$, and $a^{c_i} = a$ for every $i \geq 2$. Moreover, up to reordering the indices and up to replacing in case c_i

with c_ic_2 for $i \geq 3$, we may assume that $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| \leq 2$, $b^{c_2} = b^{1+4h_2}$ with $|b^{4h_2}| \leq 2$, and $b^{c_i} = b$ for all $i \geq 3$.
Hence, up to replacing in case b with ba, we have that G is equal to $\langle a, b, c_1, \cdots, c_r\rangle$, with $|a| = 2^n \geq 8$, $|b| = 8$ and $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{n-1}}\rangle$ and $a^b = a^{-1+4h}$ with $|a^{4h}| \leq 2$, $a^{c_1} = a^{1+4h_1}$, $b^{c_1} = b$, $a^{c_2} = a$, $b^{c_2} = b^{1+4h_2}$, and $a^{c_i} = a$, $b^{c_i} = b$ for all $i \geq 3$. Therefore, if $a^{4h_1} = 1$, then, up to replacing c_2 with $c_2b^{2h_2}$, we get that G is in $\mathscr{C}_5$.
If $|a^{4h_1}| = 2$, then up to replacing c_2 with $c_2b^{2h_2}$, we get that G is in $\mathscr{C}_6$.

Suppose now that $\langle c_1, \cdots, c_r\rangle$ is not abelian. Then, up to reordering the indices, we may assume that $\langle c_1, c_2\rangle$ is dihedral with $[c_1, c_2] = a^{2^{n-1}}$. We have that $a^{c_1} = a^{1+4h_1}$ with $|a^{4h_1}| \leq 2$, $a^{c_2} = a^{1+4h_2}$ with $|a^{4h_2}| \leq 2$, $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| \leq 2$ and $b^{c_2} = b^{1+4k_2}$ with $|b^{4k_2}| \leq 2$. Hence, up to replacing a with $ac_1^{h_2}c_2^{h_1}$ and b with $bc_1^{k_2}c_2^{k_1}$, we may assume that $\langle a, b\rangle \leq C_G(\langle c_1, c_2\rangle)$, $|a| = 2^n$, $b^4 = a^{2^{n-1}}$, $a^b = a^{-1+4h}$ with $|a^{4h}| \leq 2$. Now, for all $i \geq 3$, we have $[c_1, c_i] = a^{2^{n-1}k_{1i}}$ and $[c_2, c_i] = a^{2^{n-1}k_{2i}}$. The element $c_ic_1^{k_{2i}}c_2^{k_{1i}}$ is in the centralizer of $\langle c_i, c_j\rangle$ and, being $[\langle c_1, \ldots, c_r\rangle, \langle c_1, \ldots, c_r\rangle] \leq \langle a^{2^{n-1}}\rangle$, we have that either $c_ic_1^{k_{2i}}c_2^{k_{1i}}$ has order 2, or $c_ic_1^{k_{2i}}c_2^{k_{1i}}$ has order 4 with $(c_ic_1^{k_{2i}}c_2^{k_{1i}})^2 = a^{2^{n-1}}$. Therefore, up to replacing c_i with $c_ic_1^{k_{2i}}c_2^{k_{1i}}$ or with $c_ic_1^{k_{2i}}c_2^{k_{1i}}a^{2^{n-2}}$, we may assume that c_i has order 2 and is in the centralizer of $\langle c_1, c_2\rangle$.
This shows that $G = \langle c_1, c_2\rangle * \langle a, b, c_3, \cdots, c_r\rangle$. Since $\langle a, b, c_3, \cdots, c_r\rangle$ satisfies the same hypothesis of G and $|\langle a, b, c_3, \cdots, c_r\rangle| < |G|$, we can conclude by induction that G is in $\mathscr{C}_4$.

- Suppose $|a| = 8$, $|b| = 2^k \geq 16$ and $\langle a\rangle \cap \langle b\rangle = \langle b^{2^{k-1}}\rangle$.
  We have that $a^b = a^{-1+4h}$. In particular, we get $a^{b^2} = a$, $b^{2^{k-3}}a$ has order 4 and $G^4 = \langle a^4, b^4\rangle = \langle b^4\rangle$.

  If there exists i such that $a^{c_i} = a^5$, then $\langle b^{2^{k-3}}a, c_i\rangle$ is not metacyclic, because it contains the 3-generated abelian subgroup $\langle c_i, a^2b^{2^{k-2}}, a^4\rangle$. Therefore, we have that $a^{c_i} = a$, for all i.

  Suppose there exists $\langle c_i, c_j\rangle$ non abelian. Since $c_i^2 = 1$, we have that $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$. Now, $[c_i, c_j] \in G^4 = \langle b^4\rangle$ and $\langle b^4\rangle \in C_G(\langle c_i\rangle)$, and so $[c_i, c_j] = b^{2^{k-1}}$. The subgroup $\langle b^{2^{k-3}}a, c_i, c_j\rangle$ is isomorphic to $C_4 \times D_8$, which is not monotone, a contradiction.
  Then, we get that $\langle c_i, c_j\rangle$ is abelian for all i and j.

So, we get that $a^b = a^{-1+4h}$, $a^{c_i} = a$ for all i and $\langle c_1, \ldots, c_r\rangle$ is elementary abelian. Since $b^{c_i} = b^{1+4k_i}$ with $|b^{4k_i}| \leq 2$, up to reordering the indices and replacing perhaps c_i with $c_i c_1$, we may assume that $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| \leq 2$ and $b^{c_i} = b$ for $i \geq 2$.
Up to replacing c_1 with $c_1 b^{2^{k-2}k_1}$, we get that G is in $\mathscr{C}_5$.

- Assume now, $|a| \geq 8$, $|b| \geq 8$ and $G^4 = \langle a^4, b^4\rangle$ is not cyclic. So, we have $a^b = a^{-1+4h}$ and $\Omega_2(\langle a, b\rangle) \leq \Phi(\langle a, b\rangle)$.

  We now prove that $\langle c_i, c_j\rangle$ is abelian.
  Suppose that c_i and c_j do not commute. Then, $[c_i, c_j] \in G^4 = \langle a^4, b^4\rangle$, and $\langle a^2, b^2\rangle$ is contained in the centralizer of c_i and of c_j. In particular, this implies that $[c_i, c_j] \in \Omega_1(G^4)$.
  Since G^2 is modular and it is 2-generated, we have that $\Omega_1(\langle a, b\rangle)$ has order 4. In particular, there exists $z \in \Omega_1(\langle a^2, b^2\rangle)$ such that $\langle z\rangle \cap \langle c_i, c_j\rangle = 1$. Moreover, since G^4 is not cyclic, $z \in \langle a^4, b^4\rangle = \Phi(\langle a^2, b^2\rangle)$. Now, G^2 is a powerful group and so there exists $d \in \langle a^2, b^2\rangle$ such that $d^2 = z$. Since $\langle a^2, b^2\rangle \leq C_G(\langle c_i, c_j\rangle)$, we have that G contains the subgroup $\langle d, c_i, c_j\rangle$ isomorphic to $C_4 \times D_8$, a contradiction, because $C_4 \times D_8$ is not monotone.
  This shows that the subgroup $\langle c_1, \ldots, c_r\rangle$ is elementary abelian.

  · Suppose that $|b^2| = |a^{8h}|$.
    Since $b^{2^{k-1}}$ is not central in $\langle a, b\rangle$ and $\langle a\rangle \cap \langle b\rangle \leq Z(\langle a, b\rangle)$, we get that $\langle a\rangle \cap \langle b\rangle = 1$. Moreover, we have that $(ba)^2 = b^2 a^{4h}$ and so $\Omega_1(\langle ba\rangle) = \Omega_1(\langle a\rangle)$. If $|b^{4k_i}| = 2$, then the subgroup $\langle ab, c_i\rangle$ contains the 3-generated elementary abelian subgroup $\langle a^{2^{n-1}}, b^{2^{k-1}}, c_i\rangle$. Moreover, up to replacing c_i in case with $c_i b^{2^{k-1}}$, we may assume that $a^{4h_i} = 1$.
    Hence, G is a group in the family $\mathscr{C}_5$.

  · Suppose now $|b^2| = |a^{4h}|$. Now, we have that $(ba)^2 = b^2 a^{4h}$. If $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{n-1}}\rangle = \langle b^{2^{k-1}}\rangle$, then $|ba| = 2^{k-1}$ and $|(ba^{4h})^2| = 2^{k-2} = |a^{8h}|$. Hence, we are in the previous case.

    So, we may assume that $\langle a\rangle \cap \langle b\rangle = 1$. Now, $|ba| = 2^k$ and $(ba)^{2^{k-1}} = b^{2^{k-1}} a^{2^{n-1}}$. Then, we have that $a^{4h_i} = 1$ if and only if $b^{4k_i} = 1$. In fact, suppose that $a^{4h_i} = 1$ and $|b^{4k_i}| = 2$. Then $\langle ba, c_i\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}} a^{2^{n-1}}, b^{4k_i}, c_i\rangle$, a contradiction.

If $|a^{4h_i}| = 2$ and $b^{4k_i} = 1$, then $\langle ba, c_i\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}}a^{2^{n-1}}, a^{4h_i}, c_i\rangle$.

Therefore, if $a^{4h_i} = 1$ for every i, then G is a group in the family $\mathscr{C}_4$.

Suppose that $|a^{4h_i}| = 2$ for some i. Then $c_i b^{2^{k-2}}$ is an element of order 4, such that $\langle c_i b^{2^{k-2}}\rangle$ is central $\langle a, c_1, \dots, c_r\rangle$ and $c_i^b = c_i^{-1}$. Therefore, we have that G is in the family $\mathscr{C}_5$.

· Suppose now $|b^2| > |a^{4h}|$. We have that ba is an element of order 2^k, with $(ba)^{2^{k-1}} = b^{2^{k-1}}$.

We distinguish two cases depending on the size of the intersection $\langle a\rangle \cap \langle b\rangle$.

Suppose first that $\langle a\rangle \cap \langle b\rangle = 1$. Then, if $|a^{4h_i}| = 2$, then the subgroup $\langle ba, c_1\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}}, a^{2^{n-1}}, c_1\rangle$. Then, we get that $a^{4h_i} = 1$ for every $i \in \{1, \dots, r\}$.

If $b^{4k_i} = 1$ for every i, then G is in the family $\mathscr{C}_5$.

If $|b^{4k_i}| = 2$ for some i, then up to reordering the indices, we may assume that $|b^{4k_1}| = 2$ and, up to replacing in case c_i with c_1c_i, we may assume that c_i is central in G for every $i \geq 2$. Up to replacing c_1 with $c_1 b^{2^{k-2}}$, we get that G lies in $\mathscr{C}_5$.

Suppose now that $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{n-1}}\rangle = \langle b^{2^{k-1}}\rangle$.

If $|b^2| \geq |a|$, then we have that $a^{4h_i} = 1$. In fact, if $|a^{4h_i}| = 2$, then the subgroup $\langle b^{2^r}a, c_i\rangle$, with $|b^{2r}| = |a|$, is not metacyclic, because it contains the 3-generated elementary abelian group $\langle b^{2^{k-2}}a^{2^{n-2}}, a^{2^{n-1}}, c_i\rangle$. Hence, we have that $a^{4h_i} = 1$ for every $i \in \{1, \dots, r\}$. Up to reordering the indices and up to replacing c_i with c_ic_1 for $i \geq 2$, we may assume that $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| \leq 2$ and $b^{c_i} = b$ for every $i \geq 2$. Up to replacing c_1 with $c_1 b^{2^{k-2}k_1}$, we get that G lies in $\mathscr{C}_5$.

Suppose now that $|b^2| < |a|$.

Up to reordering the indices and up to replacing c_i with c_ic_1 for $i \geq 2$, we may assume that $a^{c_1} = a^{1+4h_1}$ with $|a^{4h_1}| \leq 2$ and $a^{c_i} = a$ for every $i \geq 2$. Up to reordering the indices and up to replacing c_i with c_ic_2 for $i \geq 3$, we may assume that $b^{c_1} = b^{1+4k_1}$ with $|b^{4k_1}| \leq 2$, $b^{c_2} = b^{1+4k_2}$ with $|b^{4k_2}| \leq 2$ and $b^{c_i} = b$ for every

$i \geq 3$.
If $a^{4h_1} = 1$, then, up to replacing c_1 with $c_1 b^{2^{k-2}k_1}$ and c_2 with $c_2 b^{2^{k-2}k_2}$, we get that G is a group in $\mathscr{C}_5$.
If $|a^{4h_1}| = 2$, then up to replacing in case b with ba we may assume that $b^{4k_1} = 1$. Up to replacing c_2 with $c_2 b^{2^{k-2}k_2}$, we get that G is in $\mathscr{C}_6$.

□

In the last part of this section we deal with non trivial monotone 2-groups with $G = H_4(G)$ and G/G^4 in $\mathscr{A}_4$. The following lemma determines the structure of a maximal subgroup of such a group G.

Lemma 5.16. *Let G be a non trivial monotone 2-group with $G = H_4(G)$, and G/G^4 isomorphic to a group in $\mathscr{A}_4$.*
Suppose that $G = A\langle b\rangle$ with AG^4/G^4 abelian of exponent 4 and $|A^2G^4/G^4| \geq 4$, $|bG^4| = 4$, $b^2G^4 \notin A^2G^4/G^4$ and $a^bG^4 = a^{-1}G^4$ for every $a \in A$.
The subgroup $H = A\langle b^2\rangle$ is modular and it does not involve Q_8.

Proof. Let $G = \langle a_1, \dots, a_s, c_1, \dots, c_r, b\rangle$, where $A = \langle a_1, \dots, a_s, c_1, \dots, c_r\rangle$ and $AG^4/G^4 = \langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_1G^4\rangle \times \langle c_rG^4\rangle$ is abelian $s \geq 2$, $|a_iG^4| = 4$, $|c_iG^4| = 2$, $|bG^4| = 4$ and $a^bG^4 = a^{-1}G^4$ for every $a \in A$.
Let H be the subgroup $\langle A, b^2\rangle = \langle a_1, \dots, a_s, c_1, \dots, c_r, b^2\rangle$.

We first show that the subgroup H is powerful.
By Lemma 2.7 and Corollary 2.8, we have that G^2 is powerful and $G^4 = \Phi(G^2)$. Since $G^2 = \langle a_1^2, \dots, a_s^2, b^2\rangle G^4$, we have that $G^2 = \langle a_1^2, \dots, a_s^2, b^2\rangle$. Moreover, $G^{2^i} = \langle a_1^{2^i}, \dots, a_s^{2^i}, b^{2^i}\rangle$. In particular, since $G^2 \leq H$, we have that $G^{2^i} \leq H^{2^{i-1}}$ for every $i \geq 1$.
The subgroup A satisfies the hypothesis of Proposition 2.14, and so A is modular and does not involve Q_8. In particular, A is powerful and, in order to conclude that H is powerful, it is sufficient to prove that $[a_i, b^2] \in H^4$, $[c_j, b^2] \in H^4$ for every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r\}$.

Since, $\langle a_1^2, \dots, a_s^2, b^4\rangle G^4/G^4 = H^2G^4/G^4$, we get that $\langle a_1^2, \dots, a_s^2, b^4\rangle G^4 = H^2G^4$. From $G^{2^i} \leq H^{2^{i-1}}$ we have that $G^4 \leq H^2$, and so $H^2G^4 = H^2$. Moreover, $G^4 = \langle a_1^4, \dots, a_s^4, b^4\rangle \leq \langle a_1^2, \dots, a_s^2, b^4\rangle$, and so we obtain $\langle a_1^2, \dots, a_s^2, b^4\rangle G^4 = \langle a_1^2, \dots, a_s^2, b^4\rangle$. Therefore, we have $\langle a_1^2, \cdots, a_s^2, b^4\rangle = H^2$. In particular, H^2 is the Frattini subgroup of a monotone group and so it is powerful. In particular, $H^{2^i} = \langle a_1^{2^i}, \dots a_s^{2^i}, b^{2^{i+1}}\rangle$.

Now, let $a \in A$ such that $a^2 \notin G^4$. The subgroup $\langle a, b\rangle$ is such that $\langle a, b\rangle G^4/G^4$ is isomorphic to K_2. Therefore, by Lemma 2.13, we have that $\langle a, b\rangle$ is non-modular metacyclic, with $\langle a^{2^i}, b^{2^i}\rangle = \langle a, b\rangle^{2^i}$. Then, there exist $x, y \in \langle a, b\rangle$ such that $\langle a, b\rangle = \langle x, y\rangle$ and $x^y = x^{-1+4h}$. Since $\langle b\rangle G^4$ is not normal in $\langle a, b\rangle G^4/G^4$, we have that $b \notin \langle x, y^2\rangle$. Hence, $b = y^j x^i$ with j odd, and so $\langle a, b\rangle = \langle x, b\rangle$, with $\langle x\rangle \trianglelefteq \langle x, b\rangle$ and $x^b = x^{-1+4k}$. Now, $x^{b^2} = x^{1-8k+16k^2}$, and so $[x, b^2] \in \langle x^8\rangle$. Now, $\langle [x, b^2]\rangle = \langle [a, b^2]\rangle$ and so we get that $[a, b^2] \in G^8 \le H^4$. Since A is generated by $\{a \in H : a^2 \notin G^4\}$, the previous argument shows that H is powerful.

Now, $H/H^4 = \langle a_1H^4\rangle \times \cdots \times \langle a_sH^4\rangle \times \langle c_1H^4\rangle \times \cdots \times \langle c_rH^4\rangle \times \langle bH^4\rangle$ is abelian with $|H/H^4| \ge 4$. In fact, from $H^4 \le G^4 \le H^2$, it follows that $|H^2/G^4| \le |H^2/H^4|$. Since $A \le H$, we have that $A^2G^4 \le H^2$, and so, being $|A^2G^4/G^4| \ge 4$, we have that $|H^2/H^4| \ge |H^2/G^4| \ge 4$. Hence, H satisfies the hypothesis of Proposition 2.14 and so H is modular and does not contain a subgroup isomorphic to Q_8. □

The next lemma states some properties of non-trivial monotone 2-groups with $G = H_4(G)$ and G/G^4 isomorphic to a group in $\mathscr{A}_4$.

Lemma 5.17. *Let G be a non-trivial monotone 2-group with $G = H_4(G)$. Suppose that $G = \langle a_1, \ldots, a_s, c_1, \ldots, c_r, b\rangle$, where $\langle a_iG^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_iG^4\rangle \times \cdots \times \langle c_rG^4\rangle$ is abelian with $s \ge 2$, $|a_iG^4| = 4$, $|c_iG^4| = 2$, $|bG^4| = 4$, $b^2G^4 \notin \langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_iG^4\rangle \times \langle c_rG^4\rangle$, $a^bG^4 = a^{-1}G^4$, for every $a \in \langle a_1, \ldots, a_s, c_1, \ldots, a_s\rangle$.*
Let A be the group $\langle a_1, \ldots a_s\rangle$, K be the group $\langle A, b^2\rangle$ and H be the group $\langle K, c_1, \ldots, c_r\rangle$.
Then, the followings hold:

1. *the group K is modular, it does not involve Q_8 and $\langle K, b\rangle^{2^i} = G^{2^i}$ for every $i \ge 1$;*
2. *the group A is modular, it does not involve Q_8 and $A^4 = G^4 \cap A$;*
3. *we may assume that $|c_i| = 2$;*
4. *we may assume that $G = K\langle c_1, \ldots, c_r\rangle$ with $\langle c_1, \ldots, c_r\rangle$ elementary abelian. Moreover, $K^2 \le C_G(c_i)$ for every i, $c_i \in N_G(\langle a\rangle)$ for every $a \in A$ and $c_i \in N_G(\langle b\rangle)$.*

Proof. By Lemma 2.7, G^2 is powerful and $\Phi(G^2) = G^4$. Since $\langle K, b\rangle^2 G^4 = G^2$, we have $\langle K, b\rangle^2 = G^2$ and so $\langle K, b\rangle^{2^i} = G^{2^i}$, for all $i \geq 1$. Moreover, $K \leq H$ and, by Lemma 5.16, the group H is modular and does not involve Q_8. This proves (1).

By Lemma 5.16, the group H is modular and does not involve Q_8. Since A is a subgroup of H, the subgroup A is modular and does not involve Q_8. In particular A is powerful, and so $A^{2^i} = \langle a_1^{2^i}, \dots, a_s^{2^i}\rangle$. Now, AG^4/G^4 is isomorphic to a direct product of s copies of C_4. Hence, we get that $A^2 = A^2(G^4 \cap A)$, i.e. $(G^4 \cap A) \leq A^4 = \Phi(A^2)$. Since A is modular, $A^2/(G^4 \cap A)$ is an elementary abelian group of order 2^s, and A^2 is at most s-generated, we get that $A^4 = (G^4 \cap A)$. Then, A/A^4 is isomorphic to a direct product of s copies of C_4. This proves (2).

Since $c_i^2 \in G^4 = \langle A^4, b^4\rangle$, and since A is powerful, there exists $a \in A \setminus A^2$ such that $c_i^2 \in \langle a, b\rangle^4$. Now, applying Lemma 5.14 to $\langle a, b, c_i\rangle$, we may assume c_i of order 2 and (3) is proved.

In order to show that c_i normalizes every element of A, it is sufficient to check that c_i normalizes every $a \in A\backslash A^2$, because for every $d \in A^2$, being A powerful, there exists $a \in A\backslash A^2$, such that $d \in \langle a\rangle$.
So let $a \in A \setminus A^2$. In particular, $a^2 \notin A^4$, because all the elements of A not in A^2 have order 4 modulo A^4.
Consider now $\langle a, c_i\rangle$. Since c_i has order 2 and $\langle a, c_i\rangle$ has a quotient isomorphic to $C_4 \times C_2$, we have that $\langle a, c_i\rangle$ is not semidihedral. Then, by Lemma 2.11, we have that c_i normalizes $\langle a\rangle$ and $a^{c_i} = a^{1+4h}$, with $|a^{4h}| \leq 2$. Therefore, we have that $c_i \in N_G(\langle a\rangle)$ and $c_i \in C_G(a^2)$, for every $a \in A\backslash A^2$. Since A^2 is generated by $\{a^2 : a \in A\backslash A^2\}$, we get that $c_i \in C_G(A^2)$.
Consider now $\langle b, c_i\rangle$. Since c_i has order 2 and $\langle b, c_i\rangle$ has a quotient isomorphic to $C_4 \times C_2$, we have that $\langle b, c_i\rangle$ is not semidihedral. By Lemma 2.11, we have that c_i normalizes $\langle b\rangle$ and $b^{c_i} = b^{1+4k}$, with $|b^{4k}| \leq 2$. In particular, we get that c_i centralizes $\langle b^2\rangle$. Since $\langle A^2, b^2\rangle$ is the Frattini subgroup of G, we have that c_i centralizes G^2. This proves the second part of (4).

Suppose now that $\langle c_i, c_j\rangle$ is not abelian. Since $c_i^2 = 1$, we get that $[c_i, c_j]^{c_i} = [c_i, c_j]^{-1}$. Now, $[c_i, c_j]$ lies in G^4 and $G^4 \leq C_G(\langle c_i, c_j\rangle)$. Then $[c_i, c_j]$ is in G^4 and has order 2. Now, A is modular, it does not involve a Q_8 and $[A : A^4] \geq 4$. Then, there exists an element d of order 4 in A such that $\langle d\rangle \cap \langle c_i, c_j\rangle = 1$. Since d is in the centralizer of $\langle c_i, c_j\rangle$, we have that G contains a subgroup isomorphic to $C_4 \times D_8$, which is not monotone.

Therefore, we have that $\langle c_i, c_j\rangle$ is abelian and also the first part of (4) is proved. $\square$

Using the previous results, in the following remark, we set up some notations useful to continue our investigation.

Remark 5.18. *Let G be a non trivial monotone 2-group with $G = H_4(G)$. Let $G = \langle a_1, \dots, a_s, c_1, \dots, c_u, b\rangle$, where $\langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_1G^4\rangle \times \cdots \times \langle c_uG^4\rangle$ is abelian with $s \geq 2$, $|a_iG^4| = 4$, $|c_iG^4| = 2$, $|bG^4| = 4$, $b^2G^4 \notin \langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_1G^4\rangle \times \langle c_uG^4\rangle$, $a^bG^4 = a^{-1}G^4$, for every $a \in \langle a_1, \dots, a_s, c_1, \dots, c_u\rangle$.*
Let K be the group $\langle a_1, \dots, a_s, b^2\rangle$, let L be the group $\langle K, b\rangle$ and let H be the group $\langle K, c_1, \dots, c_r\rangle$
Since $G = H_4(G)$, we may assume that $|b| \geq 8$. Now, by Lemma 5.17, the subgroup K is modular and does not involve Q_8.
Since $\langle a_i, b\rangle G^4/G^4$ is isomorphic to K_2, by Lemma 2.13, we have that $G^4 \cap \langle a_i, b\rangle = \langle a_i^4, b^4\rangle$. From $a_i^b(G^4 \cap \langle a_i, b\rangle) = a_i^{-1}(G^4 \cap \langle a_i, b\rangle)$, it follows that $a_i^b = a_i^{-1+4h_i}b^{4k_i}$.
Since $\langle a_i, b^{2^{k_i}}\rangle$ is modular, we get that there exists $x_i \in \langle a_i, b^{2^{k_i}}\rangle$ with $\langle x_i^2\rangle = \langle a_i^{-2+4h_i}b^{4k_i}\rangle$.
In particular, $\langle x_i, b\rangle = \langle a_i, b\rangle$ and $x_i^b = x_i^{-1+4r_i}$.

Let X be the subgroup $\langle x_1, \dots, x_s\rangle$. We have that $K = \langle X, b^2\rangle$, and so X is a modular group that does not involve Q_8.
Moreover, $X^4 = \langle x_1^4, \dots, x_s^4\rangle$ and so X/X^4 is isomorphic to a direct product of s copies of C_4. Clearly, we have that b normalizes X and $L = \langle X, b\rangle$ is such that $L/L^4 = (\langle x_1L^4\rangle \times \cdots \times \langle x_sL^4\rangle) \rtimes \langle bL^4\rangle$, where $|x_iL^4| = 4$, $|bL^4| = 4$ and $x^bL^4 = x^{-1}L^4$ for every $x \in X$.
Moreover, by Lemma 5.17, we may assume that $\langle c_1, \dots, c_u\rangle$ is elementary abelian and so G is equal to $\langle X, b\rangle\langle c_1, \dots, c_u\rangle$.

The next lemma shows that, if X and b are as in Remark 5.18, then the derived subgroup of X is contained in $\Omega_1(X)$.
By Remark 2.1, in order to show that the derived subgroup of X is contained in $\Omega_1(X)$, it is enough to prove that for every i and j in $\{1, \dots, s\}$, the commutator $[x_i, x_j] \in \Omega_1(X)$. So, we prove the following lemma.

Lemma 5.19. *Let X, b and L as in Remark 5.18. Then $|[x_i, x_j]| \leq 2$.*

Proof. Since the subgroup $\langle x_i, x_j\rangle$ is metacyclic, the subgroup $\langle [x_i, x_j]\rangle$ is the derived subgroup of $\langle x_i, x_j\rangle$.

Suppose that $|\langle [x_i, x_j]\rangle| \geq 4$. Since $\langle x_i, x_j\rangle$ is normalized by b, and $\langle x_i, x_j\rangle^{2^l} = \langle x_i^{2^l}, x_j^{2^l}\rangle$ are characteristic subgroups in $\langle x_i, x_j\rangle$, we can consider the quotient $\langle x_i, x_j, b\rangle/\langle x_i, x_j\rangle^{2^l}$ with $i \geq 4$, such that $\langle x_i, x_j, b\rangle/\langle x_i, x_j\rangle^{2^l} = \langle \bar{x_i}, \bar{x_j}, \bar{b}\rangle$, with $\bar{x_i}^{\bar{x_j}} = \bar{x_i}[\bar{x_i}, \bar{x_j}] = \bar{x_i}\bar{x_i}^{4h_i}\bar{x_j}^{4h_j}$ and $|\bar{x_i}^{4h_i}\bar{x_j}^{4h_j}| = 4$, $\bar{x_i}^{\bar{b}} = \bar{x_i}^{-1+4r_i}$, $\bar{x_j}^{\bar{b}} = \bar{x_j}^{-1+4r_j}$. Being a quotient of a monotone group $\langle \bar{x_i}, \bar{x_j}, \bar{b}\rangle$ is still monotone.

The subgroup $\langle \bar{x_i}, \bar{x_j}\rangle$ is metacyclic and modular with derived subgroup of order 4.
Therefore, there exists $\bar{x}$ and $\bar{y}$ in $\langle \bar{x_i}, \bar{x_j}\rangle$, such that $\langle \bar{x}, \bar{y}\rangle = \langle \bar{x_i}, \bar{x_j}\rangle$ and $\bar{x}^{\bar{y}} = \bar{x}^{1+4t}$, where $|\bar{x}^{4t}| = 4$. In particular, we have that $\langle \bar{x}^4, \bar{y}^4\rangle$ is central in $\langle \bar{x}, \bar{y}\rangle$ and we also have that $\langle \bar{x}^2, \bar{y}^2\rangle$ is abelian. Since $\langle \bar{x}^4, \bar{y}^4\rangle = \langle \bar{x_i}^4, \bar{x_j}^4\rangle$, we have that $\bar{x_i}^4$ and $\bar{x_j}^4$ are central in $\langle \bar{x_i}, \bar{x_j}\rangle$ and $\langle \bar{x_i}^2, \bar{x_j}^2\rangle$ is abelian.

Since $\bar{b}$ induces an automorphism on $\langle \bar{x_i}, \bar{x_j}\rangle$, from $\bar{x_i}^{\bar{x_j}} = \bar{x_i}\ \bar{x_i}^{4h_i}\bar{x_j}^{4h_j}$, we have that $(\bar{x_i}^{\bar{b}})^{\bar{x_j}^{\bar{b}}} = \bar{x_i}^{\bar{b}}(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^{\bar{b}}$.
We now show that if $\bar{b}$ inverts the element $\bar{x_j}^{4h_j}\bar{x_i}^{4h_i}$, then we get a contradiction.
In fact,

$$\begin{aligned}
(\bar{x_i}^{\bar{b}})^{\bar{x_j}^{\bar{b}}} &= (\bar{x_i}^{-1+4r_i})^{\bar{x_j}^{-1+4r_j}} \\
&= \bar{x_j}^{1-4r_j}\bar{x_i}^{-1+4r_i}\bar{x_j}^{-1+4r_j} \\
&= \bar{x_j}\bar{x_i}^{-1}\bar{x_j}^{-1}\bar{x_i}^{4r_i} \\
&= (\bar{x_i}^{-1})^{\bar{x_j}^{-1}}\bar{x_i}^{4r_i}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar{x_i}^{\bar{b}}(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^{\bar{b}} &= \bar{x_i}^{-1+4r_i}(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^{-1} \\
&= \bar{x_i}^{-1}\bar{x_i}^{4r_i}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j}.
\end{aligned}$$

Therefore, we have that $(\bar{x_i}^{-1})^{\bar{x_j}^{-1}}\bar{x_i}^{4r_i} = \bar{x_i}^{-1}\bar{x_i}^{4r_i}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j}$ and so $(\bar{x_i}^{-1})^{\bar{x_j}^{-1}} = \bar{x_i}^{-1}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j}$,
which means

$$\begin{aligned}
(\bar{x_i}^{-1}) &= (\bar{x_i}^{-1})^{\bar{x_j}}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j} \\
&= (\bar{x_i}\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^{-1}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j} \\
&= \bar{x_i}^{-1}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j}\bar{x_i}^{-4h_i}\bar{x_j}^{-4h_j}\ . \\
&= \bar{x_i}^{-1}\bar{x_i}^{8h_i}\bar{x_j}^{8h_j} \\
&= \bar{x_i}^{-1}(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^2
\end{aligned}$$

So we get that $(\bar{x_i}^{-1}) = \bar{x_i}^{-1}(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^2$, a contradiction, because $\bar{x_i}^{4h_i}\bar{x_j}^{4h_j}$ has order 4, and so $(\bar{x_i}^{4h_i}\bar{x_j}^{4h_j})^2 \neq 1$. This proves that $\bar{b}$ does not invert $\bar{x_j}^{4h_j}\bar{x_i}^{4h_i}$. In the sequel we refer to $(*)$ to recall this fact.

Since $\bar{b}$ acts as inversion on $\Omega_2(\langle \bar{x_i}\rangle)$ and on $\Omega_2(\langle \bar{x_j}\rangle)$, we have that $\bar{x_i}^{4h_i}\bar{x_j}^{4h_j}$ is neither in $\langle \bar{x_i}\rangle$ nor in $\langle \bar{x_j}\rangle$, otherwise we contradict $(*)$. We also have that $\langle x_i^4, x_j^4\rangle$ is not cyclic (otherwise $\langle x_i^4, x_j^4\rangle = \langle x_i^4\rangle$ or $\langle x_i^4, x_j^4\rangle = \langle x_j^4\rangle$, and again we contradict $(*)$).

We now show that the elements of order 2 in $\langle \bar{x_i}, \bar{x_j}\rangle$ are contained in $\langle \bar{x_i}^4, \bar{x_j}^4\rangle$.
In fact, being $\langle \bar{x_i}, \bar{x_j}\rangle/\langle \bar{x_i}^4, \bar{x_j}^4\rangle$ isomorphic to $C_4 \times C_4$ we have that there are no generators of order 2. So suppose that there exists an element $\bar{z}$ of order 2, such that $\bar{z} \in \langle \bar{x_i}^2, \bar{x_j}^2\rangle \setminus \langle \bar{x_i}^4, \bar{x_j}^4\rangle$. Since the subgroup $\langle \bar{x_i}, \bar{x_j}\rangle$ is modular metacyclic, we have that $\langle \bar{x_i}^4, \bar{x_j}^4\rangle$ is the Frattini subgroup of $\langle \bar{x_i}^2, \bar{x_j}^2\rangle$, and so $\bar{z}$ is a generator in $\langle \bar{x_i}^2, \bar{x_j}^2\rangle$. This implies that $\langle \bar{x_i}^2, \bar{x_j}^2\rangle^2 = \langle \bar{x_i}^4, \bar{x_j}^4\rangle$ is cyclic, a contradiction.

Therefore, $\Omega_1(\langle \bar{x_i}, \bar{x_j}\rangle)$ is contained in $\langle \bar{x_i}^4, \bar{x_j}^4\rangle$ and so it is central.

This implies that $|\bar{x_i}^{4h_i}| \geq 4$ and $|\bar{x_j}^{4h_j}| \geq 4$. In fact, suppose that $|\bar{x_i}^{4h_i}| \leq 2$. Since $|\bar{x_j}^{4h_j}\bar{x_i}^{4h_i}| = 4$ and $\bar{x_i}^{4h_i}$ is central we have that $|\bar{x_j}^{4h_j}| = 4$. It follows that $\bar{b}$ inverts $\bar{x_j}^{4h_j}\bar{x_i}^{4h_i}$, a contradiction to $(*)$. Using the same argument, if we suppose that $|\bar{x_j}^{4h_j}| \leq 2$, then we reach a contradiction.

Hence, we have that $|\bar{x_i}^4| \geq 4$, $|\bar{x_j}^4| \geq 4$ and $\langle \bar{x_i}^4, \bar{x_j}^4\rangle$ is not cyclic.
This means that $\langle \bar{x_i}^2, \bar{x_j}^2\rangle$ is an abelian group, with non-cyclic Frattini subgroup and such that $(\bar{x_i}^2)^{\bar{b}} = (\bar{x_i}^2)^{-1+4r_i}$ and $(\bar{x_j}^2)^{\bar{b}} = (\bar{x_j}^2)^{-1+4r_j}$. Moreover, the structure of the quotient $\langle x_i, x_j, b\rangle L^4/L^4$ guarantees that $\langle \bar{b}\rangle \cap \langle \bar{x_i}^2, \bar{x_j}^2\rangle \leq \langle \bar{x_i}^4, \bar{x_j}^4\rangle \cap \langle b^4\rangle$.

We may assume, without loss of generality, that $|\bar{x_i}^2| \geq |\bar{x_j}^2|$.
We distinguish three cases depending on the structure and the intersections of $\Omega_1(\langle \bar{b}, \bar{x_j}^2\rangle)$, $\Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle)$ and of $\Omega_1(\langle \bar{x_i}^2, \bar{x_j}^2\rangle)$.

- Suppose that $\Omega_1(\langle \bar{b}, \bar{x_j}^2\rangle) \neq \Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle)$. If $|(\bar{x_j}^2)^{4r_j-4r_i}| \geq 4$, then the subgroup $\langle \bar{x_i}^2\bar{x_j}^2, \bar{b}\rangle$ contains the 3-generated subgroup $\Omega_1(\langle x_1^2, x_j^2, b\rangle)$, a contradiction.
  Therefore, we have that $\bar{x}^{\bar{b}} = \bar{x}^{-1+4r_i}$, for every $\bar{x} \in \langle \bar{x_i}^4, \bar{x_j}^4\rangle$. In particular, $\bar{b}$ inverts every element of order 4 in $\langle \bar{x_i}^4, \bar{x_j}^4\rangle\rangle$, a contradiction to $(*)$.

- Suppose that $\Omega_1(\langle \bar{b}, \bar{x_j}^2\rangle) = \Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle)$ and $\Omega_1(\langle \bar{x_i}^2, \bar{x_j}^2\rangle) \neq \Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle)$.
  Then, we have that $\Omega_2(\langle \bar{x_j}^2\rangle) \not\leq \Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle)$. Therefore, there exists an element $\bar{x_i}^{2i_1}\bar{x_j}^{2i_2}$ such that $\Omega_1(\langle \bar{x_i}^2, \bar{x_i}^{2i_1}\bar{x_j}^{2i_2}\rangle) = \Omega_1(\langle \bar{x_i}^2, \bar{x_j}^2\rangle)$.
  If $|(\bar{x_j}^2)^{4r_j-4r_i}| \geq 4$, then the subgroup $\langle \bar{x_i}^{2i_1}\bar{x_j}^{2i_2}, \bar{b}\rangle$ contains the 3-generated subgroup $\Omega_1(\langle x_1^2, x_j^2, b\rangle)$, a contradiction. Therefore, $\bar{x}^{\bar{b}} = \bar{x}^{-1+4r_i}$, for every $\bar{x} \in \langle \bar{x_i}^4, \bar{x_j}^4\rangle$. In particular, $\bar{b}$ inverts every element of order 4 in $\langle \bar{x_i}^4, \bar{x_j}^4\rangle\rangle$, a contradiction to $(*)$.

- To conclude, suppose that $\Omega_1(\langle \bar{b}, \bar{x_j}^2\rangle) = \Omega_1(\langle \bar{x_i}^2, \bar{b}\rangle) = \Omega_1(\langle \bar{x_i}^2, \bar{x_j}^2\rangle)$.
  We have that $\langle \bar{x_i}^2, \bar{x_j}^2\rangle \cap \langle \bar{b}\rangle = \langle b^{4\gamma}\rangle$. Therefore, there exists a generator $\bar{x_i}^{2i_1}\bar{x_j}^{2i_2}$ such that $(\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{2\alpha} = \bar{b}^{4\gamma}$. It follows that for some γ_1 and α_1 such that $\langle \bar{b}^{4\gamma_1}\rangle = \langle \bar{b}^{4\gamma}\rangle = \langle (\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{2\alpha_1}\rangle = \langle (\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{2\alpha}\rangle$, the element $(\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{\alpha_1}\bar{b}^{-2\gamma_1}$ is an involution which is not in $\Omega_1(\langle \bar{b}, \bar{x_j}\rangle)$ (otherwise $b^{2\gamma} \in \langle x_i, x_j\rangle$).
  Moreover, since $\bar{x_i}^{2i_1}\bar{x_j}^{2i_2}$ is a generator, at least one of i_1 and i_2 is odd.

  Suppose that i_1 is odd. Now, if $|(\bar{x_i}^2)^{4r_i-4r_j}| \geq 4$, then $\langle \bar{x_i}^{2i_1}\bar{x_j}^{2i_2}, \bar{b}\rangle$ is not metacyclic, because it contains $\Omega_1(\langle x_i, b\rangle) \times \langle (\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{\alpha_1}\bar{b}^{-2\gamma_1}\rangle$.

  Suppose that i_2 is odd. If $|(\bar{x_j}^2)^{4r_j-4r_i}| \geq 4$, then $\langle \bar{x_i}^{2i_1}\bar{x_j}^{2i_2}, \bar{b}\rangle$ is not metacyclic, because it contains $\Omega_1(\langle x_j, b\rangle) \times \langle (\bar{x_i}^{2i_1}\bar{x_j}^{2i_2})^{\alpha_1}\bar{b}^{-2\gamma_1}\rangle$.
  This implies that, in both cases $\bar{x}^{\bar{b}} = \bar{x}^{-1+4r}$, for some r, and for every $\bar{x} \in \langle \bar{x_i}^4, \bar{x_j}^4\rangle$. In particular, the element $\bar{b}$ inverts every element of order 4 in $\langle \bar{x_i}^4, \bar{x_j}^4\rangle$, a contradiction to $(*)$.

This shows that even this case does not arise, and finally proves that $|[x_i, x_j]| \leq 2$. □

Now, the previous lemma shows that $[X, X] \leq \Omega_1(X)$ and, by construction, X/X^4 is isomorphic to a direct product of s copies of C_4. In a modular metacyclic group where the derived subgroup has order at most 2, the Frattini subgroup is central. Recalling that X is modular and using Lemma 5.19, we get that X^2 is central in X. Moreover, being $\Omega_1(X)$ contained in X^2, we have that $\Omega_1(X)$ is central in X.

In the next lemma, we study the size of the intersection of X and $\langle b\rangle$.

Lemma 5.20. *Let G, L, X and b be as in Remark 5.18.*
Then $X \cap \langle b\rangle \leq \Omega_1(\langle b\rangle)$.

Proof. In order to show that $X \cap \langle b\rangle \leq \Omega_1(\langle b\rangle)$, it is sufficient to show that there are no element of order 4 in X centralized by b.
The proof is done by induction on the exponent of X.
If $exp(X) = 4$, then X is abelian and its generators are inverted by b. Therefore the claim holds.
Suppose now that $exp(X) = 2^n \geq 8$. Let x be an element of order 4 of X. If $x^2 \notin X^{2^{n-1}}$, then x is an element of order 4 in $X/X^{2^{n-1}}$, and so, by the inductive hypothesis, x is not centralized in $X/X^{2^{n-1}}$ by b. Therefore,

b does not centralize x in X.
If $x^2 \in X^{2^{n-1}}$, then $\langle x \rangle = \Omega_2(\langle y \rangle)$, where y is an element of maximal order in X. Since b acts as inversion on X/X^4 and $X^{2^i} = \langle x_1^{2^i}, \ldots, x_s^{2^i} \rangle$, we have that b acts as inversion on every abelian section of the form $X^{2^i}/X^{2^{i+2}}$. In particular, b acts as inversion on $X^{2^{n-2}}$. Since y is an element of maximal order, we get that x is an element of order 4 in $X^{2^{n-2}}$. Hence, b inverts x. Therefore, there are no element of order 4 in X centralized by b.
It follows that $|X \cap \langle b \rangle| \leq 2$, and so $X \cap \langle b \rangle \leq \Omega_1(\langle b \rangle)$. □

In the following lemmas, we determine the subgroup $L = \langle X, b \rangle$.
Since, by Lemma 5.20, the intersection $X \cap \langle b \rangle$ has order at most 2, we study separately the two cases. Indeed, in Lemma 5.21, we study the group L when the intersection $X \cap \langle b \rangle$ is trivial. In Lemma 5.23, we study the group L when the intersection $X \cap \langle b \rangle$ has order 2.

Lemma 5.21. *Let G, L, X and b be as in Remark 5.18.*
Suppose that $X \cap \langle b \rangle = 1$.
Then L is in $\mathscr{C}_5$.

Proof. Let x be in X. Since X is normalized by b and $X \cap \langle b \rangle = 1$, we have that $\langle x, b \rangle \cap X$ is normalized by b. This intersection is cyclic and contains $\langle x \rangle$. So, we get that $\langle x \rangle$ is normalized by $\langle b \rangle$.
Since this holds for every $x \in X$, we have that b is a power automorphism of $\langle x_1, \ldots, x_s \rangle$. Moreover, $\langle x_1, \ldots, x_s \rangle$ is modular without Q_8, and so, up to renaming the generators, we may assume that either $\langle x_1, \cdots, x_s \rangle$ is abelian or $\langle x_1, \ldots, x_{s-1} \rangle$ is abelian and $x^{x_s} = x^{1+4t}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$.

If $\langle x_1, \ldots, x_s \rangle$ is abelian, then, since b acts as a power automorphism, by Lemma 1.5.4. on page 32 of [13], we have that b is a universal automorphism on X. It means that $L = \langle X, b \rangle$ is in $\mathscr{C}_5$.

Therefore, from now on, we suppose that $\langle x_1, \ldots, x_s \rangle$ is non-abelian.
We may assume that $\langle x_1, \ldots, x_{s-1} \rangle = \langle x_1 \rangle \times \cdots \times \langle x_{s-1} \rangle$ is abelian, $|x_i| \leq |x_j|$ for every $1 \leq j \leq i \leq s-1$ and $x^{x_s} = x^{1+4t}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$.
Since b is a power automorphism on the abelian group $\langle x_1, \ldots, x_{s-1} \rangle$, we have that b is a universal automorphism on $\langle x_1, \ldots, x_{s-1} \rangle$, i.e. $x^b = x^{-1+4r}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$.
Furthermore, $x_s^b = x_s^{-1+4r_s}$.
We have that $x^{bx_s} = x^{x_s b}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$, i.e. $[b, x_s]$ is in the centralizer of $\langle x_1, \ldots, x_{s-1} \rangle$. Since $\langle [b, x_s] \rangle = \langle x_s^2 \rangle$, we get that x_s acts as a non-

trivial automorphism of order 2. In particular, if $2^n = exp(\langle x_1, \ldots, x_{s-1}\rangle) = |x_1|$, then $x^{x_s} = x^{1+2^{n-1}}$ for every x in $\langle x_1, \ldots, x_{s-1}\rangle$.

We deal separately with the cases $|x_1| > |x_s|$, $|x_1| = |x_s|$, and $|x_1| < |x_s|$. Put $|x_1| = 2^n$, $|x_i| = 2^{n_i}$ (for $i \in \{2, \ldots, s-1\}$), $|x_s| = 2^m$ and $|b| = 2^k$.

1. Suppose first that $|x_s| > |x_1|$.

   Since $X \cap \langle b\rangle = 1$, we have that the 3-generated subgroup $\Omega_1(\langle x_1, x_s, b\rangle)$ is equal to $\Omega_1(\langle x_1, x_s\rangle) \times \Omega_1(\langle b\rangle)$.

   We show the following fact:
   Claim 1: for every $x \in \langle x_1, \ldots, x_{s-1}\rangle$ such that $|x| = 2^n$, we have that $\Omega_1(\langle x\rangle) = \Omega_1(\langle x_s\rangle)$.
   Suppose that $\Omega_1(\langle x\rangle) \neq \Omega_1(\langle x_s\rangle)$.
   In particular, $\Omega_1(\langle x, x_s\rangle) = \langle x_s^{2^{m-1}}, x^{2^{n-1}}\rangle$ and, since $X \cap \langle b\rangle = 1$, it follows that $\Omega_1(\langle x, x_s, b\rangle) = \langle x_s^{2^{m-1}}, x^{2^{n-1}}, b^{2^{k-1}}\rangle$.
   Note that $(xx_s^2)^b = (xx_s^2)^{-1+4r_s} x^{-4r+4r_s}$. We show that if $x^{-4r+4r_s} \neq 1$, then $\langle xx_s^2, b\rangle$ is not metacyclic. In fact, since $|x| < |x_s|$, we have that $\Omega_1(\langle xx_s^2\rangle) = \langle x_s^{2^{m-1}} x^{2^{n-1}j}\rangle$, where $j \in \{0, 1\}$. Therefore, since $x^{-4r+4r_s} \neq 1$, we have that $x^{2^{n-1}} \in \langle xx_s^2, b\rangle$. This implies that $\Omega_1(\langle x, x_s\rangle) \leq \langle xx_s^2, b\rangle$. Thus, $\langle xx_s^2, b\rangle$ contains the 3-generated elementary abelian subgroup $\Omega_1(\langle x, x_s, b\rangle)$, a contradiction.
   In the rest of this chapter, we implicitly use the previous argument, each time we need to show that a certain 2-generated group contains a 3-generated subgroup.
   Therefore, we have that $x^b = x^{-1+4r_s}$.
   Now, being $|b^2| \geq |x_s^{8r_s}| > |x^{8r_s}|$, we have that $\langle x_s, bx\rangle$ is not metacyclic. Indeed, $\Omega_1(\langle bx\rangle) = \langle b^{2^{k-1}} x^{2^{n-1}j}\rangle$ where $j \in \{0, 1\}$, $x_s^{bx} = x_s^{-1+4r_s} x^{2^{n-1}}$ and so $\langle x_s, bx\rangle$ contains the 3-generated elementary abelian group $\Omega_1(\langle x, x_s, b\rangle)$, a contradiction. This concludes the proof of *Claim 1*.

   In particular, *Claim 1* implies that $\langle x_1, \ldots, x_{s-1}\rangle^{2^{n-1}} = \Omega_1(\langle x_1\rangle) = \Omega_1(\langle x_s\rangle)$.

   Since $\langle x_1, x_s\rangle$ is a 2-generated modular group, there exists an even integer i_s such that $\Omega_1(\langle x_1, x_s\rangle) = \Omega_1(\langle x_1 x_s^{i_s}\rangle) \times \Omega_1(\langle x_s\rangle)$.
   Therefore, if $x_1^{-4r+4r_s} \neq 1$, then the subgroup $\langle x_1 x_s^{i_s}, b\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s, b\rangle)$.

It follows that $x_1^b = x_1^{-1+4r_s}$.
Since $x_s^{b^2} = x_s^{1-8r_s+16r_s^2}$, we get that $|b^2| \geq |x_s^{8r_s}|$.

We now show that if $|b^2| \geq |x_s^{4r_s}|$, then we reach a contradiction.
In fact, suppose that $|b^2| \geq |x_s^{4r_s}|$. Since $\Omega_1(\langle bx_s\rangle)$ is equal to $\langle b^{2^{k-1}} x_s^{2^{m-1}j}\rangle$, where $j \in \{0,1\}$, and $(x_1x_s^{i_s})^{bx_s} = (x_1x_s^{i_s})^{-1+4r_s}x_1^{2^{n-1}}$, we have that the subgroup $\langle x_1x_s^{i_s}, bx_s\rangle$ is not metacyclic because it contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s, b\rangle)$.

Therefore, we have $|b^2| = |x_s^{8r_s}|$. Replacing x_1 with $x_1b^{2^{k-1}}$, we get that $\langle x_1b^{2^{k-1}}, x_2, \ldots, x_s\rangle$ is abelian, and $x^b = x^{-1+4r_s}$ for every x in $\langle x_1b^{2^{k-1}}, x_2, \ldots, x_s\rangle$.
This means that L is in $\mathscr{C}_5$.

2. Suppose that $|x_1| = |x_s|$.

   Since $X \cap \langle b\rangle = 1$, we have that $\Omega_1(\langle x_1, x_s, b\rangle) = \Omega_1(\langle x_1, x_s\rangle) \times \Omega_1(\langle b\rangle)$. We distinguish two cases depending on $\Omega_1(\langle x_s\rangle)$.

   Suppose first that $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.
   Since $(x_1x_s)^b = (x_1x_s)^{-1+4r_s}x_1^{-4r+4r_s+2^{n-1}}$, if $x_1^{-4r+4r_s+2^{n-1}} \neq 1$, then the subgroup $\langle x_1x_s, b\rangle$ is not metacyclic because it contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s, b\rangle)$.
   Therefore, we have $x_1^b = x_1^{-1+4r_s+2^{n-1}}$. In particular, $x_1^{b^2} = x_1^{1-8r_s+16r_s^2}$ and so $|b^2| \geq |x_1^{8r_s}|$.
   If $|b^2| \geq |x_1^{4r_s}|$, then the subgroup $\langle x_s, bx_1\rangle$ is not metacyclic because it contains the 3-generated group $\Omega_1(\langle x_1, x_s, b\rangle)$.
   This means that we have $|b^2| = |x_1^{8r_s}|$ and so $x_sb^{2^{k-1}}$ centralizes $\langle x_1, \ldots, x_{s-1}\rangle$. Moreover, since $(x_sb^{2^{k-1}})^b = x_s^{-1+4r_s+2^{n-1}}$, we have that L is in $\mathscr{C}_5$. This concludes the case $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.

   From now on, we assume that $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle x_1\rangle)$.
   In particular, this yields $\langle x_1, \ldots, x_s\rangle^{2^{n-1}} = \Omega_1(\langle x_1\rangle)$.
   The element x_1x_s is such that $\Omega_1(\langle x_1, x_s\rangle) = \Omega_1(\langle x_1x_s\rangle) \times \Omega_1(\langle x_s\rangle)$.
   We have that $(x_1x_s)^b = (x_1x_s)^{-1+4r_s}x_1^{-1+4r+2^{n-1}}$. Now, if $x_1^{-4r+4r_s+2^{n-1}} \neq 1$, then the subgroup $\langle x_1x_s, b\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\Omega_1(\langle x_1, x_s, b\rangle)$.
   Therefore, we have that $x_1^b = x_1^{-1+4r_s+2^{n-1}}$ and so also $x^b = x^{-1+4r_s+2^{n-1}}$ for every $x \in \langle x_1, \ldots, x_{s-1}\rangle$. In particular, $x_1^{b^2} = x_1^{1-8r_s+16r_s^2}$ and so $|b^2| \geq |x_1^{8r_s}|$. Now, if $|b^2| \geq |x_1^{4r_s}|$, then the subgroup $\langle x_sx_1, bx_s\rangle$ is not metacyclic. In fact, since $(x_1x_s)^{bx_s} = (x_1x_s)^{-1+4r_s}x_1^{2^{n-1}}$ and

$\Omega_1(\langle bx_s\rangle) = \langle b^{2^{k-1}} x_s^{2^{m-1}j}\rangle$, where $j \in \{0,1\}$, we get that $\langle x_s x_1, bx_s\rangle$ contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s, b\rangle)$. Therefore, we have that $|b^2| = |x_s^{8r_s}|$. Up to replacing x_s with $x_s b^{2^{k-1}}$, we get that L is in $\mathscr{C}_5$.

3. Suppose, to conclude, that $|x_1| > |x_s|$.

   Since $X \cap \langle b\rangle = 1$, we have that $\Omega_1(\langle x_1, x_s, b\rangle) = \Omega_1(\langle x_1, x_s\rangle) \times \Omega_1(\langle b\rangle)$.

   We first prove that we may assume that $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.
   Suppose that $\Omega_1(\langle x_1\rangle) = \Omega_1(\langle x_s\rangle)$. Since $\langle x_1, x_s\rangle$ is modular and 2-generated, there exists $x_1^{2i_1}x_s$ such that $\Omega_1(\langle x_1, x_s\rangle) = \Omega_1(\langle x_1^{2i_1}x_s\rangle) \times \Omega_1(\langle x_s\rangle)$. Now, we have $(x_1^{2i_1}x_s)^b = (x_1^{2i_1}x_s)^{-1+4r}x_s^{-4r+4r_s}$. Therefore, if $x_s^{-4r+4r_s} \neq 1$, then the subgroup $\langle x_1^{2i_1}x_s, b\rangle$ is not metacyclic.
   Hence, we have that $x_1^{2i_1}x_s$ is such that $(x_1^{2i_1}x_s)^b = (x_1^{2i_1}x_s)^{-1+4r}$, $\Omega_1(\langle x_1^{2i_1}x_s\rangle) \neq \Omega_1(\langle x_1\rangle)$, and $x_1^{x_1^{2i_1}x_s} = x_1^{1+2^{n-1}}$. Therefore, up to replacing x_s with $x_1^{2i_1}x_s$ we may assume that $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.

   From now on, we assume that $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.
   Since $(x_1^2x_s)^b = (x_1^2x_s)^{-1+4r}x_s^{-4r+4r_s}$, if $x_s^{-4r+4r_s} \neq 1$, then the subgroup $\langle x_1^2x_s, b\rangle$ is not metacyclic (it contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s, b\rangle)$). This means that $x_s^b = x_s^{-1+4r}$. Since $x_1^{b^2} = x_1^{1-8r+16r^2}$, we have that $|b^2| \geq |x_1^{8r}|$.
   Now, if $|b^2| \geq |x_1^{4r}|$, then the subgroup $\langle x_s, bx_1\rangle$ is not metacyclic (it contains $\Omega_1(\langle x_1, x_s, b\rangle)$, which is 3-generated). If $|x_1^{8r_s}| = |b^2|$, then, up to replacing x_s with $x_s b^{2^{k-1}}$, we get that $\langle x_1, \ldots, x_s\rangle$ is abelian and $x^b = x^{-1+4r}$ for every $x \in X$. Therefore, L is in $\mathscr{C}_5$.

$\square$

Just for the next lemma, we do not strictly use the notation defined in Remark 5.18. In fact, to improve the presentation of the proof of Lemma 5.23, it is convenient to show a preliminary lemma where X and the automorphism of b on X are as in Remark 5.18, but the order of b is 4.

Lemma 5.22. *Let X and the automorphism of b on X be as in Remark 5.18. Suppose that the order of b is 4, and $X \cap \langle b\rangle = 1$.*
Then, there exists $\bar{X}$ a subgroup of L such that $\langle \bar{X}, b^2\rangle = \langle X, b^2\rangle$ and $L = \bar{X} \rtimes \langle b\rangle$, where $x^b = x^{-1+4r}$ for every $x \in X$ with $|X^{4r}| \leq 4$.

Proof. Let x be in X. Since X is normalized by b and $X \cap \langle b \rangle = 1$, we have that $\langle x, b \rangle \cap X$ is normalized by b. This intersection is cyclic and contains $\langle x \rangle$. So, we get that $\langle x \rangle$ is normalized by $\langle b \rangle$.
Since this holds for every $x \in X$, we have that b is a power automorphism of $\langle x_1, \ldots, x_s \rangle$. Moreover, $\langle x_1, \ldots, x_s \rangle$ is modular without Q_8, and so, up to renaming the generators, we may assume that either $\langle x_1, \cdots, x_s \rangle$ is abelian or $\langle x_1, \ldots, x_{s-1} \rangle$ is abelian and $x^{x_s} = x^{1+4t}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$.

If $\langle x_1, \ldots, x_s \rangle$ is abelian, then, since b acts as a power automorphism, by Lemma 1.5.4. on page 32 of [13], we have that b is a universal automorphism and so the lemma holds with $X = \bar{X}$.

So, suppose now that X is not abelian. Since X modular without Q_8 and b acts as a power automorphism, we may assume that $X = \langle x_1, \ldots, x_{s-1} \rangle \langle x_s \rangle$, where $\langle x_1, \ldots, x_{s-1} \rangle = \langle x_1 \rangle \times \cdots \times \langle x_{s-1} \rangle$ is abelian and $x_s^b = x_s^{-1+4r_s}$, $x^{x_s} = x^{1+4s}$, $x^b = x^{-1+4r}$, for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$.
Using Lemma 5.19 and the fact that b has order 4, we obtain that $x^{x_s} = x^{1+2^{n-1}}$ for every $x \in \langle x_1, \ldots, x_{s-1} \rangle$, $exp(\langle x_1, \ldots, x_{s-1} \rangle)^{4r} \leq 4$ and $|x_s^{4r_s}| \leq 4$. We study separately the cases $|x_1| \geq |x_s|$ and $|x_1| < |x_s|$.

- Suppose first that $|x_1| \geq |x_s|$.
  Then, there exists α such that $x_1^\alpha x_s$ satisfies $\Omega_1(\langle x_1, x_s \rangle) = \Omega_1(\langle x_1^\alpha x_s \rangle) \times \Omega_1(\langle x_1 \rangle) = \Omega_1(\langle x_1^\alpha x_s \rangle) \times \Omega_1(\langle x_s \rangle)$.
  Now, $(x_1^\alpha x_s)^b = x_1^{(-1+4r)\alpha} x_s^{-1+4r_s} = (x_1^\alpha x_s)^{-1+4r+2^{n-1}\alpha} x_s^{-4r+4r_s+2^{n-1}\alpha}$ and so we get that $x_s^b = x_s^{-1+4r+2^{n-1}\alpha}$.
  If $|x_1^{4r}| \leq 2$, then the subgroup $\langle x_1^\alpha x_s, bx_1 \rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\Omega_1(\langle x_1^\alpha x_s \rangle) \times \Omega_1(\langle x_1 \rangle) \times \langle b^2 x_1^{4r} \rangle$.
  Therefore, we have that $|x_1^{4r}| = 4$.
  Now, up to replacing x_s with $x_s b^2$, we get that $\langle x_1, \ldots, x_s b^2 \rangle$ is abelian on which b acts as a universal automorphism, and the lemma holds with $\bar{X} = \langle x_1, x_2, \ldots, x_s b^2 \rangle$.

- Suppose now that $|x_s| > |x_1|$.
  Since $\langle x_1, x_s \rangle$ is modular and 2-generated, there exists an even integer α such that $\Omega_1(\langle x_1, x_s \rangle) = \Omega_1(\langle x_1 x_s^\alpha \rangle) \times \Omega_1(\langle x_1 \rangle) = \Omega_1(\langle x_1 x_s^\alpha \rangle) \times \Omega_1(\langle x_s \rangle)$.
  In particular, $(x_1 x_s^\alpha)^b = (x_1 x_s^\alpha)^{-1+4r_s} x_1^{-4r+4r_s}$, and so we get that $x_1^b = x_1^{-1+4r_s}$ (otherwise the subgroup $\langle x_1 x_s^\alpha, b \rangle$ is not metacyclic).
  Since $|x_s^{4r_s}| \leq 4$, we get that $|x_1^{4r_s}| \leq 2$.

If $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$, then the subgroup $\langle x_s, bx_1\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\Omega_1(\langle x_s, x_1\rangle) \times \langle b^2 x_1^{4r_s}\rangle$.
Therefore, we have that $\Omega_1(\langle x_1\rangle) = \Omega_1(\langle x_s\rangle)$, and so, in particular, the subgroup $\langle x_1, \ldots, x_{s-1}\rangle$ is such that $\langle x_1, \ldots, x_{s-1}\rangle^{2^{n-1}} = \Omega_1(\langle x_1\rangle) = \Omega_1(\langle x_s\rangle)$.
If $|x_s^{4r_s}| \leq 2$, then the subgroup $\langle x_1 x_s^{\alpha}, bx_s\rangle$ contains the 3-generated elementary abelian group $\Omega_1(\langle x_1, x_s\rangle) \times \langle b^2 x_s^{4r_s}\rangle$. Therefore, we have that $|x_s^{4r_s}| = 4$, and so, up to replacing x_1 with $x_1 b^2$, we get that $\langle x_1 b^2, x_2, \ldots, x_s\rangle$ is abelian, on which b acts as a universal automorphism. So the lemma holds with $\bar{X} = \langle x_1 b^2, x_2, \ldots, x_s\rangle$.

□

Lemma 5.23. *Let G, X, b, and L be as in Remark 5.18.*
Suppose that $|X \cap \langle b\rangle| = 2$.
Then L is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$.

Proof. Let $x \in X$. Since X is normalized by $\langle b\rangle$ and $X \cap \langle b\rangle = \Omega_1(\langle b\rangle)$, we have that $\langle x, b\rangle \cap X$ is normalized by b. This intersection is contained in $\langle x, b^{2^{m-1}}\rangle$.

Since $\Omega_1(\langle b\rangle) \leq \Omega_1(X)$, we get that $\Omega_1(\langle b\rangle)$ is central in $\langle X, b\rangle$ (see Lemma 5.19), and we can consider the quotient $\langle X, b\rangle/\Omega_1(\langle b\rangle)$.
The first paragraph of this proof shows that b acts as a power automorphism on $\langle X, b\rangle/\Omega_1(\langle b\rangle)$.
Since $b\Omega_1(\langle b\rangle)$ has order at least 4 and $X/\Omega_1(\langle b\rangle) \cap \langle b\Omega_1(\langle b\rangle)\rangle = \Omega_1(\langle b\rangle)$, by Lemma 5.21 and Lemma 5.22, we may assume that $b\Omega_1(\langle b\rangle)$ acts as a non-modular universal automorphism on the abelian group $X/\Omega_1(\langle b\rangle)$.

We distinguish two cases, depending on the structure of X.

1. Suppose first that X is abelian. Hence, there exists $\{x_1, \ldots, x_s\}$ in X such that $X = \langle x_1\rangle \times \cdots \times \langle x_s\rangle$, $|x_i| \geq |x_j|$ for every i and j with $i \leq j$ and $x_i^b = x_i^{-1+4r} b^{2^{k-1}k_i}$, where $k_i \in \{0, 1\}$.
   Put $|x_1| = 2^n$, $|x_i| = 2^{n_i}$ (for $i \in \{2, \ldots, s\}$) and $|b| = 2^k$.
   We treat separately two cases depending on the size of $\langle x_1\rangle \cap \langle b\rangle$.

   (a) Suppose first that $\langle x_1\rangle \cap \langle b\rangle = 1$.
   The element $b^{2^{k-1}}$ is central in X, and so $|b^2| > exp(\langle x_1, \ldots, x_s\rangle)^{8r}$.

Since $exp(\langle x_1, \ldots, x_s\rangle) = |x_1|$, we obtain that $|x_1^{4r}| \leq |b^2|$.
We deal separately with the cases $|b^2| > |x_1^{4r}|$ and $|b^2| = |x_1^{4r}|$.
Suppose first that $|b^2| > |x_1^{4r}|$.
Since $b^{2^{k-2}}$ is central in X, up to replacing x_i with $x_i b^{2^{k-2}k_i}$, we may assume that $b^{2^{k-1}k_i} = 1$ for every $i \in \{1, \ldots, s\}$. It follows that L is in $\mathscr{C}_5$.

So, from now on, we assume that $|b^2| = |x_1^{4r}|$. In order to prove that this condition implies that L is in $\mathscr{C}_5$, we divide the proof in two parts: in *Claim 1* we study the case $b^{2^{k-1}k_1} = 1$ and in *Claim 2* we study the case $|b^{2^{k-1}k_1}| = 2$.
Claim 1: if $b^{2^{k-1}k_1} = 1$, then $b^{2^{k-1}k_i} = 1$ for every $i \in \langle 2, \cdots, s\rangle$. In particular, this would yield that $x^b = x^{-1+4r}$ for every $x \in X$, and so L is in $\mathscr{C}_5$.

Suppose that $b^{2^{k-1}k_1} = 1$, and suppose, by contradiction, that for some $i \geq 2$ we get that $|b^{2^{k-1}k_i}| = 2$. This implies that $\langle x_i\rangle \cap \langle b, x_1\rangle \neq 1$. In fact, if $\langle x_i\rangle \cap \langle b, x_1\rangle = 1$, then the subgroup $\langle x_i, bx_1\rangle$ is not metacyclic (it contains the 3-generated subgroup $\langle b^{2^{k-1}} x_1^{2^{n-1}}, x_i^{2^{n_i-1}}, b^{2^{k-1}}\rangle$).
Since $\langle x_i\rangle \cap \langle x_1\rangle = 1$ and $\Omega_1(\langle x_1, b\rangle) = \Omega_1(\langle x_1^{2^{n-1}}, b^{2^{k-1}}\rangle)$, we obtain that either $x_i^{2^{n_i-1}} = b^{2^{k-1}}$ or $x_i^{2^{n_i-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$.
In the first case, the subgroup $\langle x_1^{\alpha} x_i, bx_1\rangle$, where $|x_1^{\alpha}| = |x_i|$, is not metacyclic, because it contains the 3-generated elementary abelian group $\langle x_i^{2^{n_i-2}} b^{2^{k-2}}, x_1^{2^{k-1}}, b^{2^{k-1}}\rangle$.
On the other hand, if $x_i^{2^{n_i-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$, then $\langle x_i, bx_1\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian group $\langle x_i^{2^{n_i-2}} (bx_1)^{2^{k-2}}, x_1^{2^{k-1}}, b^{2^{k-1}}\rangle$.
This contradiction shows that there are no $i \geq 2$ such that $|b^{2^{k-1}k_i}| = 2$, and so *Claim 1* is proved.
Claim 2: if $|b^{2^{k-1}k_1}| = 2$, then L is in $\mathscr{C}_5$.
We divide the proof of the claim in three steps.

The first step consists in proving that for every i such that $\langle x_i\rangle \cap \langle b, x_1\rangle = 1$ or $|x_i| < |x_1|$, then $b^{2^{k-1}k_i} = 1$.

Suppose that there exists $i \in \{2, \ldots, s\}$ such that $\langle x_i\rangle \cap \langle b, x_1\rangle = 1$ and $|b^{2^{k-1}k_i}| = 2$. Then, the subgroup $\langle x_i, bx_1\rangle$ is not metacyclic (it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}} x_1^{2^{n-1}}, x_i^{2^{n_i-1}}, b^{2^{k-1}}\rangle$).

Suppose now that there exists $i \in \{2, \dots, s\}$ such that $\langle x_i \rangle \cap \langle b, x_1 \rangle \neq 1$, with $|x_i| < |x_1|$ and $|b^{2^{k-1}k_i}| = 2$. Since $\langle x_i \rangle \cap \langle x_1 \rangle = 1$, $\langle x_i \rangle \cap \langle b, x_1 \rangle \neq 1$ and $\Omega_1(\langle x_1, b \rangle) = \langle x_1^{2^{n-1}}, b^{2^{k-1}} \rangle$, we get that either $x_i^{2^{n_i-1}} = b^{2^{k-1}}$ or $x_i^{2^{n_i-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$. In the first case, let α be an even integer such that $|x_1^{\alpha}| = |x_i|$. The subgroup $\langle x_1^{\alpha} x_i, bx_1 \rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle x_i^{2^{n_i-2}} b^{2^{k-2}}, b^{2^{k-1}} x_1^{2^{n-1}}, b^{2^{k-1}} \rangle$. On the other hand, if $x_i^{2^{n_i-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$, then the subgroup $\langle x_i, bx_1 \rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle x_i^{2^{n_i-2}} b^{2^{k-2}}, x_i^{2^{n-1}}, b^{2^{k-1}} \rangle$. In both cases, we reach a contradiction and so the proof of the first step of *Claim 2* is concluded.

We now show that $\langle x_1, \dots, x_s \rangle^{2^{n-1}} = \Omega_1(\langle x_1 \rangle)$.
Suppose that there exists x_i such that $|x_i| = 2^n$. By construction, we have that $\Omega_1(\langle x_i \rangle) \neq \Omega_1(\langle x_1 \rangle)$.
If $\langle x_i \rangle \cap \langle b, x_1 \rangle = 1$, then, as seen in the first step of *Claim 2*, we have $b^{2^{k-1}k_i} = 1$. Now, the subgroup $\langle x_1, bx_i \rangle$ is not metacyclic (it contains the 3-generated subgroup $\langle x_1^{2^{n-1}}, b^{2^{k-1}}, x_i^{2^{n-1}} \rangle$), a contradiction.
Therefore, we obtain $\Omega_1(\langle x_i \rangle) \leq \Omega_1(\langle b, x_1 \rangle)$. Since $\langle x_i \rangle \cap \langle x_1 \rangle = 1$ and $\Omega_1(\langle x_1, b \rangle) = \langle x_1^{2^{n-1}}, b^{2^{k-1}} \rangle$, it follows that either $x_i^{2^{n-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$ or $x_i^{2^{n-1}} = b^{2^{k-1}}$.

Suppose first that $x_i^{2^{n-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$. If $|b^{2^{k-1}k_i}| = 2$, then the subgroup $\langle x_i, bx_1 \rangle$ is not metacyclic because it contains the 3-generated subgroup $\langle x_i^{2^{n-2}} b^{2^{k-2}}, b^{2^{k-1}} x_1^{2^{n-1}}, b^{2^{k-1}} \rangle$.
Hence, we get that if $x_i^{2^{n-1}} = x_1^{2^{n-1}} b^{2^{k-1}}$ then $b^{2^{k-1}k_i} = 1$. Now, being $\Omega_1(\langle bx_i \rangle) = \langle x_1^{2^{n-1}} \rangle$, the subgroup $\langle x_1, bx_i \rangle$ contains the 3-generated subgroup $\langle x_1^{2^{n-1}}, x_1^{2^{n-2}} (bx_i)^{2^{k-2}}, b^{2^{k-1}} \rangle$, a contradiction.

In particular, this shows that $x_i^{2^{n-1}} = b^{2^{k-1}}$. Now, up to replacing x_i with $x_i x_1$, we have a generator of order 2^n, such that $\Omega_1(\langle x_i x_1 \rangle) = x_1^{2^{n-1}} b^{2^{k-1}}$ and, using the argument in the previous paragraph, we reach a contradiction.
This concludes the proof of the second step of *Claim 2*.

Summarizing, the first step and the second step show that if $|b^{2^{k-1}k_1}| = 2$, then $\langle x_1, \dots, x_s \rangle$ is abelian with $\langle x_1, \dots, x_s \rangle^{2^{n-1}} =$

$\Omega_1(\langle x_1\rangle)$, and $b^{2^{k-1}k_1} = 1$ for every $i \geq 2$.
Now, since the element $b^{2^{k-2}}$ centralizes the subgroup $\langle x_2, \dots, x_s\rangle$ and $(x_1 b^{2^{k-2}})^b = (x_1 b^{2^{k-2}})^{-1+4r+2^{n-1}}$, up to replacing x_1 with $x_1 b^{2^{k-2}}$, we obtain that the group L is in $\mathscr{C}_5$.

(b) Suppose now that $\langle x_1\rangle \cap \langle b\rangle \neq 1$.
We may assume that $\langle x_1, \dots, x_s\rangle^{2^{n-1}} = \Omega_1(\langle x_1\rangle) = \Omega_1(\langle b\rangle)$. In fact, otherwise, up to reordering the indices, we are in the case 1a. Since, by construction, $\langle x_i\rangle \cap \langle x_1\rangle = 1$ for every $i \geq 2$, we have that $\langle x_i\rangle \cap \langle b\rangle = 1$. Moreover, since $b^{2^{k-2}}$ is not in X and $\Omega_1(\langle b, x_1\rangle) = \langle b^{2^{k-1}}, x_1^{2^{n-2}} b^{2^{k-2}}\rangle$, we obtain that $\langle x_i\rangle \cap \langle b, x_1\rangle = 1$ for every $i \geq 2$.
Since b^2 acts as a universal automorphism on $\langle x_1, \dots, x_s\rangle$ and $exp(\langle x_2, \dots, x_s\rangle) < |x_1|$, it follows that $b^{2^{k-2}}$ commutes with $\langle x_2, \dots, x_s\rangle$. Therefore, we may assume, up to replacing in case x_1 with $x_1 b^{2^{k-2}}$, that $b^{2^{k-1}k_1} = 1$. Since $x^{b^2} = x^{1-8r+16r^2}$, and $b^{2^{k-1}}$ centralizes X, we get that $|b^2| > |x_1^{8r}|$, i.e. $|b^2| \geq |x_1^{4r}|$.
We treat separately the cases $|b^2| > |x_1^{4r}|$ and $|b^2| = x_1^{4r}$.
Suppose first that $|b^2| > |x_1^{4r}|$. Since $b^{2^{k-2}}$ is central in X, we get that, up to replacing x_i with $x_i b^{2^{k-2}k_i}$, we may assume that $b^{2^{k-1}k_i} = 1$ for every $i \in \{1, \dots, s\}$, and so the group $\langle X, b\rangle$ is in $\mathscr{C}_5$.
From now on, we assume that $|b^2| = |x_1^{4r}|$.
From $\langle x_i\rangle \cap \langle x_1, b\rangle = 1$, it follows that $b^{2^{k-1}k_i} = 1$. In fact, if $|b^{2^{k-1}k_i}| = 2$, then the subgroup $\langle x_i, bx_1\rangle$ is not not metacyclic, because it contains the subgroup $\langle x_i^{2^{n_i-1}}, b^{2^{k-2}} x_1^{2^{n-2}}, b^{2^{k-1}}\rangle$, which is 3-generated. This shows that the group $\langle X, b\rangle$ is in $\mathscr{C}_5$.

This concludes the investigation when X is abelian. Summarizing the result, we have that if X is abelian and $|X \cap \langle b\rangle| = 2$, then $L = \langle X, b\rangle$ is a group in $\mathscr{C}_5$.

2. Suppose now that X is not abelian.

Since X is modular, and $X/\Omega_1(\langle b\rangle)$ is abelian, we have that $X = (\langle x_1\rangle \times \cdots \times \langle x_{s-1}\rangle)\langle x_s\rangle$ and $x_i^{x_s} = x_i^{1+2^{n-1}}$, where $2^n = exp(\langle x_1\rangle \times \cdots \times \langle x_{s-1}\rangle)$ and $\langle x_1, \dots, x_s\rangle^{2^{n-1}} = \Omega_1(\langle x_1\rangle) = \Omega_1(\langle b\rangle)$.
We also have, $x_i^b = x_i^{-1+4r} b^{2^{k-1}k_i}$, for every i in $\{1, \dots, s\}$.
Since $b^{2^{k-2}}$ is not in X and $\Omega_1(\langle b, x_1\rangle) = \langle b^{2^{k-2}} x_1^{2^{n-1}}, b^{2^{k-1}}\rangle$, if $x \in X$

and $\langle x\rangle \cap \langle x_1\rangle = 1$, then $\langle x\rangle \cap \langle x_1, b\rangle = 1$.
In particular, this means that for every $i \in \{2, \dots, s-1\}$, we get that $\langle x_i\rangle \cap \langle x_1, b\rangle = 1$.

We treat separately the following two cases: $\Omega_1(\langle b, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$ and $\Omega_1(\langle b, x_s\rangle) = \Omega_1(\langle b, x_1\rangle)$

(a) Suppose that $\Omega_1(\langle b, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$. We distinguish three cases depending on $|x_s|$ with respect to $|x_1|$.

i. Suppose that $|x_1| > |x_s|$.
In order to complete the investigation of this case, we distinguish two more cases depending on $\Omega_1(\langle x_s\rangle)$. Namely, in the first part, we show that if $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$, then L is in $\mathscr{C}_6$. In the second part, we deal with the case $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$.

Suppose first that $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$. Then $\langle x_s\rangle \cap \langle x_1, b\rangle = 1$, and $\Omega_1(\langle x_s, b\rangle) = \langle b^{2^{k-1}}, x_s^{2^{n-1}}\rangle$. Now, since $|b^2| \geq |x_1^{4r}|$, we deal separately with the cases $|x_1^{4r}| = |b^2|$ and $|x_1^{4r}| < |b^2|$. If $|x_1^{4r}| = |b^2|$, then, up to replacing x_s with $x_s b^{2^{k-2}}$, we have that $\langle x_1, \dots, x_s\rangle$ is abelian and we are in case (1).
Suppose now that $|b^2| > |x_1^{4r}|$. The element $b^{2^{k-2}}$ commutes with X. In particular, up to replacing x_i with $x_i b^{2^{k-2}}$, we may assume that $b^{2^{k-1}k_i} = 1$ for every $i \in \{1, \dots, s\}$. Clearly, $|b^2| < |x_1|$, otherwise the subgroup $\langle b^{2\beta} x_1, x_s\rangle$, with $|b^{2\beta}| = |x_1|$, is not metacyclic. Therefore, L is in $\mathscr{C}_6$, and this concludes the case $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$.
Suppose now that $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$. Since $\Omega_1(\langle b, x_1\rangle) \neq \Omega_1(\langle b, x_s\rangle)$, we have that there exists α even, such that $|x_1^{\alpha}| = |x_s|$, $\Omega_1(\langle x_s x_1^{\alpha}\rangle) = \langle x_s^{2^{m-2}} x_1^{2^{n-2}}\rangle$ and $\langle x_s x_1^{\alpha}\rangle \cap \langle b, x_1\rangle = 1$. Moreover, $(x_s x_1^{\alpha})^b = (x_s x_1^{\alpha})^{-1+4r} b^{2^{k-1}s_1}$, for some s_i.
Hence, up to replacing x_s with $x_s x_1^{\alpha}$, we may assume that $\langle x_s\rangle \cap \langle b\rangle = 1$, $\Omega_1(\langle x_s, b\rangle) \neq \Omega_1(\langle x_1, b\rangle)$, $x_i^b = x_i^{-1+4r} b^{2^{k-1}k_i}$ for every $i \in \{1, \dots, s\}$ and $x_i^{x_s} = x_i^{1+2^{n-1}}$ for every $i \in \{1, \dots, s-1\}$. So, we reduced to the case $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$ studied in the first paragraph of 2(a)i.

ii. Suppose that $|x_1| = |x_s|$.
We distinguish two more cases depending on $\Omega_1(\langle x_s\rangle)$. Namely, in the first part, we show that if $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$, then L

is in $\mathscr{C}_6$. In the second part, we deal with the case $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$.

Suppose first that $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$. Then $\langle x_s\rangle \cap \langle x_1, b\rangle = 1$, and $\Omega_1(\langle x_s, b\rangle) = \langle b^{2^{k-1}}, x_s^{2^{n-1}}\rangle$. Since $|b^2| \geq |x_1^{4r}|$, we deal separately with the cases $|x_1^{4r}| = |b^2|$ and $|x_1^{4r}| < |b^2|$.

If $|b^2| = |x_1^{4r}|$, then, up to replacing x_s with $x_s b^{2^{k-2}}$, we have that $\langle x_1, \ldots, x_s\rangle$ is abelian, and $x_i^b = x_i^{-1+4r} b^{2^{k-1}k_i}$. Therefore, we are in case (1).

If $|b^2| > |x_1^{4r}|$, then $b^{2^{k-2}}$ commutes with X. In particular, up to replacing x_i with $x_i b^{2^{k-2}k_i}$, we may assume that $b^{2^{k-1}k_i} = 1$ for every $i \in \{1, \ldots, s\}$, i.e. $x_i^b = x_i^{-1+4r}$ for every $i \in \{1, \ldots, s\}$. Moreover, if $|b^2| \geq |x_1|$, then the subgroup $\langle b^{2\beta} x_1, x_s\rangle$, with $|b^{2\beta}| = |x_1|$, is not metacyclic. This means that $|b^2| < |x_1|$ and so $\langle X, b\rangle$ is in $\mathscr{C}_6$.

Suppose now that $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$.

Since $\Omega_1(\langle b, x_1\rangle) \neq \Omega_1(\langle b, x_s\rangle)$, there exists α odd, such that $|x_1^\alpha| = |x_s|$, $\Omega_1(\langle x_s x_1^\alpha\rangle) = \langle x_s^{2^{m-2}} x_1^{2^{n-2}}\rangle$ and $\langle x_s x_1^\alpha\rangle \cap \langle b, x_1\rangle = 1$.

Moreover, $(x_s x_1^\alpha)^b = (x_s x_1^\alpha)^{-1+4r} b^{2^{k-1}l}$, for some l.

Therefore, up to replacing x_s with $x_s x_1^\alpha$, we may assume that $\langle x_s\rangle \cap \langle b\rangle = 1$, $\Omega_1(\langle x_s, b\rangle) \neq \Omega_1(\langle x_1, b\rangle)$, $x_i^b = x_i^{-1+4r+4t_i} b^{2^{k-1}k_i}$ for every $i \in \{1, \ldots, s\}$, $x_i^{x_s} = x_i^{1+2^{n-1}}$ for every $i \in \{1, \ldots, s-1\}$ and $|x_s| < |x_1|$. Therefore, we reduced to the case studied in the first paragraph of 2(a)i.

iii. Assume, to conclude, that $|x_1| < |x_s|$.

We now divide the proof in two part. Namely, in the first part we deal with the case $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$, whereas, in the second case we study the case $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$.

Suppose first that $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle b\rangle)$. The group $\langle x_2, \ldots, x_s\rangle$ is abelian and such that $\langle x_2, \ldots, x_s\rangle^{2^{m-1}} = \Omega_1(\langle x_s\rangle)$.

Moreover, $x^{x_1} = x^{1+2^{m-1}}$ for every $x \in \langle x_2, \ldots, x_s\rangle$, $x_i^b = x_i^{-1+4r} b^{2^{k-1}k_i}$ for every $i \in \{1, \ldots, s\}$, $\Omega_1(\langle b, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$ and $|x_s| > |x_1|$. Therefore, up to interchanging x_1 and x_s, we reduced to the case studied in 2(a)i.

So from now on, we assume that $\Omega_1(\langle x_s\rangle) \neq \Omega_1(\langle b\rangle)$.

Since $b^{2^{k-2}} \notin X$ and $\langle x_s\rangle \cap \langle b\rangle = 1$, we have that $\langle x_s\rangle \cap$

$\langle x_1, b\rangle = 1$. Hence, $\Omega_1(\langle x_s, b\rangle) = \langle b^{2^{k-1}}, x_s^{2^{n-1}}\rangle$.
Since $|x_s^{4r}| \leq |b^2|$, we distinguish the cases $|x_s^{4r}| = |b^2|$ and $|x_s^{4r}| < |b^2|$.
Suppose first that $|x_s^{4r}| < |b^2|$.
The element $b^{2^{k-2}}$ centralizes X and so, up to replacing x_i with $x_i b^{2^{k-2}k_i}$, we may assume that $x_i^b = x_i^{-1+4r}$ for every $i \in \{1, \ldots, s\}$. Moreover, if $|b^2| \geq |x_1|$, then the subgroup $\langle b^{2\beta}x_1, x_s\rangle$, with $|b^{2\beta}| = |x_1|$, is not metacyclic. Therefore, we have that $|b^2| < |x_1|$ and so $L = \langle X, b\rangle$ is in $\mathscr{C}_6$.
From now on, we assume that $|x_s^{4r}| = |b^2|$.
Since $|x_1| < |x_s|$, we have that $b^{2^{k-2}}$ commutes with x_1 and so with $\langle x_1, \ldots, x_{s-1}\rangle$. Therefore, up to replacing x_s with $x_s b^{2^{k-2}k_s}$, we may assume that $b^{2^{k-1}k_s} = 1$.
We now show that L is in $\mathscr{C}_7$. In order to do that, we divide the proof in three parts. Namely, in *Claim 1*, we prove that $|b^{2^{k-1}k_1}| = 2$. In *Claim 2*, we show that $b^{2^{k-1}k_i} = 1$ for every $i \in \{2, \ldots, s-1\}$. In the third part, we sum up the results and conclude the investigation of this case.
Claim 1: we show that $|b^{2^{k-1}k_1}| = 2$.
This would imply that $x_1^b = x_1^{-1+4r}b^{2^{k-1}} = x_1^{-1+4r+2^{n-1}}$.
Let α be an even integer such that $|x_s^\alpha| = |x_1|$. Now, we have $\Omega_1(\langle x_1x_s^\alpha\rangle) = \Omega_1(\langle bx_s\rangle)$. Moreover, we obtain that $(x_1x_s^\alpha)^{bx_s} = (x_1x_s^\alpha)^{-1+4r}x_1^{2^{n-1}}b^{2^{k-1}k_1}$.
If $b^{2^{k-1}k_1} = 1$, then the subgroup $\langle x_1x_s^\alpha, bx_s\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}}x_s^{2^{m-1}}, x_1^{2^{n-1}}, (x_1x_s^\alpha)^{2^{n-2}}(bx_s)^{2^{k-2}}\rangle$.
Hence, $|b^{2^{k-1}k_1}| = 2$, i.e. $x_1^b = x_1^{-1+4r}b^{2^{k-1}} = x_1^{4r+2^{n-1}}$. This concludes the proof of *Claim 1*.
Claim 2: we show that $b^{2^{k-1}k_i} = 1$ for every $i \in \{2, \ldots, s-1\}$.
In particular, this would imply that $x_i^b = x_i^{-1+4r}$.
For every $i \in \{2, \ldots, s-1\}$, we have that $|x_i| < 2^n$ and $\langle x_i\rangle \cap \langle x_1\rangle = 1$. Since $\Omega_1(\langle x_1\rangle) = \Omega_1(\langle b\rangle)$, we obtain that $\langle x_i\rangle \cap \langle b\rangle = 1$.
We now show that, if x_i is such that $\langle x_i\rangle \cap \langle x_s, b\rangle = 1$, then $b^{2^{k-1}k_i} = 1$.
Suppose that x_i is such that $\langle x_i\rangle \cap \langle x_s, b\rangle = 1$ and suppose,

by contradiction, that $|b^{2^{k-1}k_i}| = 2$. The subgroup $\langle x_i, bx_s\rangle$ is not metacyclic, because it contains the 3-generated subgroup $\Omega_1(\langle x_1, x_s, b\rangle)$.
We now prove that, if $\langle x_i\rangle \cap \langle x_s, b\rangle \neq 1$, then $b^{2^{k-1}k_i} = 1$.
Since $\langle x_i\rangle \cap \langle b\rangle = 1$, we have that either $x_i^{2^{n_i-1}} = x_s^{2^{m-1}}$ or $x_i^{2^{n_i-1}} = x_s^{2^{m-1}}b^{2^{k-1}}$. Therefore, there exists an even integer β such that $|x_i| = |x_s^\beta|$ and $\Omega_1(\langle x_i x_s^\beta\rangle) = \Omega_1(bx_s)$. If $|b^{2^{k-1}k_i}| = 2$, then the subgroup $\langle bx_s, x_i x_s^\beta\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}}x_s^{2^{m-1}}, (bx_s)^{2^{k-2}}(x_i)^{2^{n_i-2}}, b^{2^{k-1}}\rangle$.
Therefore, we get that $b^{2^{k-1}} = 1$ for every $i \in \{2, \ldots, s-1\}$, and this concludes the proof of *Claim 2*.
Summarizing, by *Claim 1*, we get $x_1^b = x_1^{-1+4r+2^{n-1}}$ and, by *Claim 2*, we have $x_i^b = x_i^{-1+4r}$ for every $i \in \{2, \cdots, s-1\}$. Since $|x_i| < |x_1|$ for every $i \in \{2, \cdots, s-1\}$, we obtain $x_i^b = x_i^{-1+4r+2^{n-1}}$ for every $i \in \{1, \ldots, s-1\}$.
Moreover, if $|b^2| \geq |x_1|$, then the subgroup $\langle b^{2\beta}x_1, x_s\rangle$, where $|b^{2\beta}| = |x_1|$, is not metacyclic.
Therefore, we have $|b^2| < |x_1|$, and this shows that L is in $\mathscr{C}_7$.

This concludes the investigation of the case $\Omega_1(\langle b, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$.

(b) Suppose now that $\Omega_1(\langle b, x_s\rangle) = \Omega_1(\langle b, x_1\rangle)$.
We first show that $\langle x_s\rangle \cap \langle b\rangle \neq 1$.
In fact, suppose, by contradiction, that $\langle x_s\rangle \cap \langle b\rangle = 1$. Then, since $b^{2^{k-2}} \notin X$ and $\Omega_1(\langle x_1, b\rangle) = \langle x_1^{2^{n-2}}b^{2^{k-2}}, b^{2^{k-1}}\rangle$, we get that $\langle x_s\rangle \cap \langle x_1, b\rangle = 1$. It follows that $\Omega_1(\langle b, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$, against the assumption.
This shows that $\langle x_s\rangle \cap \langle b\rangle \neq 1$.
In particular, we get that $\Omega_1(\langle x_s, b\rangle) = \langle x_s^{2^{m-2}}b^{2^{k-2}}, b^{2^{k-1}}\rangle$, and so $x_s^{2^{m-2}} \in \langle x_1, b\rangle$. Moreover, since $b^{2^{k-2}} \notin X$, we have that $b^{2^{k-2}} \notin \langle x_1, x_s\rangle$. It follows that $\Omega_1(\langle x_1, x_s\rangle) \neq \Omega_1(\langle b, x_1\rangle)$.
We deal separately with the cases $|x_1| > |x_s|$, $|x_1| = |x_s|$, and $|x_1| < |x_s|$.

i. Suppose that $|x_1| > |x_s|$. Since $\langle x_1, x_s\rangle$ is modular, there exists an even integer α such that $\Omega_1(\langle x_s x_1^\alpha, x_1\rangle) = \Omega_1(\langle x_1, x_s\rangle)$ and $\langle x_s x_1^\alpha\rangle \cap \langle x_1\rangle = 1$. Now, $(x_s x_1^\alpha)^b = (x_s x_1^\alpha)^{-1+4r}b^{2^{k-1}w}$, for some $w \in \{0,1\}$. This means that, up to replacing x_s with $x_s x_1^{\alpha}$, we are in case 2a.

ii. Suppose that $|x_1| = |x_s|$. Since $\langle x_1, x_s \rangle$ is modular, there exists an odd integer α such that $\Omega_1(\langle x_s x_1^{\alpha}, x_1 \rangle) = \Omega_1(\langle x_1, x_s \rangle)$ and $\langle x_s x_1^{\alpha} \rangle \cap \langle x_1 \rangle = 1$.
Now, $(x_s x_1^{\alpha})^b = (x_s x_1^{\alpha})^{-1+4r} b^{2^{k-1}w}$, for some $w \in \{0,1\}$. This means that, up to replacing x_s with $x_s x_1^{\alpha}$, we are in case 2a.

iii. Suppose that $|x_1| < |x_s|$. Up to interchanging x_1 and x_s, we are in case 2(b)i.

This concludes the investigation of the case $\Omega_1(\langle b, x_s \rangle) = \Omega_1(\langle b, x_1 \rangle)$. Hence, the lemma is proved.

□

The following lemmas complete the classification of the non-trivial monotone 2-groups G, with $H_4(G) = G$ and G/G^4 isomorphic to a group in $\mathscr{A}_4$. More precisely, by the previous part, the group L is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$. In Lemma 5.24, we determine the group G when L is in $\mathscr{C}_5$. In Lemma 5.25, we determine the group G when L is in $\mathscr{C}_6$. To conclude, in Lemma 5.26, we determine the group G when L is in $\mathscr{C}_7$.

Lemma 5.24. *Let G, L, X and b be as in Remark 5.18. Suppose that L is in $\mathscr{C}_5$. Then G is in $\mathscr{C}_5$ or in $\mathscr{C}_6$.*

Proof. The group L is in $\mathscr{C}_5$. Therefore, we may assume that $X = \langle x_1 \rangle \times \ldots \times \langle x_s \rangle$ where, if $i \geq j$, then $|x_i| \geq |x_j|$ and $x^b = x^{-1+4r}$, for every $x \in X$. Now, the group G is equal to $L\langle c_1, \ldots, c_u \rangle$, where $\langle c_1, \ldots, c_u \rangle$ is elementary abelian. Moreover, $\langle X, b^2, c_1, \ldots, c_u \rangle$ is modular (see Lemma 5.16).
Put $|b| = 2^k$ and $|x_1| = 2^n$.

Consider the element c_i of order 2.

For every $x \in X \setminus X^2$, we get that $\langle x, c_i \rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$. Then, $\langle x, c_i \rangle$ is not semidihedral and c_i normalizes $\langle x \rangle$ (see Lemma 2.12). In particular, since X is abelian, we get that c_i induces a power automorphism on X. So, since X is abelian, by Lemma 1.5.4. on page 32 of [13], we get that c_i acts as a universal automorphism of order at most 2 on X. Let $x^{c_i} = x^{1+4h_i}$ for every $x \in X$, where $|X^{4h_i}| \leq 2$.

The group $\langle b, c_i \rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$. Then, $\langle b, c_i \rangle$ is not semidihedral, c_i normalizes $\langle b \rangle$ and $b^{c_i} = b^{1+2^{k-1}k_i}$ with $k_i \in \{0, 1\}$.

Therefore, we have that $x^{c_i} = x^{1+4h_i}$ for every $x \in X$, where $|X^{4h_i}| \leq 2$, and $b^{c_i} = b^{1+2^{k-1}k_i}$ with $k_i \in \{0, 1\}$.

Since $x^b = x^{-1+4r}$, we get that $x^{b^2} = x^{1-8r+16r^2}$ for every $x \in X$. In particular, this implies that $|b^2| \geq exp(X^{8r})$. Since $exp(X) = |x_1|$, we have that $|b^2| \geq |x_1^{8r}|$. The rest of the proof is a case-by-case analysis depending on the order of $|b^2|$ with respect of the order $|x_1^{8r}|$ and on the size of the intersection $\langle x_1 \rangle \cap \langle b \rangle$.

- Suppose that $|b^2| = |x_1^{8r}|$.

  The element $b^{2^{k-1}}$ is not central in $\langle x_1, b \rangle$ and so $\langle x_1 \rangle \cap \langle b \rangle = 1$. Moreover, we have that $(bx_1)^2 = b^2 x_1^{4r}$ and so $\Omega_1(\langle bx_1 \rangle) = \Omega_1(\langle x_1 \rangle)$.
  If $|b^{2^{k-1}k_i}| = 2$, then the subgroup $\langle x_1 b, c_i \rangle$ contains the 3-generated elementary abelian subgroup $\langle x_1^{2^{n-1}}, b^{2^{k-1}}, c_i \rangle$. Moreover, up to replacing c_i with $c_i b^{2^{k-1}}$, we may assume that c_i is in the centralizer of X. Hence, we get that $\langle X, c_1, \ldots, c_u \rangle$ is abelian and $x^b = x^{-1+4r}$ for every $x \in \langle X, c_1, \ldots, c_u \rangle$. This proves that G is in $\mathscr{C}_5$.

- Suppose that $|b^2| = |x_1^{4r}|$ and $\langle b \rangle \cap \langle x_1 \rangle = 1$.

  We prove the following fact: if $|x_1^{4h_i}| = 2$, then $|b^{2^{k-1}k_i}| = 2$, whereas if $x_1^{4h_1} = 1$, then $b^{2^{k-1}k_i} = 1$.
  Suppose first that $|x_1^{4h_i}| = 2$ and suppose, by contradiction, that $b^{2^{k-1}k_i} = 1$. Then, the subgroup $\langle bx_1, c_i \rangle$ is not metacyclic because it contains the 3-generated subgroup $\langle x_1^{2^{n-1}}, c_i, b^{2^{k-1}} \rangle$.
  Suppose now that $x_1^{c_i} = x_1$, and suppose, by contradiction, that $|b^{2^{k-1}k_i}| = 2$. Then, the subgroup $\langle bx_1, c_i \rangle$ is not metacyclic because it contains the 3-generated subgroup $\langle x_1^{2^{n-1}}, c_i, b^{2^{k-1}} \rangle$.

  Suppose now that c_i is such that $c_i \notin C_G(\langle x_1, b \rangle)$. Then, $x_1^{c_i} = x_1^{1+4h_i}$ with $|x_1^{4h_i}| = 2$ and $b^{c_i} = b^{1+2^{k-1}}$. Now, the element $c_i b^{2^{k-2}}$ has order 4, centralizes x_1 and is inverted by b.

  It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$ for every $i \in \{1, \ldots, u\}$, the group G is in $\mathscr{C}_5$.

- Suppose that $|b^2| = |x_1^{4r}|$ and $\langle b \rangle \cap \langle x_1 \rangle \neq 1$.

We prove the following fact: if $|x_1^{4h_i}| = 2$, then $|b^{2^{k-1}k_i}| = 2$, whereas if $x_1^{4h_1} = 1$, then $b^{2^{k-1}k_i} = 1$.
Suppose first that $|x_1^{4h_i}| = 2$ and suppose, by contradiction, that $b^{2^{k-1}k_i} = 1$. Then, the subgroup $\langle bx_1, c\rangle$ is not metacyclic, because it contains the 3-generated subgroup $\langle x_1^{2^{n-2}} b^{2^{k-2}}, c_i, b^{2^{k-1}}\rangle$.
Suppose now that $x_1^{c_i} = x_1$, and suppose, by contradiction, that $|b^{2^{k-1}k_i}| = 2$. Then, the subgroup $\langle bx_1, c\rangle$ is not metacyclic because it contains the 3-generated subgroup $\langle x_1^{2^{n-2}} b^{2^{k-2}}, c_i, b^{2^{k-1}}\rangle$.

Suppose now that c_i is such that $c_i \notin C_G(\langle x_1, b\rangle)$. Then, $x_1^{c_i} = x_1^{1+4h_i}$ with $|x_1^{4h_i}| = 2$ and $b^{c_i} = b^{1+2^{k-1}}$. Now, the element $c_i b^{2^{k-2}}$ has order 4, centralizes x_1 and is inverted by b.

It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$ for every $i \in \{1, \ldots, u\}$, the group G is in $\mathscr{C}_5$.

- Suppose that $|b^2| > |x_1^{4r}|$ and $\langle x_1\rangle \cap \langle b\rangle = 1$.
  In particular, $|b^2| > |x_1^{4r}|$ implies that $\Omega_1(\langle bx_1\rangle) = \Omega_1(\langle b\rangle)$. Moreover, $b^{2^{k-2}}$ centralizes X.

  We first prove that c_i is in the centralizer of X.
  In fact, suppose, by contradiction, that there exists an $i \in \{1, \ldots, u\}$ such that $x_1^{c_i} = x_1^{1+4h_i}$ with $|x_1^{4h_i}| = 2$. The subgroup $\langle bx_1, c_i\rangle$ is not metacyclic because it contains the subgroup $\langle b^{2^{k-1}}, x_1^{2^{n-1}}, c_i\rangle$, which is 3-generated.

  It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$, the group G is in $\mathscr{C}_5$.

- To conclude, suppose that $|b^2| > |x_1^{4r}|$ and $\langle x_1\rangle \cap \langle b\rangle \neq 1$.
  In particular, we have that $\langle x_1^{2^{n-1}}\rangle = \langle b^{2^{k-1}}\rangle$, $\Omega_1(\langle bx_1\rangle) = \Omega_1(\langle b\rangle)$ and $b^{2^{k-2}}$ centralizes X.
  We distinguish two cases depending on the order of b^2 with respect to the order of x_1.

  Suppose first that $|b^2| \geq |x_1|$.
  We first prove that c_i is in the centralizer of X. In fact, suppose, by contradiction, that there exists $i \in \{1, \ldots, u\}$ such that $x_1^{c_i} = x_i^{1+4h_i}$ with $|x_1^{4h_1}| = 2$. Then, the subgroup $\langle b^{2\alpha}x_1, c_i\rangle$, where $|b^{2\alpha}| = |x_1|$, is not metacyclic, because it contains the 3-generated elementary abelian group $\langle b^{2^{k-2}} x_1^{2^{n-2}}, x_1^{2^{n-1}}, c_i\rangle$.
  It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$, the group G is in $\mathscr{C}_5$.

Suppose now that $|b^2| < |x_1|$.
If $c_i \in C_G(X)$ for every $i \in \{1, \ldots, u\}$, then, up to replacing c_i with $c_i b^{2^{k-2}k_i}$, we get that G is a group in $\mathscr{C}_5$.
Suppose now that there exists an $i \in \{1, \ldots, u\}$ such that $x_1^{c_i} = x_1^{1+4h_i}$ where $|x_1^{4h_i}| = 2$. Up to reordering the indices, we may assume that c_1 acts on X as a non-trivial automorphism of order 2. Up to replacing in case c_i with $c_i c_1$, we may assume that $c_i \in C_G(X)$ for every $i \in \{2, \ldots, u\}$.
It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$ for every $i \in \{1, \ldots, u\}$, the group G is in $\mathscr{C}_6$.

□

Lemma 5.25. *Let G, L, X and b be as in Remark 5.18. Suppose that L is in $\mathscr{C}_6$. Then G is in $\mathscr{C}_6$.*

Proof. We have that G is equal to $L\langle c_1, \cdots, c_u\rangle$, with L in $\mathscr{C}_6$ and $\langle c_1, \cdots, c_u\rangle$ elementary abelian.
Let L be $\langle X, b\rangle$. Since L is in $\mathscr{C}_6$, we may assume that $\langle X\rangle = \langle x_1, \cdots, x_s\rangle$, where $\langle x_1, \cdots, x_{s-1}\rangle$ is abelian and of exponent 2^n, $\langle x_1, \cdots, x_{s-1}\rangle^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $|b| \geq 8$, $x^b = x^{-1+4r}$ and $x^{x_s} = x^{1+2^{n-1}}$, for every $x \in \langle x_1, \cdots, x_{s-1}\rangle$, $x_s^b = x_s^{-1+4r}$ and $exp(\langle x_1, \cdots, x_s\rangle^{4r}) < |b^2| < 2^n$.
We also may assume that $|x_1| = exp(\langle x_1, \ldots, x_{s-1}\rangle)$. Let $|b| = 2^k$, $|x_s| = 2^m$, $|x_1| = 2^n$.

Consider now the element c_i.

For every $x \in X \setminus X^2$, we get that the subgroup $\langle x, c_i\rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$. Then, $\langle x, c_i\rangle$ is not semidihedral and c_i normalizes $\langle x\rangle$ (see Lemma 2.12). In particular, since X is modular not involving Q_8, we get that c_i induces a power automorphism on X. So, by Lemma 2.3.24 on page 68 of [13], the element c_i induces a universal automorphism of order at most 2 on X.

Since $\langle b, c_i\rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$, the subgroup $\langle b, c_i\rangle$ is not semidihedral. So, c_i normalizes $\langle b\rangle$ and $b^{c_i} = b^{1+2^{k-1}k_i}$ with $k_i \in \{0, 1\}$.

We distinguish two cases depending on the order of x_1 with respect to the order of x_s:

1. Suppose first that $|x_s| \leq |x_1|$.

If $c_i \in C_G(X)$ for every $i \in \{1, \ldots, u\}$, then, up to replacing c_i with $c_i b^{2^{k-2}k_i}$ for every $i \in \{1, \ldots, u\}$, we get that G is in $\mathscr{C}_6$.

So, from now on, we assume that c_i induces a non trivial automorphism of order 2 on X, for some i.
Up to reordering the indices and up to replacing in case c_i with $c_i c_1$, we may assume that $x^{c_1} = x^{1+2^{n-1}}$, for every $x \in X$, and $c_i \in C_G(X)$, for every $i \in \{2, \ldots, r\}$.

Suppose first that $|x_1| = |x_s|$.
We prove that this implies that $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle x_1\rangle)$.
In fact, suppose, by contradiction, that $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$.
Since $\Omega_1(\langle b\rangle) = \Omega_1(\langle x_1\rangle)$, $b^{2^{k-2}} \notin X$ and $\Omega_1(\langle x_1\rangle) \neq \Omega_1(\langle x_s\rangle)$, we have that $\langle x_s\rangle \cap \langle x_1, b\rangle = 1$. Now, the subgroup $\langle x_1 c_1, b x_s\rangle$ is not metacyclic because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-1}}, x_1^{2^{n-2}} b^{2^{k-2}}, x_s^{2^{n-1}}\rangle$. Therefore, $\Omega_1(\langle x_s\rangle) = \Omega_1(\langle x_1\rangle)$ and, up to replacing x_s with $x_1 x_s$, we may assume that $|x_1| > |x_s|$.

Therefore, from now on, we suppose that $|x_1| > |x_s|$. Up to replacing c_i with $c_i b^{2^{k-2}k_i}$, for every $i \geq 2$, and x_s with $x_s c_1 b^{2^{k-2}k_1}$, we get that G is in $\mathscr{C}_6$. This concludes the investigation of the case $|x_1| \geq |x_s|$.

2. Suppose now that $|x_1| < |x_s|$.

   If $\langle x_s\rangle \cap \langle b\rangle \neq 1$, then we may interchange x_1 and x_s, and we are in case 1.

   So we may assume that $\langle x_s\rangle \cap \langle b\rangle = 1$.
   In particular, since $b^{2^{k-2}} \notin X$ and $\Omega_1(\langle b, x_1\rangle) = \langle x_1^{2^{n-1}}, x_1^{2^{n-2}} b^{2^{k-2}}\rangle$, we have that $\langle x_s\rangle \cap \langle x_1, b\rangle = 1$.
   Suppose that $x_s^{c_i} = x_s^{1+4h_s}$ with $|x_s^{4h_s}| = 2$.
   Then, the subgroup $\langle x_1 c_1, b x_s\rangle$ is not metacyclic. In fact, since $|b^2| > |x_s^{4r}|$, we have that $\Omega_2(\langle b x_s\rangle) = \langle b^{2^{k-2}} x_s^{2^{n-1}j}\rangle$ where $j \in \{0, 1\}$. Hence, $\langle x_1 c_i, b x_s\rangle$ contains the 3-generated subgroup $\langle x_1^{2^{n-2}}, b^{2^{k-2}}, x_s^{2^{n-1}}\rangle$, a contradiction.
   It follows that c_i is in $C_G(X)$ for every $i \in \{1, \cdots, u\}$.
   Up to replacing c_i with $c_i b^{2^{k-2}k_i}$ for every $i \in \{1, \ldots, u\}$, we have that G is in $\mathscr{C}_6$.

$\square$

Lemma 5.26. *Let G, L, X and b be as in Remark 5.18. Suppose that L is in $\mathscr{C}_7$. Then G is in $\mathscr{C}_7$.*

Proof. We have that G is equal to $L\langle c_1, \cdots, c_u\rangle$, with L in $\mathscr{C}_7$ and $\langle c_1, \cdots, c_u\rangle$ elementary abelian.
Let L be $\langle X, b\rangle$. Since L is in $\mathscr{C}_7$, we may assume that $X = \langle x_1, \ldots, x_s\rangle$ where $\langle x_1, \ldots, x_{s-1}\rangle$ is abelian of exponent 2^n, $\langle x_1, \ldots, x_{s-1}\rangle^{2^{n-1}} = \Omega_1(\langle b\rangle)$, $x^{x_s} = x^{1+2^{n-1}}$ and $x^b = x^{-1+4r+2^{n-1}}$ for every $x \in X$, $x_s^b = x_s^{-1+4r}$ and $|x_s| > |2^n|$, $\langle x_s\rangle \cap \langle b\rangle = 1$ and $|\langle x_1, \ldots, x_{s-1}, x_s\rangle^{4r}| = |b^2| < 2^n$.

Consider the element c_i.

For every $x \in X \setminus X^2$, we get that $\langle x, c_i\rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$. Then, $\langle x, c_i\rangle$ is not semidihedral, c_i normalizes $\langle x\rangle$ and $x^{c_i} = x^{1+4h_i}$, with $|x^{4h_i}| \leq 2$. In particular, since X is modular and does not involve subgroups isomorphic to Q_8, we get that c_i induces a power automorphism on X. Therefore, c_i acts as a universal automorphism of order at most 2 on X (see Lemma 2.3.24 on page 68 of [13]).

Since $\langle b, c_i\rangle$ is metacyclic with a generator of order 2 and has a quotient isomorphic to $C_4 \times C_2$, $\langle b, c_i\rangle$ is not semidihedral. The element c_i normalizes $\langle b\rangle$ and $b^{c_i} = b^{1+2^{k-1}k_i}$ with $k_i \in \{0, 1\}$.

Let $|x_s| = 2^m$ and $|b| = 2^k$, $|x_1| = exp(\langle x_1, \ldots, x_{s-1}\rangle) = 2^n$. We note that $exp(X) = |x_s|$. Therefore, if c_i induces a non-trivial modular automorphism of order 2, then, in particular, we have that $x_s^{c_i} = x_s^{1+4h_i}$, where $|x_s^{4h_i}| = 2$.

We now show that if $x_s^{c_i} = x_s^{1+2^{m-1}}$, then $|b^{2^{k-1}k_i}| = 2$, whereas, if $x_s^{c_i} = x_s$, then $b^{2^{k-1}k_i} = 1$.
Suppose that $x_s^{c_i} = x_s^{1+2^{m-1}}$ and suppose, by contradiction, that $b^{2^{k-1}k_i} = 1$. Then, the subgroup $\langle bx_s, c_i\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-2}}x_s^{2^{m-2}}, x_s^{2^{m-1}}, c_i\rangle$.
Suppose now that $x_s^{c_i} = x_s$ and suppose, by contradiction, that $|b^{2^{k-1}k_i}| = 2$. Then, the subgroup $\langle bx_s, c_i\rangle$ is not metacyclic, because it contains the 3-generated elementary abelian subgroup $\langle b^{2^{k-2}}x_s^{2^{m-2}}, b^{2^{k-1}}, c_i\rangle$.

This implies that if c_i does not centralize $\langle X, b\rangle$, then $x_s^{c_i} = x_s^{1+4h_i}$ with $|x_s^{4h_i}| = 2$ and $|b^{2^{k-1}k_i}| = 2$.
It follows that, up to replacing c_i with $c_i b^{2^{k-2}k_i}$, the group G is in $\mathscr{C}_7$. $\square$

The following proposition sums up the results of the last part of the section, and shows that if G be a non trivial monotone 2-group with $G = H_4(G)$ and G/G^4 is in $\mathscr{A}_4$, then G is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$.

Proposition 5.27. *Let G be a non trivial monotone 2-group with $G = H_4(G)$. If G/G^4 is isomorphic to a group in $\mathscr{A}_4$, then G is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$.*

Proof. Let $G = \langle a_1, \dots, a_s, c_1, \dots, c_u, b\rangle$, where $\langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_1G^4\rangle \times \langle c_uG^4\rangle$ is abelian with $s \geq 2$, $|a_iG^4| = 4$, $|c_iG^4| = 2$, $|bG^4| = 4$, $b^2G^4 \notin \langle a_1G^4\rangle \times \cdots \times \langle a_sG^4\rangle \times \langle c_1G^4\rangle \times \langle c_uG^4\rangle$, $a^bG^4 = a^{-1}G^4$, for every $a \in \langle a_1, \dots, a_s, c_1, \dots, c_u\rangle$.

By Lemma 5.17, we may assume that $G = \langle a_1, \dots, a_s, b\rangle\langle c_1, \dots, c_u\rangle$, where $\langle c_1, \dots, c_u\rangle$ is elementary abelian.

Since $\langle a_i, b\rangle G^4/G^4$ is isomorphic to K_2, by Lemma 2.13, we have that $G^4 \cap \langle a_i, b\rangle = \langle a_i^4, b^4\rangle$. From $a_i^b(G^4 \cap \langle a_i, b\rangle) = a_i^{-1}(G^4 \cap \langle a_i, b\rangle)$, it follows that $a_i^b = a_i^{-1+4h_i}b^{2^{k-1}k_i}$.
Since $\langle a_i, b^{2^{k_i}}\rangle$ is modular, we get that there exists $x_i \in \langle a_i, b^{2^{k_i}}\rangle$ with $\langle x_i^2\rangle = \langle a_i^{-2+4h_i}b^{2^{k-1}k_i}\rangle$.
In particular, $\langle x_i, b\rangle = \langle a_i, b\rangle$ and $x_i^b = x_i^{-1+4r_i}$.
Let X be the subgroup $\langle x_1, \dots, x_s\rangle$. We have that $\langle a_1, \dots, a_s, b^2\rangle = \langle X, b^2\rangle$, and so X is a modular group that does not involve Q_8.
By Lemma 5.20, we have that $|X \cap \langle b\rangle| \leq 2$.

If $X \cap \langle b\rangle = 1$, then, by Lemma 5.21, we have that $\langle X, b\rangle$ is in $\mathscr{C}_5$, and so by Lemma 5.24, we have that $G = \langle X, b\rangle\langle c_1, \dots, c_u\rangle$ is in $\mathscr{C}_5$ or in $\mathscr{C}_6$.

If $|X \cap \langle b\rangle| = 2$, then by Lemma 5.23, we have that $\langle X, b\rangle$ is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$. Hence, by Lemma 5.24, by Lemma 5.25 and by Lemma 5.26, we have that $G = \langle X, b\rangle\langle c_1, \dots, c_r\rangle$ is in $\mathscr{C}_5$ or in $\mathscr{C}_6$ or in $\mathscr{C}_7$.

This shows that the proposition holds.

□

This concludes the analysis of the monotone 2-groups G of exponent at least 8 such that $G = H_4(G)$ and G/G^4 is a group in $\mathscr{A}_4$. Namely, such groups are in $\mathscr{C}_5$, or in $\mathscr{C}_6$, or in $\mathscr{C}_7$.

Also, this concludes the classification of the monotone 2-groups of exponent at least 8 such that $G = H_4(G)$, see Theorem 5.2.

Bibliography

[1] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.

[2] J. González-Sánchez and A. Jaikin-Zapirain. On the structure of normal subgroups of potent p-groups. *J. Algebra*, 276(1):193–209, 2004.

[3] Zvonimir Janko. New results in the theory of finite 2-groups. In *Ischia group theory 2004*, volume 402 of *Contemp. Math.*, pages 193–195. Amer. Math. Soc.

[4] Zvonimir Janko. Finite nonabelian 2-groups in which any two noncommuting elements generate a subgroup of maximal class. *Glas. Mat. Ser. III*, 41(61)(2):271–274, 2006.

[5] Bruno Kahn. A characterisation of powerfully embedded normal subgroups of a p-group. *J. Algebra*, 188(2):401–408, 1997.

[6] Thomas J. Laffey. The minimum number of generators of a finite p-group. *Bull. London Math. Soc.*, 5:288–290, 1973.

[7] Alexander Lubotzky and Avinoam Mann. Powerful p-groups. I. Finite groups. *J. Algebra*, 105(2):484–505, 1987.

[8] Andrea Lucchini. A bound on the number of generators of a finite group. *Arch. Math. (Basel)*, 53(4):313–317, 1989.

[9] Avinoam Mann. Generators of p-groups. In *Proceedings of groups—St. Andrews 1985*, volume 121 of *London Math. Soc. Lecture Note Ser.*, pages 273–281, Cambridge, 1986. Cambridge Univ. Press.

[10] Avinoam Mann. The number of generators of finite p-groups. *J. Group Theory*, 8:317–337, 2005.

[11] Avinoam Mann. The power structure of p-groups. II. *J. Algebra*, 318(2):953–956, 2007.

[12] Franco Napolitani. Gruppi finiti minimali non modulari. *Rend. Sem. Mat. Univ. Padova*, 45:229–248, 1971.

[13] Roland Schmidt. *Subgroup lattices of groups*, volume 14 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[14] Michio Suzuki. *Group theory. I*, volume 247 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1982. Translated from the Japanese by the author.

[15] Michio Suzuki. *Group theory. II*, volume 248 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1986. Translated from the Japanese.