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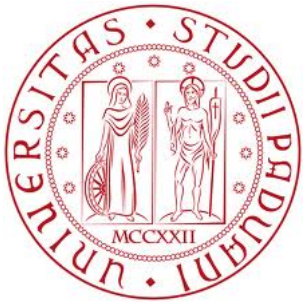

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Superstring Theories on non-maximally supersymmetric AdS backgrounds

Direttore della Scuola: Ch.mo Prof. Andrea Vitturi
Supervisore: Ch.mo Prof. Dmitri Sorokin

Dottorando: Alessandra Cagnazzo

Abstract

The subject of this thesis is Superstring Theories on Anti de Sitter backgrounds that do not have maximal supersymmetry ($AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$), a feature that introduces complications in studying these theories. In particular, in the non-maximally supersymmetric backgrounds the Green-Schwarz superstring is not fully described by a worldsheet sigma model on a corresponding supercoset space, since it has extra (non-coset) fermionic degrees of freedom associated with the broken supersymmetries. We concentrate on the study of the integrability of these theories with the aim to reveal how the non-coset fermionic modes enter into and deform the integrable structure of these string theories. We construct various (gauge-related) forms of the zero-curvature Lax connecton for the superstrings in $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ and show that in the presence of the non-coset degrees of freedom the important property of the Lax connection to be $\mathbb{Z}_4$-invariant persists. In the case of the $AdS_4 \times CP^3$ superstring we also study the string instanton wrapping a non-trivial two-cycle in CP^3 and find that it has twelve fermionic zero modes associated with 1/2 of the supersymmetry of the background, thus manifesting that this exact topologically non-trivial classical solution is 1/2 BPS.

Abstract

L'argomento di questa tesi sono le Teorie di Superstringa su spazi Anti de Sitter che non hanno supersimmetria massimale ($AdS_4 \times CP^3$ e $AdS_2 \times S^2 \times T^6$), una caretteristica che introduce molte complicazioni nello studio delle teorie stesse. In particolare, in spazi non massimamente supersimmetrici la superstringa di Green-Schwarz non è completamente descritta da un modello sigma di worldsheet sul corrispondente spazio di coset, dato che possiede ulteriori gradi di libertà fermionici (non-coset), associati alle supersimmetrie rotte. Ci concentriamo sullo studio dell'Integrabilità di queste teorie con l'obbiettivo di scoprire come i modi ferionici di non-coset entrino in e deformino la struttura integrabile di queste teorie di stringa. Costruiamo varie forme (collegate da trasformazioni di gauge) di connessioni di Lax a curvatura zero per la stringa in $AdS_4 \times CP^3$ e $AdS_2 \times S^2 \times T^6$ e mostriamo che in presenza dei gradi di libertà non-coset la $\mathbb{Z}_4$-invarianza persiste. Nel caso della superstringa in $AdS_4 \times CP^3$ studiamo anche l'istantone di stringa che si avvolge su di un ciclo non triviale in CP^3 e troviamo che ha dodici zero modi fermionici associati con 1/2 delle supersimmetrie dello spazio, quindi che questa soluzione classica, esatta e topologicamente non-banale è 1/2 BPS.

Contents

Introduction

It is not unusual in the world of research to find theories and arguments that originally were introduced to solve or to reformulate certain problems and after a while they have been reconsidered in a new unpredictable light. In the history of theoretical high energy physics the String Theory case is maybe the most striking example. Even if it may seem pedestrian, and not that original, to start the thesis with the well known "Story of the History", it is almost unavoidable if one has as a final goal to make a deeper insight into the structure of Superstring Theory. Since to reach this goal the path is long and hard, the motivations to face it have to base on deep and solid roots. For this reason we are going to briefly sketch motivations also from the historical point of view.

A bit of history

Since its brith almost fifty years ago, string theory passed through two transitions, the first, from being a candidate theory of strong interaction to the potential unifying description of all forces in nature, was characterized by a fase in which string theory was ruled out from the main stream of research, the second, in which the concept of string theory as a dual description of gauge theory took form, led to consider string theory not only as a possible "theory of everything" but also to take it into consideration as a mathematical tool in investigating gauge theories.

Let us try to go slightly more in the details of this triple process. In the 60's one of the things that were not yet understood was the nature of strong interactions and one of the first proposals to fill the lack of that theoretical description was using string theory. What was found was that the scattering amplitudes for strong interacting particles can be described by Veneziano amplitude, that turned out to appear in calculating string amplitudes. This description however was soon discredited by experiments, with the evidence of the Bjorken scaling, that lead to a parton interpretation for the high energy hadrons' scattering. This pointed out the necessity of a theory with a running coupling constant that admits asymptotic freedom.

Gross, Wilczek and Politzer in the early 70's discovered asymptotic freedom in non-Abelian gauge theories and this led to the formulation of what is known as Quantum Chromo Dynamics (QCD), $SU(N_C)$ gauge theory with $N_C = 3$ colors for each quark family. The impressive agreement with the experimental data ruled definitely out string theory as theory of strong interaction.

Then string theory had its first rebirth.

In 1974 the attention was captured by a particle-like excitation that is present in the closed string spectrum and has spin two, this string mode was identified as a graviton. This fact opened a new way of thinking about String theory. String theory became a possible unified theory of all the interactions, including gravity. At the beginning, however, the lack of phenomenological output and the impossibility of an efficient realization of the known particle physics did not attract much attention to String theory.

These theories containing gravity had a strong development in the 80's, when it was realized that Superstring theory has as a low energy limit the supersymmetric extension of Einstein's gravity, this fact allowed, at least, to soften discussions about the finiteness of this low energy limit, called Supergravity, within Superstring Theory. At the same time it has been realized that there are only five consistent ten dimensional superstring theories, known as type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$.

In 1995 the breakthrough was the discovery that five different String theories were actually five different perturbative limits of the same underlying theory, called M-theory, and therefore they are related to each other by a web of dualities. In addition M theory has an extension to 11 dimension [1, 2] which is based on the fact that Type IIA superstring theory can be regarded as a compactification of an 11 dimensional theory whose low energy limit is 11 dimentional supergravity that was costructed in 1978 by Cremmer, Julia and Scherk [3].

This unified theory, however, presents many vacua, so, if one tries to extract from string theory phenomenologically relevant models, by compactifying the extra dimensions in order to get back to the observed four dimesions, one encounters the problem of choosing among a huge landscape of string vacua and it is not obvious how to single out a particular vacuum from all the others. This plethora of vacua led to the conception of a Multiverse, one Universe for each vacua.

Then in 1997 the third phase started and sting theory began to be seen under a new, more practical in many senses, light.

In fact in 1997 Maldacena conjectured that there was a correspondence between a ten dimensional supestring theory and a four dimensional gauge theory [4]:

$$\text{type } IIB \text{ string theory on } AdS_5 \times S^5 \quad \leftrightarrow \quad \text{SYM } \mathcal{N} = 4 \;\; d = 4.$$

In the last fifteen years other examples of holographic dualities were found and explored and nowadays we speak of correspondences between a d-dimensional Conformal Field Theory and a String Theory on a $(d+1)$-dimensional Anti de Sitter space, or, more generally, between gauge theories with theories containing gravity.

Thanks to this last step string theory was recognized to be also a useful tool to study theories that, at a first glance, have nothing to share with gravity theories. In fact not only a new example of correspondence have been put forward, that is the case of Type IIA string theory on $AdS_4 \times CP^3$ background that is conjectured to be dual to Chern-Simons ABJM theory[1] [7], but also these correspondences have been used in studying physical systems. The first example was the quark-gluon plasma produced in heavy-ion collisions [8]. This was followed by holographic realizations of many phenomena of condensed matter systems: superconductivity and superfluidity [9], Fermi gas at unitarity [10], the quantum Hall effect [11], non-Fermi liquids [12], quantum phase transitions [13], exotical optical properties of materials [14]. See [15] for reviews.

Outline

The subject of this thesis is Superstring theories on Anti de Sitter backgrounds that do not have maximal supersymmetry, a feature that introduces many complications in studying these theories. In particular we shall concentrate on the study of integrability of these theories and their topologically non trivial instanton-like solutions. The characteristic of the backgrounds to break some supersymmetries, that we are going to study, has as a peculiarity that the worldsheet description of strings in such superbackgrounds is not exactly equivalent to a sigma model on a supercoset space, but has extra (non-coset) degrees of freedom associated with the broken symmetries. In fact in a maximally supersymmetric case, as it is for type IIB superstring theory on $AdS_5 \times S^5$ background, the number of supersymmetries and corresponding string fermionic modes is the same, 32, that coincides with the number of Grassmann–odd directions of the supercoset, which for $AdS_5 \times S^5$ is $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$. When some supersymmetry is broken we have to find a way to deal with the non supercoset modes of the superstring theory. This happens for the theories that we are going to study, namely type IIA $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ theories. In the absence of non-supersymmetric fermionic modes one can describe the remaining string degrees of freedom using the corresponding supercoset

[1] This example of AdS4/CFT3 correspondence is related by dimensional reduction to that of M-theory compactified on an $AdS_4 \times S^7/Z_k$ ($k = 0, 1, 2, \dots$) which is a near horizon geometry of a stack of M2-branes whose effective worldvolume description in terms of an $\mathcal{N} = 6$, $D = 3$ superconformal Chern-Simons theory was first proposed by Bagger and Lambert [5], and Gustavsson [6].

space, $\frac{OSp(6|4)}{SO(1,3)\times U(3)}$ and $\frac{PSU(1,1|2)\rtimes E(6)}{SO(1,1)\times U(1)\times SO(6)}$[2], that have respectively 24 and 8 fermionic degrees of freedom. This means that only 24 in $AdS_4 \times CP^3$ and 8 in $AdS_2 \times S^2 \times T^6$ of the 32 fermionic modes on the string worldsheet can be associated with the supercoset Grassmann–odd directions, while the 8 or 24 remaining fermionic modes do not have this group–theoretical meaning.

The presence of these non supersymmetric modes makes the study of this theory much more involved in comparison with the supercoset sigma model. In fact the two complete superstring theories, even if the bulk supergeometries have isometries $OSp(6|4)$ and $PSU(1,1|2) \times U^6(1)$, are not formulated on a supercoset. It is not always possible to eliminate the extra non supercoset fermions, so these can play a role in string dynamics. This means that, in general, the geometrical construction of the complete theory is much harder than in the coset case, since we have to understand how to incorporate the non supersymmetric fermions in the structure of the theory.

In the Green–Schwarz superstring sigma–model on $AdS_4 \times CP^3$ superspace these eight non–supercoset fermionic modes can be put to zero by partially gauge fixing the kappa–symmetry for almost all classical configurations of the string. This however is not possible when the string motion is restricted to the AdS_4 subspace [20, 21] or when the string forms a worldsheet instanton by wrapping a CP^1 cycle in CP^3 [22]. In these cases the supercoset kappa–symmetry gauge is inadmissible, and the non–coset fermions carry physical worldsheet degrees of freedom[3]. Furthermore if we look at the $AdS_2 \times S^2 \times T^6$ case the impossibility of getting rid of all the non supercoset degrees of freedom is evident, since we would need to get rid of 24 fermions but we have only a maximum of 16 Kappa-symmetry parameters at stock.

When all the worldsheet fermionic modes of the string are in one to one correspondence with the Grassmann directions of the supercoset space, and thus this fully describes the supergeometry of the supergravity solution, the classical integrability of the string theory is fully determined by the algebraic and geometrical structure of the corresponding supercoset. Conversely, in the cases that we are going to analize, even if the integrability at the classical level of the supercoset sigma model is well stated, due to the presence of the non supercoset fermions, the classical integrability of the

[2]In the $AdS_2 \times S^2 \times T^6$ case it is usefull to consider this supercoset because it captures all the 10 dimensional bosonic geometry. This can be achieved by noting that $AdS_2 \times S^2 \times \mathbb{R}^6$, with eight fermionic directions, is described by the supercoset $\frac{PSU(1,1|2)\rtimes E(6)}{SO(1,1)\times U(1)\times SO(6)}$, where the semi-direct product with $E(6)$, the Euclidean group in six dimensions, accounts for the $\mathbb{R}^6$ factor. Since $AdS_2 \times S^2 \times T^6$ is locally the same as $AdS_2 \times S^2 \times \mathbb{R}^6$, and we will only be interested in the local geometry, this gives us a (local) supercoset description of $AdS_2 \times S^2 \times T^6$.

[3]Subtleties of gauge fixing kappa-symmetry in a way consistent with the light-cone gauge in a near plane- wave limit of $AdS_4 \times CP^3$ has been discussed in [23].

complete superstring theories is still to be demonstrated. In these less supersymmetric theories we have to find a way to incorporate the non supersymmetric modes in the algebric structure of the theory. In practice, since to demonstrate classical intgrability it is sufficient to find a zero curvature connection, called **Lax Connection**, that depends on the spectral parameter, we have to find the dependence of the connection on these fermions, hoping that the flatness is preserved.

In the case of the supercoset sigma model the Lax connection is constructed with the help of the $\mathbb{Z}_4$-automorphism of the isometry of the superalgebra, that plays an important role in the applications of the integrability techniques. The hope is that the introduction of the non supercoset coordinates do not spoil this important property.

A strategy to build the Lax connection can be using the Noether currents of the isometries of the superbackground as building blocks [24, 25].When the non–coset fermions are put to zero, the Lax connections of [24, 25] are related, by a superisometry gauge transformation that depends on the spectral parameter [24], to those of the supercoset model, that have been found in [20, 26, 28, 29].

The Noether current approach though has a drawback, i.e. it is not manifestly $\mathbb{Z}_4$-invariant. If we want not only to understand how to incorporate the non coset fermions in the Lax connection but also to reveal a role of the non–coset fermionic and bosonic modes in the corresponding Bethe–ansatz techniques, it seems useful to have at hand an explicit expression which demonstrates how the $\mathbb{Z}_4$–graded supercoset Lax connection gets generalized by terms depending on the non–coset degrees of freedom. We will see how to build such a Lax connection for the theory at all orders in coset fermions and up to the second order in non coset ones. In principle the construction can be extended to all orders in the non-coset fermions, however in practice this becomes technically very complicated unless one finds a hidden underlying structure which would allow to solve the problem as it has been done in the supercoset case.

An interesting thing to note is that in the two cases, $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, the Lax connections have formally the same structure, at least to the second order in non coset fermions [30]. This is because the numbers of their target–space supersymmetries complement each other to $32 = 24 + 8$, the maximal number of *10d* type II supersymmetries, and the projectors which split 32–component fermions into 24– and 8–component ones are the same in both of the cases.

The power of integrability is that, if a theory reveals this property, one has the chance to solve the theory completely, i.e. to find infinite number of integrals of motion and if the integrability persist at the quantum level, to compute the spectrum of quantum observables, e.g. the energy spectrum. Integrability represents a remarkable short cut in the solution of the theory, in the sense that, instead of dealing with differential equations, one can find a set of integral equations that, in a certain limit, simplify to a

set of algebraic equations, the so-called asymptotic Bethe equations.

This is not the end of the story, if we really want to reveal all the features of the superstring theory on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ backgrounds, we have to take into consideration that these have non perturbative solutions, in fact in these cases there are cycles on which the strings can wrap to give instantonic configurations that can contribute to the string effective action. So we are going to consider some cases of instantonic solutions. In particular, in the case of the $AdS_4 \times CP^3$ superstring we study the string instanton wrapping a non-trivial two-cycle in CP^3 and find that it has twelve fermionic zero modes associated with 1/2 of the supersymmetry of the background thus manifesting that this exact topologically non-trivial classical solution is 1/2 BPS.

In Chapters 1 and 2 we will set up the background on which we are going to move both in theoretical aspects and in the notation and formalism that we are going to refer to in the rest of the thesis. In Chapter 3 we will look at geometrical aspects of the spaces, $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ superspaces, that we are going to work with, also looking at the geometries of the corresponding super-coset spaces. In Chapter 4 we will consider, first of all, the equations of motion of sigma models on $\frac{OSp(6|4)}{SO(1,3)\times U(3)}$ and $\frac{PSU(1,1|2)\rtimes E(6)}{SO(1,1)\times U(1)\times SO(6)}$ supercoset spaces, and then we will look at the complete type IIA superstring theory on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, writing the equations of motion, and presenting their expressions first up to the second order in all fermions, both coset and non-coset, and then to all orders in coset fermions, but still to the second order in non coset ones. In Chapters 5 and 6 we will move to consider the problem of integrability. In Chapter 5 we will review what was done in the case of supercoset sigma models. In Chapter 6 we will include the non-coset degrees of freedom in the integrable structures of the theories. We will see how these fermions enter in the Lax connections, first of all reviewing the result up to the second order in all fermions and then writing the Lax connection to all orders in coset fermions and to the second order in non coset one. Finally in Chapter 7 we will consider topologically non-trivial configurations of superstrings on the backgrounds of interest, in particular we will explicitly compute the instanton solution wrapping a CP^1 cycle in CP^3 in the $AdS_4 \times CP^3$ case.

Chapter 1

Setting the background: Theoretical motivations

In this chapter we will review some basic theoretical arguments that will be useful for understanding the core of the thesis.

1.1 AdS/ CFT correspondence

This section can be considered as part of the motivations of this work, in fact, even if in this thesis we are not going to directly study the AdS/CFT correspondence problems, this argument will be the background on which we are going to move, considering that we are going to work with superstring theories that admit a dual field theory, and what we will see has as a final outcome to enrich the Holographic dictionary of the correspondences, by mean of a deeper knowledge of the theory on the string side. There are many reviews on the subject, to cite a few [31, 32, 33, 34, 35]

In general it was found that there is a correspondence between d-dimensional gauge theories, the boundary theory, and d+1 dimensional theory containing gravity on an Anti de Sitter space, the bulk theory.

The Lagrangian of the CFT with external sorces can be written in the following schematic form:

$$\mathcal{L} + \mathcal{L}_J = \mathcal{L} + \sum_A \mathcal{O}_A J_A \tag{1.1.1}$$

where $\mathcal{O}$ are a basis of operators in the CFT and J are their sources, that can be interpreted as boundary values of fields in the bulk. Let us now introduce the partition

function $Z_{CFT}[J]$ of the field theory

$$Z_{CFT}[J] = \int \mathcal{D}x e^{i\frac{S_{CFT}[x]}{\hbar}}\,, \tag{1.1.2}$$

that is the generating function of the vacuum correlators of local operators in the CFT, i.e.:

$$< \prod_n \mathcal{O}_n >= \prod_n \frac{\delta}{\delta J_n} \ln Z\,. \tag{1.1.3}$$

What the duality says is that one can obtain $Z[J]$ from gravitational computation, in fact at the base of the correspondence there is the relation between the partition function of the CFT and the partition function of gravity (or string theory) evaluated at the boundary:

$$Z_{CFT}[J] = Z_{str}[\phi = J] \sim e^{S_{grav}}|_{\text{EOM, }\phi=J}\,, \tag{1.1.4}$$

Where ϕ is some bulk field in the gravity theory, that assumes J value at the boundary. This manifests the basic idea that one can perform the computation in the more suitable side, usually the one in which one can realize the perturbative regime, to extract information on the other side.

The ideas of holographic duality were first put forward by 't Hooft and Polyakov [16][17], noticing that Feynman Diagrams in field theories possesses a double expansions, that is a typical behavior in string theory.

Some concrete examples of this correspondence were proposed by Maldacena in 1997 [4]. In particular he stated that there is a correspondence between two theories, one that contains gravity and lives in 10 dimensions and another one that is a gauge theory in 4 dimensions:

$$\text{type } IIB \text{ string theory on } AdS_5 \times S^5 \quad \leftrightarrow \quad SU(N) \;\; \text{SYM } \mathcal{N} = 4 \;\; d = 4.$$

In this correspondence, that is a weak/strong coupling correspondence, i.e.

$$\lambda = \frac{R^4}{\alpha'^2} \quad \text{and} \quad g_{str} = \frac{4\pi\lambda}{N}\,, \tag{1.1.5}$$

where λ is the 't Hooft coupling of the Super Yang Mills theory, R is the AdS_5 radius, α' is the slope of the Regge trajectory and g_{str} is the coupling constant of the string theory, one finds a one to one relation between the energy spectrum of the string theory and the spectrum of scaling dimensions for planar operators in the CFT.

The fact that the perturbative regime of one theory corresponds to the strong coupling of the other theory represents the power of the correspondence, but it is also the main obstacle to prove it.

To evidence the correspondence Maldacena recurred to geometrical consideration, by means of extended objects called Dp-branes. These are p dimensional objects on which the open strings can end. The attached open string describes via its oscillations a gauge field that lives on the brane and its fermionic counterpart.

Now if we take a stuck of N Dp-branes we notice that the strings can stretch from one brane to another, this means that the string ends, that are atteched to one or two different branes, can be labeled by two indices, each one that goes from 1 to N. Given this, it is not hard to recognize on the branes a $U(N)$ gauge theory. On the other hand the p dimensional extended objects carry a charge which induces a (p+1)-form gauge field with a (p+2)-form field strenght. The p+2-form thus acts as a flux which curves the background. Indeed it is possible to find corresponding solutions of the supergravity low energy limit of string theory that carry fluxes. If we analyze the solutions we discover that there is an event horizon, which corresponds to a p dimensional black brane, and we can look at the near horizon geometry.

For the first conjecture Maldacena took a stack of coincident $D3$-branes and considered what happens on the surface of the branes and what was the backreaction of the branes on the background. He noticed that looking at the gauge theory on the branes this is an $U(N)$[1] $\mathcal{N} = 4$ Super Yang Mills Theory in four dimensions, since the worldvolume of a D3-brane has 4 dimensions, and the near horizon geometry is $AdS_5 \times S^5$. In the original paper of Maldacena was already present the idea that this correspondence was not an isolated accident, but he studied similar setups for theories on $AdS_2 \times S^2$, $AdS_3 \times S^3$ and, for what concern 11 dimensional spaces, $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

A great deal of attention has been given, in later times, to another example of correspondence: the correspondence between a 10 dimensional type IIA superstring theory on $AdS_4 \times CP^3$ and a three dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory (known as ABJM theory). The geometrical considerations to reach the conclusion of a correspondence between these two theories are analogous to those that Maldacena gave in 1997, but this time taking coincident $M2$-branes.

Today the AdS/CFT is also applied to study problems of various physical systems, an example is the application in condensed matter theory. In dualities applied in condensed matter problems the general idea is that the masses of the fields in AdS correspond to the conformal dimensions of the operators in the gauge part:

[1]The $U(1)$ factor of the $U(N) \sim SU(N) \times U(1)$ gauge theory corresponds to the center of mass of the stuck of branes. The $U(1)$ vector supermultiplet is decoupled from the rest. The theory in the AdS bulk describes the $SU(N)$ part of the gauge theory, while the $U(1)$ sector corresponds to topological B-fields in AdS (see [18] and [19], page 58 for more detailed discussion of this point).

AdS	Gauge theory
mass (m)	conformal dimension of the operators (Δ)

$$m^2 R_{AdS}^2 = \Delta(\Delta - 4) \tag{1.1.6}$$

where R_{AdS} is the AdS radius.

1.1.1 Large N limit and gauge/gravity correspondence

In this subsection we are going to consider some intuitive hints of the correspondence between string and gauge theories. Similarities were already noticed when string theory was still considered as a candidate for strong interactions. The fact that the string theory on flat space predicts that mass squared of hadronic resonances is in a linear relation with their spins (Regge trajectories), as actually happens, led to consider this for a while as the theory of strong interaction. Though afterwords QCD came into the stage to become "the theory of strong interactions", the flux tube that stretches between two quarks can be interpreted in a certain limit as a string with a tension.

On the other hand, the large N limit of non-Abelian gauge theories proposed by 't Hooft in 1974 [12] in relation to QCD has revealed the relation of the topologiacal structure of the 1/N expansion of gauge theory diagrams to that of the dual string theory. We shall shortly return to more detailed discussion of ths point.

All these considerations are quite heuristic and can give to string theory, under some assumptions, the role of an effective strong interaction theory. However this is not what exactly AdS/CFT states, in fact in the correspondence the gauge theory and the string theory are actually two different realizations of the same theory. Even if this argument is not directly applied to QCD, that does not posses conformal symmetry, it can be helpful to study it, considering the generality of some quantities that can be traced in SYM theory (e.g the shear viscosity), for a review see [33]. Of course nowadays the practical usefulness of dualities have found new applications also for other physical systems, e.g. condensed matter ones.

The chance to compare gauge and string theories came from the use of the large N limit ('t Hooft limit). This is in fact a limit in which the two theories resemble each other.

The large N limit can be performed in $SU(N)$, $SO(N)$, and $Sp(N)$ gauge theories, and it consist in taking the rank of the group to be infinite ($N \to \infty$), while one introduces a new gauge coupling $\lambda = g_{YM}^2 N$, that has to remain finite. In this limit one can perform a new expansion around $\lambda = 0$ and to classify the Feynman graphs through their genera. In fact graphs whose lines do not overlap can be drown on a

surface of genus zero, while when overlaps occur the genus of the canvas surface grows. In the large N limit the higher genera graph can be neglected this is why it is often called **planar limit**.

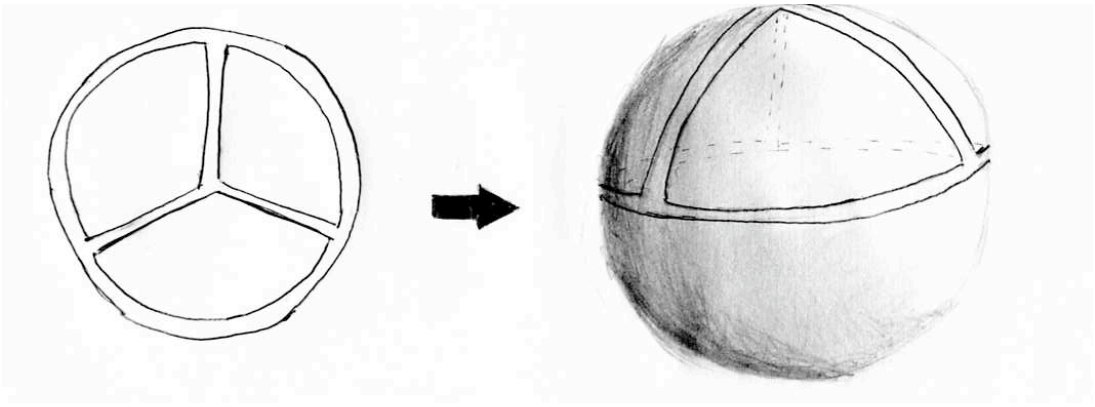


Figure 1.1: planar graph (QCD like graph)

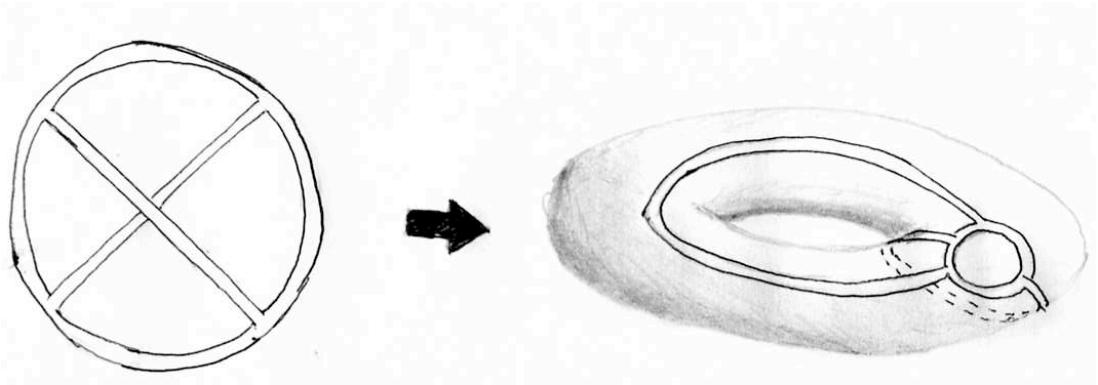


Figure 1.2: genus 1 graph (QCD like graph)

The interesting fact is that we introduced a two dimensional structure in the gauge theory, that reproduce the expansion in genera of the string theory, in which the genus growth is due to interactions ($g_{str} \sim 1/N$).

As we have seen there are many clues of an actual correpondence between gauge and gravity theories. What is now an on going work is to find a way of testing and studying these correspondences, also moving away from the planar limit. One of the most important features is the existence of Integrability.

1.2 Integrability

The term Integrability is widely used in the field of Physics and Mathematics, so, before entering into the details of its meaning in string theory and its well known importance in the study of string theories and Gauge/Gravity correspondences, it is important to review where the concept of Integrability comes from and why we can speak of integrable systems when we consider superstring theories.

Integrability is a property that is natural to trace in two dimensional theories (for review see e.g. the books [36], [37]). If we have an integrable model for an interacting theory it means that we can find an infinite number of conserved charges Q_i in involution.

For a classical theory this means that Q_i sutisfy:

$$\{Q_i, Q_j\} = \{Q_i, H\} = 0\,, \tag{1.2.7}$$

where H is the Hamiltonian of the theory. Here we have used Poisson brackets, that, taking two functions f an g, are defined:

$$\{f, g\} = \sum_{i=1}^{n} \{\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\} \tag{1.2.8}$$

where $q_1, ..., q_n$ $p_1, ..., p_n$ are the coordinates of the phase space, that sutisfy:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}. \tag{1.2.9}$$

From (1.2.7) we can see that Q_is are integrals of motion:

$$\frac{dQ_i}{dt} = \{H, Q_i\} = 0. \tag{1.2.10}$$

If we move to consider a quantum system the Poisson brackets get replaced by the (anti)commutators:

$$[Q_i, Q_j] = [Q_i, H] = 0\,. \tag{1.2.11}$$

Moreover the concept of integrability in the field of mathematical physics is deeply linked with the concept of solvability of the theory, in fact for an integrable theory in principle one can solve its equations of motion and find e.g. the spectrum of the states. To find the spectrum is not to find a simple analitical expression for it, but rather to find a set of equations that enable us, if we manage to solve them, to have the spectrum as a solution of them. The advantage is that for integrable systems what we have to do is to solve a system of integrable equations, that, under certain assumptions, can be reduced to algebraic equations, rather than differential ones.

Approaching a theory one has to determine first of all if this theory is integrable at the classical level and thus if it can be classically solved. In some cases the integrability survives at the quantum level, if this happen one has a chance to find the full spectrum of the quantum observables of the theory.

1.3 Integrability in String Theory and its dual

When integrability shows up in string theory it is not quite a surprise, in fact many two dimensional sigma models on coset spaces show integrability, at least at the classical level. What is rather astonishing is that such a property arises in gauge theories in three and four dimensions. This feature was at first noticed in the case of the planar $\mathcal{N} = 4$ SYM in four dimensions. The analitical solvability of the theory is addressable to the fact that we are dealing with an integrable theory. The presence of integrability in this gauge theory can be explained with the peculiarity that it possesses a dual integrable string theory. Analyzing the situation more carefully we can realize that dealing with integrability means really to test the correspondence, with integrability techniques we can have the same predictions on the both sides of the correspondence.

Integrability techniques allow us to study physical systems without the employment of perturbative calculations, that can introduce many complications in the computations. When a model turns out to be integrable, this represents a remarkable shortcut to the knowledge of the observables of the theory.

When we consider a Gauge/Gravity duality, we have to study integrability on the both sides of the correspondence. To see if the String Theory is classically integrable we should find a Lax connection $L(\mathtt{x})$ that satisfies the zero curvature condition. In the field theory side the integrability techniques are based on the fact that one can apply a spin chain realization of the theory and, as in the case of the Heisenberg spin chain, one can use the Bethe Ansätz to solve the problem.

1.3.1 Integrability in String Theory

As we already said, we have to find a Lax connection $L(\mathtt{x})$, such that:

$$dL(\mathrm{x}) - L(\mathrm{x})L(\mathrm{x}) = 0, \tag{1.3.12}$$

when the equations of motion of the system are satisfied, and vice versa the zero curvature condition implies the equations of motion. Strategies to build this connection will be pointed out in more details later. The Lax connection depends on a spectral paremeter $\mathtt{x}$.

We can use this connection to define a monodromy matrix as a function of the spectral parameter:

$$\mathcal{M}(\mathrm{x}) = P \exp\left(\oint L(\mathrm{x})\right). \tag{1.3.13}$$

The eigenvalues of the monodromy matrix form an infinite set of conserved currents.

To describe the classical solution of a theory we can use spectral curves, the spectral curve is a complex curve defined by the eigenvalues of the monodromy matrix. In particular if we diagonalize $\mathcal{M}(\mathtt{x})$ we get as eigenvalues the quasi-momenta, and we may define the spectral curves in terms of these. If we now consider the spectral curves of a finite genus, i.e. the finite-gap equations, we notice that those encode the solution of the model. This is the strategy that one adopts in studying the classical integrability of the superstring theories that we are going to analyze, even if in this work we are at the early stages of the process.

An important example of integrable superstring theory is Type IIB superstring on $AdS_5 \times S^5$. It not only was the first non-trivial example to appear and the one that is more understood right now, but it also does not have difficulties that show up, when we deal with superstring theories on less supersymmetric backgrounds. The key of the demonstration of the integrability of such a theory, which is maximally supersymmetric, is that it is described by a non-linear-sigma-model on the supercoset

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}, \tag{1.3.14}$$

which has $AdS_5 \times S^5$ as the bosonic part, and 32 fermionic directions. In this particular case the integrability is also conserved at the quantum level. This, of course, does not mean that we already have the full solution of the theory in our hand, neither that we have the full test of the correspondence, but that we have a chance, via integrability techniques, to rich it.

The picture becomes more complicated if we deal with less supersymmetric theories. The process that we have just described, in this case, as we have said, is still at the very early stages, in fact the problem is the lack of knowledge of the Lax connection and, as a consequence, of the integrability, even at the classical level, in the cases in which the kappa gauge fixing is not allowed or not enough to eliminate non-coset degrees of freedom and bring us back to a supercoset sigma-model.

What we intend to do, in the core of this thesis, is to address and develope the problem of integrability for $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, that are backgrounds that break some of the supersymmetries.

Before to go on it is worth to consider a simple, but very instructive, example of integrable model. For the very moment we are not going to consider supersymmetric

theories, that will be taken into consideration later.

1.3.2 Classical Integrability: a simple bosonic example

Let us take a nonlinear bosonic sigma model on a coset symmetric space G/H and $k \in G/H$ to be a coset element [26].

We can define the left invariant Cartan forms pulled back on the 2-dimensional surface

$$J = k^{-1}dk \tag{1.3.15}$$

that can be decompose according to the decomposition of the Lie algebra $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$

$$J = H + P, \tag{1.3.16}$$

the differential d in our conventions acts from the right, this means for example that for two one-forms a and b we have:

$$d(ab) = a(db) - (da)b. \tag{1.3.17}$$

The decomposition of the Lie algebra gives the following commutation relation:

$$[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}, \quad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{H}, \quad [\mathcal{P}, \mathcal{H}] \subseteq \mathcal{P} \tag{1.3.18}$$

this is a decomposition for the algebra that reflect the $\mathbb{Z}_2$-grading ($\mathcal{H}$ has $\mathbb{Z}_2$-grading 0 and $\mathcal{P}$ has grading 1).

The Lagrangian of the model is then

$$Tr(P_I P^I) \tag{1.3.19}$$

where $I = 0, 1$ labels the two dimensional coordinates.

In order to construct the Lax connection of the model we can start either with the left invariant form (1.3.15), or with the right invariant Cartan forms:

$$j = dkk^{-1} = kJk^{-1} \tag{1.3.20}$$

We have to notice that for a generic $f = kFk^{-1}$ the following relation holds:

$$df = kdFk^{-1} - j \wedge f - f \wedge j \tag{1.3.21}$$

Let us notice that in this case $p = kPk^{-1}$ is the Noether current for the global symmetry:

$$d * p = 0, \tag{1.3.22}$$

where $*p$ is the worldsheet Hodge dual of p, and that J sutisfies the Maurer-Cartan equation:

$$dJ - J \wedge J = 0 \tag{1.3.23}$$

that decompose into:

$$\begin{aligned} dH &= H \wedge H + P \wedge P, \\ dP &= P \wedge H + H \wedge P. \end{aligned} \tag{1.3.24}$$

this implies, introducing $h = kHk^{-1}$, that:

$$\begin{aligned} dh &= h \wedge h + p \wedge p - p \wedge h - h \wedge p, \\ dp &= -2p \wedge p. \end{aligned} \tag{1.3.25}$$

This means that $2p$ is both flat and conserver and can be used to bild Lax connection:

$$l = 2\alpha p + 2\beta * p, \tag{1.3.26}$$

that turns out to be flat

$$dl - l \wedge l = 4(\alpha^2 - \alpha - \beta^2)p \wedge p = 0 \tag{1.3.27}$$

with:

$$\alpha = \frac{1}{2}(1 \pm \cosh \mathtt{x}), \quad \beta = \tfrac{1}{2} \sinh \mathtt{x} \tag{1.3.28}$$

where we have introduced the spectral parameter $\mathtt{x}$.

This can be associated with the left invariant connection via a transformation:

$$L = k^{-1}lk + k^{-1}dk \tag{1.3.29}$$

Till this moment we have not considered any supersymmety, we will see a supersymmetric example when we will enter more in the details of the thesis.

1.3.3 Spin chain realization of a CFT

For completeness we sketch here integrability techniques based on spin chain representation for the CFT.

The first time that this structure for a gauge theory was pointed out was in 2002 by Minahan and Zarembo [38], and it was for the $\mathcal{N} = 4$ $D = 4$ SYM theory.

The main consideration was that the local operators can be mapped in spin chain states and the operator which measures the planar one-loop anomalous dimension corresponds to a spin chain Hamiltonian, thus one can write an effective Lagrangian for the theory that is a Lagrangian of a one dimensional spin chain with nearest neighbor interaction.

The first thing to do is to individualize the set of eigenvectors for the dilation operator, that gives anomalous dimension as eigenvalue. To do this one performs the so called BMN limit [27], singling out those operators that have finite $\Delta - J$ value, when J, the R-charge, goes to infinity. This limit gives infinite size trace operators [2], and taking as a ground state the operator that has $\Delta - J = 0$

$$Tr(ZZZZZZ.....ZZZ) \tag{1.3.30}$$

where Z is a field of dimension 1 and R-charge 1. Then we can build a tower of operators starting to insert field X with dimension 1 and no R-charge in the operators

$$Tr(ZXZ...ZXZ.....ZZZ) \quad \text{where } \Delta - J = \#\,\text{of}\,X \tag{1.3.31}$$

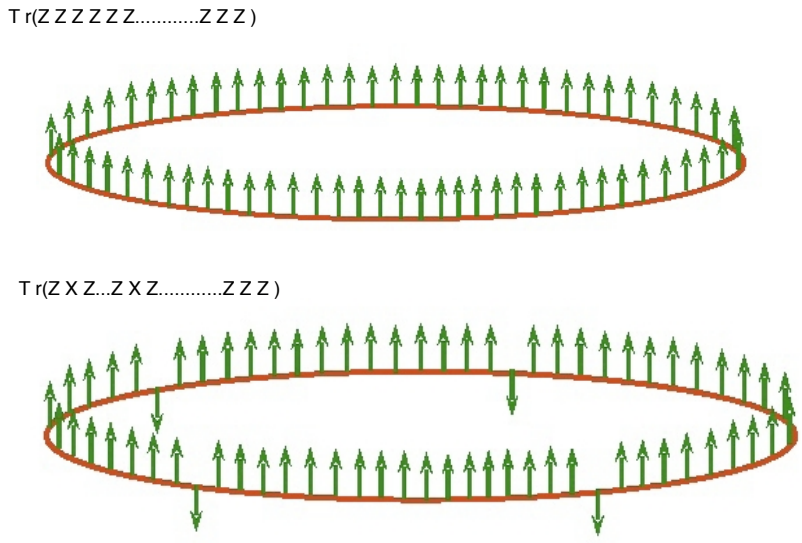
These are in general not eigenvectors of the dilation operator, but we can sum them in order to get a state with definite dimension.

The great intuition of Minahan and Zarembo was to find the analogy in the planar limit for these operators with a spin chain system, stating that the ground state corresponds to a spin chain with all the spins up, and the excited (magnons, that are labelled with X) states would correspond to the progressive spin flip at some sites of the spin chain, since these are trace operators the only important thing is the relative position of the exitation. They found that the action of the dilation operator on these infinite chains (since $J \to \infty$ in the planar limit) was analogous of an effective spin Hamiltonian.

In this way we can rewrite the CFT theory as a spin chain theory, i.e. as an integrable system. This explains why in the planar limit of SYM $\mathcal{N} = 4$ is exactly solvable, it is an integrable theory!

It is also worth to emphasize that, as was at first done for Heisenberg Hamiltonians, to get the spectrum of gauge theories that admit a spin chain representation, one can use Bethe ansatz techniques.

[2] In general the effective Hamiltonian is an $SO(6)$ spin chain Hamiltonian and the structure is rather more involved. This simple structure is what one can find if one restrict to an $SU(2)$ or an $SL(2)$ sector of the theory. Nevertheless this is a quite instructive framework to understand the idea, even if it is not the more general one



More recently the spin chain representation has been found also for ABJM theory, thus people are trying to reconstruct the same considerations done for AdS_5/SYM in the case of $AdS_4/ABJM$[3] [39].

What one gets studying the spin chain representations of the gauge theories are asymptotic Bethe equations, that turn out to be not enough to study the complete theory, this means not perturbatively, but at all the intermediate values of the coupling constant. To try to study the complete theory many people are developing what is called Thermodynamic Bethe ansatz or Y-systems.

Having used integrability techniques on both sides, one can compare the results. In this sense using Integrability we are not only computing the spectra of the theories, but also giving a test of the correspondence which we briefly sketch in the next subsection.

1.3.4 The general idea of integrability for AdS/CFT

Integrability is, for the examples of dualities that up to now are known, a property that is confined to the planar limit. Thus it helps in solving the theory at the planar level, in which the string theory is free.

[3] Another technique that allow to compare the results coming from the two sides of both the correspondences, also trying to move from the plain wave limit, is the recursion to Landau-Lifshitz models [40, 41, 42, 43].

The main idea of integrability applied in a duality scenario is to find expressions for physical quantities that are valid for every value of the coupling constant λ. In fact in gauge/gravity correspondence one tries to match the perturbative regime of the gauge theory with the perturbative regime of the gravity theory, but this is not easy to do. Integrability has the advantage of giving the scaling dimension of the local operator as a function of λ. In fact when we say that we are going to solve the theory with integrability, this means to find equations of the form:

$$f(\Delta, \lambda) = 0 \tag{1.3.32}$$

where Δ is the scaling dimension. As one can verify the spectral equations contain the coupling constant in functional form. This means that with integrability we try to access a theory over the complete range of the coupling constant λ. At the end the hope is to find the behavior of the theory at intermediate values of couplings for which neither the perturbative regimes are valid.

1.4 Istantons

Usually when we study a theory we make recursion to perturbative expansions. But it can be that not all the solutions of the theory are captured by that expansions, we have to directly study these non-perturbative solutions in order to see whether they contribute to the effective action of the theory. This is the case of the instantons [44, 45, 46].

Istantons are usually defined as classical solutions of the equations of motion that have a finite non zero action, infact, since these contribute to the path integral with a weight

$$e^{-\frac{S_{cl}}{\hbar}}, \tag{1.4.33}$$

where S_{cl} is the classical action of the theory, they can not be found in a perturbative expansion. This means that, if instantons are present, they can represent an essential quantum correction to the classical behaviour of the theory.

Around an instanton solution there are quantum fluctuations. The subtlety in studying instantons is that one has to take into account normalizable solutions of the linearized equations for the fluctuations, that are called **zero modes**. These modes have to be computed and studied separately from the non-zero modes.

Let us sketch how zero modes appear in perturbation theory. Once a classical instanton solution is found, one can study quantum fluctuations above this solution, in doing this we can write a solution

$$\phi = \phi_{cl}(\gamma) + \xi \tag{1.4.34}$$

where ξ is the quantum correction to the classical solution, we denote with γ the parameters on which the classical solution depends, they are usually called collective coordinates and they are associated to the invariances of the action. We can expand the action:

$$S = S_{cl} + \int \xi M \xi + O(\xi^3). \tag{1.4.35}$$

The zero modes are eigenstates with a null eigenvalue of the M matrix. The linear term vanishes due to the equations of motion

$$\frac{\delta S_{cl}}{\delta \phi_{cl}} = 0, \tag{1.4.36}$$

that also imply:

$$0 = \partial_\gamma \frac{\delta S_{cl}}{\delta \phi_{cl}} = \int \frac{\delta^2 S_{cl}}{\delta \phi_{cl} \delta \phi_{cl}} \partial_\gamma \phi_{cl}. \tag{1.4.37}$$

This means that, differentiating with respect to the collective coordinates the instanton solution, one can find a zero mode, since $\partial_\gamma \phi$ is an eigenvector with null eigenvalue of $M = \delta^2 S / \delta\phi\delta\phi|_{classical}$. This means that for each collective coordinate we have one zero mode, so counting the parameters on which the instantonic solution depends one can know the number of zero modes corresponding to that classical solution.

It is not a surprise to have the number of zero modes to be equal to the number of collective coordinates, in fact, being M the second derivative of the action, in principle, perturbing the least action solution we go to higher values for the action and this gives modes with a positive eigenvalue, the only chance to have a zero eigenvalue is perturbing the action in an invariant way, this means along collective coordinates.

Instanton effects are present in many theories, an example is the tunneling between two vacua in Minkowski space-time, that classically is forbidden but quantum mechanically can occur. The strategy to compute these instantons is to go to the Euclidean space, by a Wick rotation, in fact in this passage the potential is turned upside down and now to go from a vacua to another is classically admitted. In this way one computes the non perturbative effects in the Euclidean space and, in order to see the physical effect that can arise, one has to continue them to the Minkowski one.

In this thesis we shall study fermionic zero modes of the worldsheet instanton in $AdS_4 \times CP^3$ string theory, that are normalizable solutions of the Dirac equation. In order to do this we are going to directly study the solution of the fermionic equations of motion of the theory.

1.4.1 Instantons in CP^n bosonic sigma model

This example will be useful to be compared with the instanton-like solution computed in the last chapter of this thesis.

The instanton solution in the $O(3)$ (or CP^1) sigma–model was first found in [47] and then generalized to the case of the CP^n sigma–models in [48, 49, 50]. The instanton solution in the supersymmetric CP^1 sigma–model was first discussed in [51]. See [52, 53, 54] for a review and references on this subject.

The action of a CP^n bosonic sigma model can be written as:

$$S = \frac{1}{g^2} \int d^2x \ \overline{D_\mu \sigma_a} D^\mu \sigma^a \tag{1.4.38}$$

where σ_a are $n+1$ complex fields for which is valid:

$$\bar{\sigma}_a \sigma^a = 1. \tag{1.4.39}$$

and $D_\mu = \partial_\mu - \bar{\sigma}\partial_\mu\sigma$.

We can now introduce complex 2-dimensional coordinates:

$$z = x_1 + ix_2 \qquad \bar{z} = x_1 - ix_2 \tag{1.4.40}$$

It is not hard to show that the action can be rewritten in the following form:

$$S = \frac{2}{g^2} \int d^2x \overline{D_z \sigma_a} D_z \sigma^a + \overline{D_{\bar{z}} \sigma_a} D_{\bar{z}} \sigma^a \tag{1.4.41}$$

We can introduce the topological charge q:

$$q = \frac{2}{c} \int d^2x \overline{D_z \sigma_a} D_z \sigma^a - \overline{D_{\bar{z}} \sigma_a} D_{\bar{z}} \sigma^a \,, \tag{1.4.42}$$

c is a normalization factor that ensures that q takes only integer values. In the form (1.4.41) it is easy to see that local minima for the action are obtained when:

$$D_z \sigma^a = 0 \,, \ D_{\bar{z}} \sigma^a = 0 \tag{1.4.43}$$

that gives solutions with a finite action:

$$S_I = \frac{c}{g^2} |q| \,. \tag{1.4.44}$$

We can rewrite the action in terms of n fields ϕ_γ by going from the variables

$$(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \tag{1.4.45}$$

to

$$(\phi_1, \ldots, \phi_n, \sigma_{n+1}) \tag{1.4.46}$$

performing the following field redefinition:

$$(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) = (\sigma_{n+1}\phi_1, \ldots, \sigma_{n+1}\phi_n, \sigma_{n+1}). \tag{1.4.47}$$

The conditions (1.4.43) translate into:

$$\partial_{\bar{z}}\phi_\gamma(z, \bar{z}) = 0 \quad (\text{or } \partial_z\phi_\gamma(z, \bar{z}) = 0)\,, \quad \gamma = 1, ..., n \tag{1.4.48}$$

that means that an (anti-)instanton solution is given by a (anti-)holomorphic function. The solution to this form must sutisfy also the condition $\phi \to \phi_0$ at $|x| \to \infty$, chosing ϕ_0 to be 1, in principle one can choose any point of CP^n since this is homogeneous, ϕ can be written in the rational form:

$$\phi_\gamma = \left[\prod_{j=1}^{q}(z - a_j^0)\right]^{-1} \prod_{j=1}^{q}(z - a_j^\gamma)\,. \tag{1.4.49}$$

Therefore each instanton solution with the topological charge q is characterized by $2(n+1)q$ real parameters (or collective coordinates). This means, in view of the relation (1.4.37), that each CP^n instanton of unit charge has 2(n+1) zero modes.

Now we have reviewed all the main theoretical aspects that we need to understand and contextualize the rest of the thesis, but, before to get into the main discussion, we still need to fix some notation and to look at the formalism that we have chosen to use, this will be the subject of the next Chapter.

Chapter 2

Setting the background: Main Notation and Formalism

2.1 Superfields

In this thesis I will use the superfield formalism, that is based on the fact that one can ensuit all the information about bosonic and fermionic fields in one single field, that due to the supersimmetric nature of its components is called Superfield.

To be more specific let us take a set of supersymmetric coordinates $Z^{\mathcal{M}} = (X^M, \Theta^\mu)$, where X^M are the bosonic coordinates and Θ^μ are the fermionic ones. In the case of a D dimensional theory we would have:

$$M = 0, 1, ..., D-1$$
$$\mu = 1, 2, ..., 2^{[D/2]},$$

where $[\cdot]$ stands for the integer part.
One can thus introduce a superfield:

$$\Psi(X, \Theta), \tag{2.1.1}$$

that transform covariantly under susy transformation of $Z^{\mathcal{M}}$. In superfield spergravity supersymmetric transformations are part of the local diffeomorphisms of the superspace, so the superspace description of supergravity uses the same quantities of Genaral Relativity in the frame-like formalism. So using this formalism, the geometry of the Supergravity backgrond can be described using the Supervielbeins

$$\mathcal{E}^{\mathcal{A}}(X, \Theta) = dZ^{\mathcal{M}} \mathcal{E}^{\mathcal{A}}_{\mathcal{M}}(Z) = (\mathcal{E}^A(X, \Theta) = e^A(X) + \text{terms containing } \Theta, \mathcal{E}^\alpha(X, \Theta))$$

(2.1.2)

and the Superconnection

$$\Omega^{AB}(X,\Theta) = \omega^{AB}(X) + \text{terms containing } \Theta, \tag{2.1.3}$$

where e^A and ω^{AB} are respectively the bosonic vielbein and spin connection. We usually use the first half of the latin alphabet to denote the flat tangent space indices and the second half to denote the curved ones.

The covariant objects which describe superspace geometry are torsion:

$$T^{\mathcal{A}} \equiv \nabla E^{\mathcal{A}} \tag{2.1.4}$$

and curvature

$$R^{AB} \equiv d\Omega^{AB} + \Omega^{A}{}_{C}\Omega^{CB} \tag{2.1.5}$$

Note that in contrast to General Relativity in supergravity torsion is non-zero.

A drawback of the formalism is that using Superfields we introduce far too many degrees of freedom and, in order to get rid of such a redundancy, we have to impose some **Supergravity constraints**.

In 10 and 11 dimensions the most essential constraint which eventually removes (together with partial gauge fixing superdiffeomorphisms) all the auxiliary fields and puts the theory on the mass shell is on the vector component of the torsion. The torsion constraint of type IIA supergravity that we are going to use is:

$$T^A = dE^A + \Omega^{A}{}_{B}E^B = -i\mathcal{E}\Gamma^A\mathcal{E} + i\mathcal{E}^A\,\mathcal{E}\lambda + \frac{1}{3}\mathcal{E}^A\,\mathcal{E}^B\,\partial_B\,\phi\,, \tag{2.1.6}$$

moreover we will have also a constraint on the NS-NS three-form superfield strength:

$$H_3 = dB_2 = -i\mathcal{E}^A\,\mathcal{E}\Gamma_A\Gamma_{11}\mathcal{E} + i\mathcal{E}^B\mathcal{E}^A\,\mathcal{E}\Gamma_{AB}\Gamma_{11}\lambda + \frac{1}{3!}\mathcal{E}^C\mathcal{E}^B\mathcal{E}^A\,H_{ABC}\,. \tag{2.1.7}$$

We are going to introduce what are the Γ matrices in the next section, the other elements entering these expressions, λ and ϕ, are the dilatino and dilaton superfield (See section 3.2 for more details).

2.2 Gamma Matrices

We will widely use the Gamma Matrices, that in a D-dimensional space time are defined as satisfying the Clifford Algebra:

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \quad A, B = 0, 1, ..., D-1 \tag{2.2.8}$$

where η is the Minkowski flat metric:

$$\eta = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.2.9}$$

The dimension of the Gamma matrices is $2^{[D/2]} \times 2^{[D/2]}$.

It is also usefull to define

$$\Gamma^{A_1..A_n} = \Gamma^{[A_1}....\Gamma^{A_n]} \tag{2.2.10}$$

as a totally antisymmetrized product of the matrices (the square brakets denotes the total antisymmetization)

If D is even we can define:

$$\Gamma^{(D+1)} = i^{D(D+1)/2+1}\Gamma^0\Gamma^1\dots\Gamma^{D-1}, \tag{2.2.11}$$

that is a matrix that has the property of anti commuting with the other gamma matrices:

$$\{\Gamma^{(D+1)}, \Gamma^A\} = 0. \tag{2.2.12}$$

2.3 Fierz identities

The basic Fierz identity for the $D = 11$ gamma-matrices we use says that

$$\Gamma^{\hat{A}}_{(\alpha\beta}(\Gamma_{\hat{A}\hat{B}})_{\gamma\delta)} = 0\,, \tag{2.3.13}$$

where $\hat{A} = 0, \dots, 10$. In $D = 10$ notation this becomes the identities

$$\Gamma^{A}_{(\alpha\beta}(\Gamma_{A}\Gamma_{11})_{\gamma\delta)} = 0 \tag{2.3.14}$$

and

$$\Gamma^{A}_{(\alpha\beta}(\Gamma_{AB})_{\gamma\delta)} + \Gamma^{11}_{(\alpha\beta}(\Gamma_{11}\Gamma_{B})_{\gamma\delta)} = 0\,. \tag{2.3.15}$$

We can also expand fermion bilinears in a Fierz basis as follows:

$$\begin{aligned} \Theta^\alpha\,\Theta^\beta &= \frac{1}{32}\mathcal{C}^{\alpha\beta}\,\Theta\Theta - \frac{1}{32\cdot 2}(\Gamma_{AB}\Gamma_{11})^{\alpha\beta}\,\Theta\Gamma^{AB}\Gamma_{11}\Theta - \frac{1}{32\cdot 3!}\Gamma^{\alpha\beta}_{ABC}\,\Theta\Gamma^{ABC}\Theta \\ &\quad + \frac{1}{32\cdot 3!}(\Gamma_{ABC}\Gamma_{11})^{\alpha\beta}\,\Theta\Gamma^{ABC}\Gamma_{11}\Theta + \frac{1}{32\cdot 4!}\Gamma^{\alpha\beta}_{ABCD}\,\Theta\Gamma^{ABCD}\Theta\,. \end{aligned} \tag{2.3.16}$$

where $\mathcal{C}$ is the charge conjugation matrix.

Beside these relations we have to remember the Fierz Identity:

$$\Gamma^{A}_{(\alpha\beta}\Gamma_{A\gamma)\delta} = 0. \tag{2.3.17}$$

2.4 The Green-Shwarz Superstring

Now we focus on Superstring Theories in a D=10 dimensional space. The formalism that we are going to refer to is the Green-Schwarz one. The Green-Schwarz superstring action in a generic supergravity background can be written as:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\,\sqrt{-h}\,h^{IJ}\,\mathcal{E}_I{}^A\mathcal{E}_J{}^B\eta_{AB} - \frac{1}{2\pi\alpha'}\int B_2\,, \tag{2.4.18}$$

where ξ^I $(I, J = 0, 1)$ are the worldsheet coordinates, $h_{IJ}(\xi)$ is an intrinsic worldsheet metric, $\mathcal{E}_J{}^A$ are worldsheet pullbacks of target superspace vector supervielbeins and B_2 is the pull–back of the NS–NS 2–form:

$$B_2(\xi) = \frac{1}{2}d\xi^I d\xi^J \partial_I Z^{\mathcal{N}}\partial_J Z^{\mathcal{M}} B_{\mathcal{M}\mathcal{N}}(Z) \tag{2.4.19}$$

Such an action possesses a local fermionic symmetry that is called κ-symmetry and acts in the following way:

$$\begin{aligned}\delta_\kappa Z^{\mathcal{M}}\mathcal{E}^\alpha_{\mathcal{M}} &= \frac{1}{2}(1+\Gamma)^\alpha_\beta\kappa^\beta(\xi),\\ \delta_\kappa Z^{\mathcal{M}}\mathcal{E}^A_{\mathcal{M}} &= 0,\end{aligned} \tag{2.4.20}$$

where $\kappa^\alpha(\xi)$ is a 32-component spinor parameter and $\frac{1}{2}(1+\Gamma)$ is a spinor projector matrix ($\Gamma^2 = 1$). This symmetry has the remarcable property that allows us to get rid of 16 of the 32 fermionic degrees of freedom of the superstring and thus can be used to partially gauge fix the theory, why this can be useful will be more clear afterwords when we will examin explicitly some Superstring Theories, but by now will be clear to the reader that this symmetry can be used to simplyfy the form of the action.

In general we can find the equations of motion from the Green-Schwarz action. The fermionic equations of motion take the following

$$(1-\Gamma)\left[G^{IJ}\,\mathcal{E}_J{}^A\,\Gamma_A\mathcal{E}_I + \lambda\right] = 0\,, \tag{2.4.21}$$

The equations of motion of the string bosonic modes is

$$\nabla_I\left(\sqrt{-G}\,G^{IJ}\mathcal{E}_{JA}\right) + \sqrt{-G}\,G^{IJ}\mathcal{E}_J{}^{\mathcal{B}}\,T_{\mathcal{B}A}{}^D\,\mathcal{E}_{ID} + \frac{1}{2}\varepsilon^{IJ}\,\mathcal{E}_J{}^{\mathcal{B}}\,\mathcal{E}_I{}^{\mathcal{C}}\,H_{\mathcal{C}\mathcal{B}A} = 0. \tag{2.4.22}$$

At this point we have fixed the language that we are going to use in the thesis and we are ready to go into the geometrical and mathematical details of type IIA Superstring theories on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ backgrounds.

Chapter 3

$AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ supergeometries

Before to go on, it is usefull to recall the form of the geometry of the backgrounds that we are going to deal with. The geometry of $AdS_4 \times CP^3$ is known to all orders in all the fermions and it was for the first time explicitly worked out in [21]. For $AdS_2 \times S^2 \times T^6$ we know the explicit form of the geometry only up to the second order in non supercoset fermions [30]. In both these cases we have some fermionic modes along directions that preserve the supersymmetry (24 in the case of $AdS_4 \times CP^3$ and 8 in the case of $AdS_2 \times S^2 \times T^6$) the other fermionic modes are non-supersymmetric. The two theories can be reduced or truncated to supercoset sigma models in a consistent way, as we will show later.

The theory on $AdS_4 \times CP^3$ reproduces a sigma model on the $\frac{OSp(6|4)}{U(3) \times SO(1,3)}$ supercoset, that has only 24 fermionic coordinates, if we get rid of the fermions along the non supersymmetric directions, that is why we are often going to refer to them as non-supercoset coordinates.

For $AdS_2 \times S^2 \times T^6$ in order to get a sigma model on $\frac{PSU(1,1|2) \rtimes E(6)}{SO(1,1) \times U(1) \times SO(6)}$, that has only 8 fermions, we have to get rid of the 24 non-supersymmetric fermions.

$\mathbf{AdS_4 \times CP^3}$:
Parametrized by the coordinates x^m $(m = 0, 1, 2, 3)$ and $y^{m'}$ $(m' = 4, 5, 6, 7, 8, 9)$. Its vielbeins are $e^a = dx^m \, e_m{}^a(x)$ $(a = 0, 1, 2, 3)$, and $e^{a'}(y) = dy^{m'} e_{m'}{}^{a'}(y)$. The AdS_4 curvature is

$$R_{ab}{}^{cd} = \frac{8}{R^2} \, \delta^c_{[a} \, \delta^d_{b]} \, , \qquad R^{ab} = -\frac{4}{R^2} \, e^a \, e^b \, , \tag{3.0.1}$$

where R is the CP^3 radius or twice the AdS_4 radius, and the CP^3 curvature is

$$R_{a'b'}{}^{c'd'} = -\frac{2}{R^2}\,(\delta^{c'}_{[a'}\,\delta^{d'}_{b']} + J_{[a'}{}^{c'}\,J_{b']}{}^{d'} + J_{a'b'}\,J^{c'd'})\,, \tag{3.0.2}$$

where $J^{a'b'}$ is the Kähler form on CP^3.

$\mathbf{AdS_2 \times S^2 \times T^6}$:
Parametrized by the coordinates x^m $(m = 0, 1)$, $x^{\hat{m}}$ $(\hat{m} = 2, 3)$ and $y^{m'}$ $(m' = 4, 5, 6, 7, 8, 9)$. Its vielbeins are $e^a = dx^m\, e_m{}^a(x)$ $(a = 0, 1)$, $e^{\hat{a}} = dx^{\hat{m}}\, e_{\hat{m}}{}^{\hat{a}}(\hat{x})$ $(\hat{a} = 2, 3)$ and $e^{a'}(y) = dy^{a'}$. The AdS_2 curvature is

$$R_{ab}{}^{cd} = \frac{8}{R^2}\,\delta^c_{[a}\,\delta^d_{b]}\,, \qquad R^{ab} = -\frac{4}{R^2}\,e^a\,e^b\,, \tag{3.0.3}$$

where R is twice the AdS_2 (or S^2) radius, and the S^2 curvature is

$$R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = -\frac{8}{R^2}\,\delta^{\hat{c}}_{[\hat{a}}\,\delta^{\hat{d}}_{\hat{b}]}\,, \qquad R^{\hat{a}\hat{b}} = \frac{4}{R^2}\,e^{\hat{a}}\,e^{\hat{b}}\,. \tag{3.0.4}$$

The reason to take R as twice the AdS_2 radius will be clear later, when we will write $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ elements in a unified way. We will often combine the AdS_2 and S^2 indices as $\underline{a} = 0, 1, 2, 3$, i.e. $\underline{a} = (a, \hat{a})$.

The $D = 10$ gamma–matrices satisfy

$$\{\Gamma^A,\, \Gamma^B\} = 2\eta^{AB}\,, \qquad \Gamma^A = (\Gamma^a,\, \Gamma^{a'})\,, \qquad a = 0, 1, 2, 3 \qquad a' = 4, \cdots, 9\,. \tag{3.0.5}$$

We also define

$$\begin{aligned} \gamma_5 &= i\Gamma^{0123} \\ \gamma_7 &= i\Gamma^{456789} \\ \Gamma_{11} &= \gamma_5\gamma_7\,, \end{aligned} \tag{3.0.6}$$

all of which square to one.

We can write

$$\Gamma^a = \gamma^a \otimes \mathbf{1}\,, \qquad \Gamma^{a'} = \gamma^5 \otimes \gamma^{a'}\,, \tag{3.0.7}$$

where γ^a and $\gamma^{a'}$ are the 4-dimensional and 6-dimensional gamma matrices.

For $AdS_2 \times S^2 \times T^6$ case the 4×4 matrices can be represented in terms of 2×2 AdS_2 gamma–matrices ρ^a $(a = 0, 1)$ and the matrices $\rho^{\hat{a}}$ associated with S^2 as

$$\gamma^a = \rho^a \otimes \mathbf{1}, \qquad \gamma^{\hat{a}} = \gamma \otimes \rho^{\hat{a}}\,, \qquad \gamma = \rho^0\rho^1\,. \tag{3.0.8}$$

The charge conjugation matrix is denoted $\mathcal{C}$. The matrices $\mathcal{C}$, $\mathcal{C}\Gamma_{\hat{A}\hat{B}\hat{C}}$ and $\mathcal{C}\Gamma_{\hat{A}\hat{B}\hat{C}\hat{D}}$ are anti-symmetric while $\mathcal{C}\Gamma_{\hat{A}}$, $\mathcal{C}\Gamma_{\hat{A}\hat{B}}$ and $\mathcal{C}\Gamma_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}$ are symmetric, where the indices are eleven dimensional, $\hat{A} = (A, 11)$.

Finally we introduce a spinor projection operator which projects onto an 8-dimensional subspace of the 32-dimensional space of spinors as follows

$$\mathcal{P}_8 = \frac{1}{8}(2 - iJ_{a'b'}\Gamma^{a'b'}\gamma^7)\,, \tag{3.0.9}$$

where $J_{a'b'}$ is the Kähler form on CP^3 or T^6. The complementary projection operator which projects onto a 24-dimensional subspace is given by

$$\mathcal{P}_{24} = 1 - \mathcal{P}_8 = \frac{1}{8}(6 + iJ_{a'b'}\Gamma^{a'b'}\gamma^7)\,. \tag{3.0.10}$$

These are the projectors that allowed us to project out the non supercoset fermions from the supercoset ones.

We have to emphasize the fact that the role of the two projectors is interchanged in the two theories, in this sense we can say that the two theories are "dual" to each other.

Whenever is possible we will use a **unified** way of presenting the results, by using the following notation:

	$OSp(6\|4)$	$PSU(1,1\|2) \rtimes E(6)$
γ_*	γ^5	$\Gamma^{01}\gamma^7$
$\mathcal{P}$	$\mathcal{P}_{24}$	$\mathcal{P}_8$
R	$2R_{AdS_4}$	$2R_{AdS_2}$

. (3.0.11)

So we can write the $OSp(6|4)$ and the $PSU(1,1|2) \rtimes E(6)$ superalgebras in a compact way:

$$\begin{aligned}
[P_A, P_B] &= -\frac{1}{2}R_{AB}{}^{CD}M_{CD}\,, \qquad [M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A \\
[M_{AB}, M_{CD}] &= \eta_{AC}M_{BD} + \eta_{BD}M_{AC} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC}\,,
\end{aligned} \tag{3.0.12}$$

$$[P_A, Q] = \frac{i}{R}Q\gamma_*\Gamma_A\mathcal{P} \qquad [M_{AB}, Q] = -\frac{1}{2}Q\Gamma_{AB}\mathcal{P}\,, \tag{3.0.13}$$

$$\{Q, Q\} = 2i(\mathcal{P}\Gamma_A\mathcal{P})\,P_A + \frac{R}{4}(\mathcal{P}\Gamma^{AB}\gamma_*\mathcal{P})R_{AB}{}^{CD}M_{CD}\,. \tag{3.0.14}$$

In terms of the generators this algebra has a $\mathbb{Z}_4$-grading structure, in fact we can see that M_{AB} has grading 0 under $\mathbb{Z}_4$-automorphism, P_A has grading 2, while, defining $Q_1 = \frac{1}{2}Q(1-\Gamma^{11})$ and $Q_2 = \frac{1}{2}Q(1+\Gamma^{11})$, it is easy to see that these have respectively grading 1 and 3. Schematically we can write the $\mathbb{Z}_4$-grading structure of the theory as follows:

$$\begin{gathered}[M_0, M_0] \sim M_0, \quad [M_0, P_2] \sim P_2, \quad [P_2, P_2] \sim M_0, \\ [M_0, Q_1] \sim Q_1, \quad [M_0, Q_3] \sim Q_3, \quad [P_2, Q_1] \sim Q_3, \quad [P_2, Q_3] \sim Q_1, \\ \{Q_1, Q_1\} \sim P_2, \quad \{Q_3, Q_3\} \sim P_2, \quad \{Q_1, Q_3\} \sim M_0.\end{gathered} \tag{3.0.15}$$

3.1 Supercoset geometry

The supercoset geometry, in both the cases, is given by

$$E^A = e^A + i\vartheta\Gamma^A c(\mathcal{M}^2) D\vartheta \tag{3.1.16}$$

$$E^\alpha = \left(s(\mathcal{M}^2) D\vartheta\right)^\alpha \tag{3.1.17}$$

$$\Omega^{AB} = \omega^{AB} + \frac{R}{4}\vartheta\Gamma^{CD}\gamma_* c(\mathcal{M}^2) D\vartheta \, R_{CD}{}^{AB} \tag{3.1.18}$$

where $\vartheta = \mathcal{P}\Theta$,

$$s(\mathcal{M}^2) = \frac{\sinh \mathcal{M}}{\mathcal{M}} \tag{3.1.19}$$

$$c(\mathcal{M}^2) = 2\frac{\cosh \mathcal{M} - 1}{\mathcal{M}^2} \tag{3.1.20}$$

while

$$D\vartheta = \mathcal{P}(\nabla^{(\omega)} + \frac{i}{R} e^A \gamma_* \Gamma_A)\vartheta \tag{3.1.21}$$

is the Killing derivative and

$$\mathcal{M}^2 = -\frac{2}{R}(\mathcal{P}\gamma_*\Gamma_A\vartheta)(\vartheta\Gamma^A\mathcal{P}) - \frac{R}{8} R_{AB}{}^{CD}(\Gamma_{CD}\vartheta)(\vartheta\Gamma^{AB}\gamma_*)\,. \tag{3.1.22}$$

These elements are the components of the Cartan form valued in the isometry supergroup G (*i.e.* $OSp(6|4)$ or $PSU(1,1|2) \rtimes E(6)$)

$$K = g^{-1}dg(X,\vartheta) = \frac{1}{2}\Omega_0^{AB} M_{AB} + E^A P_A + Q_\alpha E^\alpha\,, \qquad g(X,\vartheta) \in G/H\,. \tag{3.1.23}$$

For this Cartan form we have the relation

$$dK = KK \tag{3.1.24}$$

that is called Maurer Cartan equation.

Splitting the Maurer-Cartan equation in linearly independent components one finds the superspace torsion

$$\nabla E^A = -iE\Gamma^A E \tag{3.1.25}$$
$$\nabla E = \frac{i}{R}E^A\left(\mathcal{P}\gamma_*\Gamma_A E\right) \tag{3.1.26}$$

and curvature

$$d\Omega^{AB} + \Omega^{AC}\Omega_C{}^B = \left(\frac{1}{2}E^D E^C - \frac{R}{4}E\Gamma^{CD}\gamma_* E\right) R_{CD}{}^{AB}\,. \tag{3.1.27}$$

3.2 Including non-supercoset modes

Now we are going to see what happens when we include the non-supercoset coordinates in the geometry. From now on we will indicate the non supercoset fermions as:

$$\upsilon = (1-\mathcal{P})\Theta \tag{3.2.28}$$

3.2.1 $AdS_4 \times CP^3$

As already said, we know the full $AdS_4 \times CP^3$ supergeometry, that was worked out in [21].

The supervielbeins have the following form

$$\mathcal{E}^{a'}(x,y,\vartheta,\upsilon) = e^{\frac{1}{3}\phi(\upsilon)} \left(E^{a'}(x,y,\vartheta) + 2i\upsilon \frac{\sinh m}{m} \gamma^{a'}\gamma^5 E(x,y,\vartheta) \right) ,$$

$$\begin{aligned} \mathcal{E}^{a}(x,y,\vartheta,\upsilon) = e^{\frac{1}{3}\phi(\upsilon)} & \left(E^{b}(x,y,\vartheta) + 4i\upsilon\gamma^{b} \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) \Lambda_b{}^{a}(\upsilon) \\ & - e^{-\frac{1}{3}\phi(\upsilon)} \frac{R^2}{kl_p} \left(A(x,y,\vartheta) - \frac{4}{R}\upsilon\,\varepsilon\gamma^5 \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) E_7{}^{a}(\upsilon) , \end{aligned}$$

$$\mathcal{E}^{\alpha i}(x,y,\vartheta,\upsilon) = e^{\frac{1}{6}\phi(\upsilon)} \left(\frac{\sinh \mathcal{M}}{\mathcal{M}} D\upsilon \right)^{\beta j} S_{\beta j}{}^{\alpha i}(\upsilon) - ie^{\phi(\upsilon)} \mathcal{A}_1(x,y,\vartheta,\upsilon) \left(\gamma^5 \varepsilon \lambda(\upsilon)\right)^{\alpha i} ,$$

$$\mathcal{E}^{\alpha a'}(x,y,\vartheta,\upsilon) = e^{\frac{1}{6}\phi(\upsilon)} E^{\gamma b'}(x,y,\vartheta) \left(\delta_\gamma{}^{\beta} - \frac{8}{R} \left(\gamma^5 \upsilon \frac{\sinh^2 m/2}{m^2} \right)_{\gamma i} \upsilon^{\beta i} \right) S_{\beta b'}{}^{\alpha a'}(\upsilon) . \tag{3.2.29}$$

The functions of υ appearing in these expressions, m, $\mathcal{M}$, $\Lambda_a{}^b$, $E_7{}^a$ and $S_{\underline{\alpha}}^{\underline{\beta}}$, the dilaton ϕ, dilatino λ and RR one–form $\mathcal{A}_1$ are given below. Contracted spinor indices have been suppressed, *e.g.* $(\upsilon\varepsilon\gamma^5)_{\alpha i} = \upsilon^{\beta j}\varepsilon_{ji}\gamma^5_{\beta\alpha}$, where $\varepsilon_{ij} = -\varepsilon_{ji}$, $\varepsilon_{12} = 1$ is the $SO(2)$ invariant tensor. The covariant derivative of υ is defined as

$$D\upsilon = \left(d + \frac{i}{R} E^a(x,y,\vartheta)\,\gamma^5\gamma_a - \frac{1}{4}\Omega^{ab}(x,y,\vartheta)\,\gamma_{ab} \right) \upsilon . \tag{3.2.30}$$

The type IIA RR one–form gauge superfield is

$$\begin{aligned} \mathcal{A}_1(x,y,\vartheta,\upsilon) = R\, e^{-\frac{4}{3}\phi(\upsilon)} & \left[\left(A(x,y,\vartheta) - \frac{4}{R}\upsilon\,\varepsilon\gamma^5 \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) \frac{R}{kl_p} \Phi(\upsilon) \right. \\ & \left. + \frac{1}{kl_p} \left(E^{\hat{a}}(x,y,\vartheta) + 4i\upsilon\gamma^{\hat{a}} \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) E_{7\hat{a}}(\upsilon) \right] . \end{aligned} \tag{3.2.31}$$

The RR four-form and the NS–NS three-form superfield strengths are given by

$$\begin{aligned} F_4 &= d\mathcal{A}_3 - \mathcal{A}_1\, H_3 = -\frac{1}{4!}\mathcal{E}^d\mathcal{E}^c\mathcal{E}^b\mathcal{E}^a \left(\frac{6}{kl_p}\, e^{-2\phi}\Phi\varepsilon_{abcd} \right) - \frac{i}{2}\mathcal{E}^B\mathcal{E}^A\mathcal{E}^{\underline{\beta}}\mathcal{E}^{\underline{\alpha}} e^{-\phi}(\Gamma_{AB})_{\underline{\alpha\beta}} , \\ H_3 &= dB_2 = -\frac{1}{3!}\mathcal{E}^c\mathcal{E}^b\mathcal{E}^a(\frac{6}{kl_p}e^{-\phi}\varepsilon_{abcd}E_7{}^d) - i\mathcal{E}^A\mathcal{E}^{\underline{\beta}}\mathcal{E}^{\underline{\alpha}}(\Gamma_A\Gamma_{11})_{\underline{\alpha\beta}} + i\mathcal{E}^B\mathcal{E}^A\mathcal{E}^{\underline{\alpha}}(\Gamma_{AB}\Gamma^{11}\lambda)_{\underline{\alpha}} \end{aligned} \tag{3.2.32}$$

and the corresponding gauge potentials are

$$B_2 = b_2 + \int_0^1 dt\, i_\Theta H_3(x, y, t\Theta)\,, \qquad \Theta = (\vartheta, \upsilon) \tag{3.2.33}$$

$$\mathcal{A}_3 = a_3 + \int_0^1 dt\, i_\Theta \left(F_4 + \mathcal{A}_1 H_3\right)(x, y, t\Theta)\,, \tag{3.2.34}$$

where b_2 and a_3 are the purely bosonic parts of the gauge potentials and i_Θ means the inner product with $\Theta^{\underline{\alpha}}$. Note that b_2 is pure gauge while a_3 is the RR three-form potential of the bosonic background.

The dilaton superfield $\phi(\upsilon)$, which depends only on the eight fermionic coordinates corresponding to the broken supersymmetries, is

$$e^{\frac{2}{3}\phi(\upsilon)} = \frac{R}{kl_p}\sqrt{\Phi^2 + E_7{}^a\, E_7{}^b\, \eta_{ab}}\,. \tag{3.2.35}$$

where we introduced the eleven-dimensional Planck length $l_p = e^{\frac{1}{3}<\phi>}\sqrt{\alpha'}$ and the Chern–Simons level k.

The value of the dilaton at $\upsilon = 0$ is

$$e^{\frac{2}{3}\phi(\upsilon)}|_{\upsilon=0} = e^{\frac{2}{3}\phi_0} = \frac{R}{kl_p}\,. \tag{3.2.36}$$

The fermionic field $\lambda^{\alpha i}(\upsilon)$ describes the non-zero components of the dilatino superfield and is given by the equation [55]

$$\lambda_{\alpha i} = -\frac{i}{3} D_{\alpha i}\,\phi(\upsilon)\,. \tag{3.2.37}$$

The other quantities appearing in this section are:

$$R\,(\mathcal{M}^2)^{\alpha i}{}_{\beta j} = 4(\varepsilon\upsilon)^{\alpha i}(\upsilon\varepsilon\gamma^5)_{\beta j} - 2(\gamma^5\gamma^a\upsilon)^{\alpha i}(\upsilon\gamma_a)_{\beta j} - (\gamma^{ab}\upsilon)^{\alpha i}(\upsilon\gamma_{ab}\gamma^5)_{\beta j}\,, \tag{3.2.38}$$

$$(m^2)^{ij} = -\frac{4}{R}\upsilon^i\,\gamma^5\,\upsilon^j\,, \tag{3.2.39}$$

$$\begin{aligned}
\Lambda_a{}^b &= \delta_a{}^b - \frac{R^2}{k^2 l_p^2}\cdot\frac{e^{-\frac{2}{3}\phi}}{e^{\frac{2}{3}\phi} + \frac{R}{kl_p}\Phi}\, E_{7a}\, E_7{}^b\,, \\
S_{\underline{\beta}}{}^{\underline{\alpha}} &= \frac{e^{-\frac{1}{3}\phi}}{\sqrt{2}}\left(\sqrt{e^{\frac{2}{3}\phi} + \frac{R}{kl_p}\Phi} - \frac{R}{kl_p}\frac{E_7{}^a\,\Gamma_a\Gamma_{11}}{\sqrt{e^{\frac{2}{3}\phi} + \frac{R}{kl_p}\Phi}}\right)_{\underline{\beta}}{}^{\underline{\alpha}}
\end{aligned} \tag{3.2.40}$$

$$
\begin{aligned}
E_7{}^a(\upsilon) &= -\frac{8i}{R}\,\upsilon\gamma^a\,\frac{\sinh^2\mathcal{M}/2}{\mathcal{M}^2}\,\varepsilon\,\upsilon\,,\\
\Phi(\upsilon) &= 1+\frac{8}{R}\,\upsilon\,\varepsilon\gamma^5\,\frac{\sinh^2\mathcal{M}/2}{\mathcal{M}^2}\,\varepsilon\upsilon\,.
\end{aligned}
\tag{3.2.41}
$$

It will be useful for the rest of the discussion to consider the second order expansion in υ of this geometry. First of all we introduce the following notation:

$$
\hat{E}^A = E^A + 2i\upsilon\Gamma^A E\,. \tag{3.2.42}
$$

Performing this expansion the supergeometry becomes

$$
\begin{aligned}
\mathcal{E}^A &= c^2(1-\frac{1}{R}\upsilon\gamma^5\upsilon)(\hat{E}^A + i\upsilon\Gamma^A D\upsilon) + \mathcal{O}(\upsilon^3)\\
\mathcal{P}\mathcal{E} &= c(1-\frac{1}{2R}\upsilon\Gamma^b\gamma^5\upsilon\,\Gamma_b + \frac{1}{R}\upsilon\Gamma^b\gamma^7\upsilon\,\Gamma_b\Gamma_{11} + \frac{1}{2R}\upsilon\upsilon\,\gamma_5)E + \mathcal{O}(\upsilon^3)\\
(1-\mathcal{P})\mathcal{E} &= cD\upsilon + \mathcal{O}(\upsilon^3)\\
\lambda &= \frac{2i}{cR}\gamma^5\upsilon + \mathcal{O}(\upsilon^3)\\
\Omega^{AB} &= \Omega^{AB}_{coset} + \frac{2}{R}\Big(-\delta^A_{a'}\delta^B_{b'}\upsilon\Gamma^{a'b'}\gamma_5 E + \delta^A_a\delta^B_b\,\upsilon\Gamma^{ab}\gamma_5 D\upsilon + \frac{i}{R}\delta^A_{a'}\delta^B_{b'}\,E^c\,\upsilon\Gamma^{a'b'}{}_c\upsilon\\
&\quad - \frac{2i}{R}\delta^{[A}_{a'}\delta^{B]}_b\,E^{c'}\,\upsilon\Gamma^{a'b}{}_{c'}\upsilon\Big) + \mathcal{O}(\upsilon^3)\\
H_{abc} &= -\frac{12i}{c^2R^2}\upsilon\Gamma_{abc}\Gamma_{11}\upsilon + \mathcal{O}(\upsilon^3)\,.
\end{aligned}
\tag{3.2.43}
$$

where

$$
D\upsilon = (\nabla + \frac{i}{R}E^a\,\gamma_5\Gamma_a)\upsilon \tag{3.2.44}
$$

and

$$
c = e^{\frac{1}{6}\phi_0}. \tag{3.2.45}
$$

In order to explicitly compute the spin connection

$$
\Omega^{AB} = \Omega^{AB}_{coset} + \Omega^{AB}_v \tag{3.2.46}
$$

we can use the definition of the torsion

$$
T^A = \nabla\mathcal{E}^A = d\mathcal{E}^A + \Omega^{AB}\mathcal{E}_B = \nabla_{cos}\mathcal{E}^A + \Omega^{AB}_v\,.\mathcal{E}_B \tag{3.2.47}
$$

We know that:

$$
T^A = -i\mathcal{E}\Gamma^A\mathcal{E} + i\mathcal{E}^A\,\mathcal{E}\lambda + \frac{1}{3}\mathcal{E}^A\,\mathcal{E}^B\nabla_B\phi\,. \tag{3.2.48}
$$

It is worthwile to recall that in the Supercoset sigma model we have:

$$\nabla_{cos} E^A = T^A_{coset} = -iE\Gamma^A E \tag{3.2.49}$$

Considering the second order expansion, where

$$\phi \;\; = -\tfrac{3}{R}\upsilon\gamma^5\upsilon + \phi_0\,, \tag{3.2.50}$$

$$\lambda_{\alpha i} = -\frac{i}{3} D_{\alpha i}\,\phi(\upsilon) = \frac{2i}{R}(\gamma^5\upsilon)_{\alpha i} \tag{3.2.51}$$

we have

$$-i\mathcal{E}\Gamma^A\mathcal{E} = -iE\Gamma^A E - 2iE\Gamma^A\mathcal{E}_v - i\mathcal{E}_v\Gamma^A\mathcal{E}_v \tag{3.2.52}$$

where

$$\mathcal{E}_v = e^{\frac{1}{6}\phi_0}\left(\nabla_{cos}\upsilon + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^B\upsilon E_B - \frac{1}{2R}E\upsilon\gamma^5\upsilon + \frac{1}{R}\mathcal{P}_6\Gamma_A\Gamma^{11}E\upsilon\Gamma^A\gamma^7\upsilon + \frac{2}{R}\mathcal{P}_6\Gamma_A\gamma^5 v_i v^i E\right). \tag{3.2.53}$$

$$\begin{aligned}
-2iE\Gamma^A\mathcal{E}_v \;\; &= \;\; -2iE\Gamma^A\left(\nabla_{cos}\upsilon + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^B\upsilon E_B\right) + i\frac{1}{R}E\Gamma^A E\upsilon\gamma^5\upsilon + \\
&\quad -4\left(\frac{i}{R}\right)\upsilon\gamma^5\Gamma^B\mathcal{P}_6\Gamma^A E\upsilon\Gamma_B E - \frac{iR}{16}E\Gamma_{DE}\gamma^5 E\,\upsilon\Gamma^A\Gamma_{11}\Gamma^{BC}\upsilon\, R_{BC}{}^{DE} = \\
&= \;\; \nabla_{cos}\left(2i\upsilon\Gamma^A E - \frac{1}{R}E^A\upsilon\gamma^5\upsilon\right) + 2i\frac{i}{R}\upsilon\Gamma^{[A}\gamma^5\Gamma^{B]}EE_B + \\
&\quad -\frac{iR}{16}E\Gamma_{DE}\gamma_* E\,\upsilon\Gamma^A\Gamma_{11}\Gamma^{BC}\upsilon\, R_{BC}{}^{DE} + \\
&\quad -4i\left(\frac{1}{R}\right)\upsilon\gamma^5\Gamma^B\mathcal{P}_6\Gamma^A E\upsilon\Gamma_B E - 2\frac{i}{R}\upsilon\gamma^5\nabla_{cos}\upsilon E^A
\end{aligned} \tag{3.2.54}$$

where we have use the fact that due to the Fierz identities we can write:

$$\begin{aligned}
&-\frac{i}{R}E\Gamma^A E\upsilon\gamma^5\upsilon + 2\frac{i}{R}E\Gamma^A\mathcal{P}_6\Gamma^B\Gamma^{11}E\upsilon\Gamma_B\gamma^7\upsilon + 4\frac{i}{R}E\Gamma^A\gamma^5 v_i v^i E = \\
= \;\; &4\frac{i}{R}E\Gamma^A\Gamma^B\gamma^7\upsilon\upsilon\Gamma_B\Gamma^{11}E + \frac{i}{R}E\Gamma^A E\upsilon\gamma^5\upsilon + \frac{iR}{16}E\Gamma_{DE}\gamma^5 E\,\upsilon\Gamma^A\Gamma_{11}\Gamma^{BC}\upsilon\, R_{BC}{}^{DE}
\end{aligned} \tag{3.2.55}$$

$$-i\mathcal{E}_v\Gamma^A\mathcal{E}_v \;\; = \;\; -i\left(\nabla_{cos}\upsilon + \frac{i}{R}\upsilon\Gamma^B\gamma^5\mathcal{P}_2 E_B\right)\Gamma^A\left(\nabla_{cos}\upsilon + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^C\upsilon E_C\right) =$$

$$
\begin{aligned}
&= \nabla_{cos}(iv\Gamma^A\nabla_{cos}v) - i\left(\frac{i}{R}\right)^2 v\Gamma^B\gamma^5\mathcal{P}_2\Gamma^A\mathcal{P}_2\gamma^5\Gamma^C v E_C E_B + \\
&\quad -i\left(\frac{i}{R}\right) v\Gamma^B\gamma^5\mathcal{P}_2\Gamma^A\nabla_{cos}v E_B - i\left(\frac{i}{R}\right)\nabla_{cos}v\Gamma^A\gamma^5\mathcal{P}_2\Gamma^B v E_B + iv\Gamma^A\nabla_{cos}\nabla_{cos}v = \\
&= \nabla_{cos}\left(iv\Gamma^A\left(\nabla_{cos}v + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^C v E_C\right)\right) - 2i\left(\frac{i}{R}\right) v\Gamma^{[B}\gamma^5\mathcal{P}_2\Gamma^{A]}\nabla_{cos}v E_B \\
&\quad -i\left(\frac{i}{R}\right)^2 v\Gamma^B\gamma^5\mathcal{P}_2\Gamma^A\mathcal{P}_2\gamma^5\Gamma^C v E_C E_B + iv\Gamma^A\nabla_{cos}\nabla_{cos}v
\end{aligned} \tag{3.2.56}
$$

$$
i\mathcal{E}^A\,\mathcal{E}\lambda = 2i\frac{i}{R}v\gamma^5\left(\nabla_{cos}v + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^B v E_B\right)E^A \tag{3.2.57}
$$

in the above calculation we used the Supercoset Maurer Cartan condition and the Fierz identities. So

$$
\begin{aligned}
T^A &= \nabla_{cos}\left(E^A + 2iv\,\Gamma^A E + iv\Gamma^A\left(\nabla_{cos}v + \frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^B v E_B\right) - \frac{1}{R}E^A v\gamma^5 v\right) + \\
&\quad +2\left(\frac{i}{R}v\Gamma^{[A}\gamma^5\Gamma^{B]}E + \frac{1}{R}v\Gamma^{[A}\mathcal{P}_2\gamma^5\Gamma^{B]}\left(\nabla_{cos}v + \frac{i}{R}\mathcal{P}_2\Gamma^C v E_C\right)\right)E_B + \\
&\quad +4i\left(\frac{i}{R}\right)v\Gamma^{[A}\gamma^5\Gamma^{B]}Ev\Gamma_B E - \frac{2i}{R^2}v\Gamma^A\epsilon v J^{b'}{}_{c'}E^{c'}E_{b'} = \\
&= \nabla_{cos}\left(\mathcal{E}^A\right) + 2\left(\frac{i}{R}\right)v\Gamma^{[A}\gamma^5\Gamma^{B]}E\mathcal{E}_{vB} + 2\left(\frac{i}{R}v\Gamma^{[A}\gamma^5\Gamma^{B]}E\right. \\
&\quad \left.+\frac{1}{R}v\Gamma^{[A}\mathcal{P}_2\gamma^5\Gamma^{B]}\left(\nabla_{cos}v + \frac{i}{R}\mathcal{P}_2\Gamma^C v E_C\right)\right)E_B - \frac{2i}{R^2}v\Gamma^A\epsilon v J^{b'}{}_{c'}E^{c'}E_{b'}
\end{aligned} \tag{3.2.58}
$$

Thus we finally get at the first oder in v:

$$
\Omega_v^{AB(1)} = 2i\frac{i}{R}v\Gamma^{[A}\gamma^5\Gamma^{B]}E \tag{3.2.59}
$$

The second order v-corrections to the spin connection are:

$$
\Omega_v^{ab(2)} = \frac{2}{R}v\gamma^5\Gamma^{ab}Dv \tag{3.2.60}
$$

$$
\Omega_v^{ab'(2)} = -\frac{2i}{R^2}v\Gamma^a\epsilon v J^{b'}{}_{c'}E^{c'} \tag{3.2.61}
$$

$$\Omega_v^{a'b'(2)} = \frac{2i}{R^2} v\Gamma^b \epsilon v J^{a'}{}_{c'} E^{c'} \tag{3.2.62}$$

$$\Omega_v^{a'b'(2)} = \frac{2i}{R^2} v\Gamma^c \epsilon v J^{a'b'} E_c \tag{3.2.63}$$

3.2.2 $AdS_2 \times S^2 \times T^6$

The $AdS_2 \times S^2 \times T^6$ supergeometry is known to all orders in the coset fermions, but only up to the second order in v, it was for the first time found in [30]:

$$\begin{aligned}
\mathcal{E}^A &= c^2((1+\frac{1}{R}v\gamma_* v)\hat{E}^A + iv\Gamma^A Dv) + \mathcal{O}(v^3) \\
\mathcal{P}\mathcal{E} &= c\mathcal{P}(1+\frac{1}{2R}v\Gamma^B\gamma_*\gamma_7 v\,\Gamma_B\gamma_7 - \frac{1}{2R}v\Gamma^B\gamma_*\Gamma_{11}v\,\Gamma_B\Gamma_{11} - \frac{1}{4R}v\Gamma^{BC}\gamma_* v\,\Gamma_{BC})E + \mathcal{O}(v^3) \\
(1-\mathcal{P})\mathcal{E} &= c(Dv + (1-\mathcal{P})(\frac{1}{2R}v\Gamma^B\gamma_*\gamma_7 v\,\Gamma_B\gamma_7 - \frac{1}{2R}v\Gamma^B\gamma_*\Gamma_{11}v\,\Gamma_B\Gamma_{11} - \frac{1}{4R}v\Gamma^{BC}\gamma_* v\,\Gamma_{BC} \\
&\quad - \frac{1}{2R}v\Gamma^{\underline{b}c'}\Gamma_{11}v\,\gamma_*\Gamma_{\underline{b}c'}\Gamma_{11} - \frac{1}{4R}v\Gamma^{\underline{b}\underline{c}d'}v\,\gamma_*\Gamma_{\underline{b}\underline{c}d'})E + \mathcal{O}(v^3) \\
\lambda &= -\frac{2i}{cR}\gamma_* v + \mathcal{O}(v^3)\,,
\end{aligned} \tag{3.2.64}$$

where

$$Dv = (\nabla + \frac{i}{R}E^B\,(1-\mathcal{P})\Gamma_B\gamma_*)v\,. \tag{3.2.65}$$

The spin–connection takes the form

$$\begin{aligned}
\Omega^{AB} &= \Omega^{AB}_{coset} - \frac{2}{R}\delta^{[A}_{a'}\delta^{B]}_{b'}v\gamma_*\Gamma^{a'b'}E - \frac{1}{R}v\Gamma^{[A}(1-\mathcal{P})\Gamma^{B]}\gamma_* Dv - \frac{1}{R}v\gamma_*\Gamma^{[A}(1-\mathcal{P})\Gamma^{B]}Dv \\
&\quad + \frac{i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b'}E^C\,v\Gamma^{a'}(1-\mathcal{P})\Gamma^{b'}\gamma_*\Gamma_C\gamma_* v + \frac{i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b'}E^{c'}\,v\Gamma^{a'}\mathcal{P}\Gamma^{b'c'}v \\
&\quad - \frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{\underline{b}}E^{c'}\,v\Gamma^{a'}(1-\mathcal{P})\Gamma^{\underline{b}}\Gamma_{c'}v + \frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b}E^{\hat{c}}\,v\Gamma^{a'b}{}_{\hat{c}}v \\
&\quad - \frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{\hat{b}}E^{c}\,v\Gamma^{a'\hat{b}}{}_{c}v - \frac{2i}{R^2}\delta^{[A}_{a}\delta^{B]}_{\underline{b}}E^{c'}\,v\Gamma^{a\underline{b}}{}_{c'}v + \mathcal{O}(v^3)\,.
\end{aligned} \tag{3.2.66}$$

With this choice of the spin–connection it is not hard to verify that the torsion constraint is satisfied, i.e.

$$T^A \equiv \nabla\mathcal{E}^A = -i\mathcal{E}\Gamma^A\mathcal{E} + i\mathcal{E}^A\,\mathcal{E}\lambda\,, \tag{3.2.67}$$

the way to proceed is analogous to $AdS_4 \times CP^3$.

The NS–NS three–form field strength is

$$H_{ABC} = \frac{6i}{c^2 R^2}\Big(\upsilon\gamma_*\Gamma_{ABC}\Gamma_{11}\gamma_*\upsilon - \delta^{\underline{a}}_{[A}\,\upsilon\gamma_*\Gamma_B\mathcal{P}\Gamma_{C]}\Gamma_{\underline{a}}\gamma_*\Gamma_{11}\upsilon - \delta^d_{[A}\delta^e_B\,\upsilon\Gamma_{C]de}\Gamma_{11}\upsilon + \delta^{\hat{d}}_{[A}\delta^{\hat{e}}_B\,\upsilon\Gamma_{C]\hat{d}\hat{e}}\Gamma_{11}\upsilon\Big) + \mathcal{O}(\upsilon^3)\,. \tag{3.2.68}$$

The demonstration that this actually is a closed form, that is what together with the torsion constraint ensure us to have a supergravity solution (up to the second order in non coset fermions), is given in Appendix C.

Chapter 4

Superstring Theories on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$

4.1 Supercoset Equations of Motions

First of all, even if we already clarified that the supercoset sigma-model is not sufficient for our purpose to describe the full theory, we will present the equations of motions of the string reduced to the supercoset, in fact this will be useful for understanding the integrability of the supercoset theory and to compare the situation with what happens when the non supercoset degrees of freedom are taken into account.

In our conventions the super–coset equations of motion have the following form

$$\begin{aligned} \Psi_{0\alpha} &= i * E^A \, (\Gamma_A E)_\alpha - i E^A \, (\Gamma_A \Gamma_{11} E)_\alpha = 0 \,, & (4.1.1) \\ \mathcal{B}_0^A &= \nabla * E^A - i E \Gamma^A \Gamma_{11} E = 0 \,, & (4.1.2) \end{aligned}$$

the various elements entring this formula were defined in section 3.1.

For completeness we mention also the second order expansion in fermions of (4.1.2) and (4.1.1):

$$\begin{aligned} &\nabla_I \left[\sqrt{-h}\, h^{IJ} e_J{}^A + i\vartheta (\sqrt{-h}\, h^{IJ} - \varepsilon^{IJ}\Gamma_{11}) \big(\Gamma^A \nabla_J \vartheta + \tfrac{2i}{R}\, e_J{}^B \Gamma^A \mathcal{P} \gamma^* \Gamma_B \vartheta \big) \right] \\ &\quad - \tfrac{i}{4} \vartheta \, (\sqrt{-h}\, h^{IJ} - \varepsilon^{IJ}\Gamma_{11}) \Gamma_D{}^{BC} \vartheta \, R_{BCE}{}^A \, e_I{}^D e_J{}^E = 0 \qquad (4.1.3) \end{aligned}$$

$$(\sqrt{-h}\, h^{IJ} - \varepsilon^{IJ}\Gamma_{11})\, e_I{}^A \Gamma_A \mathcal{P} \big(\nabla_J \vartheta + \tfrac{i}{R} e_J{}^B \gamma^* \Gamma_B \vartheta \big) = 0 \,, \qquad (4.1.4)$$

4.2 Theory up to the second order in fermions

The action for the GS superstring in a bosonic supergravity background (with zero NS–NS flux and constant dilaton ϕ) has the following form up to quadratic order in fermions, both coset and non-coset [56, 57]:

$$S = -T \int \left(\frac{1}{2} * e^A e_A + i * e^A \, \Theta \Gamma_A \mathcal{D} \Theta - i e^A \, \Theta \Gamma_A \hat{\Gamma} \, \mathcal{D} \Theta \right) , \tag{4.2.5}$$

where $e^A(X)$ $(A = 0, 1, \cdots, 9)$ are worldsheet pull–backs of the background vielbein one–forms and

$$\mathcal{D}\Theta = (\nabla - \frac{1}{8} e^A \not{F} \Gamma_A)\Theta \, , \tag{4.2.6}$$

is the fermionic vielbein to the lowest order in fermions Θ. Here $\nabla = d + \omega$ is the covariant derivative containing the spin connection of the background space–time,

$$\hat{\Gamma} = \begin{cases} \Gamma_{11} & \text{(IIA)} \\ \sigma^3 & \text{(IIB)} \end{cases} \tag{4.2.7}$$

and the coupling to the RR fields is given in terms of the matrix

$$\not{F} = e^{\phi} \begin{cases} -\frac{1}{2} \Gamma^{AB} \Gamma_{11} F_{AB} + \frac{1}{4!} \Gamma^{ABCD} F_{ABCD} & \text{(IIA)} \\ i \sigma^2 \Gamma^A F_A - \frac{1}{3!} \sigma^1 \Gamma^{ABC} F_{ABC} + \frac{i}{2 \cdot 5!} \sigma^2 \Gamma^{ABCDE} F_{ABCDE} & \text{(IIB)} \end{cases} \tag{4.2.8}$$

in the type IIA and type IIB case, respectively. The σs are the Pauli matrices.

There is a caveat when we compare (4.2.5) with what one gets from (2.4.18) by substituting the expressions for the vielbeins and the NS–NS two–form, respectively for $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, keeping only terms up to quadratic order in fermions, in fact from (2.4.18) we get the following action:

$$\begin{aligned} S &= -\frac{e^{\frac{2}{3}\phi_0}}{4\pi\alpha'} \int d^2\xi \, \sqrt{-h} \, h^{IJ} \, \left(e_I{}^A e_J{}^B \eta_{AB} \right) \\ &- \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi \, \Theta (\sqrt{-h} \, h^{IJ} - \varepsilon^{IJ} \Gamma_{11}) \left[i \, e_I{}^A \Gamma_A \mathcal{D}_J \Theta \right] \\ &+ \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi \, \sqrt{-h} \, h^{IJ} \, e_I{}^{a'} \, \nabla_J (i \, \Theta \mathcal{P} \Gamma_{a'} (1 - \mathcal{P}) \Theta) \, , \end{aligned} \tag{4.2.9}$$

where we have adopted the unified notation.

The first two lines of this action coincide with the action (4.2.5) but the action (4.2.9) also has the third term. It appeared because of our choice of parametrization of

the $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ superspaces. It is not hard to see that the last term in (4.2.9) can be canceled (modulo higher order terms in fermions) by making the following shift of the bosonic coordinates $y^{m'}$ of CP^3 or T^6

$$y^{m'} = \hat{y}^{m'} + i\Theta\mathcal{P}\Gamma^{a'}(1-\mathcal{P})\Theta\, e_{a'}{}^{m'}(\hat{y}). \tag{4.2.10}$$

After this field redefinition the two forms of the string action become equivalent.

Considering the case of $AdS_4 \times CP^3$, the fluxes are:

$$\begin{aligned} F_2 &= \frac{e^{-\phi}}{R} J_2\,, \\ F_4 &= -\frac{6e^{-\phi}}{R} e^a e^b e^c e^d \epsilon_{abcd} \end{aligned} \tag{4.2.11}$$

and we get:

$$\begin{aligned} S &= -\frac{e^{\frac{2}{3}\phi_0}}{4\pi\alpha'} \int d^2\xi\, \sqrt{-h}\, h^{IJ} \left(e_I{}^a e_J{}^b \eta_{ab} + e_I{}^{a'} e_J{}^{b'}\, \delta_{a'b'} \right) \\ &- \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi\, \Theta(\sqrt{-h}\, h^{IJ} - \varepsilon^{IJ}\Gamma_{11}) \big[i\, e_I{}^A \Gamma_A \nabla_J \Theta - \frac{1}{R} e_I{}^A e_J{}^B \Gamma_A \mathcal{P}_{24} \gamma^5 \Gamma_B \Theta \big] \end{aligned} \tag{4.2.12}$$

In the $AdS_2 \times S^2 \times T^6$ case supported by the fluxes [1]:

$$\begin{aligned} F_2 &= -\frac{e^{-\phi}}{R} e^b e^a \varepsilon_{ab}\,, \\ F_4 &= -\frac{e^{-\phi}}{R} e^{\hat{b}} e^{\hat{a}} \varepsilon_{\hat{a}\hat{b}} J_2\,, \end{aligned} \tag{4.2.13}$$

this gives:

$$\begin{aligned} S &= -\frac{e^{\frac{2}{3}\phi_0}}{4\pi\alpha'} \int d^2\xi\, \sqrt{-h}\, h^{IJ} \left(e_I{}^a e_J{}^b \eta_{ab} + e_I{}^{\hat{a}} e_J{}^{\hat{b}}\, \delta_{\hat{a}\hat{b}} + e_I{}^{a'} e_J{}^{b'}\, \delta_{a'b'} \right) \\ &- \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi\, i\vartheta \left(\sqrt{-h} h^{ij} - \varepsilon^{ij}\Gamma_{11} \right) e_i{}^{\underline{a}} \Gamma_{\underline{a}} \left(\nabla_j + \frac{1}{R} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_{\underline{a}} e_j{}^{\underline{a}} \right) \vartheta \\ &- \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi\, i\vartheta \left(\sqrt{-h} h^{ij} - \varepsilon^{ij}\Gamma_{11} \right) \Gamma_{a'} \nabla_j \upsilon\, \partial_i y^{a'} + i\upsilon \left(\sqrt{-h} h^{ij} - \varepsilon^{ij}\Gamma_{11} \right) \Gamma_{a'} \nabla_j \vartheta\, \partial_i y^{a'} \end{aligned}$$

[1] There are several $AdS_2 \times S^2 \times T^6$ solutions with different fluxes [25], but we chose this one that enables us to easily compare this case to the $AdS_4 \times CP^3$ one.

$$
\begin{aligned}
- \quad & \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi \, \frac{i}{R}\vartheta \left(\sqrt{-h}h^{ij} - \varepsilon^{ij}\Gamma_{11}\right) e_i{}^{\underline{a}}\Gamma_{\underline{a}}\, \mathcal{P}_8\gamma\Gamma_{11}\Gamma_{a'}\upsilon\, \partial_j y^{a'} \qquad (4.2.14)\\
- \quad & \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'} \int d^2\xi \, i\upsilon \left(\sqrt{-h}h^{ij} - \varepsilon^{ij}\Gamma_{11}\right) \not{e}_i \nabla_j \upsilon \\
+ \quad & \frac{i}{R}\,\upsilon \left(\sqrt{-h}h^{ij} - \varepsilon^{ij}\Gamma_{11}\right) \Gamma_{a'}\mathcal{P}_8\gamma\Gamma_{11}\Gamma_{b'}\upsilon\, \partial_i y^{a'}\partial_j y^{b'},
\end{aligned}
$$

where

$$
\not{e}_i = \Gamma_{\underline{a}} e_i{}^{\underline{a}}(x) + \Gamma_{a'}\partial_i y^{a'}, \qquad \nabla_j = \partial_j + \frac{1}{4}\,\Gamma_{\underline{ab}}\,\omega_j{}^{\underline{ab}}(x)\,, \qquad (4.2.15)
$$

and $\omega_i{}^{\underline{ab}}(x)$ and $e_i{}^{\underline{a}}(x)$ are the worldsheet pull-backs of the spin connection and the local frame in $AdS_2 \times S^2$.

Going back to the unified notation, the bosonic equations of motion to the second order in Θ are then

$$
\begin{aligned}
\nabla \Big(*(e^A + i\Theta\Gamma^A\mathcal{D}\Theta) + i\Theta\Gamma^A\hat{\Gamma}\,\mathcal{D}\Theta - \tfrac{i}{8} * e^B\,\Theta\Gamma^A \not{F}\Gamma_B\Theta - \tfrac{i}{8} e^B\,\Theta\Gamma^A\hat{\Gamma}\not{F}\Gamma_B\Theta\Big) \\
- \tfrac{i}{4} * e^B e^E\,\Theta\Gamma_B{}^{CD}\Theta\, R_{CDE}{}^A + \tfrac{i}{4} e^B e^E\,\Theta\Gamma_B{}^{CD}\hat{\Gamma}\Theta\, R_{CDE}{}^A = 0\,, \qquad (4.2.16)
\end{aligned}
$$

and the fermionic equations (linear in Θ) are

$$
(*e^A\,\Gamma_A - e^A\,\Gamma_A\hat{\Gamma})\,\mathcal{D}\Theta = 0\,. \qquad (4.2.17)
$$

If the background has bosonic isometries, generated by Killing vectors $K_A(X)$, the worldsheet model has the corresponding conserved Noether current one–form of the following generic form (see [24] for more details)

$$
J_{\mathcal{B}} = J^A K_A + J^{AB}\nabla_A K_B = e^A K_A + \text{fermions}\,, \qquad (4.2.18)
$$

The J^A and J^{AB} terms in the current have the following form

$$
J^A = e^A + i\Theta\Gamma^A\mathcal{D}\Theta - \frac{i}{8}e^B\,\Theta\Gamma^A\not{F}\Gamma_B\Theta + i\Theta\Gamma^A\hat{\Gamma} * \mathcal{D}\Theta - \frac{i}{8} * e^B\,\Theta\Gamma^A\hat{\Gamma}\not{F}\Gamma_B\Theta\,, \quad (4.2.19)
$$

$$
J^{AB} = \frac{1}{4}(\Gamma^{AB}\Theta)^\alpha * i_\alpha\mathcal{L} = -\frac{i}{4}e^C\,\Theta\Gamma^{AB}{}_C\Theta + \frac{i}{4} * e^C\,\Theta\Gamma^{AB}{}_C\hat{\Gamma}\Theta\,, \qquad (4.2.20)
$$

where $\mathcal{L}$ is the superstring Lagrangian in (4.2.5).

In those backgrounds that preserve some supersymmetries one also has a conserved supersymmetry current:

$$
J_{susy} = \frac{i}{2R}(e^A\,\Theta\Gamma_A\Xi - *e^A\,\Theta\Gamma_A\hat{\Gamma}\Xi)\,, \qquad (4.2.21)
$$

R is the dimension–of–length constant associated with the curvature radius of the background.

We recall some useful basic properties of Killing vectors of a D–dimensional symmetric space G/H. First of all $K = dX^M\, K_M$ satisfy the Maurer–Cartan equations

$$dK = -2K \wedge K\ , \qquad dK \wedge K = K \wedge dK = -2K \wedge K \wedge K\ . \tag{4.2.22}$$

The following relations also hold

$$\begin{gathered}
[\nabla_A, \nabla_B] K_C = -R_{ABC}{}^D\, K_D\,, \qquad \nabla_A K_B = [K_A, K_B], \\
\nabla_A \nabla_B K_C = [\nabla_A K_B,\, K_C] + [K_B, \nabla_A K_C] = [\nabla_A K_B,\, K_C] - [\nabla_A K_C,\, K_B] = -2R_{A[BC]}{}^D K_D, \\
[\nabla_A K_B,\, K_C] = [[K_A,\, K_B],\, K_C] = -R_{ABC}{}^D\, K_D, \\
\Big[[K_A, K_B], [K_C, K_D]\Big] = R_{AB[C}{}^F\, [K_{D]}, K_F] - R_{CD[A}{}^F\, [K_{B]}, K_F]\,,
\end{gathered}$$

The Killing spinors satisfy the equation:

$$\nabla \Xi - \frac{1}{8} e^A \not{F} \Gamma_A \Xi = 0\,. \tag{4.2.23}$$

4.3 Theory up to the second order in non-coset fermions

In this section we will present the expansion to the second order in non-supersymmetric fermions, this means that we will extend the result in the previous section to all orders in the coset fermions. This expansion will be later used to extend the results for integrability that previously were found to the second order in all the fermions[2].

The fermionic field equations are

$$\Psi_\alpha \equiv i * \mathcal{E}^A\, (\Gamma_A \mathcal{E})_\alpha - i \mathcal{E}^A\, (\Gamma_A \Gamma_{11} \mathcal{E})_\alpha + \frac{i}{2} * \mathcal{E}^A \mathcal{E}_A\, \lambda_\alpha + \frac{i}{2} \mathcal{E}^A \mathcal{E}^B\, (\Gamma_{AB} \Gamma_{11} \lambda)_\alpha = 0\,, \tag{4.3.24}$$

and the bosonic field equations are

$$\begin{aligned}
\mathcal{B}^A \;\equiv\; & \bar{\nabla} * \mathcal{E}^A + i * \mathcal{E}^A\, \mathcal{E} \lambda + \frac{1}{3} (* \mathcal{E}^A \mathcal{E}^B\, \bar{\nabla}_B \phi - * \mathcal{E}^B \mathcal{E}_B\, \bar{\nabla}^A \phi) - i \mathcal{E} \Gamma^A \Gamma_{11} \mathcal{E} - 2i \mathcal{E}^B\, \mathcal{E} \Gamma^A{}_B \Gamma_{11} \lambda \\
& + \frac{1}{2} \mathcal{E}^C \mathcal{E}^B\, H^A{}_{BC} = 0\,.
\end{aligned} \tag{4.3.25}$$

[2]So far a complete Lax connection to all orders in non-coset fermions has been constructed (in a certain kappa-symmetry gauge [58]) only in the AdS_4 sub-sector of the $AdS_4 \times CP^3$ superstring [24].

This is the way to rewrite in differential form notation the equations (2.4.21) and (2.4.22). The star $*$ denotes the Hodge dual operation on the worldsheet and $\nabla * \mathcal{E}^A = d * \mathcal{E}^A + * \mathcal{E}^B \Omega_B{}^A$ is the pull–back on the worldsheet of the target–superspace covariant derivative.

Introducing the explicit form of the supergeometry elements for $AdS_4 \times CP^3$ the fermionic equation of motion is given by:

$$\begin{aligned}
\Psi &= i * E^A (\Gamma_A E) - i E^A (\Gamma_A \Gamma_{11} E) \\
&\quad -i * (2i\upsilon\Gamma^A E + i\upsilon\Gamma^A D\upsilon)(\Gamma_A E) + \frac{i}{R} * E^A \upsilon\gamma^5\upsilon(\Gamma_A E) + \frac{i}{2R} * E^A \upsilon\Gamma^b\gamma^5\upsilon(\Gamma_A\Gamma_b E) \\
&\quad -\frac{i}{R} * E^A \upsilon\Gamma^b\gamma^7\upsilon(\Gamma_A\Gamma_b\Gamma_{11} E) - \frac{i}{2R} * E^A \upsilon\upsilon(\Gamma_A\gamma_5 E) + i(2i\upsilon\Gamma^A E + i\upsilon\Gamma^A D\upsilon)(\Gamma_A\Gamma_{11} E) \\
&\quad -\frac{i}{R} E^A \upsilon\gamma^5\upsilon(\Gamma_A\Gamma_{11} E) - \frac{i}{2R} E^A \upsilon\Gamma^b\gamma^5\upsilon(\Gamma_A\Gamma_{11}\Gamma_b E) - \frac{i}{R} E^A \upsilon\Gamma^b\gamma^7\upsilon(\Gamma_A\Gamma_b E) \\
&\quad +\frac{i}{2R} E^A \upsilon\upsilon(\Gamma_A\gamma_7 E) + i * (E^A + 2i\upsilon\Gamma^A E)(\Gamma_A D\upsilon) - i(E^A + 2i\upsilon\Gamma^A E)(\Gamma_A\Gamma_{11} D\upsilon) \\
&\quad +\frac{1}{R} * (E^A + 2i\upsilon\Gamma^A E)(E_A + 2i\upsilon\Gamma_A E)\gamma^5\upsilon + \frac{1}{R}(E^A + 2i\upsilon\Gamma^A E)(E^B + 2i\upsilon\Gamma^B E)(\Gamma_{AB}\Gamma_{11}\gamma^5\upsilon)
\end{aligned} \tag{4.3.26}$$

whereas the bosonic one is:

$$\begin{aligned}
B &= \nabla * E^A - iE\Gamma^A\Gamma_{11} E \\
&\quad +\nabla * \left[(1 - \frac{1}{R}\upsilon\gamma^5\upsilon)(\hat{E}^A + i\upsilon\Gamma^A D\upsilon)\right] - \frac{2}{R} E^A D\upsilon\,\gamma^5\upsilon - 2iE\Gamma_A\Gamma^{11} D\upsilon - iD\upsilon\Gamma^A\Gamma^{11} D\upsilon \\
&\quad +2iE\Gamma^A\Gamma^{11}\left(1 + \frac{1}{2R}\upsilon\Gamma^b\gamma^5\upsilon\,\Gamma_b + \frac{1}{R}\upsilon\Gamma^b\gamma^7\upsilon\,\Gamma_b\Gamma_{11} + \frac{1}{2R}\upsilon\upsilon\,\gamma_5\right) E + \frac{2}{R} * \hat{E}_B \delta^A_{a'}\delta^B_{b'}\upsilon\Gamma^{a'b'}\gamma_5 E \\
&\quad -\frac{2}{R} * E_B\left(\delta^A_a\delta^B_b\,\upsilon\Gamma^{ab}\gamma_5 D\upsilon + \frac{i}{R}\delta^A_{a'}\delta^B_{b'}\,E^c\,\upsilon\Gamma^{a'b'}{}_c\upsilon - \frac{2i}{R}\delta^{[A}_{a'}\delta^{B]}_b\,E^{c'}\,\upsilon\Gamma^{a'b}{}_{c'}\upsilon\right) \\
&\quad +\frac{4}{R}\hat{E}^B E\Gamma^A{}_B\gamma^7\upsilon + \frac{4}{R} D\upsilon\,\Gamma^A{}_B\gamma^7\upsilon - \frac{6i}{R^2} E^c E^b \upsilon\Gamma_{abc}\Gamma_{11}\upsilon.
\end{aligned} \tag{4.3.27}$$

For $AdS_2 \times S^2 \times T^6$ we get:

$$\begin{aligned}
\Psi &= i * E^A (\Gamma_A E) - i E^A (\Gamma_A \Gamma_{11} E) \\
&\quad -i * (2i\upsilon\Gamma^A E + i\upsilon\Gamma^A D\upsilon)(\Gamma_A E) - \frac{i}{2R} * E^A \upsilon\Gamma^B\gamma_*\gamma_7\upsilon(\Gamma_A\Gamma_B\gamma_7 E) \\
&\quad +\frac{i}{2R} * E^A \upsilon\Gamma^B\gamma_*\Gamma_{11}\upsilon(\Gamma_A\Gamma_B\Gamma_{11} E)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{4R} * E^A \upsilon\Gamma^{BC}\gamma_*\upsilon\,(\Gamma_A\Gamma_{BC}E) - \frac{1}{2R} * E^A\upsilon\Gamma^{\underline{b}c'}\Gamma_{11}\upsilon\,\gamma_*(\Gamma_A(1-\mathcal{P})\Gamma_{\underline{b}c'}\Gamma_{11}E)\\
&-\frac{1}{4R} * E^A\upsilon\Gamma^{\underline{bc}d'}\upsilon\,\gamma_*(\Gamma_A(1-\mathcal{P})\Gamma_{\underline{bc}d'})E) + i(2i\upsilon\Gamma^A E + i\upsilon\Gamma^A D\upsilon)\,(\Gamma_A\Gamma_{11}E)\\
&+\frac{i}{2R}E^A\,\upsilon\Gamma^B\gamma_*\gamma_7\upsilon\,(\Gamma_A\Gamma^{11}\Gamma_B\gamma_7 E) + \frac{i}{2R}E^A\,\upsilon\Gamma^B\gamma_*\Gamma_{11}\upsilon\,(\Gamma_A\Gamma_B E)\\
&-\frac{i}{4R}E^A\upsilon\Gamma^{BC}\gamma_*\upsilon\,(\Gamma_A\Gamma^{11}\Gamma_{BC}E) + \frac{1}{2R}E^A\upsilon\Gamma^{\underline{b}c'}\Gamma_{11}\upsilon\,\gamma_*(\Gamma_A\Gamma^{11}(1-\mathcal{P})\Gamma_{\underline{b}c'}\Gamma_{11}E)\\
&+\frac{1}{4R}E^A\upsilon\Gamma^{\underline{bc}d'}\upsilon\,\gamma_*(\Gamma_A\Gamma^{11}(1-\mathcal{P})\Gamma_{\underline{bc}d'})E) + i*(E^A+2i\upsilon\Gamma^A E)\,(\Gamma_A D\upsilon)\\
&-i(E^A+2i\upsilon\Gamma^A E)\,(\Gamma_A\Gamma_{11}D\upsilon)\\
&+\frac{1}{R}*(E^A+2i\upsilon\Gamma^A E)(E_A+2i\upsilon\Gamma_A E)\,\gamma_*\upsilon + \frac{1}{R}(E^A+2i\upsilon\Gamma^A E)(E^B+2i\upsilon\Gamma^B E)\,(\Gamma_{AB}\Gamma_{11}\gamma_*\upsilon)
\end{aligned}
\tag{4.3.28}
$$

and

$$
\begin{aligned}
B \;=\;& \nabla * E^A - iE\Gamma^A\Gamma_{11}E\\
&+\nabla*\left[(1-\frac{1}{R}\upsilon\gamma_*\upsilon)(\hat{E}^A+i\upsilon\Gamma^A D\upsilon)\right] - \frac{2}{R}E^A D\upsilon\,\gamma_*\upsilon - 2iE\Gamma_A\Gamma^{11}D\upsilon - iD\upsilon\Gamma^A\Gamma^{11}D\upsilon\\
&+2iE\Gamma^A\Gamma^{11}\left(1+\frac{1}{2R}\upsilon\Gamma^B\gamma_*\gamma_7\upsilon\,\Gamma_B\gamma_7 - \frac{1}{2R}\upsilon\Gamma^B\gamma_*\Gamma_{11}\upsilon\,\Gamma_B\Gamma_{11} - \frac{1}{4R}\upsilon\Gamma^{BC}\gamma_*\upsilon\,\Gamma_{BC}\right)E\\
&2iE\Gamma_A\Gamma^{11}(1-\mathcal{P})\left(-\frac{1}{2R}\upsilon\Gamma^{\underline{b}c'}\Gamma_{11}\upsilon\,\gamma_*\Gamma_{\underline{b}c'}\Gamma_{11} - \frac{1}{4R}\upsilon\Gamma^{\underline{bc}d'}\upsilon\,\gamma_*\Gamma_{\underline{bc}d'}\right)E + \frac{2}{R}*\hat{E}_B\delta^A_{a'}\delta^B_{b'}\upsilon\Gamma^{a'b'}\gamma_*E\\
&-\frac{2}{R}*E_B\left(-\frac{1}{R}\upsilon\Gamma^{[A}(1-\mathcal{P})\Gamma^{B]}\gamma_*D\upsilon - \frac{1}{R}\upsilon\gamma_*\Gamma^{[A}(1-\mathcal{P})\Gamma^{B]}D\upsilon\right.\\
&+\frac{i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b'}E^C\,\upsilon\Gamma^{a'}(1-\mathcal{P})\Gamma^{b'}\gamma_*\Gamma_C\gamma_*\upsilon + \frac{i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b'}E^{c'}\,\upsilon\Gamma^{a'}\mathcal{P}\Gamma^{b'c'}\upsilon\\
&-\frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{\underline{b}}E^{c'}\,\upsilon\Gamma^{a'}(1-\mathcal{P})\Gamma^{\underline{b}}\Gamma_{c'}\upsilon + \frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{b}E^{\hat{c}}\,\upsilon\Gamma^{a'b}{}_{\hat{c}}\upsilon\\
&\left.-\frac{2i}{R^2}\delta^{[A}_{a'}\delta^{B]}_{\hat{b}}E^{c}\,\upsilon\Gamma^{a'\hat{b}}{}_{c}\upsilon - \frac{2i}{R^2}\delta^{[A}_{a}\delta^{B]}_{\underline{b}}E^{c'}\,\upsilon\Gamma^{a\underline{b}}{}_{c'}\upsilon\right)\\
&+\frac{4}{R}\hat{E}^B E\Gamma^A{}_B\Gamma^{11}\gamma_*\upsilon + \frac{4}{R}D\upsilon\,\Gamma^A{}_B\Gamma^{11}\gamma_*\upsilon\\
&-\frac{3i}{R^2}E^C E^B\Big(\upsilon\gamma_*\Gamma_{ABC}\Gamma_{11}\gamma_*\upsilon - \delta^{\underline{a}}_{[A}\,\upsilon\gamma_*\Gamma_B\mathcal{P}\Gamma_{C]}\Gamma_{\underline{a}}\gamma_*\Gamma_{11}\upsilon - \delta^d_{[A}\delta^e_B\,\upsilon\Gamma_{C]de}\Gamma_{11}\upsilon\\
&+\delta^{\hat{d}}_{[A}\delta^{\hat{e}}_B\,\upsilon\Gamma_{C]\hat{d}\hat{e}}\Gamma_{11}\upsilon\Big) = 0.
\end{aligned}
\tag{4.3.29}
$$

Note that the first row of each equation of motion has the same form of the correspond-

ing supercoset equation of motion.

4.4 Truncation to the supercoset Sigma Model

One may wonder now if there exists a consistent truncation of these two theories from the full theory to those on a supercoset. For the $AdS_2 \times S^2 \times T^6$ case a consistent truncation is possible to the supercoset $\frac{PSU(1,1|}{SO(1,1)\times U(1)}$, whose bosonic part is $AdS_2 \times S^2$, in fact we see that switching off the non-supercoset degrees of freedom (24 fermionic modes and 6 bosonic directins along T^6), one gets the sigma model on the supercoset and its equations of motion. This happens because in the action of the full theory the non supercoset coordinates never appear linearly (see the form of the action (4.2.14)). This ensures us that to recover the supercoset sigma-model it is enough to put non-supercoset degrees of freedom to zero.

This is not valid in the case of $AdS_4 \times CP^3$, where the non-coset fermions appear linearly in the action, however a consistent truncation to the $\frac{OSp(6|4)}{SO(1,3)\times U(3)}$ supercoset exists thanks to the Kappa-symmetry of the action. On the other hand, in the $AdS_4 \times CP^3$ action the coset fermions and the CP^3 coordinates always appear at least quadratically. This allows one to put the coset fermions and CP^3 modes to zero and reduce the theory to a non-supersymmetric sigma-model on AdS_4 [21, 24, 58] .

The gauge fixing consists in introducing a projection which removes some of the fermions:

$$\Theta = \frac{1}{2}(1 \pm \gamma)\Theta \tag{4.4.30}$$

and in order to have a gauge fixing consistent with the Kappa-symmetry one has to take γ either to be equal to Γ appearing in the Kappa-projector, either to not commute with this. This is quite easy to understand, in fact to lowest order in fermions Θ transforms under kappa–symmetry as

$$\delta_\kappa \Theta = \frac{1}{2}(1 + \Gamma)\kappa\,, \tag{4.4.31}$$

where $\frac{1}{2}(1+\Gamma)$ is a projection matrix and $\kappa(\xi)$ is an arbitrary spinor parameter. It is then clear that if the two projectors coincide, we can pick a κ such that $\frac{1}{2}(1+\Gamma)\Theta = 0$, or equivalently $\Theta = \frac{1}{2}(1-\Gamma)\Theta$. In the case when the two projection operators do not coincide a kappa–symmetry variation of the gauge–fixing condition $\frac{1}{2}(1\mp\gamma)\Theta = 0$ which leaves it intact gives

$$0 = \frac{1}{4}(1\mp\gamma)(1+\Gamma)\kappa = \frac{1}{8}(1+\Gamma)(1\mp\gamma)(1+\Gamma)\kappa \mp \frac{1}{8}[\gamma,\Gamma](1+\Gamma)\kappa = \mp\frac{1}{8}[\gamma,\Gamma](1+\Gamma)\kappa\,, \tag{4.4.32}$$

where in the last step we made use of the initial equation. This means that to have the variation of gauge fixing vanishing if and only if all independent Kappa-paramenters are put to zero the we have to choose $[\gamma, \Gamma] \neq 0$.

It is clear that kappa-symmetry is not enough for the $AdS_2 \times S^2 \times T^6$ theory, in fact, using the kappa symmetry, one can get rid at most of 16 ferminic degrees of freedom, but in this case the sigma model possesses 24 non-coset fermionic ones. Note that, since in this case there are only 8 coset fermions, there are classical sectors of the string theory in which kappa-symmetry gauge fixing can completely remove them.

In the case of $AdS_4 \times CP^3$ a partial kappa symmetry gauge fixing can lead to the supercoset model, but this is not always possible, for example this fails if one considers a classical string moving entirely in AdS_4, or in the case, that we are going to consider, of an instanton wrapping a two-cycle in CP^3. In these cases the projector $\mathcal{P}_8$ which singles out 8 non-coset fermions commutes with Γ, and hence kappa-symmetry cannot eliminate all of them, as explained above. This explains why, if we would like to find all the physical degrees of freedom of the theory and check its integrability, it is not enough to check the integrability of the supercoset sigma-model, to say that the full theory is integrable, but we need to explore how the non supercoset modes enters in the integrable structure of the theory, and how these modes modify the Lax Connection.

Chapter 5

Integrability of a supercoset sigma model

Let us start the study of the integrable structures of the $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ superstrings by reviewing the form of the Lax connections of their supercoset subsectors:

$$\frac{OSp(6|4)}{U(3) \times SO(1,3)} \quad \text{and} \quad \frac{PSU(1,1|2) \times E(6)}{SO(1,1) \times U(1) \times SO(6)}\,, \tag{5.0.1}$$

it is worth to mention that this study was at first proposed in [26] for

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \tag{5.0.2}$$

whose bosonic part is

$$\frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} = AdS_5 \times S^5. \tag{5.0.3}$$

In these cases the superalgebra shows a $\mathbb{Z}_4$ decomposition, this, as already mentioned, is one of the fundamental points of the study of the integrability, and it is the key point in the construction of the Lax connection in the supercoset sigma model case. All what we need to know to find a flat connection is, in fact, that we can decompose the Cartan form and corresponding Maurer-Cartan equations and equations of motion according to the $\mathbb{Z}_4$-grading of the algebra.

Let us take K to be a supercoset element and re-adopt the unified notation for the two supercosets (5.0.1). If we consider the grading of the superalgebra (3.0.15), we can write

$$K^{-1}dK = \Omega^{AB}\, M_{AB} + E^A\, P_A + E\, Q_1 + E\, Q_3\,. \tag{5.0.4}$$

It can be found that in this case a zero curvature Lax connection can be written as:

$$L_{coset} = \frac{1}{2}\Omega_0^{AB} M_{AB} + (1+\alpha_1) E^A P_A + \alpha_2 * E^A P_A + Q(\beta_2 + \beta_1 \Gamma_{11}) E \,, \tag{5.0.5}$$

Considering the grading of the Maurer-Cartan equations, given in (3.1.25), (3.1.26) and (3.1.27), and the equations of motion (4.1.1) and (4.1.2), we get:

$$\begin{aligned} dL_{coset} - L_{coset} L_{coset} \;=\; & -i(1+\alpha_1-\beta_1^2+\beta_2^2) E\Gamma^A E P_A + i(\alpha_2 - 2\beta_1\beta_2) E\Gamma^A \Gamma^{11} E P_A \\ & + \frac{1}{2}(-2\alpha_1 - \alpha_1^2 + \alpha_2^2) E^D E^C R_{CD}{}^{AB} M_{AB} \\ & - \frac{R}{4}(1-\beta_1^2-\beta_2^2) E\Gamma^{CD}\gamma^* E R_{CD}{}^{AB} M_{AB} \\ & + \frac{i}{R}(\alpha_1\beta_2 + \alpha_2\beta_1) Q\gamma^*\Gamma_A E E^A \\ & + \frac{i}{R}((1+\alpha_1)\beta_1 + \alpha_2\beta_2) Q\gamma^*\Gamma_A\Gamma^{11} E E^A = 0 \,, \end{aligned} \tag{5.0.6}$$

where:

$$\begin{aligned} \alpha_1 &= \frac{2\mathtt{x}^2}{1-\mathtt{x}^2} \,, \\ \alpha_2^2 &= \alpha_1^2 + 2\alpha_1 \,, \\ \beta_1 &= \mp\sqrt{\frac{\alpha_1}{2}} \,, \\ \beta_2 &= \pm\frac{\alpha_2}{\sqrt{2\alpha_1}} \,. \end{aligned} \tag{5.0.7}$$

It is very useful for further analysis to specify the properties of the 32×32 matrix

$$V = \beta_2 + \beta_1 \Gamma_{11} \,, \tag{5.0.8}$$

which enters the Lax connection (5.0.5). It is easily seen to satisfy the relations

$$V^2 = 1 + \alpha_1 - \alpha_2 \Gamma_{11} \,, \quad V V^\dagger = \beta_2^2 - \beta_1^2 = 1 \,, \quad (V^\dagger)^\alpha{}_\beta = -(\mathcal{C} V^{\mathrm{T}} \mathcal{C})^\alpha{}_\beta = (\beta_2 - \beta_1\Gamma_{11})^\alpha{}_\beta \,, \tag{5.0.9}$$

where $\mathcal{C}$ denotes the anti-symmetric charge-conjugation matrix. This means that we can rewrite the Lax connection in the following way:

$$L_{coset} = \frac{1}{2}\Omega_0^{AB} M_{AB} + (1+\alpha_1) E^A P_A + \alpha_2 * E^A P_A + QVE \,. \tag{5.0.10}$$

The explicit dependence of the supercoset Lax curvature on the left–hand sides of the supercoset field equations (4.1.1) and (4.1.2) looks as follows

$$dL_{coset} - L_{coset} \wedge L_{coset} = \alpha_2(\mathcal{B}_0^A P_A - \frac{1}{R} Q V^\dagger \gamma_\star \Psi_0)\,, \tag{5.0.11}$$

where again $\gamma_\star$ stands for γ^5 in the $AdS_4 \times CP^3$ case and for $\Gamma^{01}\gamma^7$ in the $AdS_2 \times S^2 \times T^6$ case, $\mathcal{B}_0$ and Ψ_0 are the right-hand-sides of the equations of motion of the supercoset model given in (4.1.1). This means that the Lax connection is flat on the mass shell.

If one performs a Gauge transformation we get another form of the Lax connection:

$$\mathcal{L}_{coset} = K L_{coset} K^{-1} - dK K^{-1} \tag{5.0.12}$$

The coset Lax connection can be rewritten in the following form:

$$\begin{aligned} \mathcal{L}_{coset} &= K\Big(\alpha_1 E^A P_A + \alpha_2 * E^A P_A + \beta_1 Q\Gamma_{11} E + (1+\beta_2) QE\Big) K^{-1} \\ &= K\Big(\alpha_1\, E^A\, P_A + (1+\beta_2)\, QE + (\beta_1 - \frac{\alpha_2}{2})\, Q\Gamma_{11} E\Big) K^{-1} + \alpha_2\, * J_{coset}\,, \end{aligned} \tag{5.0.13}$$

with

$$J_{coset}(X,\vartheta) = K(X,\vartheta)\,\Lambda\, K^{-1}(X,\vartheta) \doteq K(X,\vartheta)\left(E^A P_A + \frac{1}{2} Q\Gamma_{11} * E\right) K^{-1}(X,\vartheta) \tag{5.0.14}$$

being the conserved current associated to the isometry.

5.1 $\mathbb{Z}_4$ symmetry of the supercoset Lax

The Lax connection (5.0.5) is $\mathbb{Z}_4$-invariant. In fact the $\mathbb{Z}_4$ transformation acts on the generators in the following way:

$$\Omega(T) \equiv \Omega^{-1} T \Omega \qquad \Omega(M_{AB}) = M_{AB}, \qquad \Omega(P_A) = -P_A, \qquad \Omega(Q) = -iQ\Gamma_{11}, \qquad \Omega^4 = 1\,. \tag{5.1.15}$$

Note that in the $AdS_2 \times S^2 \times T^6$ case the T^6 translation generators $P_{a'}$ also have $\mathbb{Z}_4$–grading one, as those of $AdS_2 \times S^2$.

It is easy to check that the Lax connection (5.0.5) is invariant under the $\mathbb{Z}_4$–transformations of the generators (5.1.15) accompanied by the inversion of the spectral parameter

$$\Omega(\mathtt{x}) = \frac{1}{\mathtt{x}}\,, \tag{5.1.16}$$

which implies that

$$\alpha_1 \to -\alpha_1 - 2\,, \qquad \alpha_2 \to -\alpha_2\,, \qquad V \to i\Gamma_{11}V\,. \tag{5.1.17}$$

Namely,

$$\Omega(L_{coset}(\mathtt{x})) = \Omega^{-1} L_{coset}\left(\frac{1}{\mathtt{x}}\right)\Omega = L_{coset}(\mathtt{x})\,.$$

Chapter 6

Classical Integrability of $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$

In this chapter we will see how to study the classical integrability of these less supersymmetric theories, the adopted approach is to start to expand the theories in the fermions, seeing at first what happens at the second order in all fermions. Once we have computed the Lax connection to this second order, this can suggest us what is the deviation that the non coset fermions introduce in the coset form of the Lax connection, and then move to consider the expansion only in the non-supercoset fermions. The way to proceed was first to deal with the $AdS_4 \times CP^3$ case, with the aim to complete the proof of the integrability of the full theory, and then move to consider the $AdS_2 \times S^2 \times T^6$ case that, as already emphasized, is"dually analogous" to the $AdS_4 \times CP^3$ one.

6.1 What happens up to the second order in fermions?

The form for the Lax connection for certain second order type IIA/B string theories was found in [24, 25], this can be constructed in terms of the components of the conserved current (4.2.18) and the supercurrent (4.2.21):

$$\Lambda = \alpha_1 e^A K_A + \alpha_2 * J_{\mathcal{B}} + \alpha_2^2 J^{AB} \nabla_A K_B + \alpha_1 \alpha_2 * J^{AB} \nabla_A K_B + -\alpha_2 \beta_1 J_{susy} + \alpha_2 \beta_2 * J_{susy} \,, \tag{6.1.1}$$

where the coefficients are those of the previous section (see equations (5.0.7)).

To show that this has a zero curvature we can use, as usual, the conservation of the currents:

$$d * J_{\mathcal{B}} = 0, \qquad d * J_{susy} = 0, \tag{6.1.2}$$

the first of these two conditions, considering the orthogonality of the Killing vectors K_A and $\nabla_A K_B = [K_A, K_B]$, implies that the following equations hold separately

$$(\nabla * J^{AB} - *J^{[A} e^{B]}) K_A \, K_B = 0 \,, \tag{6.1.3}$$

$$\nabla * J^A - 2 R_{BCD}{}^A * J^{CD} e^B = 0 \,. \tag{6.1.4}$$

Sketching the calculation of the curvature, we have:

$$\begin{aligned} d\Lambda - \Lambda\Lambda \;=\; & \alpha_2^2 (\nabla J^{AB} + (J^A - e^A) e^B) \nabla_A K_B + \alpha_2^2 (\beta_2^2 - \beta_1^2) J_{susy}^2 - \alpha_2 \beta_1 dJ_{susy} \\ & + \alpha_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2)(J_{\mathcal{B}} J_{susy} + J_{susy} J_{\mathcal{B}}) \\ & - \alpha_2 (\alpha_1 \beta_2 + \alpha_2 \beta_1)(J_{\mathcal{B}} * J_{susy} + * J_{susy} J_{\mathcal{B}}) = \\ & = \alpha_2^2 \left(\left[\nabla J^{AB} + (J^A - e^A) e^B \right] \nabla_A K_B + J_{susy}^2 \right) \\ & - \alpha_2 \beta_1 \left(dJ_{susy} + 2 (J_{\mathcal{B}} J_{susy} + J_{susy} J_{\mathcal{B}}) \right) \,. \end{aligned} \tag{6.1.5}$$

Using the form of the supersymmetry current (4.2.21) and the equations of motion we get

$$dJ_{susy} = \frac{i}{8R} (e^A e^B \, \Theta \Gamma_A \not{\nabla} \Gamma_B \Xi - * e^A e^B \, \Theta \Gamma_A \hat{\Gamma} \not{\nabla} \Gamma_B \Xi) \,. \tag{6.1.6}$$

Using the form (4.2.20) of J^{AB} we find that

$$\begin{aligned} \nabla J^{AB} \;=\; & -\frac{i}{2} e^C \, \Theta \Gamma^{AB}{}_C \nabla \Theta + \frac{i}{2} * e^C \, \Theta \Gamma^{AB}{}_C \hat{\Gamma} \nabla \Theta \\ = \; & i e^{[B} \, \Theta \Gamma^{A]} \nabla \Theta - i * e^{[B} \, \Theta \Gamma^{A]} \hat{\Gamma} \nabla \Theta - \frac{i}{16} e^C e^D \, \Theta \Gamma^{AB} \Gamma_C \not{\nabla} \Gamma_D \Theta \\ & + \frac{i}{16} * e^C e^D \, \Theta \Gamma^{AB} \Gamma_C \hat{\Gamma} \not{\nabla} \Gamma_D \Theta \,, \end{aligned} \tag{6.1.7}$$

where we have again made use of the equations of motion. We can further rewrite this as

$$\nabla J^{AB} + e^{[A} (J^{B]} - e^{B]}) = -\frac{i}{16} e^C e^D \, \Theta \Gamma_C \Gamma^{AB} \not{\nabla} \Gamma_D \Theta + \frac{i}{16} * e^C e^D \, \Theta \Gamma_C \Gamma^{AB} \hat{\Gamma} \not{\nabla} \Gamma_D \Theta \,. \tag{6.1.8}$$

If we consider $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ case the curvature of the Lax indeed vanishes.

6.1.1 Other forms of Lax connection

As we have already mentioned, other forms of Lax connection can be found by performing a gauge transformation:

$$\mathcal{L} = g^{-1} \Lambda \, g + g^{-1} dg \,. \tag{6.1.9}$$

where $g \in OSp(6|4)$ or $\in PSU(1,1|2) \rtimes E(6)$.

If one wishes to know which is the gauge transformation that relates the Lax connection (6.1.1) constructed above, when $v = 0$, to the supercoset Lax connection in eq. (5.0.13) , so that

$$\mathcal{L}_{coset} = g^{-1}\Lambda|_{v=0}\, g + g^{-1}dg\,. \tag{6.1.10}$$

the g, that gives the answer, is:

$$g(X,\vartheta;\alpha_2,\beta_1,\beta_2) \;=\; k(X)\, e^{\frac{\alpha_2 R}{16}\vartheta\Gamma^{AB}\gamma^*\Gamma^{11}\vartheta\, R_{AB}{}^{CD}M_{CD}}\, e^{-\beta_1\vartheta\Gamma_{11}Q}\, e^{-(1+\beta_2)\vartheta Q}\, k^{-1}(X) \tag{6.1.11}$$

One can perform this transformation without setting v to zero to get a new form of the Lax connection to the second order in all fermions:

$$\begin{aligned}
\mathcal{L} \;=\;& \mathcal{L}_{coset} + \alpha_2 * 2iv\Gamma^A D\vartheta K_A + \alpha_2 2iv\Gamma^A\Gamma^{11}D\vartheta K_A + \\
&+\alpha_2\beta_2\frac{i}{R}2iv\Gamma^{11}\Gamma^A\mathcal{P}\Gamma_B\vartheta K_A e^B - \alpha_2\beta_2\frac{i}{R}*2iv\Gamma^A\mathcal{P}\Gamma_B\vartheta K_A e^B + \\
&-\alpha_2\beta_1\frac{i}{R}2iv\Gamma^A\mathcal{P}\Gamma_B\vartheta K_A e^B + \alpha_2\beta_1\frac{i}{R}*2iv\Gamma^{11}\Gamma^A\mathcal{P}\Gamma_B\vartheta K_A e^B + \\
&+\alpha_2\beta_2\frac{i}{4}v\Gamma^{ABC}\vartheta\nabla_B K_C e_A - \alpha_2\beta_2\frac{i}{4}*v\Gamma^{11}\Gamma^{ABC}\vartheta\nabla_B K_C e_A + \\
&+\alpha_2\beta_1\frac{i}{4}v\Gamma^{11}\Gamma^{ABC}\vartheta\nabla_B K_C e_A - \alpha_2\beta_1\frac{i}{4}*v\Gamma^{ABC}\vartheta\nabla_B K_C e_A \\
&-\alpha_2\beta_1\frac{i}{R}v\Gamma^A\Xi e_A - \alpha_2\beta_1\frac{i}{R}*v\Gamma^{11}\Gamma^A\Xi e_A + \alpha_2\beta_2\frac{i}{R}*v\Gamma^A\Xi e_A + \alpha_2\beta_2\frac{i}{R}v\Gamma^{11}\Gamma^A\Xi e_A + \\
&+\alpha_2 * iv\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e^B\right)K_A - \alpha_2 iv\Gamma^{11}\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e^B\right)K_A + \\
&+(\alpha_2+\alpha_1\alpha_2)\frac{i}{4}*v\Gamma_A{}^{BC}v\nabla_B K_C e^A - (\alpha_2+\alpha_1\alpha_2)\frac{i}{4}v\Gamma^{11}\Gamma_A{}^{BC}v\nabla_B K_C e^A + \\
&+\alpha_2^2\frac{i}{4}v\Gamma_A{}^{BC}v\nabla_B K_C e^A - \alpha_2^2\frac{i}{4}*v\Gamma^{11}\Gamma_A{}^{BC}v\nabla_B K_C e^A.
\end{aligned} \tag{6.1.12}$$

where $\mathcal{L}_{coset}$ is defined in (5.0.13).

Since this is a gauge transformation, it does not destroy the zero curvature condition. One can also explicitly check that this is a zero curvature Lax connection, in fact when we introduce the non-supercoset degrees of freedom $\mathcal{L}_{coset}$ is not the proper Lax connection anymore, since it is not flat, due to the dependence of the equations of motion on v:

$$d\mathcal{L}_{coset} \;-\; \mathcal{L}_{coset}\wedge\mathcal{L}_{coset} = -\alpha_2\left[+\nabla_I\left(iv(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma^A\left(\nabla_J\theta + 2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\theta e_J^B\right) + \right.\right.$$

$$
\begin{aligned}
&+ \quad i\vartheta(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma^A \left(\nabla_J v + 2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B v e_J^B\right) + \\
&+ \quad iv(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma^A \left(\nabla_J v + 2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B v e_J^B\right)\Big) + \\
&+ \quad \frac{i}{2}v(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma_D{}^{BC}\vartheta R_{BCD}{}^A e_I^B e_J^D + \\
&+ \quad \frac{i}{4}v(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma_D{}^{BC} v R_{BCD}{}^A e_I^B e_J^D\Big] K_A + \\
&+ \quad \alpha_2\beta_1\left[\Xi(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma_A \left(\nabla_J v + \frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e_J^B\right) e_I^A\right] + \\
&+ \quad \alpha_2\beta_2\left[\Xi(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma_A \left(\nabla_J v + \frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e_J^B\right) e_I^A\right] + \\
&+ \quad \alpha_2\beta_1\left[\vartheta\gamma^*\{\Xi,\Xi\}(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma_A \left(\nabla_J v + \frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e_J^B\right) e_I^A\right] + \\
&+ \quad \alpha_2\beta_2\left[\vartheta\gamma^*\{\Xi,\Xi\}(\sqrt{-h}h^{IJ} - \epsilon^{IJ}\Gamma^{11})\Gamma_A \left(\nabla_J v + \frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e_J^B\right) e_I^A\right]
\end{aligned}
\tag{6.1.13}
$$

.

It is not hard to see that these v-dependent terms in (6.1.13) are cancelled by the v-dependent terms in the curvature of (6.1.12).

As we did in the coset case (see (5.0.10)) we would like to build a Lax connection in a form in which the check of its $\mathbb{Z}_4$-invariance would be more straightforward. To this end we perform one more gauge transformation of $\mathcal{L}$ (6.1.12):

$$
L = K^{-1}\mathcal{L}K - K^{-1}dK = L_{coset} + L_v \tag{6.1.14}
$$

where $K(X,\vartheta) = k(X)e^{\vartheta Q}$. The Lax connection that we get is:

$$
\begin{aligned}
L \;=\; & L_{coset} + \alpha_2 * \left(2iv\Gamma^A D\vartheta\right) P_A + \alpha_2\left(2iv\Gamma^{11}\Gamma^A D\vartheta\right) P_A + \alpha_2\beta_1\frac{i}{R}Q\gamma^*\Gamma^A v e_A + \\
& -\alpha_2\beta_1\frac{i}{R} * Q\gamma^*\Gamma^{11}\Gamma^A v e_A - \alpha_2\beta_2\frac{i}{R} * Q\gamma^*\Gamma^A v e_A + \alpha_2\beta_2\frac{i}{R}Q\gamma^*\Gamma^{11}\Gamma^A v e_A \\
& +\alpha_2 * iv\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e^B\right) P_A - \alpha_2 iv\Gamma^{11}\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}\gamma^*\Gamma_B v e^B\right) P_A + \\
& -(\alpha_2 + \alpha_1\alpha_2)\frac{i}{8} * v e^A\Gamma_A{}^{BC} v R_{BC}{}^{DE} M_{DE} + (\alpha_2 + \alpha_1\alpha_2)\frac{i}{8}v\Gamma^{11}e^A\Gamma_A{}^{BC} v R_{BC}{}^{DE} M_{DE} + \\
& +\alpha_2^2\frac{i}{8} * v\Gamma^{11}e^A\Gamma_A{}^{BC} v R_{BC}{}^{DE} M_{DE} - \alpha_2^2\frac{i}{8}v e^A\Gamma_A{}^{BC} v R_{BC}{}^{DE} M_{DE}
\end{aligned}
\tag{6.1.15}
$$

We can see that $dL + L \wedge L = 0$ up to the second order.

Note that:

$$\Omega(L(\mathtt{x})) = \Omega^{-1} L\left(\frac{1}{\mathtt{x}}\right) \Omega = L(\mathtt{x})\,.$$

(the action of Ω was defined in (5.1.15) and (5.1.16)) and thus, in this form, the Lax connection is $\mathbb{Z}_4$-invariant.

We can also find which is the transformation that relates (6.1.1) directly with (6.1.15). The non–straightforward relation between the two connections is realized by the following gauge transformation depending on the spectral parameter and accompanied by the shift in the X–dependence of Λ

$$\Lambda(X^M + i\upsilon\Gamma^M\vartheta, \Theta) = \mathcal{G}_{\mathbf{x}}(X,\Theta)\, L(X,\Theta)\, \mathcal{G}_{\mathbf{x}}^{-1}(X,\Theta) - d\mathcal{G}_{\mathbf{x}}\mathcal{G}_{\mathbf{x}}{}^{-1}(X,\Theta)\,, \qquad (6.1.16)$$

where both sides are truncated to quadratic order in fermions, $\Gamma^M = \Gamma^A\, e_A{}^M(X)$ and $\mathcal{G}_{\mathbf{x}}(X,\Theta)$ is an isometry supergroup element depending on the spectral parameter $\mathbf{x}$, which in the exponential parametrization has the following form

$$\mathcal{G}_{\mathbf{x}}(X^A,\Theta) = \; e^{(X^A + i\upsilon\Gamma^A(1-V^2)\vartheta)P_A}\, e^{QV\vartheta}\, h(i_\Delta\, \Omega_0^{AB})\,, \qquad (6.1.17)$$

where

$$h(i_\Delta\, \Omega_0^{AB}) = e^{-\frac{1}{2}i\upsilon\Gamma^C(1-V^2)\vartheta\, \Omega_{0C}{}^{AB}(X,\vartheta)M_{AB}}$$

is a compensating gauge transformation in the stability subgroup H (*i.e.* $SO(1,3) \times U(3)$ or $SO(1,1) \times SO(2)$) of the superisometry group G.

Note that in contrast to (6.1.15) the Lax connection (6.1.1) is not directly invariant under the $\mathbb{Z}_4$–transformations (5.1.15), (5.1.16) and (5.1.17). In particular, its first (α_1–dependent) term acquires the shift $-2e^A\Omega(gP_Ag^{-1}|_{\vartheta=0})$. To get back Λ in its initial form the $\mathbb{Z}_4$–transformed Lax connection

$$\Omega(\Lambda(\mathtt{x})) = \Omega^{-1}\Lambda(\frac{1}{\mathtt{x}})\,\Omega$$

should undergo a compensating gauge transformation $\mathcal{G}_\Omega$ and one finds

$$\Lambda = \mathcal{G}_\Omega\, \Omega(\Lambda)\, \mathcal{G}_\Omega^{-1} - \mathcal{G}_\Omega d\mathcal{G}_\Omega^{-1}, \qquad \text{where} \qquad \mathcal{G}_\Omega = \mathcal{G}_{\mathbf{x}}\Omega^{-1}\mathcal{G}_{\mathbf{x}}^{-1}\Omega\,,$$

$\mathcal{G}_{\mathbf{x}}(X,\Theta)$ is the same as in (6.1.16) and Λ is evaluated at $X^M + i\upsilon\Gamma^M\vartheta$. Of course, this gauge transformation, which also affects the spectral parameter $\mathtt{x}$, is nothing but a different form of the relation (6.1.16) taking into account the $\mathbb{Z}_4$–invariance of L.

6.2 Integrability of the theory up to the second order in υ

Now we shall generalize this to all orders in ϑ.

Since the computation in the two cases, $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, present some subtleties and differences we will first proceed in analyzing the $AdS_4 \times CP^3$ case, and then we will move to consider the discrepancies in the $AdS_2 \times S^2 \times T^6$ case. This can also be consider as a tribute to the way to proceed that during the last year we used in our research.

6.2.1 $AdS_4 \times CP^3$

As we saw in the expansion to the second oreder in all fermions the curvature of the coset Lax connection is not zero due to the terms in the equations of motion that depend on the non- supercoset fermions. First of all we can analyze the situation at the first order in non supersymmetric fermions. The curvature of the coset Lax connection is:

$$\begin{aligned} dL_{coset} - L_{coset} \wedge L_{coset} \;=\; & -\alpha_2 \left[\nabla_I \left[2i\sqrt{-h}h^{IJ}\upsilon\Gamma^A E_J\right]\right. \\ & +2\frac{i}{R}\upsilon(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^A\gamma^5\mathcal{P}_6\Gamma^B E_I E_{JB}+ \\ & \left.+\,2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right]P_A + \\ & +\alpha_2\beta_1\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\left(2i\upsilon\,\Gamma^A E_I\right)\Gamma_A E_J+\right.\right. \\ & \left.\left.+\,\left(E_I^A\right)\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right]\right]+ \\ & -\alpha_2\beta_2\frac{i}{R}\left[Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_A E_J 2i\upsilon\,\Gamma^A E_I+\right.\right. \\ & \left.\left.+\,E_I^A\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right]\right] \end{aligned} \tag{6.2.18}$$

In writing the term proportional to the P_A generator we made use of this relation:

$$\begin{aligned} &(B^A+\upsilon\Gamma^A\gamma^5(1-\mathcal{P})\Psi)|_{\text{first order in }\upsilon} = \nabla * \hat{E}^A - iE\Gamma^A\Gamma^{11}E - 2iE\Gamma^A\Gamma^{11}\nabla\upsilon \\ &-\frac{2}{R}E^B\upsilon\Gamma^A\Gamma^{11}\mathcal{P}\gamma^5\Gamma_B E - \frac{2}{R}*E^B\upsilon\Gamma^A\mathcal{P}\gamma^5\Gamma_B E + \frac{2}{R}E^B\upsilon\Gamma^A\mathcal{P}\gamma^5\Gamma_B\Gamma^{11}E\,, \end{aligned} \tag{6.2.19}$$

we are allowed to use a combination of bosonic and fermionic equations of motion, since we are going to ask for the curvature to vanish on the mass shell.

Let us see that it is sufficient, for ensuring zero curvature, to generalize the extra pieces that were found in the $O(\Theta^2)$ case (see equation (6.1.15)), introducing the full supercoset supervielbeins.

$$L = L_{coset} + \alpha_2 * \left(2iv\Gamma^A E\right) P_A + \alpha_2 \left(2iv\Gamma^{11}\Gamma^A E\right) P_A \tag{6.2.20}$$

Then the curvarture takes the form:

$$\begin{aligned} dL - L\wedge L \;=\; & -\alpha_2\left[2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^A\gamma^5\mathcal{P}_6\Gamma^B E_I E_{JB}\right]P_A + \\ & +\alpha_2\beta_1\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\left(E_I^A\right)\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right]\right] + \\ & +\alpha_2\beta_2\frac{i}{R}\left[Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[E_I^A\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right]\right] + \\ & +(\alpha_1\alpha_2+\alpha_2)\left[\left(iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^A E_J\right)E_I^C\right]R_{AC}{}^{DE}M_{DE} + \\ & +\alpha_2^2\left[\left(iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^A E_J\right)E_I^C\right]R_{AC}{}^{DE}M_{DE} \end{aligned} \tag{6.2.21}$$

To cancel the terms containing the coefficients β_1 and β_2 we include futher terms in the Lax connection:

$$\begin{aligned} L \;=\; & L_{coset} + \alpha_2 * \left(2iv\Gamma^A E\right) P_A + \alpha_2\left(2iv\Gamma^{11}\Gamma^A E\right)P_A + \alpha_2\beta_1\frac{i}{R}Q\gamma^5\Gamma^A v E_A + \\ & -\alpha_2\beta_1\frac{i}{R} * Q\gamma^7\Gamma^A v E_A + \alpha_2\beta_2\frac{i}{R} * Q\gamma^5\Gamma^A v E_A - \alpha_2\beta_2\frac{i}{R}Q\gamma^7\Gamma^A v E_A \end{aligned} \tag{6.2.22}$$

and the curvature becomes:

$$\begin{aligned} & dL - L\wedge L = -\alpha_2\left[2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^A\mathcal{P}_6\gamma^5\Gamma^B E_I E_{JB}\right]P_A + \\ & +(\alpha_1\alpha_2+\alpha_2)\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^A E_J\right)E_I^C\right]R_{AC}{}^{DE}M_{DE} + \\ & +\alpha_2^2\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^A E_J\right)E_I^C\right]R_{AC}{}^{DE}M_{DE} \\ & +\left(\alpha_2\beta_1^2-\alpha_2\beta_2(\beta_2+1)+\alpha_2\beta_2\right)\left[2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^B\mathcal{P}_6\gamma^5\Gamma^A E_I E_{JB}\right]P_A + \\ & +(\alpha_2\beta_1\beta_2-\alpha_2\beta_1(\beta_2+1)+\alpha_2\beta_1)\left[2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^B\mathcal{P}_6\gamma^7\Gamma^A E_I E_{JB}\right]P_A + \end{aligned}$$

$$
\begin{aligned}
&+\left(\alpha_2\beta_1^2+\alpha_2\beta_2(\beta_2+1)-\alpha_2\beta_2\right)\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\Gamma^{BC}E_J\right)E_I^A\right]R_{BC}{}^{DE}M_{DE}\\
&+\left(\alpha_2\beta_1\beta_2+\alpha_2\beta_1(\beta_2+1)-\alpha_2\beta_1\right)\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\Gamma^{11}\Gamma^{BC}E_J\right)E_I^A\right]R_{BC}{}^{DE}M_{DE}\\
&=-\left(\alpha_1\alpha_2+\alpha_2\right)\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{BC}\Gamma_AE_J\right)E_I^A\right]R_{BC}{}^{DE}M_{DE}\\
&-\alpha_2^2\left[\left(2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^{BC}\Gamma_AE_J\right)E_I^A\right]R_{BC}{}^{DE}M_{DE}=\mathcal{O}(v^2)
\end{aligned}
\tag{6.2.23}
$$

In the last passage we have used the fermionic equation of motion (4.3.24) that ensures us that the remaining contribution is of the second order in v.

At least at this order it was not necessary to introduce terms with a different structure with respect to those found in the previous section at the quadratic order in Θ.

Let us see, if adding terms of the second order in v to the full supercoset Lax connection, this is still true and let us see if it is possible to find which is the form of the Lax connection up to the second order in v.
What is left from the first order computation is:

$$
\begin{aligned}
dL^{(1)}-L^{(1)}\wedge L^{(1)} \quad &=-\alpha_2\left[\nabla_I\left(iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^A\left(\nabla_Jv+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_Bv E_J^B\right)\right)\right.\\
&+\frac{i}{4}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma_D{}^{BC}vR_{BCD}{}^AE_I^BE_J^D\\
&\left.+\frac{4}{R}v\sqrt{-h}h^{IJ}\Gamma^A\mathcal{P}_6\gamma^5\Gamma^BE_Iv\Gamma^{11}\Gamma_BE_J\right]P_A+\\
&+\alpha_2\beta_1\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_AE_Jiv\,\Gamma^A\left(\nabla_Iv+\frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^Bv E_{IB}\right)+\right.\right.\\
&\Gamma_A\left(\nabla_Jv+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^Bv E_{JB}\right)2iv\Gamma^AE_I+\\
&\left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_Jv\gamma^5v+\frac{1}{R}\mathcal{P}_6\Gamma_B\Gamma^{11}E_Jv\Gamma^B\gamma^7v+\frac{2}{R}\mathcal{P}_6\gamma^5v_iv^iE_J\right)\right]+\right.\\
&+\alpha_2\beta_2\frac{i}{R}\left[Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_AE_Jiv\,\Gamma^A\left(\nabla_Iv+\frac{i}{R}\mathcal{P}_2\gamma^5\Gamma^Bv E_{IB}\right)+\right.\right.\\
&\Gamma_A\left(\nabla_Jv+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^Bv E_{JB}\right)2iv\Gamma^AE_I+\\
&\left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_Jv\gamma^5v+\frac{1}{R}\mathcal{P}_6\Gamma_B\Gamma^{11}E_Jv\Gamma^B\gamma^7v+\frac{2}{R}\mathcal{P}_6\gamma^5v_iv^iE_J\right)\right]+\right.\\
&-\left(\alpha_1\alpha_2+\alpha_2\right)\left[2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{BC}\left(\Gamma_AE_J2iv\,\Gamma^AE_I+\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)E_I^A\Big)\Big]\,R_{BC}{}^{DE}M_{DE}\\
&-\alpha_2^2\Big[2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^{BC}\left(\Gamma_AE_J2iv\,\Gamma^AE_I+\right.\\
&\left.\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)E_I^A\right)\Big]\,R_{BC}{}^{DE}M_{DE}\\
&-\frac{1}{4R}\alpha_2^2v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma_A\mathcal{P}_6\gamma^5\Gamma^{CD}\mathcal{P}_6\Gamma_B\upsilon R_{CD}{}^{EF}M_{EF}\\
&+\alpha_2^24\epsilon^{IJ}v\Gamma^AE_Iv\Gamma^B\Gamma^{11}E_JR_{AB}{}^{CD}M_{CD}\\
&+(\alpha_2^2\beta_1-\alpha_2^2\beta_2)\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^AE_I\right.\\
&\left.+\,Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^AE_I\right]\\
&-\alpha_2\beta_1Q\gamma^7\Gamma^Av\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}v\Gamma^AE_J\right]\right.\\
&+2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^{[A}\gamma^5\Gamma^{B]}E_IE_{JB}+\\
&\left.+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right)+\\
&-\alpha_2\beta_2Q\gamma^5\Gamma^Av\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}v\Gamma^AE_J\right]\right.\\
&+2\frac{i}{R}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^{[A}\gamma^5\Gamma^{B]}E_IE_{JB}+\\
&\left.+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\right)
\end{aligned}
\tag{6.2.24}
$$

This time we used

$$
\begin{aligned}
(B^A+v\Gamma^A\gamma^5(1-\mathcal{P})\Psi-\frac{c}{R}v\gamma^5vB^{a'}\delta^A_{a'})|_{\text{second order in }\upsilon}&=\nabla*\hat{E}^A-iE\Gamma^A\Gamma^{11}E-2iE\Gamma^A\Gamma^{11}\nabla\upsilon\\
&-\frac{2}{R}E^Bv\Gamma^A\Gamma^{11}\mathcal{P}\gamma^5\Gamma_BE-\frac{2}{R}*E^Bv\Gamma^A\mathcal{P}\gamma^5\Gamma_BE+\frac{2}{R}E^Bv\Gamma^A\mathcal{P}\gamma^5\Gamma_B\Gamma^{11}E\\
\nabla*\left(iv\Gamma^A\left(\nabla_J\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_Bv\upsilon E_J^B\right)\right)&+\nabla\left(iv\Gamma^A\Gamma^{11}\left(\nabla v+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_Bv\upsilon E^B\right)\right)\\
+\frac{i}{4}v\Gamma_D{}^{BC}vR_{BCD}{}^AE^B*E^D+\frac{i}{4}v\Gamma^{11}\Gamma_D{}^{BC}vR_{BCD}{}^AE^B*E^D&+\frac{4}{R}v\Gamma^A\mathcal{P}_6\gamma^5\Gamma^BE*v\Gamma^{11}\Gamma_BE.
\end{aligned}
\tag{6.2.25}
$$

We have to add $O(v^2)$-terms to the Lax connection:

$$L = L^{(1)} + \alpha_2 * iv\Gamma^A \left(\nabla v + 2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B v E^B\right) P_A - \alpha_2 iv\Gamma^{11}\Gamma^A \left(\nabla v + 2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B v E^B\right) P_A\,, \tag{6.2.26}$$

that gives

$$\begin{aligned}
dL - L\wedge L \;=\; & -\alpha_2\left[\frac{i}{4}v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma_D{}^{BC}vR_{BCD}{}^A E_I^B E_J^D\right.\\
& \left.+\frac{4}{R}v(\sqrt{-h}h^{IJ})\Gamma^A\mathcal{P}_6\gamma^5\Gamma^B E_I v\Gamma^{11}\Gamma_B E_J\right]P_A +\\
& +\alpha_2\beta_1\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^A E_I+\right.\right.\\
& \left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_J v\gamma^5 v+\frac{1}{R}\mathcal{P}_6\Gamma_B\Gamma^{11}E_J v\Gamma^B\gamma^7 v+\frac{2}{R}\mathcal{P}_6\gamma^5 v_i v^i E_J\right)\right]+\right.\\
& -\alpha_2\beta_2\frac{i}{R}\left[Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^A E_I+\right.\right.\\
& \left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_J v\gamma^5 v+\frac{1}{R}\mathcal{P}_6\Gamma_B\Gamma^{11}E_J v\Gamma^B\gamma^7 v+\frac{2}{R}\mathcal{P}_6\gamma^5 v_i v^i E_J\right)\right]+\right.\\
& +\frac{1}{2}\left(\alpha_1\alpha_2+\alpha_2\right)\left[2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{BC}\left(\Gamma_A E_J 2iv\,\Gamma^A E_I+\right.\right.\\
& \left.\left.+\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^E\upsilon E_{JE}\right)E_I^A\right)\right]R_{BC}{}^{DE}M_{DE}\\
& +\frac{1}{2}\alpha_2^2\left[2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^{BC}\left(\Gamma_A E_J 2iv\,\Gamma^A E_I+\right.\right.\\
& \left.\left.+\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^E\upsilon E_{JE}\right)E_I^A\right)\right]R_{BC}{}^{DE}M_{DE}\\
& +\alpha_2^2\beta_2\frac{i}{R}Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}(1+\mathcal{P}_6)\Gamma^B\gamma^5\upsilon E_{JB}\right)2iv\Gamma^A E_I\\
& +\alpha_2^2\beta_1\frac{i}{R}Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}(1+\mathcal{P}_6)\Gamma^B\gamma^5\upsilon E_{JB}\right)2iv\Gamma^A E_I\\
& -\frac{1}{4R}\alpha_2^2 v(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma_A\mathcal{P}_6\gamma^5\Gamma^{CD}\mathcal{P}_6\Gamma_B v R_{CD}{}^{EF}M_{EF}\\
& +\alpha_2^2 4\epsilon^{IJ}v\Gamma^A E_I v\Gamma^B\Gamma^{11}E_J R_{AB}{}^{CD}M_{CD}\\
& -\alpha_2\beta_1 Q\gamma^7\Gamma_A v\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}\upsilon\Gamma^A E_J\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{i}{R}\upsilon(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^{[A}\gamma^5\Gamma^{B]}E_IE_{JB}+\\
&+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\Bigg)+\\
&-\alpha_2\beta_2Q\gamma^5\Gamma_A v\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}\upsilon\Gamma^AE_J\right]\right.\\
&+2\frac{i}{R}\upsilon(\sqrt{-h}h^{IJ}+\epsilon^{IJ}\Gamma^{11})\Gamma^{[A}\gamma^5\Gamma^{B]}E_IE_{JB}+\\
&+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)\Bigg)\\
&-\frac{1}{2}\alpha_1(1+\alpha_2)2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^A\left(\nabla_J\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E_J^B\right)E_I^CR_{AC}{}^{DE}M_{DE}\\
&-\frac{1}{2}\alpha_2^2 2iv(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma^{11}\Gamma^A\left(\nabla_J\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E_J^B\right)E_I^CR_{AC}{}^{DE}M_{DE}\\
&+2\alpha_2\beta_2\epsilon^{IJ}Q\gamma^7\Gamma_AE_Jiv\Gamma^A\nabla_I\upsilon+2\alpha_2\beta_1\epsilon^{IJ}Q\gamma^5\Gamma_AE_Jiv\Gamma^A\nabla_I\upsilon\,. \qquad (6.2.27)
\end{aligned}
$$

The structure of (6.2.27) prompts us to add to the Lax connection more terms which have already been obtained at the second order (see eq. (6.1.16)):

$$
\begin{aligned}
L \;=\;& L^{(1)}+\alpha_2*iv\Gamma^A\left(\nabla\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E^B\right)-\alpha_2iv\Gamma^{11}\Gamma^A\left(\nabla\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E^B\right)+\\
&-(\alpha_2+\alpha_1\alpha_2)\frac{i}{8}*\upsilon E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}+(\alpha_2+\alpha_1\alpha_2)\frac{i}{8}\upsilon\Gamma^{11}E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}+\\
&+\alpha_2^2\frac{i}{8}*\upsilon\Gamma^{11}E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}-\alpha_2^2\frac{i}{8}\upsilon E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}\,, \qquad (6.2.28)
\end{aligned}
$$

that gives the curvature:

$$
\begin{aligned}
dL-L\wedge L \;=\;& -\alpha_2\left[\frac{4}{R}\upsilon\sqrt{-h}h^{IJ}\Gamma^A\mathcal{P}_6\gamma^5\Gamma^BE_I\upsilon\Gamma^{11}\Gamma_BE_J\right]P_A+\\
&+\alpha_2\beta_1\frac{i}{R}\left[Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^AE_I+\right.\right.\\
&\left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_J\upsilon\gamma^5\upsilon+\frac{1}{R}\mathcal{P}_6\Gamma_B\Gamma^{11}E_J\upsilon\Gamma^B\gamma^7\upsilon+\frac{2}{R}\mathcal{P}_6\gamma^5v_iv^iE_J\right)\right]\right]+\\
&-\alpha_2\beta_2\frac{i}{R}\left[Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\left[\Gamma_A\left(\nabla_J\upsilon+\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma^B\upsilon E_{JB}\right)2iv\Gamma^AE_I+\right.\right.\\
&\left.\left.+E_I^A\Gamma_A\left(-\frac{1}{2R}E_J\upsilon\gamma^5\upsilon+\frac{1}{R}\mathcal{P}_6\Gamma_A\Gamma^{11}E_J\upsilon\Gamma^A\gamma^7\upsilon+\frac{2}{R}\mathcal{P}_6\gamma^5v_iv^iE_J\right)\right]\right]+
\end{aligned}
$$

$$
\begin{aligned}
&-(\alpha_1\alpha_2+\alpha_2)\left[iv(\sqrt{-h}h^{IJ})\Gamma_A\Gamma^{BC}\left(E_J 2iv\,\Gamma^A E_I\right)\right]R_{BC}{}^{DE}M_{DE}\\
&-\alpha_2^2\left[iv(\sqrt{-h}h^{IJ})\Gamma^{11}\Gamma_A\Gamma^{BC}E_J 2iv\,\Gamma^A E_I\right]R_{BC}{}^{DE}M_{DE}\\
&+\alpha_2^2\beta_2\frac{i}{R}Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}(1+\mathcal{P}_6)\Gamma^B\gamma^5 v E_{JB}\right)2iv\Gamma^A E_I\\
&+\alpha_2^2\beta_1 Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}(1+\mathcal{P}_6)\Gamma^B\gamma^5 v E_{JB}\right)2iv\Gamma^A E_I\\
&+\alpha_2^2\beta_1\frac{R}{8}(E^C v(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_C{}^{DE}\Gamma^{11}v)R_{DE}{}^{AB}Q\Gamma_{AB}\Gamma^{11}E\\
&+\alpha_2^2\beta_2\frac{R}{8}(E^C v(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_C{}^{DE}\Gamma^{11}v)R_{DE}{}^{AB}Q\Gamma_{AB}E\\
&+\alpha_2\beta_1 Q\gamma^7\Gamma_A v\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}v\Gamma^A E_J\right]+2\frac{1}{R}v(\sqrt{-h}h^{IJ})\Gamma^{[A}\gamma^5\Gamma^{B]}E_I E_{JB}+\right.\\
&\left.+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J v\right)\right)+\\
&+\alpha_2\beta_2 Q\gamma^5\Gamma_A v\left(\nabla_I\left[2i\sqrt{-h}h^{IJ}v\Gamma^A E_J\right]+2\frac{1}{R}v(\sqrt{-h}h^{IJ})\Gamma^{[A}\gamma^5\Gamma^{B]}E_I E_{JB}+\right.\\
&\left.+2i\epsilon^{IJ}E_I\Gamma^A\Gamma^{11}\left(\nabla_J v\right)\right)\\
&+2\alpha_2\beta_2\epsilon^{IJ}Q\gamma^7\Gamma_A E_J iv\Gamma^A\nabla_I v+2\alpha_2\beta_1\epsilon^{IJ}Q\gamma^5\Gamma_A E_J iv\Gamma^A\nabla_I v
\end{aligned}
\tag{6.2.29}
$$

What has been done is not sufficient to ensure a zero curvature Lax connection, we have to do a step further, adding contributions that are, at list, of the third order in all the fermions. Such terms have not appeared in the second order Lax connection of equation (6.1.16).

Using the Fierz identieties, with a bit of algebra one can simplify (6.2.29) and get:

$$
\begin{aligned}
dL-L\wedge L \;=\;& -\alpha_2\left[\frac{4}{R}v\sqrt{-h}h^{IJ}\Gamma^A\mathcal{P}_6\gamma^5\Gamma^B E_I v\Gamma^{11}\Gamma_B E_J\right]P_A+\\
&-\frac{1}{2}(\alpha_1\alpha_2+\alpha_2)\left[iv(\sqrt{-h}h^{IJ})\Gamma_A\Gamma^{BC}E_J 2iv\,\Gamma^A E_I\right]R_{BC}{}^{DE}M_{DE}\\
&-\frac{1}{2}\alpha_2^2\left[iv(\sqrt{-h}h^{IJ})\Gamma^{11}\Gamma_A\Gamma^{BC}E_J 2iv\,\Gamma^A E_I\right]R_{BC}{}^{DE}M_{DE}\\
&+\alpha_2\beta_1\frac{i}{R}\nabla_J\left[Q\gamma^7(\sqrt{-h}h^{IJ})\Gamma_A v 2iv\Gamma^A E_I\right]+\\
&-\alpha_2\beta_2\frac{i}{R}\nabla_J\left[Q\gamma^5(\sqrt{-h}h^{IJ})\Gamma_A v 2iv\Gamma^A E_I\right]+\\
&+\alpha_2\beta_1\frac{i}{R}Q\gamma^7(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}\mathcal{P}_6\Gamma^B\gamma^5 v E_{IB}\right)2iv\Gamma^A E_J
\end{aligned}
$$

$$-\alpha_2\beta_2 Q\gamma^5(\sqrt{-h}h^{IJ}-\epsilon^{IJ}\Gamma^{11})\Gamma_A\left(\frac{i}{R}\mathcal{P}_6\Gamma^B\gamma^5\upsilon E_{IB}\right)2i\upsilon\Gamma^A E_J \tag{6.2.30}$$

In order to cancel these terms we can introduce terms of the form $\upsilon\Gamma^A\gamma^5 Q 2i\upsilon\Gamma_A * E$ and $\upsilon\Gamma^A\gamma^7 Q 2i\upsilon\Gamma_A * E$.
The resulting zero curvature $O(v^2)$-Lax connection is:

$$\begin{aligned}
L &= (\alpha_1+1)E^A P_A+\alpha_2 * E^A P_A+\beta_1 Q\Gamma_{11}E+\beta_2 QE+\Omega^{AB}M_{AB}\\
&+\alpha_2 * \left(2i\upsilon\Gamma^A E\right)P_A+\alpha_2\left(2i\upsilon\Gamma^{11}\Gamma^A E\right)P_A+\alpha_2\beta_1\frac{i}{R}Q\gamma^5\Gamma^A\upsilon E_A+\\
&-\alpha_2\beta_1\frac{i}{R}*Q\gamma^7\Gamma^A\upsilon E_A+\alpha_2\beta_2\frac{i}{R}*Q\gamma^5\Gamma^A\upsilon E_A-\alpha_2\beta_2\frac{i}{R}Q\gamma^7\Gamma^A\upsilon E_A+\\
&+\alpha_2 * i\upsilon\Gamma^A\left(\nabla\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E^B\right)P_A-\alpha_2 i\upsilon\Gamma^{11}\Gamma^A\left(\nabla\upsilon+2\frac{i}{R}\mathcal{P}_6\gamma^5\Gamma_B\upsilon E^B\right)P_A+\\
&-(\alpha_2+\alpha_1\alpha_2)\frac{i}{8}*\upsilon E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}+(\alpha_2+\alpha_1\alpha_2)\frac{i}{8}\upsilon\Gamma^{11}E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}+\\
&+\alpha_2^2\frac{i}{8}*\upsilon\Gamma^{11}E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}-\alpha_2^2\frac{i}{8}\upsilon E^A\Gamma_A{}^{BC}\upsilon R_{BC}{}^{DE}M_{DE}+\\
&-\alpha_2\beta_1\frac{i}{R}*Q\gamma^7\Gamma^A\upsilon 2i\upsilon\Gamma_A E+\alpha_2\beta_2\frac{i}{R}*Q\gamma^5\Gamma^A\upsilon 2i\upsilon\Gamma_A E
\end{aligned} \tag{6.2.31}$$

Summarizing the whole process we can rewrite the curvature of the Lax connection as:

$$\begin{aligned}
dL-LL &= \frac{\alpha_2}{c^3}\Big[\frac{1}{4}(\upsilon(V^\dagger)^2\Gamma^{CD}\Psi+\frac{ic}{2}\mathcal{B}^A\,\upsilon\Gamma_A{}^{CD}V^2\upsilon)\,R_{CD}{}^{EF}M_{EF}\\
&+c\mathcal{B}^A\,P_A-\frac{2i}{R}\upsilon\gamma_5\Gamma^a\Psi\,P_a-\frac{c}{R}\mathcal{B}^{a'}\,\upsilon\gamma_5\upsilon\,P_{a'}\\
&-\frac{1}{R}QV^\dagger\gamma_5\Psi-\frac{1}{2R^2}QV^\dagger\Psi\,\upsilon\upsilon-\frac{1}{2R^2}QV^\dagger\gamma_5\Gamma_a\Psi\,\upsilon\Gamma_a\gamma_5\upsilon-\frac{1}{R^2}QV^\dagger\gamma_7\Gamma_a\Psi\,\upsilon\Gamma^a\gamma^7\upsilon\\
&+\frac{3\alpha_2}{2R^2}(QV\gamma^7\Gamma_a\Psi\,\upsilon\Gamma^a\gamma^5\upsilon+QV\gamma^5\Gamma_a\Psi\,\upsilon\Gamma^a\gamma^7\upsilon)-\frac{ic}{R}\mathcal{B}^A\,QV^\dagger\gamma_*\Gamma_A\upsilon\Big],
\end{aligned} \tag{6.2.32}$$

where we used the coefficients defined in (5.0.8). As in the coset case, the curvature is proportional to the equations of motions and, hence, vanishes on the mass shell.

6.2.2 $AdS_2\times S^2\times T^6$

A similar construction can be carried out in the $AdS_2\times S^2\times T^6$ case, paying attention to the fact that the role of the projectors this time is interchanged and that γ^* com-

mutes with six dimensional gamma matrices and with the gamma matrices of AdS_2 but anticommutes with the matrices in S^2.

The Lax connection in this case has the form:

$$\begin{aligned}
L &= (\alpha_1+1)E^A P_A + \alpha_2 * E^A P_A + \beta_1 Q\Gamma_{11}E + \beta_2 QE + \Omega^{AB}M_{AB} \\
&+\alpha_2 * \left(2iv\Gamma^A E\right) P_A - \alpha_2\left(2iv\Gamma^{11}\Gamma^A E\right) P_A - \alpha_2\beta_1\frac{i}{R}Q\gamma^5\Gamma^A v\mathcal{E}_A + \\
&-\alpha_2\beta_1\frac{i}{R} * Q\gamma^7\Gamma^A v\mathcal{E}_A - \alpha_2\beta_2\frac{i}{R} * Q\gamma^*\Gamma^A v\mathcal{E}_A - \alpha_2\beta_2\frac{i}{R}Q\gamma^*\Gamma^{11}\Gamma^A v\mathcal{E}_A + \\
&+\alpha_2 * iv\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}_6\gamma^*\Gamma_B v E^B\right) P_A - \alpha_2 iv\Gamma^{11}\Gamma^A\left(\nabla v + 2\frac{i}{R}\mathcal{P}_6\gamma^*\Gamma_B v E^B\right) P_A + \\
&-\alpha_2\beta_1\frac{i}{R}Q\gamma^*\Gamma^A v 2iv\Gamma_A E + \alpha_2\beta_2\frac{i}{R}Q\gamma^*\Gamma^{11}\Gamma^A v 2iv\Gamma_A E \\
&-(\alpha_2+\alpha_1\alpha_2)\frac{i}{8} * vE^A\Gamma_A{}^{BC}vR_{BC}{}^{DE}M_{DE} - (\alpha_2+\alpha_1\alpha_2)\frac{i}{8}v\Gamma^{11}E^A\Gamma_A{}^{BC}vR_{BC}{}^{DE}M_{DE} + \\
&+\alpha_2^2\frac{i}{8} * v\Gamma^{11}E^A\Gamma_A{}^{BC}vR_{BC}{}^{DE}M_{DE} + \alpha_2^2\frac{i}{8}vE^A\Gamma_A{}^{BC}vR_{BC}{}^{DE}M_{DE}
\end{aligned} \tag{6.2.33}$$

If we look at the curvature of the Lax connection this time we have a slightly different structure:

$$\begin{aligned}
dL - LL &= \frac{\alpha_2}{c^3}\Big[\frac{1}{4}(\upsilon(V^\dagger)^2\Gamma^{\underline{cd}}\Psi + \frac{ic}{2}\mathcal{B}^{\underline{a}}\upsilon\Gamma_{\underline{a}}{}^{\underline{cd}}V^2\upsilon)\, R_{\underline{cd}}{}^{\underline{ef}}M_{\underline{ef}} + c\mathcal{B}^A P_A \\
&+ \frac{2i}{R}\upsilon\gamma_\star\Gamma^A(1-\mathcal{P})\Psi\, P_A + \frac{c}{R}\mathcal{B}^B\,\upsilon\Gamma^A{}_B\gamma_\star\upsilon\, P_A + \frac{c}{R}\mathcal{B}^B\,\upsilon\Gamma^A\mathcal{P}\Gamma_B\gamma_\star\upsilon\, P_A \\
&- \frac{1}{R}QV^\dagger\gamma_\star\Psi - \frac{ic}{R}\mathcal{B}^A\, QV^\dagger\gamma_\star\Gamma_A\upsilon - \frac{1}{4R^2}QV^\dagger\gamma_\star\Gamma_{BC}\Psi\,\upsilon\Gamma^{BC}\gamma_\star\upsilon \\
&- \frac{1}{2R^2}QV^\dagger\gamma_\star\gamma_7\Gamma_B\Psi\,\upsilon\Gamma^B\gamma_\star\gamma_7\upsilon - \frac{1}{2R^2}QV^\dagger\gamma_\star\Gamma_B\Gamma_{11}\Psi\,\upsilon\Gamma^B\gamma_\star\Gamma_{11}\upsilon \\
&- \frac{1}{2R^2}QV^\dagger\gamma_\star\Gamma_{\underline{a}b'}\gamma_5\gamma_\star\Psi\,\upsilon\Gamma^{\underline{a}b'}\gamma_5\upsilon - \frac{1}{4R^2}QV^\dagger\gamma_\star\Gamma_{\underline{abc'}}\gamma_\star\Psi\,\upsilon\Gamma^{\underline{abc'}}\upsilon \\
&+ \frac{\alpha_2}{2R^2}\Big(\upsilon\Gamma^{\underline{a}}\gamma^5\upsilon\, QV\gamma^7\gamma_\star\Gamma_{\underline{a}}\gamma_\star\Psi + \upsilon\Gamma^{\underline{a}}\gamma^7\upsilon\, QV\gamma^5\gamma_\star\Gamma_{\underline{a}}\gamma_\star\Psi \\
&+ 2\upsilon\Gamma^{a'}\gamma_\star\Gamma_{11}\upsilon\, QV\Gamma_{a'}\gamma_\star\Psi + 2\upsilon\Gamma^{a'}\gamma_\star\upsilon\, QV\Gamma^{11}\Gamma_{a'}\gamma_\star\Psi\Big)\Big] + \mathcal{O}(\upsilon^3)\,,
\end{aligned} \tag{6.2.34}$$

Also in this case we have that the curvature vanishes on shell.

6.3 Some properties of the Lax connection

We have thus found that for the two theories, that we are studying, the Lax connections have the same form and can be summarize as follows:

$$L = L_{coset}(X, \vartheta) + \alpha_2 L'(X, \vartheta, \upsilon)\,, \tag{6.3.35}$$

with

$$\begin{aligned} L' \;=\; & -\frac{i}{R} Q\gamma_\star \left[*(E^A + 2i\upsilon\Gamma^A E)\,\Gamma_A V\upsilon - (E^A + 2i\upsilon\Gamma^A E)\,\Gamma_A\Gamma_{11} V\upsilon\right] \\ & -\frac{i}{R} Q\gamma_\star \left[i(\upsilon\Gamma^A\Gamma_{11}E)\,\Gamma_A V\upsilon + i(\upsilon\Gamma^A E)\,\Gamma_A\Gamma_{11} V\upsilon\right] \\ & + (2i\upsilon\Gamma^A * E + i\upsilon\Gamma^A * \nabla\upsilon - \frac{2}{R} * E^B\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon)\,P_A \\ & + (2i\upsilon\Gamma^A\Gamma_{11}E + i\upsilon\Gamma^A\Gamma_{11}\nabla\upsilon - \frac{2}{R}E^B\,\upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B\upsilon)\,P_A \\ & + \frac{i}{8}(*E^C\,\upsilon\Gamma_C{}^{DE}V^2\upsilon - E^C\,\upsilon\Gamma_C{}^{DE}\Gamma_{11}V^2\upsilon)\,R_{DE}{}^{AB}\,M_{AB}\,, \end{aligned} \tag{6.3.36}$$

6.3.1 $\mathbb{Z}_4$–invariance

The Lax connection (6.3.35) with L_{coset} and L' given, respectively, in (5.0.5) and (6.3.36) is invariant under the $\mathbb{Z}_4$–transformations (5.1.15) of the isometry generators and the inversion of the spectral parameter (5.1.16) and (5.1.17). In fact performing a $\mathbb{Z}_4$-transformation we have:

$$L_{coset} \to L_{coset}\,, \quad L' \to -L'\,, \quad \alpha_2 \to -\alpha_2\,. \tag{6.3.37}$$

This demonstrates that the contribution of the non–coset fermions does not spoil the $\mathbb{Z}_4$–symmetry of the *supercoset* Lax connection which is of crucial importance for the derivation of the algebraic curve and the Bethe ansatz equations, both classical and quantum [59, 60, 62, 61]. The $\mathbb{Z}_4$–invariance induces the corresponding conjugation symmetry of the monodromy matrix (1.3.13) of the Lax connection

$$\Omega^{-1}\,M(1/\mathtt{x})\,\Omega = M(\mathtt{x})$$

used for the construction of the algebraic curve.

6.3.2 The Lax connection and conserved currents

Let us now present an interesting observation: how the Lax connection $L = L_{coset} + \alpha_2 L'$ can be related to the conserved current associated with the superisometries. First of all notice that in the limit $\alpha_2 = \epsilon \to 0$, in which

$$\alpha_1 = \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^4)\,, \qquad \beta_1 = -\frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2)\,, \qquad \beta_2 = 1 + \mathcal{O}(\epsilon^2)\,, \tag{6.3.38}$$

and

$$V = \beta_2 + \beta_1\Gamma_{11} \to 1, \qquad V^\dagger = \beta_2 - \beta_1\Gamma_{11} \to 1\,,$$

the Lax connection reduces to

$$L = K + \mathcal{O}(\epsilon)\,, \tag{6.3.39}$$

where $K(X,\vartheta)$ is the supercoset Cartan form introduced in (4.2.22).

In fact, the term denoted by $\mathcal{O}(\epsilon)$ in the (gauge–transformed) Lax connection is the worldsheet Hodge dual of a superstring conserved current J associated with the background superisometries, namely

$$*J = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(gLg^{-1} - dgg^{-1}\right) = g\lim_{\epsilon\to 0}\frac{L-K}{\epsilon}g^{-1}\,, \tag{6.3.40}$$

where $g(X,\vartheta)$ is the superisometry group element determining K in (4.2.22).

The conservation of J, i.e. $d*J = 0$, follows from the flatness of the Lax connection and the Cartan form

$$dL - LL = 0\,, \qquad dK - KK = 0\,. \tag{6.3.41}$$

Indeed, in view of (6.3.39), we have

$$\begin{aligned} d*J &= g\lim_{\epsilon\to 0}\frac{dL-dK}{\epsilon}g^{-1} - *JgKg^{-1} - gKg^{-1}*J \\ &= g\lim_{\epsilon\to 0}\frac{(L-K)K + K(L-K)}{\epsilon}g^{-1} - *JgKg^{-1} - gKg^{-1}*J = 0 \end{aligned} \tag{6.3.42}$$

Note that in the case of the supercoset Lax connection (5.0.5) (when $\upsilon = 0$), the current constructed in this way coincides with the conserved current of the G/H sigma–model considered in [20, 24, 26]

$$J_{coset} = g(E^A P_A - \frac{1}{2}Q\Gamma_{11} * E)g^{-1}\,. \tag{6.3.43}$$

We can now write the correction (6.3.36) to the Lax connection in terms of (V-transformed) components of the conserved current as

$$L' = g^{-1} * (\tilde{J} - \tilde{J}_{coset})g\,, \tag{6.3.44}$$

where $\tilde{J}$ and $\tilde{J}_{coset}$ are, respectively, the complete conserved current (to second order in υ) (6.3.40) and the conserved current of the supercoset model (6.3.43), and the tilde means that in their expressions we perform the following substitutions of spinorial quantities

$$E \to V^\dagger E, \qquad \nabla\upsilon \to V^\dagger \nabla\upsilon \qquad \text{and} \qquad \upsilon \to V\upsilon.$$

For instance,

$$\tilde{J}_{coset} = g(E^A P_A - \frac{1}{2} Q\Gamma_{11} V^\dagger * E)g^{-1}\,. \tag{6.3.45}$$

Whether this fact is of some deeper significance remains to be understood. Perhaps, a better understanding of this could lead to a proposal for the complete Lax connection to all orders in the non–coset fermions. We leave this problem for future analysis.

6.3.3 Lax connection and kappa–symmetry

The Green–Schwarz formulation of the superstring is invariant under the local fermionic transformations (2.4.20) (2.4.21) of the target–space coordinates $Z^{\mathcal{M}} = (X^M, \Theta^\mu)$ where the Γ matrix that appears in the spinor projection matrix is:

$$\Gamma = \frac{1}{2\sqrt{-\det g_{IJ}}}\,\epsilon^{IJ}\,\mathcal{E}_I{}^A\,\mathcal{E}_J{}^B\,\Gamma_{AB}\,\Gamma_{11}, \qquad \Gamma^2 = 1\,, \tag{6.3.46}$$

and g_{IJ} is an induced worldsheet metric.

The string equations of motion (4.3.24) and (4.3.25) transform into each other under the kappa–symmetry variations. Since the condition for the Lax connection to have zero–curvature is in one to one correspondence with the equations of motion, it is natural to assume that on the mass–shell the Lax connection should be invariant under the kappa–symmetry transformations, at least, modulo a gauge transformation. This is indeed so in the case of the supercoset sigma–models (see e.g. [20]). The explicit check that also the non–coset Lax connection (6.3.35), (6.3.36) possesses this property would be somewhat cumbersome, but fortunately one should not do this, because there is a simple generic proof that makes this fact evident. Indeed, since any Lax curvature depends on the left–hand–sides of the equations of motion (as *e.g.* in (5.0.11) and (6.2.34)), its variation under (2.4.20) and (2.4.21) also depends on the field equations and hence vanishes on–shell. This means that kappa–variation of the Lax connection leaves its curvature zero and, therefore, the kappa–transformed Lax connection is related to the initial one by a corresponding infinitesimal gauge transformation taking values in the isometry superalgebra.

Chapter 7

Non perturbative solutions: Instantons

In the theories that we are studying also string instantons are present. The importance of looking at these solutions is that these can contribute as non perturbative corrections to the string effective action.

7.1 Instantons on $AdS_4 \times CP^3$

This kind of corrections may be relevant also in the study of AdS/CFT correspondence, in fact for the theory on $AdS_4 \times CP^3$ this effect seems to have a manifestation in dual field theory [64].

Usually, for simplicity, when analyzing fermionic zero modes of the instanton solutions, one first restricts the consideration to the second order in fermions and then tries to infer whether the solution persists to all orders. So to study the string instantons we take the quadratic action (4.2.5) in which we should perform a Wick rotation of the worldsheet and the target space to Euclidean signature. The Wick rotation basically consists in replacing $\sqrt{-h}$ and $\sqrt{-G}$, respectively with $\sqrt{h}$ and $\sqrt{G}$, replacing ε^{IJ} with $-i\,\varepsilon^{IJ}$ and taking into account that the fermions Θ become complex spinors, since there are no Majorana spinors in ten-dimensional Euclidean space. However, the complex conjugate spinors do not appear in the Wick rotated action and, hence, the number of the fermionic degrees of freedom formally remains the same as before the Wick rotation. Note also that the Euclidean γ^5 is defined as $\gamma^5 = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$, where γ^4 is the Wick rotated γ^0. So $(\gamma^5)^2 = 1$ as in the case of Minkowski signature.

Thus, after the redefinition (4.2.10) and the Wick rotation the action takes the

following form

$$\begin{aligned} S_E &= \frac{e^{\frac{2}{3}\phi_0}}{4\pi\alpha'}\int d^2\xi\,\sqrt{h}\,h^{IJ}\,\Big(e_I{}^a e_J{}^b\delta_{ab}+e_I{}^{a'}e_J{}^{b'}\,\delta_{a'b'}\Big) \qquad (7.1.1)\\ &+ \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'}\int d^2\xi\,\Theta(\sqrt{h}\,h^{IJ}+i\varepsilon^{IJ}\Gamma_{11})\big[i\,e_I{}^A\Gamma_A\nabla_J\Theta-\frac{1}{R}e_I{}^A e_J{}^B\Gamma_A\mathcal{P}_6\gamma^5\Gamma_B\Theta\big] \end{aligned}$$

and the kappa–symmetry matrix Γ gets replaced by

$$\Gamma=-\frac{i}{2\sqrt{\det G_{IJ}}}\,\varepsilon^{IJ}\,\mathcal{E}_I{}^A\,\mathcal{E}_J{}^B\,\Gamma_{AB}\,\Gamma_{11},\qquad \Gamma^2=1\,. \qquad (7.1.2)$$

7.1.1 String instanton wrapping a two–sphere inside CP^3

We are interested in a string whose worldsheet wraps a topologically non–trivial two–cycle inside CP^3 and thus is a stringy counterpart of the instantons of two–dimensional CP^n sigma–models. To be topologically non–trivial this two–cycle should have a non–zero pull–back on its worldsheet of the Kähler two–form $J_2=\frac{1}{2}\,e^{b'}\,e^{a'}\,J_{a'b'}$ of CP^3. Such a two–cycle is a $CP^1\simeq S^2$ subspace of CP^3. To identify it, it is convenient to consider the form of the Fubini–Study metric on CP^3 given in [76]

$$ds^2=R^2\big[\frac{1}{4}\big(d\theta^2+\sin^2\theta(d\varphi+\frac{1}{2}\sin^2\alpha\,\sigma_3)^2\big)+\sin^2\frac{\theta}{2}\,d\alpha^2+\frac{1}{4}\sin^2\frac{\theta}{2}\,\sin^2\alpha(\sigma_1^2+\sigma_2^2+\cos^2\alpha\,\sigma_3^2)\big]\,, \qquad (7.1.3)$$

where $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$ and $0\le\alpha\le\frac{\pi}{2}$, and $\sigma_1,\sigma_2,\sigma_3$ are three left-invariant one-forms on $SU(2)$ obeying $d\sigma_1=-\sigma_2\sigma_3$ etc. (see Appendix B for more details). Notice that with this choice of the CP^3 coordinates, θ and φ parameterize a two–sphere of radius $\frac{R}{2}$. This two–sphere is topologically non–trivial and associated to the Kähler form on CP^3. The string instanton wraps this sphere. For instance, if it wraps the sphere once θ and φ can be identified with the string worldsheet coordinates, while all other CP^3 as well as AdS_4 coordinates are worldsheet constants in this case. Thus the pullback on the string instanton of the metric (7.1.3) of CP^3 (of radius R) is the metric of the sphere of radius $R/2$

$$ds^2=\frac{R^2}{4}\,(d\theta^2+\sin^2\theta\,d\varphi^2)\,. \qquad (7.1.4)$$

In this coordinate system the S^2 vielbein e^i and the spin connection $\omega^{ij}_{S^2}$ $(i,j=1,2)$ can be chosen in the form

$$e^1=\frac{R}{2}d\theta\,,\qquad e^2=\frac{R}{2}\sin\theta\,d\varphi\,,\qquad \omega^{12}_{S^2}=\cos\theta d\varphi\,, \qquad (7.1.5)$$

and the S^2 curvature is

$$R^{ij} = d\omega^{ij}_{S^2} = \frac{4}{R^2}\, e^i\, e^j\,. \tag{7.1.6}$$

7.1.2 Bosonic part of the instanton solution

The bosonic part of the Wick rotated string action (7.1.1) is

$$S_E = \frac{T}{2}\int d^2\xi\,\sqrt{h}\,h^{IJ}\,e_I{}^i e_J{}^j\,\delta_{ij}\,, \tag{7.1.7}$$

where $T = \frac{e^{\frac{2}{3}\phi_0}}{2\pi\alpha'}$ and e^i are the vielbeins on S^2. To discuss the instanton solution of this CP^1 sigma model it is convenient to introduce complex coordinates both on the worldsheet and in target space (see [52] for a review of instantons in two–dimensional sigma models). In the conformal gauge $\sqrt{h}\,h^{IJ} = \delta^{IJ}$ and in the $(z,\bar z)$ coordinate system on the worldsheet the action takes the form

$$S_E = \frac{T}{2}\int d^2z\, e_z{}^i e_{\bar z}{}^j\,\delta_{ij}\,. \tag{7.1.8}$$

To introduce complex coordinates on the target sphere it is convenient to describe it as CP^1. The Fubini-Study metric on CP^1 is

$$ds^2_{CP^1} = \frac{d\zeta\, d\bar\zeta}{(1+|\zeta|^2)^2}\,. \tag{7.1.9}$$

If we choose ζ to be

$$\zeta = \tan\frac{\theta}{2}\, e^{i\varphi}\,, \tag{7.1.10}$$

eq. (7.1.9) takes the form of the metric on S^2 of radius $\frac{1}{2}$

$$ds^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\varphi^2)\,. \tag{7.1.11}$$

In the ζ, $\bar\zeta$ coordinate system the string action takes the following form (which is similar to that of the $O(3)$–sigma model)

$$S_E = \frac{TR^2}{4}\int d^2z\,\frac{|\partial\zeta|^2 + |\bar\partial\zeta|^2}{(1+|\zeta|^2)^2}\,. \tag{7.1.12}$$

It is now obvious that a local minimum is attained if $\bar\partial\zeta = 0$ or $\partial\zeta = 0$, *i.e.* the embedding is given by a holomorphic function $\zeta = \zeta(z)$ for the instanton or by an

anti–holomorphic function $\zeta = \zeta(\bar{z})$ for the anti–instanton. The remaining part of the action can be shown to be a topological invariant, namely,

$$S_I = \pi\, n\, T R^2 = n\, \frac{R^2_{CP^3}}{2\alpha'}, \tag{7.1.13}$$

where n is the topolgical charge of the instanton and $R_{CP^3} = e^{\frac{1}{3}\phi_0}\, R$ is the CP^3 radius in the string frame.

What we have just reproduced is the classical instanton solution of the two–dimensional $O(3)$ sigma–model [47] or rather its extension to CP^3 [49, 48, 50] which in terms of the Fubini–Studi coordinates ζ^a $(a = 1, 2, 3)$ of CP^3 (see eq. (D.0.1) of Appendix D) has the form

$$\zeta^a = \zeta^a(z) \quad \text{or} \quad \zeta^a = \zeta^a(\bar{z})\,.$$

The difference with the CP^n models is that in our case the string action is also invariant under worldsheet reparametrization. This means that every classical string solution must satisfy the Virasoro constraints implying that the worldsheet bosonic physical fields are associated with the string oscillations transverse to the worldsheet. For the instanton solution the string excitations along AdS_4 are zero and the Virasoro constraints have the following form in the conformal gauge

$$\left(\delta^{ab}(1+|\zeta|^2) - \zeta^b\, \bar{\zeta}^a\right)\, \frac{\partial\zeta^a\, \partial\bar{\zeta}^b}{(1+|\zeta|^2)^2} = 0\,. \tag{7.1.14}$$

We see that the Virasoro constraints are identically satisfied by the (anti)instanton solution.

Let us note that though in the $AdS_4 \times CP^3$ background the purely bosonic components of the NS–NS 3–form field strength H_3 are zero, the NS–NS 2–form may have non–zero expectation values proportional to the Kähler two–form on CP^3, $B_2 = \frac{\alpha'}{R^2}\, a\, J_{a'b'}\, e^{a'}\, e^{b'}$, where a plays the role of a constant axion. For such a two–form, $H_3 = dB_2$ is zero since J_2 is the closed (but not exact) form, $dJ_2 = 0$. In this case also the Wess–Zumino part of the (Wick rotated) string action (2.4.18) will contribute to the instanton action, which becomes

$$S_I = n\,(\pi\, R^2\, T - ia) = n\,(\frac{R^2_{CP^3}}{2\alpha'} - ia). \tag{7.1.15}$$

A similar situation one has in the case of string instantons on Calabi–Yau spaces [63, 77]. In the context of the AdS_4/CFT_3 correspondence, the co–homologically non–trivial B_2 field appears from the string side in the generalization of the ABJM model to include

gauge groups of a different rank proposed in [78] (see the Summary below for more discussion of this point).

Finally, we note that the bosonic string instanton has twelve zero modes. Four of them correspond to the directions along AdS_4 and, as has been demonstrated in Subsection 1.4.1, eight are the instanton zero modes on CP^3 [53]. We are now in a position to proceed with the study of the fermionic zero modes carried by the string instanton. We shall see that their number is also twelve.

7.2 Fermionic zero modes of the string instanton on CP^3

Now we re-introduce the fermionic modes up to the second order.

7.2.1 Restriction to the instanton solution

As we have discussed in Section 7.1.1, the instanton solution is supported on the CP^3 two–dimensional subspace whose tangent space is characterized *e.g.* by the first two values of the CP^3 tangent space index $a' = 1, 2, \ldots, 6$. Restricting to this solution we have

$$e_I{}^a = 0\,, \qquad e_I{}^{a'} = (e_I{}^i, e_I{}^{\tilde{a}} = 0)\,, \qquad J_{ij} = \varepsilon_{ij}\,, \qquad i = 1, 2 \quad \text{and} \quad \tilde{a} = 3, 4, 5, 6\,. \tag{7.2.16}$$

It will be convenient to choose the CP^3 gamma matrices as follows

$$\gamma^{a'} = (\rho^i \otimes \mathbf{1}, \rho^3 \otimes \gamma^{\tilde{a}})\,, \qquad \gamma_7 = -\rho^3 \otimes \gamma_{\tilde{5}}\,, \qquad \gamma_{\tilde{5}} = \frac{1}{4!}\varepsilon_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}\gamma^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}\,, \tag{7.2.17}$$

where $(\rho^1, \rho^2, \rho^3) = (\sigma^1, \sigma^3, -\sigma^2 = i\varepsilon)$ are the (re–labeled) Pauli matrices so that $\rho^1\rho^2 = i\rho^3$, and $\gamma^{\tilde{a}}$ are 4×4 Dirac gamma matrices corresponding to the four–dimensional subspace of CP^3 orthogonal to the instanton CP^1 and $\gamma_{\tilde{5}}^2 = 1$.

The Wick rotated kappa–symmetry projection matrix (7.1.2) then reduces to

$$\Gamma = i\frac{e^{\frac{2}{3}\phi_0}}{2\sqrt{G}}\,\varepsilon^{IJ}e_I{}^i e_J{}^j\rho_{ij}\rho^3\gamma_5\gamma_{\tilde{5}} = -\frac{\det e_I{}^i}{\sqrt{\det e_I^i e_J^j \delta_{ij}}}\gamma_5\gamma_{\tilde{5}} = -\gamma_5\gamma_{\tilde{5}} \tag{7.2.18}$$

and the fermionic part of the Euclidean action (7.1.1) becomes

$$S_F = T\int d^2\xi\,\sqrt{g}\,g^{IJ}\,\Theta(1-\Gamma)\gamma_5\left[i\,e_I{}^i\rho_i\nabla_J\Theta - \frac{1}{R}e_I{}^i e_J{}^j\rho_i\mathcal{P}_6\rho_j\Theta\right], \tag{7.2.19}$$

where the metric g_{IJ} is defined:

$$g_{IJ} = e_I{}^A\, e_J{}^B\, \eta_{AB} = e^{-\frac{2}{3}\phi_0}\, G_{IJ}|_{\Theta=0} \tag{7.2.20}$$

Note that in our case the fermionic terms of this two–dimensional theory differ from those of the conventional $2d$ supersymmetric $O(3) \sim CP^1$ (or in general CP^n) sigma–model (see [52, 54] for a review and references). For comparison, the CP^n sigma–model Lagrangian is

$$L_{CP^n} = G_{a\bar{b}}(\zeta,\bar{\zeta})\,\Big(\partial_I \bar{\zeta}^{\bar{b}}\,\partial_I \zeta^a + i\Psi^{\dagger\bar{b}}\rho^I D_I \Psi^a\Big) - \frac{1}{2}\,R_{a\bar{b}c\bar{d}}(\Psi^{\dagger b}\Psi^a)\,(\Psi^{\dagger d}\Psi^c)\,, \tag{7.2.21}$$

where now $\zeta^a(\xi)$ $(a,\bar{a}=1,\ldots,n)$ are the complex CP^n coordinates and Ψ^a and $\Psi^{\dagger\bar{a}}$ are independent complex $2n$–component spinor fields, $G_{a\bar{b}}(\zeta,\bar{\zeta})$ is the Kähler (Fubini-Study) metric on CP^n (see eq. (D.0.1) of Appendix B for the CP^3 case), $D_I\,\Psi^a = \partial_I\,\Psi^a + \Gamma^a_{bc}\,\partial_I\,\zeta^b\,\Psi^c$ and Γ^a_{bc} and $R_{a\bar{b}c\bar{d}}$ are the CP^n Christoffel symbol and curvature, respectively.

In view of the form of the quadratic action (7.2.19) and of the fermionic equation (4.1.4) it is natural to impose on the fermionic fields the conventional kappa–symmetry gauge–fixing condition

$$\frac{1}{2}(1+\Gamma)\Theta = \frac{1}{2}(1-\gamma_5\gamma_{\tilde{5}})\Theta = 0\,, \tag{7.2.22}$$

which means that the fermions split into two sectors according to their chiralities in AdS_4 and in the four–dimensional subspace of CP^3 orthogonal to the instanton CP^1

$$\Theta_+ : \quad \gamma_5\Theta_+ = \gamma_{\tilde{5}}\Theta_+ = \Theta_+\,, \qquad \Theta_- : \quad \gamma_5\Theta_- = \gamma_{\tilde{5}}\Theta_- = -\Theta_-\,. \tag{7.2.23}$$

Using the form of the CP^3 gamma–matrices (7.2.17) we find that

$$J = -iJ_{a'b'}\gamma^{a'b'}\gamma_7 = -2\gamma_{\tilde{5}} + iJ_{\tilde{a}\tilde{b}}\gamma^{\tilde{a}\tilde{b}}\gamma_{\tilde{5}}\rho^3 = -2\gamma_{\tilde{5}} + 2\rho^3\tilde{J}(1-\gamma_{\tilde{5}})\,, \tag{7.2.24}$$

where

$$\tilde{J} = -\frac{i}{4}J_{\tilde{a}\tilde{b}}\gamma^{\tilde{a}\tilde{b}} = -\frac{i}{8}J_{\tilde{a}\tilde{b}}\gamma^{\tilde{a}\tilde{b}}(1-\gamma_{\tilde{5}}) \qquad \tilde{J}^2 = \frac{1}{2}(1-\gamma_{\tilde{5}})\,. \tag{7.2.25}$$

So, the supersymmetry projection matrices $\mathcal{P}_2$ and $\mathcal{P}_6$ become

$$\begin{aligned} \mathcal{P}_2 &= \frac{1}{8}(2+J) = \frac{1}{4}(1+\rho^3\tilde{J})(1-\gamma_{\tilde{5}}) \\ \mathcal{P}_6 &= \frac{1}{8}(6-J) = \frac{1}{4}\big(3+\gamma_{\tilde{5}}-\rho^3\tilde{J}(1-\gamma_{\tilde{5}})\big)\,. \end{aligned} \tag{7.2.26}$$

Their action on the two sets of the chiral fermions is

$$\mathcal{P}_6\Theta_+ = \Theta_+ = \vartheta_+\,, \qquad \mathcal{P}_2\Theta_+ = \upsilon_+ = 0\,, \tag{7.2.27}$$

$$\mathcal{P}_6\Theta_- = \frac{1}{2}(1-\rho^3\tilde{J})\Theta_- = \vartheta_-\,, \qquad \mathcal{P}_2\Theta_- = \frac{1}{2}(1+\rho^3\tilde{J})\Theta_- = \upsilon\,. \tag{7.2.28}$$

Note that from eqs. (7.2.27) and (7.2.28) it follows that all the eight ϑ_+ are fermions which correspond to unbroken supersymmetries of the $AdS_4 \times CP^3$ superbackground, while in the Θ_- sector four fermions (ϑ_-) correspond to unbroken supersymmetries and other four (υ) to the broken ones. Note also that since for the instanton configuration the kappa–symmetry projector (7.2.18) commutes with the 'supersymmetry' projectors (7.2.26), it is not possible to choose the kappa–symmetry gauge–fixing condition which would put to zero all the eight 'broken–supersymmetry' fermions. In terms of the fields ϑ_+, ϑ_- and υ the fermionic action (7.2.19) takes the form

$$S_F = 2T\int d^2\xi\, \det e\, \left[i\,\vartheta_+ e_i{}^I\,\rho^i\nabla_I\vartheta_+ - \frac{2}{R}\vartheta_+\vartheta_+ - 2\big(i\,\upsilon\, e_i{}^I\,\rho^i\nabla_I\vartheta_- - \frac{1}{R}\upsilon\upsilon\big)\right], \tag{7.2.29}$$

where $e_i{}^I$ is the inverse vielbein on S^2.

For the instanton configuration the fermionic equation (4.1.4) reduces to the following ones

$$e_i{}^I\,\rho^i\nabla_I\vartheta_+ + \frac{2i}{R}\vartheta_+ = 0\,, \tag{7.2.30}$$

$$e_i{}^I\,\rho^i\nabla_I\vartheta_- + \frac{2i}{R}\upsilon = 0\,, \tag{7.2.31}$$

$$e_i{}^I\,\rho^i\nabla_I\upsilon = 0\,. \tag{7.2.32}$$

From the form of the action (7.2.29) and the equation of motion (7.2.31) it follows that the field υ can be regarded as an auxiliary one, which can be expressed in terms of a derivative of ϑ_-. However, for the analysis of the solutions of eqs. (7.2.30)–(7.2.32) it is more convenient to consider it as an independent variable satisfying the Dirac equation (7.2.32).

The covariant derivative ∇_I

$$\nabla_J\Theta = \left\{ \begin{array}{l} \nabla_J\upsilon = (\partial_J - \frac{1}{4}\omega_J{}^{ab}\gamma_{ab} - 2iA_J\gamma_7)\upsilon \\ \nabla_J\vartheta = (\partial_J - \frac{1}{4}\omega_J{}^{ab}\gamma_{ab} - \frac{1}{4}\omega_J{}^{a'b'}\gamma_{a'b'})\vartheta \end{array} \right. , \tag{7.2.33}$$

contains the pullback on the instanton two–sphere of the CP^3 spin connection.

$$\nabla_I\vartheta_\pm = (\partial_I - \frac{1}{4}\,\omega_I{}^{a'b'}\,\gamma_{a'b'})\,\vartheta_\pm\,. \tag{7.2.34}$$

Computing the pullback of the CP^3 connection, substituting it into the Dirac equations and taking into account the projection properties of the spinors we get the fermionic equations in the following form

$$e_i{}^I \rho^i \nabla_I^{S^2} \vartheta_+ + \frac{2i}{R}\vartheta_+ = e_i{}^I \rho^i (\nabla_I^{S^2}\vartheta_+ + \frac{i}{R}\, e_I{}^j \rho_j \vartheta_+) \;=\; 0\,, \tag{7.2.35}$$

$$e_i{}^I \rho^i (\nabla_I^{S^2} + i\tilde{A}_I\, \rho^3)\vartheta_- + \frac{2i}{R}\upsilon \;=\; 0\,, \tag{7.2.36}$$

$$e_i{}^I \rho^i\, (\nabla_I^{S^2} - i\tilde{A}_I\, \rho^3)\, \upsilon \;=\; 0\,, \tag{7.2.37}$$

where $\nabla_{S^2} = d - \frac{1}{4}\omega_{S^2}^{ij}\,\rho_{ij}$ is the intrinsic covariant derivative on the sphere of radius $R_{S^2} = R/2$ with curvature $R_{S^2}^{ij} = d\omega_{S^2}^{ij} = \frac{4}{R^2} e^i\, e^j$ and $\tilde{A}$ can be interpreted as the electromagnetic potential induced by a magnetic monopole of charge $g = -1/2$ placed at the center of the sphere. This is due to the fact that

$$F = d\tilde{A} = \frac{1}{R^2}\, e^i e^j \varepsilon_{ij} = \frac{1}{2}\, e^i e^j\, F_{ji} \quad \Rightarrow \quad F_{ij} = -\frac{2}{R^2}\,\varepsilon_{ij} = \frac{g}{R_{S^2}^2}\varepsilon_{ij}\,. \tag{7.2.38}$$

Note that $\frac{1}{4}\,\omega_{S^2}^{ij}\varepsilon_{ij}$ and $\tilde{A}$ are equivalent up to a total derivative term

$$\tilde{A} = \frac{1}{4}\,\omega_{S^2}^{ij}\varepsilon_{ij} + d\Lambda\,.$$

In our parametrization of CP^3 (see Appendix D) and for a given embedding of S^2 in CP^3, ω_{S^2} and $\tilde{A}$ have the following form in terms of the angular coordinates on S^2

$$\omega_{S^2}^{12} = \cos\theta d\varphi\,, \qquad \tilde{A} = \frac{1}{2}(1+\cos\theta)\, d\varphi\,. \tag{7.2.39}$$

We are now in a position to analyze the solutions of the fermionic equations (7.2.35)–(7.2.37). Eq. (7.2.35) has the form of the Dirac equation for a fermion of mass $\frac{2}{R}$. It is the product of $e_i{}^I\rho^i$ with the Killing spinor equation on the sphere

$$(\nabla_I^{S^2} + \frac{i}{R}\, e_I{}^j \rho_j)\vartheta_+ = 0\,. \tag{7.2.40}$$

The Killing spinor equation on S^2 for a two–component spinor has two non–trivial solutions [79]. Our ϑ_+ spinors carry four (independent) external indices in addition to the S^2–spinor index. Therefore, eq. (7.2.40) has eight solutions which are obviously solutions of the Dirac equation (7.2.35). These are actually the only regular eigenspinors of the Dirac operator on the sphere with the eigenvalue $-2i/R$ [65]. Thus, in the Θ_+ sector the string instanton has eight fermionic zero modes which are the solutions of the

Killing spinor equation (7.2.40). In spherical coordinates they have the explicit form [66]

$$\vartheta_+ = e^{-\frac{i}{2}\theta\rho^1} e^{\frac{i}{2}\varphi\rho^3}\epsilon_+ = \left(\cos\frac{\theta}{2} - i\rho^1 \sin\frac{\theta}{2}\right)\left(\cos\frac{\varphi}{2} + i\rho^3 \sin\frac{\varphi}{2}\right)\epsilon_+ \,, \tag{7.2.41}$$

where ϵ_+ is an arbitrary constant spinor satisfying the chirality conditions $\gamma_5\epsilon_+ = \gamma_{\tilde{5}}\epsilon_+ = \epsilon_+$.

Let us now proceed with the analysis of the third fermionic equation (7.2.37). As we have already mentioned, this equation describes the electric coupling of the fermionic field υ to the monopole potential on the sphere. The electric charge of υ is $e = \pm 1$ for $\upsilon = \pm\rho_3\upsilon$, *i.e.* when υ is a chiral/anti–chiral two–dimensional spinor, respectively. The analysis in [67] then tells us that there are non–trivial solutions of the charged Dirac equation (7.2.37) of positive chirality when $ge \geq 1/2$ and of negative chirality when $ge \leq -1/2$. Since we are in the opposite situation, there are no non–trivial solutions in our case and hence $\upsilon = 0$.

If $\upsilon = 0$, eq. (7.2.36) implies that ϑ_- should satisfy the massless Dirac equation

$$e_i{}^I \rho^i (\nabla_I^{S^2} + i\tilde{A}_I\, \rho^3)\vartheta_- = e_i{}^I \rho^i (\partial_I + \frac{i}{2}\rho_3 \partial_I \varphi)\vartheta_- = 0 \,. \tag{7.2.42}$$

We observe that the electric charge of ϑ_- is opposite to that of υ, *i.e.* it is $e = \mp 1$ depending on whether ϑ_- is chiral or anti–chiral two–dimensional spinor, *i.e.* whether $\vartheta_- = \pm\rho_3\vartheta_-$. Now we are in the situation in which the requirement of [67] for the Dirac equation (7.2.42) to have non–trivial solutions is saturated, *i.e.* in our case for ϑ_- of positive ρ^3–chirality $ge = 1/2$ and for ϑ_- of negative ρ^3–chirality $ge = -1/2$. By the Atiyah–Singer index theorem there is one solution for each ρ^3–chirality of ϑ_-. The general solution of (7.2.42) has actually a very simple form

$$\vartheta_- = \frac{1}{2}\, e^{-\frac{i}{2}\rho_3\,\varphi}\left[(1+\rho^3)\lambda_-(\zeta) + (1-\rho^3)\,\mu_-(\bar{\zeta})\right] , \tag{7.2.43}$$

where $\lambda_-(\zeta)$ and $\mu_-(\bar{\zeta})$ are holomorphic and anti–holomorphic spinors in the projective coordinates ζ and $\bar{\zeta}$ of $S^2 \simeq CP^1$ which are anti–chiral in the directions transverse to the instanton, *i.e.* $\lambda_- = -\gamma_5\lambda_-$, $\lambda_- = -\gamma_{\tilde{5}}\lambda_-$ $\mu_- = -\gamma_5\mu_-$ and $\mu_- = -\gamma_{\tilde{5}}\mu_-$. For the anti–instanton the solution takes the same form but with anti–holomorphic $\lambda_-(\bar{\zeta})$ and holomorphic $\mu_-(\zeta)$.

In [67] it has been shown that the only normalizable solutions of the Dirac equation (7.2.42) are those with constant λ_- and μ_- in (7.2.43). This allows us to conclude that in the ϑ_- sector the string instanton has four zero modes characterized by eq. (7.2.43)

with constant λ_- and μ_-.[1] Note that for $\lambda_- = const$ and $\mu_- = const$ the spinor (7.2.43) is the solution of the stronger equation

$$(\partial_I + \frac{i}{2}\rho_3\partial_I\varphi)\vartheta_- = 0\,. \tag{7.2.44}$$

This equation is the projection on the instantonic sphere of the $AdS_4 \times CP^3$ Killing spinor equation for ϑ_-.

To summarize, when $\upsilon = 0$ and in view of the form of the fermionic supervielbeins $E^{\alpha a'}$ $(a' = 1, \ldots, 6)$ [2] of the supercoset $OSp(6|4)/U(3) \times SO(1,3)$, the non–linear fermionic equation of motion (2.4.21) as well as the linear one (4.1.4) involve the pull–back on the string worldsheet of the $AdS_4 \times CP^3$ Killing spinor operator

$$\mathcal{D}\vartheta = D_{24}\vartheta = \mathcal{P}_6\,(d + \frac{i}{R}\,e^a\gamma^5\gamma_a + \frac{i}{R}e^{a'}\gamma_{a'} - \frac{1}{4}\omega^{ab}\gamma_{ab} - \frac{1}{4}\omega^{a'b'}\gamma_{a'b'})\vartheta\,, \tag{7.2.45}$$

which acts on the 24 fermions ϑ associated with the supersymmetry of $AdS_4 \times CP^3$. Therefore, if ϑ are the 24 Killing spinors on $AdS_4 \times CP^3$ they solve not only the linearized equations (4.1.4) but also the complete fermionic equations (2.4.21). In the case of the string instanton considered above, the kappa–symmetry projector reduces the number of solutions of the pulled–back Killing spinor equation by half, leaving us with the twelve physical fermionic zero modes. It should also be noted that these fermionic zero modes do not contribute to the bosonic equations (2.4.22). This guarantees that the bosonic instanton solution does not have a back reaction from the fermionic modes.

We should note that the Dirac equations (7.2.35)–(7.2.37) may have (non–normalizable) solutions which are not the Killing spinors (as *e.g.* eq. (7.2.43) with non–constant λ and μ). However, these other fermionic modes would modify the string field equations at higher order in fermions. In particular, they would produce a non–trivial contribution to the bosonic field equations (2.4.22), *i.e.* back–react on the form of the purely bosonic instanton and, hence, should be discarded.

Let us stress once again that, as we have shown, for the instanton solution considered above the kappa–symmetry cannot eliminate all the eight fermions υ associated with the supersymmetries broken in the $AdS_4 \times CP^3$ background. Therefore, even if among the instanton fermionic zero modes there is no υ–modes, the fluctuations around the instanton solution will have four physical fermionic degrees of freedom corresponding to the target–space supersymmetries broken by the $AdS_4 \times CP^3$ background.

[1] Remember that the eight–component spinor ϑ_- satisfies the additional projection condition (7.2.28) which reduces the number of its components to four.

[2] To avoid confusion, let us note that the index a' on spinors is different from the same index on bosonic quantities.

7.2.2 Fermionic zero modes and supersymmetry

Let us discuss in more detail how the fermionic zero modes are related to supersymmetry of the $AdS_4 \times CP^3$ superbackground and, correspondingly, of the superstring action. At the linearized level in fermions the supersymmetry part of the $OSp(6|4)$ transformations acts as follows

$$\begin{aligned} \delta\vartheta &= \epsilon\,, \\ \delta\upsilon &= 0\,, \\ \delta X^M e_M{}^A(X) &= -i\epsilon\Gamma^A\vartheta\,, \end{aligned} \tag{7.2.46}$$

where $\epsilon \equiv \mathcal{P}_6\,\epsilon(X)$ are 24 supersymmetry parameters of $OSp(6|4)$ satisfying the $AdS_4 \times CP^3$ Killing spinor equation

$$\mathcal{D}\epsilon = \nabla\epsilon + \frac{i}{R} e^A\,\mathcal{P}_6\gamma^5\Gamma_A\epsilon = 0 \tag{7.2.47}$$

with the explicit form of $\mathcal{D}$ given in eq. (7.2.45). Note that, at the leading order in fermions, the eight fermions υ are not subject to the supersymmetry transformations. The action of the isometry group $OSp(6|4)$ on these fermions is such that it takes the form of induced $SO(1,3) \times U(1)$ rotations with parameters depending on X, ϑ and the $OSp(6|4)$ parameters

$$\delta\upsilon = \frac{1}{4}\,\Lambda_{AB}(\epsilon, X, \vartheta)\,\Gamma^{AB}\,\upsilon\,. \tag{7.2.48}$$

Thus the first nontrivial term in the supersymmetry variation of υ is quadratic in fermionic fields which is beyond the linear approximation we are interested in.

It is not hard to see that the quadratic string action (4.2.9) is invariant under the supersymmetry transformations (7.2.46) (up to quadratic order in fermions). At the same time, the action (7.1.1), which is obtained from (4.2.9) by the shift (4.2.10) of the CP^3 coordinates, is invariant under the supersymmetry with the transformations of the *shifted* bosonic coordinates being

$$\delta\hat{X}^M e_M{}^A(\hat{X}) = -i\epsilon\Gamma^A\vartheta - i\epsilon\Gamma^A\upsilon = -i\epsilon\Gamma^A\Theta\,. \tag{7.2.49}$$

Let us now briefly recall how the target–space supersymmetry gets converted into world-sheet supersymmetry upon elimination of the un–physical fermionic degrees of freedom by gauge fixing kappa–symmetry. A more detailed discussion of such a "transmutation" of supersymmetry and its partial breaking in the Green–Schwarz formulation of superstrings and superbranes the reader may find *e.g.* in [68, 69, 70, 71].

If we impose on the fermionic fields $\Theta = (\vartheta,\, \upsilon)$ a kappa–symmetry gauge condition as *e.g.* the one we have used for studying the instanton solution, eq. (7.2.22),

$$\frac{1}{2}(1+\Gamma_0)\,\Theta = 0\,, \tag{7.2.50}$$

the kappa–symmetry gauge–fixing condition will not be invariant under all the twenty–four supersymmetries (7.2.46) but only under half of them satisfying the condition

$$\epsilon_{br} = \frac{1}{2}(1-\Gamma_0)\,\epsilon\,. \tag{7.2.51}$$

In eqs. (7.2.50) and (7.2.51) we denoted the gauge–fixing projector by Γ_0 to distinguish it from the more general projector matrix Γ that appears in the kappa–symmetry transformations (2.4.20),(2.4.21) and (6.3.46).

The target–space supersymmetries with the parameter ϵ_{br} are those which are spontaneously broken by the presence of the string. The reason is that the remaining twelve fermionic fields $\vartheta = \frac{1}{2}(1-\Gamma_0)\,\vartheta$ get shifted by these transformations and hence behave as Volkov–Akulov goldstinos [72, 73].

The supersymmetries which remain unbroken and which become worldsheet supersymmetries are identified as follows. The gauge fixing condition (7.2.50) is not invariant under the supersymmetry transformations (7.2.46) with the parameter $\epsilon_w = \frac{1}{2}(1+\Gamma_0)\,\epsilon$. However, this can be cured by an appropriate compensating kappa–symmetry transformation that (at the leading order in fermions) satisfies the condition

$$-\frac{1}{4}\,\mathcal{P}_6(1+\Gamma_0)(1+\Gamma)\,\kappa = \epsilon_w \equiv \frac{1}{2}(1+\Gamma_0)\,\epsilon. \tag{7.2.52}$$

This condition relates the components of the κ–symmetry parameter appearing in the transformation of ϑ to the supersymmetry parameter ϵ_w. Since kappa–symmetry is the worldsheet fermionic symmetry which can actually be identified with the conventional local worldsheet supersymmetry [74], eq. (7.2.52) thus converts the unbroken target–space supersymmetries into worldsheet supersymmetry. Note that eq. (7.2.52) does not involve the part of the kappa–symmetry transformation acting on the υ–fermions since they are singled out with the complementary projector $\mathcal{P}_2$. This part of kappa–symmetry is fixed by the gauge condition $\frac{1}{2}(1+\Gamma_0)\upsilon = 0$ (see eq. (7.2.50)).

As a result, (at a leading order in fermions and bosons) under the broken and unbroken supersymmetries the worldsheet fermionic fields remaining after the gauge-fixing (7.2.50) $\Theta \equiv \frac{1}{2}\,(1-\Gamma_0)\Theta = (\vartheta, \upsilon)$ and the bosonic fields $\hat{X}^M$ transform as follows

$$\delta\upsilon \;=\; 0\,, \qquad \delta\vartheta = \epsilon_{br} + \frac{1}{4}\,\mathcal{P}_6\,(1-\Gamma_0)(1+\Gamma)\kappa\,, \tag{7.2.53}$$

$$\delta\hat{X}^M\, e_M{}^i(\hat{X}) = -i\epsilon_{br}\Gamma^i\Theta - \delta_\kappa\vartheta^{\underline{\alpha}}\, E_{\underline{\alpha}}{}^i(\hat{X},\Theta) + \mathcal{O}(\epsilon,\Theta,\hat{X})\,, \tag{7.2.54}$$

$$\delta\hat{X}^M\, e_M{}^\perp(\hat{X}) = -i\epsilon_{w}\Gamma^\perp\Theta - \delta_\kappa\vartheta^{\underline{\alpha}}\, E_{\underline{\alpha}}{}^\perp(\hat{X},\Theta) + \mathcal{O}(\epsilon,\Theta,\hat{X})\,, \tag{7.2.55}$$

where $i = 0, 1$ and $\perp = 2, \ldots, 9$ indicate the directions parallel and orthogonal to the string worldsheet, respectively, the second terms in (7.2.54) and (7.2.55) come from the compensating kappa–symmetry transformation (2.4.20) that at the linearized level is just $-i\epsilon\Gamma^A\Theta$, and $O(\epsilon,\Theta,\hat{X})$ stand for terms which are non–linear in fields (and their derivatives).

It is instructive to notice that the leading (linear) term in the supersymmetry transformations of $\hat{X}^M$ along the directions trasverse to the string worldsheet contains the parameter of the unbroken supersymmetries, while along the worldsheet the linear term contains the broken supersymmetry parameter. This reflects the fact that the bosonic excitations transversal to the classical string configuration and kappa–gauge fixed fermionic fields are associated with worldsheet physical fields forming supermultiplets under the unbroken supersymmetry. At the same time the supersymmetry transformations along the string worldsheet can be compensated by an appropriated worldsheet reparametrization.

For the instanton solution under consideration we have $\Gamma = \Gamma_0 = -\gamma_5\gamma_{\bar{5}}$ and $\upsilon = 0$. So the supersymmetry transformations (at the leading order) become

$$\delta\upsilon = 0 + \ldots\,, \qquad \delta\vartheta = \epsilon_{br} + \ldots\,, \tag{7.2.56}$$

$$\delta\hat{X}^M\, e_M{}^\perp(X) = -2i\epsilon_w\Gamma^\perp\vartheta\, + \ldots\,, \tag{7.2.57}$$

where the dots stand for higher order terms in fermions.

Under the unbroken supersymmetry transformations the fermionic zero modes induce an (isometry) transformation of the string coordinates in the transverse directions which results in a shift of the bosonic parameters characterizing the string instanton. This is analogous to the supersymmetry transformations of the 'collective coordinates' of the CP^n sigma–model instanton [52].

From eqs. (7.2.56) and (7.2.57) we also see that if we start from the purely bosonic instanton solution discussed in Section 7.1.1, we can find at least part of the instanton fermionic zero modes by looking at the variation of the fermionic fields under supersymmetry. The form of the supersymmetry transformations implies that the bosonic instanton configuration is 1/2 BPS. Namely, the string instanton solution with $\Theta = 0$ is invariant under the twelve supersymmetries ϵ_w. Fermionic zero modes are generated by the target–space supersymmetries (with the parameter ϵ_{br}) which are broken by the string configuration, as we have already discussed in the end of Section 7.2.1 where we have also directly shown that the instanton does not have other fermionic zero modes associated with the fields υ. Note that the latter could not be obtained from the purely

bosonic solution by a supersymmetry transformation since the corresponding variation of υ is proportional to υ itself (see eq. (7.2.48)).

Let us now compare our $AdS_4 \times CP^3$ superstring worldsheet theory (which has 12 unbroken worldsheet supersymmetries) with the supersymmetry properties of the conventional $\mathcal{N} = (2,2)$[3] supersymmetric CP^n sigma–model (described by the Lagrangian (7.2.21)) and with its instanton solutions [52, 54].

The supersymmetry transformations in the CP^n sigma–model have the following form

$$\delta\zeta^a = \bar{\epsilon}\Psi^a\,, \qquad \delta\Psi^a = i\rho^I\epsilon\,\partial_I\zeta^a + \cdots\,, \qquad (a = 1,\ldots,n) \tag{7.2.58}$$

where ϵ is now a constant complex two–component spinor parameter of $\mathcal{N} = (2,2)$ supersymmetry and the dots stand for the terms non–linear in the fields. The CP^n sigma–model is also invariant under superconformal transformations [75]

$$\delta\zeta^a = \bar{\eta}(z,\bar{z})\,\Psi^a\,, \qquad \delta\Psi^a = i\rho^I\eta(z,\bar{z})\,\partial_I\zeta^a + \cdots\,, \qquad (a = 1,\ldots,n) \tag{7.2.59}$$

The superconformal transformations are similar to the rigid supersymmetries (7.2.58) but with the complex two–component spinor parameters whose chiral and anti–chiral components are, respectively, holomorphic and anti–holomorphic $\eta(z,\bar{z}) = (\eta_+(z), \eta_-(\bar{z}))$.

The superconformal symmetry of the CP^n sigma–model (which is broken by quantum anomalies [52]) is in a sense a counterpart of the spontaneously broken part of the target–space supersymmetry of the superstring action.

If one starts from the purely bosonic instanton solution of the CP^n sigma–model

$$\bar{\partial}\zeta^a = 0 \quad \text{or} \quad \partial\zeta^a = 0 \quad \text{and} \quad \Psi = 0 \tag{7.2.60}$$

one can then generate solutions of the fermionic field equations and find the corresponding fermionic zero modes by considering the supersymmetry transformations (7.2.58) and (7.2.59) of Ψ. In this way, taking into account that for the instanton the fields ζ^a are either holomorphic or anti–holomorphic, one finds that only half of the supersymmetry transformations (7.2.58) and of the superconformal transformations (7.2.59) are non–trivial, those with the parameters ϵ and η being (anti)chiral $2d$ spinors. The fermionic zero modes obtained in this way are (anti)holomorphic (anti)chiral $2d$ spinor fields. We observe that in contrast to the case of the string instanton whose fermionic zero modes are generated by the spontaneously broken supersymmetry transformations, in the case of the CP^n sigma–model half of the fermionic zero modes are generated by the rigid supersymmetry transformations and another half by the superconformal symmetry.

[3] $\mathcal{N}$ labels the real number of left– and right–handed worldsheet supersymmetries.

7.2.3 Instantons in $AdS_2 \times S^2 \times T^6$

The case of instantons in $AdS_2 \times S^2 \times T^6$ is still to be systematically studied, so below we will only briefly mention properties of some of them. There are several possible instanton configurations in this case, in fact the string can wrap the S^2 sphere of the metric but it can also wrap a 2-cycle in T^6 or both.

If we concentrate on the simplest case, the S^2 case, what we know is that an instanton in a bosonic sigma model on S^2 has 4 zero modes [52], to these one has to add the zero modes corresponding to the directions of AdS_2 and T^6, so the total number of bosonic zero modes for the case of $AdS_2 \times S^2 \times T^6$ is 12.

As far as the fermions are concerned, it is easy to see that, restricting to the instanton solution, the equations of motion of the coset and the non coset fermions decouple, and if we impose the gauge $\frac{1}{2}(1+\Gamma)\Theta = 0$, with Γ being the same that appears in the projector of the kappa symmetry (7.1.2), these become:

$$e_I^{\hat{a}}\Gamma_{\hat{a}}\nabla^I_{S^2}\vartheta + i\frac{2}{R}\vartheta = 0. \tag{7.2.61}$$

$$e_I^{\hat{a}}\Gamma_{\hat{a}}\nabla^I_{S^2}v = 0. \tag{7.2.62}$$

The equation (7.2.62) has no solutions but the trivial one $v = 0$ [67], the equation (7.2.61) is solved by the Killing spinors on S^2. The S^2 Killing spinor has 2 components. In our case the gauge fixed ϑ has 4 components, which correspond to two copies of the Killing spinor. So we expect to have 4 fermionic zero modes. Moreover, since also in this case the Killing spinors (in the absence of the non-coset fermions) solve the fermionic field equations to all orders, the above instanton solution should be valid to all orders in fermions.

We expect that the situation in the cases of the instantons wrapping a two-cycle in T^6 or cycles in both, S^2 and T^6, will be more involved.

We reserve the analysis and interpretation of these non-perturbative solutions for future work.

Conclusions

In this thesis we have studied and enlarged the knowledge of Type IIA Superstring theories on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$. These are theories on non maximal supersymmatric backgrounds, that hence could not be fully described by sigma models on supercoset spaces $\frac{OSp(6|4)}{SO(1,3)\times U(3)}$ and $\frac{PSU(1,1|2)}{SO(1,1)\times U(1)}$. For this reason we have studied the full structure of these Superstring theories with a particular attention to the non coset degrees of freedom.

The study of these theories has been brought on, as much as possible, in a parallel way for the two cases, this was possible since the two theories present remarkable similarities. In fact they are in a certain sense complementary or dual to each other, since for $AdS_4 \times CP^3$ there are 24 coset ferminic degrees of freedom and 8 non-coset ones, for $AdS_2 \times S^2 \times T^6$ the situation is the opposite, with only 8 supersymmetries preserved. Furthermore the projectors that splits the fermions in two sets $(24 + 8)$ are, for both the theories, of the same form. This led us to write the $OSp(6|4)$ and $PSU(1,1|2) \rtimes E(6)$ algebras in a unified way.

The first problem that we have addressed was Integrability. As we have seen, one can, in a rather easy way, show the Classical Integrability of the sigma models on $\frac{OSp(6|4)}{SO(1,3)\times U(3)}$ and $\frac{PSU(1,1|2)\rtimes E(6)}{SO(1,1)\times U(1)\times SO(6)}$. The problem is that it is not always possible to truncate our theories to the corresponding coset sigma model. For this reason we have studied how to include in the integrable structures of these theories the non coset degrees of freedom.

To demonstrate that a theory is classically integrable one has to find a Lax connection with zero curvature. There are two possible ways of proceeding, starting from the right invariant currents (Noether currents), or from the left invariant ones.

The construction of the Lax connection using the Noether currents, that has been carried on up to the second order in all fermions, has the lack to be non manifestly $\mathbb{Z}_4$ invariant. We have not only proposed a construction that allowed us to write a Lax connection that is invariant under $\mathbb{Z}_4$-transformation, but we have also been able to write an extension of the complete supercoset Lax connection up to the second order in the non coset fermions. We have written the Lax connections for the two theories in

a unified way:

$$\begin{aligned} L &= \frac{1}{2}\Omega_0^{AB} M_{AB} + (1+\alpha_1) E^A P_A + \alpha_2 * E^A P_A + QVE \\ &\quad - \frac{i}{R} Q\gamma_\star \left[*(E^A + 2i\upsilon\Gamma^A E)\,\Gamma_A V\upsilon - (E^A + 2i\upsilon\Gamma^A E)\,\Gamma_A\Gamma_{11} V\upsilon \right] \\ &\quad - \frac{i}{R} Q\gamma_\star \left[i(\upsilon\Gamma^A\Gamma_{11} E)\,\Gamma_A V\upsilon + i(\upsilon\Gamma^A E)\,\Gamma_A\Gamma_{11} V\upsilon \right] \\ &\quad + (2i\upsilon\Gamma^A * E + i\upsilon\Gamma^A * \nabla\upsilon - \frac{2}{R} * E^B\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon)\,P_A \\ &\quad + (2i\upsilon\Gamma^A\Gamma_{11} E + i\upsilon\Gamma^A\Gamma_{11}\nabla\upsilon - \frac{2}{R} E^B\,\upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B\upsilon)\,P_A \\ &\quad + \frac{i}{8}(*E^C\,\upsilon\Gamma_C{}^{DE} V^2\upsilon - E^C\,\upsilon\Gamma_C{}^{DE}\Gamma_{11} V^2\upsilon)\,R_{DE}{}^{AB}\,M_{AB} + \mathcal{O}(v^3)\,. \end{aligned}$$

In the last part of the thesis we have moved to consider possible non perturbative corrections to the effective action of the two theories. For the $AdS_4 \times CP^3$ case there is the possibility of a string instanton wrapping a CP^1 cycle in the CP^3 part of the space. For $AdS_2 \times S^2 \times T^6$ there are several possible configuations, either a string wrapping S^2 or an instanton in the T^6 part or both.

We have thus found that the instanton in the $AdS_4 \times CP^3$ theory has twelve fermionic zero modes. They are associted with $\frac{1}{2}$ of the supersymmetries of the background broken by the instanton configuration, thus demonstrating that the instanton configuration is $\frac{1}{2}$ BPS.

Future directions

The work, that has been presented in this thesis, is a good starting point for further study of type IIA Superstring theories on $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$ and their dual theories.

What now one has to do is to understand the classical integrability of the full theories, i.e. seeing if it is possible to include in the structure of the Lax connection the non-supercoset fermions to all orders, preserving the zero curvature condition. Moreover in the case of $AdS_2 \times S^2 \times T^6$ we still have to compute the full geometry, since up to now we know it only up to the second order in the non-supersymmetric fermions.

We found a form for the Lax connection that is manifestly $\mathbb{Z}_4$-invariant, characteristic that in the previous examples (supercoset sigma models) turned out to be fundamental for the formulation of Bethe ansatz equations. What would be now interesting to know is if, going to all orders in all the fermions, this invariance would be preserved.

Another thing, that one has to look at is whether there exist Bethe equations that take into account the non-supercoset degrees of freedom. This would allow us to better understand the integrable structure of these string theories and, hopefully, also their CFT duals.

In particular, in the case of $AdS_2 \times S^2 \times T^6$ we hope that the study of the superstring theory would allow us to know something more concrete on the dual one-dimensional CFT, whose nature up to now is mysterious.

For what concern the instanton-like solutions of the theories, that we have spoke about, there is still some work to do. For the theory on $AdS_4 \times CP^3$ one should still study possible instanton corrections to the effective action (see also [80] and [81]) and compare the result with what was found in the CFT side [64], looking at a possible enrichment of the holographic dictionary.

In the $AdS_2 \times S^2 \times T^6$ theory we have still to analyze all the instantonic solutions and their zero modes, looking at the amount of supersymmetry that each istantonic solution preserves.

There is still a lot of work to dissolve all the questions concerning these and other cases of less supersymmetric theories, but the bases, that have been built in this thesis, represent a consistent step towards the final goal.

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Appendix A

Curvature of the Lax connection

The curvature of the Lax connection (6.3.35), (6.3.36) computed to quadratic order in the non–coset fermions υ can be split into three pieces corresponding to the generators in the superalgebra, M_{AB}, P_A and Q:

$$dL - LL = (dL - LL)_M + (dL - LL)_P + (dL - LL)_Q \,. \tag{A.0.1}$$

With a bit of work one finds that, to order υ^2,

$$\begin{aligned}
&(dL - LL)_M = \\
&\quad \frac{\alpha_2}{4}\Big[i * \hat{E}^A\, \upsilon\Gamma^{CD}\Gamma_A V^2 E + i * E^A\, \upsilon\Gamma^{CD}\Gamma_A V^2 \nabla\upsilon - i\hat{E}^A\, \upsilon\Gamma^{CD}\Gamma_A\Gamma_{11} V^2 E \\
&\quad - iE^A\, \upsilon\Gamma^{CD}\Gamma_A\Gamma_{11} V^2\nabla\upsilon + 4\upsilon\Gamma^C V^2 E\, \upsilon\Gamma^D\Gamma_{11} E + 4\upsilon\Gamma^C E\, \upsilon\Gamma^D\Gamma_{11} V^2 E \\
&\quad - \upsilon\Gamma^A\Gamma_{11} E\, E\Gamma^{CD}\mathcal{P}\Gamma_A V^2\upsilon - \upsilon\Gamma^A E\, E\Gamma^{CD}\mathcal{P}\Gamma_A\Gamma_{11} V^2\upsilon + \frac{i}{2}\nabla * E^A\, \upsilon\Gamma_A{}^{CD} V^2\upsilon \\
&\quad - \frac{1}{2} E\Gamma^A E\, \upsilon\Gamma_A{}^{CD}\Gamma_{11} V^2\upsilon + \frac{4}{R} * E^C E^A\, \upsilon\Gamma^D V^2\mathcal{P}\gamma_\star\Gamma_A\upsilon - \frac{4}{R} E^C E^A\, \upsilon\Gamma^D V^2\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_A\upsilon \\
&\quad - \frac{\alpha_2}{R} * E^B E^A\, \upsilon\Gamma_B\mathcal{P}\gamma_\star\Gamma^{CD}\mathcal{P}\Gamma_A\Gamma_{11}\upsilon - \frac{\alpha_2}{R} E^B E^A\, \upsilon\Gamma_B\mathcal{P}\gamma_\star\Gamma^{CD}\mathcal{P}\Gamma_A\upsilon\Big] R_{CD}{}^{EF} M_{EF} \,,
\end{aligned} \tag{A.0.2}$$

where we have again introduced $\hat{E}^A = E^A + 2i\upsilon\Gamma^A E$ to shorten the expressions. The terms in the Lax curvature proportional to P_A are

$$\begin{aligned}
&(dL - LL)_P = \\
&\quad \alpha_2\Big[\nabla * (\hat{E}^A + i\upsilon\Gamma^A\nabla\upsilon) - iE\Gamma^A\Gamma_{11} E - 2iE\Gamma^A\Gamma_{11}\nabla\upsilon - i\nabla\upsilon\Gamma^A\Gamma_{11}\nabla\upsilon \\
&\quad - \frac{2}{R} E^B\, \upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B E - \frac{2}{R} E^B\, \upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B\nabla\upsilon - \frac{2}{R} * \hat{E}^B\, E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{R} * E^B\, \nabla\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon + \frac{2}{R}\hat{E}^B\, E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon + \frac{2}{R}E^B\, \nabla\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon \\
&-\frac{2}{R} * E^B\, \upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\nabla\upsilon + \frac{2i}{R}E\Gamma^B E\,\upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B\upsilon - \frac{2i}{R}E\Gamma^B\Gamma_{11}E\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon \\
&+\frac{iR}{16}E\Gamma_{DE}\gamma_\star E\,\upsilon\Gamma^A\Gamma_{11}\Gamma^{BC}\upsilon\, R_{BC}{}^{DE} - \frac{i}{8}E^F E^D\,\upsilon\Gamma^{ABC}\Gamma_{11}\upsilon\, R_{BCDF} \\
&-\frac{i}{4}E^F E^D\,\upsilon\Gamma_D{}^{BC}\Gamma_{11}\upsilon\, R_{BCF}{}^A + \frac{i}{4}E^B * E^F\,\upsilon\Gamma_F{}^{DE}\upsilon\, R_{DEB}{}^A - \frac{2i}{R}\upsilon\Gamma^B\Gamma_{11}E\,E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon \\
&-\frac{2i}{R}\upsilon\Gamma^B E\,E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon - \frac{2}{R}(\nabla * E^B - iE\Gamma^B\Gamma_{11}E)\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon\Big]\, P_A\,. \qquad \text{(A.0.3)}
\end{aligned}
$$

Finally the terms proportional to Q in the Lax curvature become

$$
\begin{aligned}
&(dL - LL)_Q = \\
&\quad \alpha_2\frac{i}{R}\Big[(\hat{E}^A + i\upsilon\Gamma^A\nabla\upsilon)\,QV^\dagger\gamma_\star\Gamma_A\Gamma_{11}E + \hat{E}^A\,QV^\dagger\gamma_\star\Gamma_A\Gamma_{11}\nabla\upsilon \\
&\quad -*(\hat{E}^A + i\upsilon\Gamma^A\nabla\upsilon)\,QV^\dagger\gamma_\star\Gamma_A E - *\hat{E}^A\,QV^\dagger\gamma_\star\Gamma_A\nabla\upsilon \\
&\quad -\frac{i}{R}\hat{E}^A\hat{E}^B\,QV^\dagger\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon - \frac{i}{R} * \hat{E}^A\hat{E}^B\,QV^\dagger\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\upsilon \\
&\quad -\frac{2}{R} * E^B\,E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon\,QV^\dagger\gamma_\star\Gamma_A\upsilon + \frac{2}{R} * E^B\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon\,QV^\dagger\gamma_\star\Gamma_A E \\
&\quad -\frac{2}{R} * E^B\,\upsilon\Gamma^A E\,QV^\dagger\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\upsilon + \frac{2}{R}E^B\,E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon\,QV^\dagger\gamma_\star\Gamma_A\upsilon \\
&\quad +\frac{2}{R}E^B\,\upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B\upsilon\,QV^\dagger\gamma_\star\Gamma_A E + \frac{2}{R}E^B\,\upsilon\Gamma^A E\,QV^\dagger\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon \\
&\quad -\frac{1}{R}E^B\,\upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B E\,QV^\dagger\gamma_\star\Gamma_A\upsilon - \frac{1}{R}E^B\,\upsilon\Gamma^A\mathcal{P}\gamma_\star\Gamma_B E\,QV^\dagger\gamma_\star\Gamma_A\Gamma_{11}\upsilon \\
&\quad -\frac{1}{R}E^B\,\upsilon\Gamma_A E\,QV^\dagger\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma^A\Gamma_{11}\upsilon - \frac{1}{R}E^B\,\upsilon\Gamma^A\Gamma_{11}E\,QV^\dagger\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma_A\upsilon \\
&\quad +\frac{R}{16}(*E^C\,\upsilon\Gamma_C{}^{DE}\upsilon - E^C\,\upsilon\Gamma_C{}^{DE}\Gamma_{11}\upsilon)\,R_{DE}{}^{AB}\,QV^\dagger\Gamma_{AB}E\Big] \\
&\quad +\alpha_2^2\frac{i}{R^2}\Big(*E^B\,\upsilon\Gamma^A E\,(2QV\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon - QV\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma^A\Gamma_{11}\upsilon) \\
&\quad +*E^B\,\upsilon\Gamma^A\Gamma_{11}E\,(2QV\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\upsilon - QV\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma_A\upsilon) \\
&\quad -E^B\,\upsilon\Gamma^A E\,(2QV\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\upsilon - QV\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma^A\upsilon) \\
&\quad -E^B\,\upsilon\Gamma^A\Gamma_{11}E\,(2QV\gamma_\star\Gamma_A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon - QV\gamma_\star\Gamma_B\gamma_\star\mathcal{P}\Gamma_A\Gamma_{11}\upsilon) \\
&\quad -\frac{R^2}{16}(*E^C\,\upsilon\Gamma_C{}^{DE}\upsilon - E^C\,\upsilon\Gamma_C{}^{DE}\Gamma_{11}\upsilon)\,R_{DE}{}^{AB}\,QV\Gamma_{AB}\Gamma_{11}E
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{R^2}{16}(*E^C\, \upsilon\Gamma_C{}^{DE}\Gamma_{11}\upsilon - E^C\, \upsilon\Gamma_C{}^{DE}\upsilon)\, R_{DE}{}^{AB}\, QV\Gamma_{AB}E\Big) \\
&- \alpha_2\frac{i}{R}\Big(\nabla * \hat{E}^A - iE\Gamma^A\Gamma_{11}E - 2iE\Gamma^A\Gamma_{11}\nabla\upsilon - \frac{2}{R}E^B\, \upsilon\Gamma^A\Gamma_{11}\mathcal{P}\gamma_\star\Gamma_B E \\
&+ \frac{2}{R}E^B\, E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\Gamma_{11}\upsilon - \frac{2}{R} * E^B\, E\Gamma^A\mathcal{P}\gamma_\star\Gamma_B\upsilon\Big)\, QV^\dagger\gamma_\star\Gamma_A\upsilon\,, \qquad \text{(A.0.4)}
\end{aligned}
$$

where we've used the fact that

$$
2(\Gamma^A\Gamma_{11}E)_\alpha(\Gamma_A E)_\beta + 2(\Gamma_A E)_\alpha(\Gamma^A\Gamma_{11}E)_\beta + (\Gamma^A\Gamma_{11})_{\alpha\beta}\, E\Gamma_A E + (\Gamma^A)_{\alpha\beta}\, E\Gamma_A\Gamma_{11}E = 0 \qquad \text{(A.0.5)}
$$

and

$$
\begin{aligned}
&\upsilon\Gamma^A\nabla\upsilon\, QV^\dagger\gamma_\star\Gamma_A\Gamma_{11}E + \upsilon\Gamma^A\Gamma_{11}\nabla\upsilon\, QV^\dagger\gamma_\star\Gamma_A E + E\Gamma^A\Gamma_{11}\nabla\upsilon\, QV^\dagger\gamma_\star\Gamma_A\upsilon \\
&+ E\Gamma^A\nabla\upsilon\, QV^\dagger\gamma_\star\Gamma_A\Gamma_{11}\upsilon + \upsilon\Gamma^A\Gamma_{11}E\, QV^\dagger\gamma_\star\Gamma_A\nabla\upsilon + \upsilon\Gamma_A E\, QV^\dagger\gamma_\star\Gamma^A\Gamma_{11}\nabla\upsilon = 0 \qquad \text{(A.0.6)}
\end{aligned}
$$

which follow from the basic Fierz identity (2.3.14).

Appendix B

Gamma-matrix identities

Some useful gamma-matrix identities are ($a = 0, 1, 2, 3$)

$$\Gamma^{abc} = -i\varepsilon^{abcd}\Gamma_d\gamma^5 \tag{B.0.1}$$

$$\Gamma^{ab} = -\frac{i}{2}\varepsilon^{abcd}\Gamma_{cd}\gamma^5 \tag{B.0.2}$$

$$\Gamma^{a} = \frac{i}{6}\varepsilon^{abcd}\Gamma_{bcd}\gamma^5 \tag{B.0.3}$$

and some useful identities involving the projection operators are

$$\mathcal{P}_8\Gamma^{a'b'c'}\mathcal{P}_{24} = -3iJ^{[a'b'}\mathcal{P}_8\Gamma^{c']}\gamma_7\mathcal{P}_{24} \tag{B.0.4}$$

$$\mathcal{P}_8\Gamma^{a'b'}\mathcal{P}_8 = iJ^{a'b'}\gamma^7\mathcal{P}_8 \tag{B.0.5}$$

$$\Gamma^{a'}\mathcal{P}_8\Gamma_{a'} = 2\mathcal{P}_{24} \tag{B.0.6}$$

$$\Gamma^{[a'}\mathcal{P}_8\Gamma^{b']} = \frac{1}{2}\mathcal{P}_{24}\Gamma^{a'b'}\mathcal{P}_{24} + \frac{i}{2}J^{a'b'}\gamma_7\mathcal{P}_{24} \tag{B.0.7}$$

$$\mathcal{P}_{24}(\delta^{b'}_{a'} + iJ_{a'}{}^{b'}\gamma_7)\Gamma_{b'}\mathcal{P}_{24} = 0 \tag{B.0.8}$$

$$\mathcal{P}_8(\delta^{b'}_{a'} - iJ_{a'}{}^{b'}\gamma_7)\Gamma_{b'}\mathcal{P}_{24} = 0\,. \tag{B.0.9}$$

$$\mathcal{P}_{24}\Gamma_{[A}\mathcal{P}_{24}\gamma^5\Gamma_{B]}\mathcal{P}_{24} = -\frac{R^2}{8}R_{AB}{}^{CD}\mathcal{P}_{24}\Gamma_{CD}\gamma^5\mathcal{P}_{24}\,. \tag{B.0.10}$$

B.1 Projection of the Fierz identities

It is useful to project this identity in various ways using our projection operators. In the $AdS_4 \times CP^3$ case when $\upsilon = \mathcal{P}_8\upsilon$ we get

$$\upsilon^\alpha\,\upsilon^\beta = \frac{1}{8}(\mathcal{P}_8\mathcal{C})^{\alpha\beta}\,\upsilon\upsilon + \frac{1}{8}(\gamma^5\mathcal{P}_8)^{\alpha\beta}\,\upsilon\gamma^5\upsilon - \frac{1}{8}(\Gamma_a\gamma^5\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^a\gamma^5\upsilon$$

$$
\begin{aligned}
&+\frac{1}{8}(\Gamma_a\gamma^7\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^a\gamma^7\upsilon-\frac{1}{16}(\Gamma_{ab}\gamma^7\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^{ab}\gamma^7\upsilon\\
&-\frac{1}{32\cdot 3!}(\mathcal{P}_8\Gamma_{a'b'c'}\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^{a'b'c'}\upsilon+\frac{1}{32\cdot 3!}(\mathcal{P}_8\Gamma_{a'b'c'}\Gamma_{11}\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^{a'b'c'}\Gamma_{11}\upsilon\\
&+\frac{1}{32\cdot 3!}(\mathcal{P}_8\Gamma_a\Gamma_{b'c'd'}\mathcal{P}_8)^{\alpha\beta}\,\upsilon\Gamma^a\Gamma^{b'c'd'}\upsilon
\end{aligned}
\tag{B.1.11}
$$

and it follows from this expression that

$$
\begin{aligned}
(\Gamma^{a'}\upsilon)^\alpha\,(\Gamma_{a'}\upsilon)^\beta &= -\frac{1}{4}(\mathcal{P}_{24}\mathcal{C})^{\alpha\beta}\,\upsilon\upsilon-\frac{1}{4}(\gamma^5\mathcal{P}_{24})^{\alpha\beta}\,\upsilon\gamma^5\upsilon-\frac{1}{4}(\Gamma_a\gamma^5\mathcal{P}_{24})^{\alpha\beta}\,\upsilon\Gamma^a\gamma^5\upsilon\\
&\quad-\frac{1}{4}(\Gamma_a\gamma^7\mathcal{P}_{24})^{\alpha\beta}\,\upsilon\Gamma^a\gamma^7\upsilon-\frac{1}{8}(\Gamma_{ab}\gamma^7\mathcal{P}_{24})^{\alpha\beta}\,\upsilon\Gamma^{ab}\gamma^7\upsilon\,.
\end{aligned}
\tag{B.1.12}
$$

Similarly we have in the $AdS_2\times S^2\times T^6$ case when $\upsilon=\mathcal{P}_{24}\upsilon$ that

$$
\begin{aligned}
(\mathcal{P}_8\Gamma^{a'}\upsilon)^\alpha\,(\Gamma_{a'}\upsilon)^\beta &= -\frac{1}{4}(\mathcal{P}_8\mathcal{C})^{\alpha\beta}\,\upsilon\upsilon-\frac{1}{4}(\mathcal{P}_8\gamma_5)^{\alpha\beta}\,\upsilon\gamma_5\upsilon-\frac{1}{4}(\mathcal{P}_8\Gamma^{\underline{a}}\gamma_5)^{\alpha\beta}\,\upsilon\Gamma_{\underline{a}}\gamma_5\upsilon\\
&\quad-\frac{1}{4}(\mathcal{P}_8\Gamma^{\underline{a}}\gamma_7)^{\alpha\beta}\,\upsilon\Gamma_{\underline{a}}\gamma_7\upsilon-\frac{1}{8}(\mathcal{P}_8\Gamma_{\underline{ab}}\gamma_7)^{\alpha\beta}\,\upsilon\Gamma^{\underline{ab}}\gamma_7\upsilon\\
&\quad-\frac{1}{4}(\mathcal{P}_8\Gamma_{\underline{a}b'}\gamma_5)^{\alpha\beta}\,\upsilon\Gamma^{\underline{a}b'}\gamma_5\upsilon-\frac{1}{8}(\mathcal{P}_8\Gamma_{\underline{abc}'})^{\alpha\beta}\,\upsilon\Gamma^{\underline{abc}'}\upsilon\,.
\end{aligned}
\tag{B.1.13}
$$

These identities were used in many places in the calculation of the curvature of the Lax connection.

Appendix C

Check of the closure of H in $AdS_2 \times S^2 \times T^6$

The NS–NS three-form superfield strength in the $AdS_2 \times S^2 \times T^6$ background is given by

$$\begin{aligned} H &= -i\mathcal{E}^A\,\mathcal{E}\Gamma_A\Gamma_{11}\mathcal{E} + i\mathcal{E}^B\mathcal{E}^A\,\mathcal{E}\Gamma_{AB}\Gamma_{11}\lambda + \frac{1}{3!}\mathcal{E}^C\mathcal{E}^B\mathcal{E}^A\,H_{ABC} \\ &= -ic^2\mathcal{E}^A\,E\Gamma_A\Gamma_{11}E - 2ic^4\hat{E}^A\,E\Gamma_A\Gamma_{11}D\upsilon - ic^4E^A\,D\upsilon\Gamma_A\Gamma_{11}D\upsilon \\ &\quad + \frac{ic^4}{R}E^A\Big(\upsilon\Gamma^B\gamma_\star\gamma_7\upsilon\,E\Gamma_{AB}\gamma_5E - \upsilon\Gamma^B\gamma_\star\Gamma_{11}\upsilon\,E\Gamma_{AB}E + \upsilon\Gamma^{AB}\gamma_\star\upsilon\,E\Gamma_B\Gamma_{11}E \\ &\quad - \upsilon\Gamma^{\underline{bc}'}\Gamma_{11}\upsilon\,E\gamma_\star\Gamma_A\Gamma_{\underline{bc}'}E + \frac{1}{2}\upsilon\Gamma^{\underline{bcd}'}\upsilon\,E\Gamma_{11}\gamma_\star\Gamma_A\Gamma_{\underline{bcd}'}E\Big) + \frac{2c^4}{R}\hat{E}^B\hat{E}^A\,E\Gamma_{AB}\Gamma_{11}\gamma_\star\upsilon \\ &\quad + \frac{2c^4}{R}E^BE^A\,D\upsilon\Gamma_{AB}\Gamma_{11}\gamma_\star\upsilon + \frac{c^6}{3!}E^CE^BE^A\,H_{ABC} + \mathcal{O}(\upsilon^3)\,, \end{aligned} \tag{C.0.1}$$

where $\mathcal{E}^A$ and H_{ABC} are given is (3.2.64) and (3.2.68) respectively, and $\hat{E}^A = E^A + 2i\upsilon\Gamma^AE$. We wish to demonstrate that this form is indeed closed.

After a bit of algebra using the torsion equation (2.1.6) and Fierz identities one finds

$$\begin{aligned} \frac{1}{c^4}dH &= \frac{2i}{R}\hat{E}^B\,E\Gamma^AE\,E\Gamma_{AB}\Gamma_{11}\gamma_\star\upsilon + \frac{2i}{R}\hat{E}^B\,E\Gamma_{AB}E\,E\Gamma^A\Gamma_{11}\gamma_\star\upsilon - \frac{2i}{R}\hat{E}^B\,E\Gamma_{11}E\,E\Gamma_B\gamma_\star\upsilon \\ &\quad + \frac{4i}{R}E^{b'}\,\upsilon\Gamma^{a'}E\,E\Gamma_{a'b'}\Gamma_{11}\gamma_\star\nabla\upsilon + \frac{8i}{R}E^B\,E\Gamma^{a'}D\upsilon\,E\Gamma_{a'B}\Gamma_{11}\gamma_\star\upsilon - \frac{2i}{R}E^{\underline{a}}\,E\Gamma_{\underline{a}}\Gamma_{11}E\,D\upsilon\gamma_\star\upsilon \\ &\quad + \frac{2i}{R}E^{\underline{b}}\,\upsilon\Gamma^{\underline{a}}D\upsilon\,E\Gamma_{\underline{a}}\Gamma_{11}\gamma_\star\Gamma_{\underline{b}}E - i\hat{E}^B\,(\Omega_{AB} - \Omega_{0AB})\,E\Gamma^A\Gamma_{11}E - 2iE^A\,D\upsilon\Gamma_A\Gamma_{11}\nabla D\upsilon \\ &\quad - 2iE^B\,(\Omega_{AB} - \Omega_{0AB})\,E\Gamma^A\Gamma_{11}D\upsilon - \frac{4}{R}E^CE^B\,(\Omega_{AB} - \Omega_{0AB})\,E\Gamma^A{}_C\Gamma_{11}\gamma_\star\upsilon \end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{R^2}E^{a'}E^{\underline{c}}\,\upsilon\Gamma^{b'}E\,E\Gamma_{a'b'\underline{c}}\Gamma_{11}\upsilon+\frac{i}{R}E^{A}\,\nabla\Big(\upsilon\Gamma^{B}\gamma_\star\gamma_7\upsilon\,E\Gamma_{AB}\gamma_5 E-\upsilon\Gamma^{B}\gamma_\star\Gamma_{11}\upsilon\,E\Gamma_{AB}E\\
&+\upsilon\Gamma^{AB}\gamma_\star\upsilon\,E\Gamma_{B}\Gamma_{11}E-\upsilon\Gamma^{\underline{b}c'}\Gamma_{11}\upsilon\,E\gamma_\star\Gamma_{A}\Gamma_{\underline{b}c'}E+\frac{1}{2}\upsilon\Gamma^{\underline{b}cd'}\upsilon\,E\Gamma_{11}\gamma_\star\Gamma_{A}\Gamma_{\underline{b}cd'}E\Big)\\
&+d\left(\frac{2}{R}E^{B}E^{A}\,D\upsilon\Gamma_{AB}\Gamma_{11}\gamma_\star\upsilon+\frac{c^2}{6}E^{C}E^{B}E^{A}\,H_{ABC}\right)\,. \qquad \text{(C.0.2)}
\end{aligned}
$$

The terms in the first line vanish due to the Fierz identity (2.3.15). Using the expressions for Ω^{AB} and H_{ABC} given in (3.2.66) and (3.2.68) and simplifying further this becomes

$$
\begin{aligned}
&-\frac{1}{R^2}E^{C}E^{b'}\,E\Gamma^{A}E\,\upsilon\gamma_\star\Gamma_{b'}\Gamma_{A}\Gamma_{11}(1-P)\Gamma_{C}\gamma_\star\upsilon+\dots\\
&+\frac{2}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma_{\underline{c}}\gamma_\star\Gamma_{a'}\upsilon\,E\Gamma_{11}\gamma_\star\Gamma_{b'}\Gamma_{a'}\upsilon+\frac{2}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma_{\underline{c}}\Gamma_{11}\gamma_\star\Gamma_{a'}\upsilon\,E\gamma_\star\Gamma_{b'}\Gamma_{a'}\upsilon\\
&-\frac{4}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma^{a'}\upsilon\,E\Gamma_{b'\underline{c}}\Gamma_{11}\Gamma_{a'}\upsilon-\frac{8}{R^2}E^{B}E^{\underline{c}}\,E\Gamma_{\underline{c}}\gamma_\star\gamma^{a'}\upsilon\,E\Gamma_{B}\Gamma_{11}\gamma_\star\Gamma_{a'}\upsilon\\
&-\frac{2}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma_{\underline{c}}{}^{\hat{a}}E\,\upsilon\Gamma_{\hat{a}b'}\Gamma_{11}\upsilon-\frac{2}{R^2}E^{b'}E^{c}\,E\gamma_7 E\,\upsilon\Gamma_{cb'}\gamma_5\upsilon-\frac{1}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma^{\underline{a}}\Gamma_{11}E\,\upsilon\Gamma_{\underline{ca}b'}\upsilon\\
&-\frac{2}{R^2}E^{B}E^{\underline{c}}\,E\Gamma^{\underline{a}}\Gamma_{11}E\,\upsilon\gamma_\star\Gamma_{\underline{ca}B}\gamma_\star\upsilon+\frac{2}{R^2}E^{B}E^{\underline{c}}\,E\Gamma^{\underline{a}}E\,\upsilon\gamma_\star\Gamma_{\underline{ca}B}\Gamma_{11}\gamma_\star\upsilon\\
&-\frac{1}{R^2}E^{b'}E^{\underline{c}}\,E\Gamma^{\underline{a}}E\,\upsilon\Gamma_{\underline{ca}b'}\Gamma_{11}\upsilon+\frac{2i}{R}E^{a'}\,E\Gamma^{b\hat{c}}E\,\upsilon\Gamma_{a'b\hat{c}}\gamma_\star\Gamma_{11}\nabla\upsilon+\frac{2i}{R}E^{\underline{a}}\,E\Gamma_{\underline{ab}}E\,\upsilon\gamma_\star\Gamma^{\underline{b}}\Gamma_{11}\nabla\upsilon\\
&+\frac{2i}{R}E^{\underline{a}}\,E\Gamma_{\underline{ab}}\gamma_5 E\,\upsilon\gamma_\star\Gamma^{\underline{b}}\gamma_7\nabla\upsilon+\frac{4i}{R}E^{b'}\,\upsilon\Gamma^{a'}E\,E\Gamma_{a'b'}\Gamma_{11}\gamma_\star\nabla\upsilon+\frac{8i}{R}E^{B}\,E\Gamma^{a'}\nabla\upsilon\,E\Gamma_{a'B}\Gamma_{11}\gamma_\star\upsilon\\
&+\frac{4i}{R}E^{b'}\,\upsilon\gamma_\star\Gamma_{a'b'}E\,E\Gamma^{a'}\Gamma_{11}\nabla\upsilon-\frac{3i}{R}E^{a'}\,E\Gamma^{\underline{b}}E\,\upsilon\gamma_\star\Gamma_{a'\underline{b}}\Gamma_{11}\nabla\upsilon-\frac{3i}{R}E^{a'}\,E\Gamma^{\underline{b}}\Gamma_{11}E\,\upsilon\gamma_\star\Gamma_{a'\underline{b}}\nabla\upsilon\\
&-\frac{i}{R}E^{a'}\,E\Gamma^{\underline{b}}E\,\upsilon\Gamma_{a'\underline{b}}\gamma_\star\Gamma_{11}\nabla\upsilon+\frac{i}{R}E^{a'}\,E\Gamma^{\underline{b}}\Gamma_{11}E\,\upsilon\Gamma_{a'\underline{b}}\gamma_\star\nabla\upsilon-\frac{2i}{R}E^{\underline{b}}\,E\Gamma^{\underline{a}}E\,\upsilon\gamma_\star\Gamma_{\underline{b}}\Gamma_{\underline{a}}\Gamma_{11}\nabla\upsilon\\
&-\frac{2i}{R}E^{\underline{b}}\,E\Gamma^{\underline{a}}\Gamma_{11}E\,\upsilon\gamma_\star\Gamma_{\underline{b}}\Gamma_{\underline{a}}\nabla\upsilon+\frac{2i}{R}E^{A}\,E\Gamma_{11}E\,\upsilon\gamma_\star\Gamma_{A}\nabla\upsilon-\frac{2i}{R}E^{A}\,E\gamma_7 E\,\upsilon\gamma_\star\gamma_5\Gamma_{A}\nabla\upsilon\\
&+\frac{4i}{R}E^{a'}\,E\gamma_7 E\,\upsilon\gamma_\star\gamma_5\Gamma_{a'}\nabla\upsilon\,, \qquad \text{(C.0.3)}
\end{aligned}
$$

where the ellipsis in the first line denote three terms which, together with the previous term, cancel due to the Fierz identitity (2.3.14). Using the Fierz identity in (B.1.13) the terms with two bosonic vielbeins can be seen to cancel and we are left with

$$
\begin{aligned}
&\frac{2i}{R}E^{a'}\,E\Gamma^{b\hat{c}}E\,\upsilon\Gamma_{a'b\hat{c}}\gamma_\star\Gamma_{11}\nabla\upsilon+\frac{2i}{R}E^{\underline{a}}\,E\Gamma_{\underline{ab}}E\,\upsilon\gamma_\star\Gamma^{\underline{b}}\Gamma_{11}\nabla\upsilon+\frac{2i}{R}E^{\underline{a}}\,E\Gamma_{\underline{ab}}\gamma_5 E\,\upsilon\gamma_\star\Gamma^{\underline{b}}\gamma_7\nabla\upsilon\\
&+\frac{4i}{R}E^{b'}\,\upsilon\Gamma^{a'}E\,E\Gamma_{a'b'}\Gamma_{11}\gamma_\star\nabla\upsilon+\frac{8i}{R}E^{B}\,E\Gamma^{a'}\nabla\upsilon\,E\Gamma_{a'B}\Gamma_{11}\gamma_\star\upsilon+\frac{4i}{R}E^{b'}\,\upsilon\gamma_\star\Gamma_{a'b'}E\,E\Gamma^{a'}\Gamma_{11}\nabla\upsilon
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3i}{R} E^{a'}\, E\Gamma^{\underline{b}} E\, \upsilon\gamma_\star \Gamma_{a'\underline{b}} \Gamma_{11} \nabla\upsilon - \frac{3i}{R} E^{a'}\, E\Gamma^{\underline{b}} \Gamma_{11} E\, \upsilon\gamma_\star \Gamma_{a'\underline{b}} \nabla\upsilon - \frac{i}{R} E^{a'}\, E\Gamma^{\underline{b}} E\, \upsilon\Gamma_{a'\underline{b}} \gamma_\star \Gamma_{11} \nabla\upsilon \\
&+ \frac{i}{R} E^{a'}\, E\Gamma^{\underline{b}} \Gamma_{11} E\, \upsilon\Gamma_{a'\underline{b}} \gamma_\star \nabla\upsilon - \frac{2i}{R} E^{\underline{b}}\, E\Gamma^{\underline{a}} E\, \upsilon\gamma_\star \Gamma_{\underline{b}} \Gamma_{\underline{a}} \Gamma_{11} \nabla\upsilon - \frac{2i}{R} E^{\underline{b}}\, E\Gamma^{\underline{a}} \Gamma_{11} E\, \upsilon\gamma_\star \Gamma_{\underline{b}} \Gamma_{\underline{a}} \nabla\upsilon \\
&+ \frac{2i}{R} E^{A}\, E\Gamma_{11} E\, \upsilon\gamma_\star \Gamma_A \nabla\upsilon - \frac{2i}{R} E^{A}\, E\gamma_7 E\, \upsilon\gamma_\star \gamma_5 \Gamma_A \nabla\upsilon + \frac{4i}{R} E^{a'}\, E\gamma_7 E\, \upsilon\gamma_\star \gamma_5 \Gamma_{a'} \nabla\upsilon\,. \qquad \text{(C.0.4)}
\end{aligned}
$$

We now use the fact that

$$
\begin{aligned}
&\frac{2i}{R} E^{\underline{a}}\, E\Gamma_{\underline{ab}} E\, \upsilon\gamma_\star \Gamma^{\underline{b}} \Gamma_{11} \nabla\upsilon + \frac{2i}{R} E^{\underline{a}}\, E\Gamma_{\underline{ab}} \gamma_5 E\, \upsilon\gamma_\star \Gamma^{\underline{b}} \gamma_7 \nabla\upsilon \\
&= \frac{2i}{R} E^{\underline{a}}\, E\Gamma_{\underline{a}} \Gamma_B E\, \upsilon\gamma_\star \Gamma^B \Gamma_{11} \nabla\upsilon + \ldots + \frac{2i}{R} E^{\underline{a}}\, E\gamma_5 \Gamma_{\underline{a}} \Gamma_B E\, \upsilon\gamma_\star \Gamma^B \Gamma_{11} \gamma_5 \nabla\upsilon + \ldots \\
&\quad + \frac{4i}{R} E^{\underline{a}}\, \upsilon\gamma_\star \Gamma^{b'} \Gamma_{\underline{a}} E\, \nabla\upsilon \Gamma_{b'} \Gamma_{11} E - \frac{4i}{R} E^{\underline{a}}\, \upsilon\gamma_\star \Gamma^{b'} \Gamma_{\underline{a}} \Gamma_{11} E\, \nabla\upsilon \Gamma_{b'} E \\
&\quad - \frac{2i}{R} E^{\underline{a}}\, E\Gamma_{11} E\, \upsilon\gamma_\star \Gamma_{\underline{a}} \nabla\upsilon - \frac{2i}{R} E^{\underline{a}}\, E\gamma_7 E\, \upsilon\gamma_\star \Gamma_{\underline{a}} \gamma_5 \nabla\upsilon \\
&= \frac{4i}{R} E^{\underline{a}}\, \upsilon\gamma_\star \Gamma^{b'} \Gamma_{\underline{a}} E\, \nabla\upsilon \Gamma_{b'} \Gamma_{11} E - \frac{4i}{R} E^{\underline{a}}\, \upsilon\gamma_\star \Gamma^{b'} \Gamma_{\underline{a}} \Gamma_{11} E\, \nabla\upsilon \Gamma_{b'} E \\
&\quad - \frac{2i}{R} E^{\underline{a}}\, E\Gamma_{11} E\, \upsilon\gamma_\star \Gamma_{\underline{a}} \nabla\upsilon - \frac{2i}{R} E^{\underline{a}}\, E\gamma_7 E\, \upsilon\gamma_\star \Gamma_{\underline{a}} \gamma_5 \nabla\upsilon\,, \qquad \text{(C.0.5)}
\end{aligned}
$$

where the first ellipsis denote the 5 terms which together with the previous term cancel due to the Fierz identity (2.3.14) and similarly for the second ellipsis. This leaves us with the following terms in dH

$$
\begin{aligned}
&- \frac{2i}{R} E^{B}\, E\Gamma^{A} E\, \nabla\upsilon \Gamma_A \Gamma_{11} \Gamma_B \gamma_\star \upsilon + \ldots - \frac{i}{R} E^{b'}\, E\Gamma^{A} E\, \nabla\upsilon \Gamma_A \Gamma_{11} \Gamma_{b'} \gamma_\star \upsilon + \ldots \\
&- \frac{i}{R} E^{b'}\, E\Gamma^{A} E\, \nabla\upsilon \gamma_\star \Gamma_{b'} \Gamma_A \Gamma_{11} \upsilon + \ldots + \frac{2i}{R} E^{b'}\, E\gamma_\star \Gamma_{b'} \Gamma^{A} E\, \nabla\upsilon \Gamma_A \Gamma_{11} \upsilon + \ldots \\
&+ \frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'} \upsilon\, \nabla\upsilon \Gamma^{a'} \Gamma_{11} E + \frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'} \Gamma_{11} \upsilon\, \nabla\upsilon \Gamma^{a'} E \\
&+ \frac{2i}{R} E^{a'}\, E\Gamma^{b\hat{c}} E\, \upsilon\Gamma_{a'b\hat{c}} \gamma_\star \Gamma_{11} \nabla\upsilon - \frac{2i}{R} E^{b'}\, E\gamma_\star \Gamma_{11} E\, \nabla\upsilon \Gamma_{b'} \upsilon - \frac{2i}{R} E^{b'}\, E\gamma E\, \nabla\upsilon \Gamma_{b'} \gamma_5 \upsilon \\
&+ \frac{2i}{R} E^{b'}\, E\Gamma_{11} E\, \upsilon\gamma_\star \Gamma_{b'} \nabla\upsilon + \frac{2i}{R} E^{b'}\, E\gamma_7 E\, \upsilon\gamma_\star \Gamma_{b'} \gamma_5 \nabla\upsilon\,. \qquad \text{(C.0.6)}
\end{aligned}
$$

Again the ellipsis denotes terms which cancel together with the previous term due to (2.3.14). Using the relation

$$
\frac{2i}{R} E^{a'}\, E\Gamma^{b\hat{c}} E\, \upsilon\Gamma_{a'b\hat{c}} \gamma_\star \Gamma_{11} \nabla\upsilon
$$

$$
\begin{aligned}
&= -\frac{i}{2R} E^{b'}\, E\gamma_\star \Gamma^{\underline{ca}} \Gamma_{11} E\, \nabla\upsilon \Gamma_{\underline{ac}b'}\upsilon - \frac{i}{2R} E^{b'}\, E\gamma_\star\gamma_5 \Gamma^{\underline{ca}} \Gamma_{11} E\, \nabla\upsilon \Gamma_{\underline{ac}b'}\gamma_5\upsilon \\
&= -\frac{i}{2R} E^{b'}\, E\gamma_\star \Gamma^{\underline{c}} \Gamma^A \Gamma_{11} E\, \nabla\upsilon \Gamma_A \Gamma_{\underline{c}b'}\upsilon + \ldots - \frac{i}{2R} E^{b'}\, E\gamma_\star\gamma_5 \Gamma^{\underline{c}} \Gamma^A \Gamma_{11} E\, \nabla\upsilon \Gamma_A \Gamma_{\underline{c}b'}\gamma_5\upsilon + \ldots \\
&\quad - \frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'}\upsilon\, \nabla\upsilon \Gamma^{a'} \Gamma_{11} E - \frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'} \Gamma_{11}\upsilon\, \nabla\upsilon \Gamma^{a'} E \\
&\quad + \frac{2i}{R} E^{b'}\, E\gamma_7 E\, \nabla\upsilon \Gamma_{b'}\gamma_\star\gamma_5\upsilon + \frac{2i}{R} E^{b'}\, E\Gamma_{11} E\, \nabla\upsilon \Gamma_{b'}\gamma_\star\upsilon \\
&\quad + \frac{2i}{R} E^{b'}\, E\gamma E\, \nabla\upsilon \Gamma_{b'}\gamma_5\upsilon + \frac{2i}{R} E^{b'}\, E\gamma_\star \Gamma_{11} E\, \nabla\upsilon \Gamma_{b'}\upsilon \\
&= -\frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'}\upsilon\, \nabla\upsilon \Gamma^{a'} \Gamma_{11} E - \frac{4i}{R} E^{b'}\, E\gamma_\star \Gamma_{a'b'} \Gamma_{11}\upsilon\, \nabla\upsilon \Gamma^{a'} E \\
&\quad + \frac{2i}{R} E^{b'}\, E\gamma_7 E\, \nabla\upsilon \Gamma_{b'}\gamma_\star\gamma_5\upsilon + \frac{2i}{R} E^{b'}\, E\Gamma_{11} E\, \nabla\upsilon \Gamma_{b'}\gamma_\star\upsilon \\
&\quad + \frac{2i}{R} E^{b'}\, E\gamma E\, \nabla\upsilon \Gamma_{b'}\gamma_5\upsilon + \frac{2i}{R} E^{b'}\, E\gamma_\star \Gamma_{11} E\, \nabla\upsilon \Gamma_{b'}\upsilon\,, \qquad \text{(C.0.7)}
\end{aligned}
$$

we see that also the last remaining seven terms in dH cancel. This completes the proof that the NS–NS three-form we have constructed for $AdS_2 \times S^2 \times T^6$ is indeed closed.

Appendix D

CP^3 Geometry

The Fubini-Study metric on CP^3 is

$$ds^2 = \rho^{-2} d\bar{\zeta}_a d\zeta^a - \rho^{-4} \zeta^a d\bar{\zeta}_a \, \bar{\zeta}_b d\zeta^b \,, \tag{D.0.1}$$

where ζ^a are three complex numbers and $\rho^2 = 1 + \zeta^a \bar{\zeta}_a$. Real coordinates adapted to the $U(3)$ isotropy group can be introduced as follows [76]

$$\begin{aligned} \zeta^1 &= \tan\frac{\theta}{2} \, \sin\alpha \, \sin\frac{\vartheta}{2} \, e^{i(\psi-\chi)/2} e^{i\varphi} \\ \zeta^2 &= \tan\frac{\theta}{2} \, \cos\alpha \, e^{i\varphi} \\ \zeta^3 &= \tan\frac{\theta}{2} \, \sin\alpha \, \cos\frac{\vartheta}{2} \, e^{i(\psi+\chi)/2} e^{i\varphi} \,, \end{aligned} \tag{D.0.2}$$

where $0 \leq \theta, \vartheta \leq \pi$, $0 \leq \varphi, \chi \leq 2\pi$, $0 \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \psi \leq 4\pi$. In these coordinates the metric becomes

$$ds^2 = \frac{1}{4}\big(d\theta^2 + \sin^2\theta (d\varphi + \frac{1}{2}\sin^2\alpha \, \sigma_3)^2\big) + \sin^2\frac{\theta}{2} \, d\alpha^2 + \frac{1}{4}\sin^2\frac{\theta}{2} \, \sin^2\alpha (\sigma_1^2 + \sigma_2^2 + \cos^2\alpha \, \sigma_3^2) \,, \tag{D.0.3}$$

where

$$\begin{aligned} \sigma_1 &= \sin\psi \, d\vartheta - \cos\psi \, \sin\vartheta \, d\chi \\ \sigma_2 &= \cos\psi \, d\vartheta + \sin\psi \, \sin\vartheta \, d\chi \\ \sigma_3 &= d\psi + \cos\vartheta \, d\chi \end{aligned} \tag{D.0.4}$$

are three left-invariant one-forms on $SU(2)$ obeying $d\sigma_1 = -\sigma_2\sigma_3$ etc. Notice that with this choice of coordinates θ and φ parameterize a two-sphere of radius $\frac{1}{2}$. This

two-sphere is topologically non-trivial and associated to the Kähler form on CP^3. We choose the CP^3 vielbeins as follows

$$\begin{aligned}
e^1 &= \frac{1}{2}d\theta \\
e^2 &= \frac{1}{2}\sin\theta(d\varphi + \frac{1}{2}\sin^2\alpha\,\sigma_3) \\
e^3 &= -\frac{1}{2}\sin\frac{\theta}{2}\,\sin\alpha\,\sigma_2 \\
e^4 &= \frac{1}{2}\sin\frac{\theta}{2}\,\sin\alpha\,\sigma_1 \\
e^5 &= \sin\frac{\theta}{2}\,d\alpha \\
e^6 &= \frac{1}{4}\sin\frac{\theta}{2}\,\sin(2\alpha)\,\sigma_3\,.
\end{aligned} \tag{D.0.5}$$

Using the fact that

$$\begin{aligned}
de^1 &= 0 \\
de^2 &= 2\cot\theta\,e^2\,e^1 + 2\cot\frac{\theta}{2}\,e^6\,e^5 + 2\cot\frac{\theta}{2}\,e^4\,e^3 \\
de^3 &= \cot\frac{\theta}{2}\,e^3\,e^1 + \frac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^3\,e^5 + \frac{4}{\sin\frac{\theta}{2}\,\sin(2\alpha)}\,e^6\,e^4 \\
de^4 &= \cot\frac{\theta}{2}\,e^4\,e^1 + \frac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^4\,e^5 + \frac{4}{\sin\frac{\theta}{2}\,\sin(2\alpha)}\,e^3\,e^6 \\
de^5 &= \cot\frac{\theta}{2}\,e^5\,e^1 \\
de^6 &= \cot\frac{\theta}{2}\,e^6\,e^1 + \frac{2\cot(2\alpha)}{\sin\frac{\theta}{2}}\,e^6\,e^5 + \frac{2\cot\alpha}{\sin\frac{\theta}{2}}\,e^4\,e^3
\end{aligned} \tag{D.0.6}$$

one can show that the connection can be taken to be

$$\begin{aligned}
\omega^{12} &= 2\cot\theta\,e^2 & \omega^{34} &= \cot\tfrac{\theta}{2}\,e^2 - 2\tfrac{1+\sin^2\alpha}{\sin\frac{\theta}{2}\sin(2\alpha)}\,e^6 \\
\omega^{1\tilde{a}} &= \cot\tfrac{\theta}{2}\,e^{\tilde{a}} & \omega^{35} &= -\tfrac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^3 \\
\omega^{23} &= -\cot\tfrac{\theta}{2}\,e^4 & \omega^{36} &= \tfrac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^4 \\
\omega^{24} &= \cot\tfrac{\theta}{2}\,e^3 & \omega^{45} &= -\tfrac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^4 \\
\omega^{25} &= -\cot\tfrac{\theta}{2}\,e^6 & \omega^{46} &= -\tfrac{\cot\alpha}{\sin\frac{\theta}{2}}\,e^3 \\
\omega^{26} &= \cot\tfrac{\theta}{2}\,e^5 & \omega^{56} &= \cot\tfrac{\theta}{2}\,e^2 + 2\tfrac{\cot(2\alpha)}{\sin\frac{\theta}{2}}\,e^6\,,
\end{aligned} \tag{D.0.7}$$

where $\tilde{a} = 3, 4, 5, 6$. The curvature of CP^3 is

$$R^{a'b'} = d\omega^{a'b'} + \omega^{a'}{}_{c'}\omega^{c'b'} = (\delta^{a'}_{c'}\delta^{b'}_{d'} + J_{c'}{}^{a'}J_{d'}{}^{b'})\, e^{c'}e^{d'} + J^{a'b'}J_{c'd'}\, e^{c'}e^{d'}\,, \tag{D.0.8}$$

where $J_{a'b'}$ are the components of the Kähler form with $J_{12} = J_{34} = J_{56} = 1$.

The $U(1)$ part of the connection is

$$A = \frac{1}{8}J_{a'b'}\omega^{a'b'} = \frac{1}{2}(\cot\theta\, e^2 + \cot\frac{\theta}{2}\, e^2 - 3\frac{\tan\alpha}{2\sin\frac{\theta}{2}}\, e^6) = \cot\theta\, e^2 + \frac{1}{4}d\varphi - \frac{\tan\alpha}{2\sin\frac{\theta}{2}}\, e^6\,, \tag{D.0.9}$$

and it is easy to verify that it's derivative is proportional to the Kähler form

$$dA = 2\, e^1\, e^2 + 2\, e^3\, e^4 + 2\, e^5\, e^6\,. \tag{D.0.10}$$

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