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DOTTORATO DI RICERCA IN

MATEMATICA

X CICLO

PERTURBATION ANALYSIS OF THE CONFORMAL SEWING PROBLEM AND RELATED PROBLEMS

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Riassunto

In questa tesi sviluppiamo due problemi connessi di analisi funzionale non lineare: l'analisi di tipo perturbativo del problema di cucitura conforme negli spazi di Schauder e di Roumieu che, nella nostra formulazione, richiede come prerequisito lo studio di un secondo problema, ovvero l'analiticità dell'integrale singolare di Cauchy negli spazi di Schauder. Nel Capitolo II, consideriamo l'integrale singolare di Cauchy

$$\boldsymbol{C}[\phi, f](\cdot) \equiv \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t)-\phi(\cdot)}\,dt = \frac{1}{2\pi i}\mathrm{p.\ v.}\ \int_{\phi} \frac{f\circ\phi^{(-1)}(\xi)}{\xi-\phi(\cdot)}\,d\xi$$

dove la curva semplice orientata ϕ e la funzione densità f sono entrambe definite nel bordo orientato $\partial\mathbb{D}$ del disco unitario aperto $\mathbb{D}$. Nonostante temi di ricerca come le proprietà dell'operatore lineare $\boldsymbol{C}[\phi,\cdot]$ ed il calcolo numerico della funzione $\boldsymbol{C}[\phi, f]$ siano stati approfonditamente studiati a partire dal secolo scorso in vista di numerose applicazioni alle equazioni integrali e ai problemi al contorno (cf. *e.g.* Muskhelishvili (1953) and Gakhov (1966)), l'analisi della dipendenza funzionale di $\boldsymbol{C}[\phi, f]$ da entrambe gli argomenti, ed in particolare da ϕ, sembra cominciare solo di recente (cf. Introduzione Cap. II). Questo nuovo argomento di ricerca può essere applicato allo studio di tipo perturbativo di problemi non lineari in cui compaia l'integrale singolare di Cauchy. Nel Capitolo II estendiamo un risultato di analiticità di Coifman & Meyer (1983b) ad un contesto di spazi di Schauder. Assumiamo che ϕ e f appartengano ad uno spazio di Schauder, che chiameremo $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$, costituito dalle funzioni di classe $\mathcal{C}^{m,\alpha}$ su $\partial\mathbb{D}$, dove m è un numero naturale positivo ed $\alpha \in\,]0,1[$. È ben noto che sotto queste ipotesi su ϕ e f, la funzione $\boldsymbol{C}[\phi, f](\cdot)$ è di classe $\mathcal{C}^{m,\alpha}$. Dimostrando esistenza ed unicità di soluzione per un problema al contorno di tipo ellittico ed applicando il Teorema della Funzione Implicita, otteniamo che l'operatore $\boldsymbol{C}[\cdot,\cdot]$ è reale analitico. Successivamente calcoliamo tutti i differenziali di $\boldsymbol{C}[\cdot,\cdot]$ e proviamo che $\boldsymbol{C}[\cdot,\cdot]$ è analitico in senso complesso. Questo risultato di Lanza & Preciso (1998) è stato applicato nella seconda parte della tesi ed in un altro problema di perturbazione relativo ad una equazione integrale non lineare (cf. Lanza & Rogosin (1997)).

Nel Capitolo III, introduciamo il problema di cucitura conforme associato ad uno shift ϕ di $\partial\mathbb{D}$, *i.e.* un omeomorfismo di $\partial\mathbb{D}$ in sé. Tale problema consiste nella ricerca di una coppia di funzioni (F, G) definite in $\mathbb{D}$ e $\mathbb{C}\setminus \mathrm{cl}\,\mathbb{D}$, rispettivamente,

tali che le loro estensioni continue a $\mathrm{cl}\,\mathbb{D}$ e $\mathbb{C}\setminus\mathbb{D}$, $\widetilde{F}$ e $\widetilde{G}$ rispettivamente, soddisfino

$$\widetilde{F}(\phi(t)) = \widetilde{G}(t)$$

per ogni $t \in \partial\mathbb{D}$. Una semplice condizione di normalizzazione e risultati noti assicurano che il problema di cucitura conforme abbia un'unica soluzione (F, G) e indichiamo con $(\boldsymbol{F}[\cdot], \boldsymbol{G}[\cdot])$ la coppia di operatori che mappa lo shift ϕ nella traccia su $\partial\mathbb{D}$ di tale soluzione. Le proprietà di regolarità degli operatori $\boldsymbol{F}[\cdot]$ e $\boldsymbol{G}[\cdot]$ in spazi di funzioni regolari possono essere usate per ottenere informazioni sulle funzioni $\boldsymbol{F}[\phi]$ e $\boldsymbol{G}[\phi]$, che in generale risultano solo implicitamente determinate come soluzioni di equazioni integrali. Una tale analisi può avere un parziale interesse nello studio degli spazi di Teichmüller, che costituiscono un importante argomento nella teoria geometrica delle funzioni (cf. Nag (1996)). Quindi ci siamo proposti di trovare spazi di funzioni regolari rispetto ai quali gli operatori $\boldsymbol{F}[\cdot]$ e $\boldsymbol{G}[\cdot]$ siano analitici. Prima studiamo la regolarità di tali operatori negli spazi di Schauder $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$, con $m \geq 1$, $\alpha \in]0, 1[$. Utilizzando un metodo classico di equazioni integrali con shift già applicato al problema di cucitura conforme, mostriamo che $\boldsymbol{G}[\phi]$ e $\boldsymbol{F}[\phi] = \boldsymbol{G}[\phi] \circ \phi^{(-1)}$ appartengono a $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ quando ϕ appartiene $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$. In tale contesto di spazi di Schauder, usando l'analiticità dell'integrale singolare di Cauchy (cf. Cap. II) ed applicando il Teorema della Funzione Implicita ad un'opportuna equazione integrale, mostriamo che $\boldsymbol{G}[\cdot]$ si estende ad un operatore analitico in senso complesso. Proviamo poi che questo contesto di spazi di Schauder non è sufficiente per ottenere un'estensione analitica dell'operatore $\boldsymbol{F}[\cdot]$. Infatti una condizione naturale per avere $\boldsymbol{F}[\cdot]$ analitico si rivela essere l'appartenenza di ϕ ad uno spazio di funzioni reali analitiche di $\partial\mathbb{D}$ in $\mathbb{C}$. Nel Capitolo IV introduciamo dei ben noti spazi di funzioni reali analitiche, vale a dire gli spazi di Roumieu associati all'operatore di differenziazione. In questo contesto dimostriamo che gli operatori $\boldsymbol{G}[\cdot]$ e $\boldsymbol{F}[\cdot]$ si estendono ad operatori analitici in senso complesso utilizzando i risultati di regolarità per l'operatore di composizione e di inversione di Lanza (1994 e 1996b).

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Abstract

In this dissertation, we develop two related problems in the nonlinear functional analysis. The first is the analyticity of the Cauchy singular integral in Schauder spaces which is motivated by the second problem, namely the perturbation analysis of the conformal sewing problem in Schauder and Roumieu spaces. In Chapter II, we consider the Cauchy singular integral

$$\boldsymbol{C}[\phi, f](\,\cdot\,) \equiv \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t)-\phi(\cdot)}\,dt = \frac{1}{2\pi i}\mathrm{p.\ v.}\ \int_{\phi} \frac{f\circ\phi^{(-1)}(\xi)}{\xi-\phi(\cdot)}\,d\xi$$

where the oriented simple closed curve ϕ and the density function f are both defined on the counterclockwise oriented boundary $\partial\mathbb{D}$ of the plane unit disk $\mathbb{D}$. Although the linear operator $\boldsymbol{C}[\phi, \cdot]$, for a fixed ϕ, and the numerical computation of $\boldsymbol{C}[\phi, f]$ have been extensively studied for the last century, in view to several applications to integral equations and boundary value problems (cf. *e.g.* Muskhelishvili (1953) and Gakhov (1966)), the analysis of the nonlinear functional dependence of $\boldsymbol{C}[\phi, f]$ upon both its arguments seems to be a subject analyzed only more recently (see Introduction Ch. II). This new subject of research finds application in the nonlinear problems of perturbation nature which involve the Cauchy singular integral. In Chapter II we extend the analyticity result for the operator $\boldsymbol{C}[\cdot, \cdot]$ of Coifman & Meyer (1983b) to a Schauder spaces setting. We assume that both ϕ and f belong to a Schauder space, say $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C})$, of complex-valued function of class $\mathcal{C}^{m,\alpha}$ on $\partial\mathbb{D}$, with m a positive natural number and $\alpha \in\,]0, 1[$. As it is well-known, under such conditions on ϕ and f, the function $\boldsymbol{C}[\phi, f](\cdot)$ is also of class $\mathcal{C}^{m,\alpha}$. By proving the unique solvability of a boundary value problem of elliptic nature in $\mathbb{D}$ and by applying Implicit Function Theorem to a suitable functional equation, we show the real analyticity of $\boldsymbol{C}[\cdot, \cdot]$. Then we show the complex analyticity of $\boldsymbol{C}[\cdot, \cdot]$ and we compute all its differentials. This result of Lanza & Preciso (1998) will be applied in the second part of this dissertation and in another perturbation problem associated to a nonlinear integral equation (cf. Lanza & Rogosin (1997)).

In Chapter III, we introduce the conformal sewing problem associated to a shift ϕ of $\partial\mathbb{D}$, *i.e.* a homeomorphism of $\partial\mathbb{D}$ to itself. It consists in finding a pair of conformal functions (F, G) defined in $\mathbb{D}$ and $\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}$, respectively, such that their continuous extensions to $\mathrm{cl}\,\mathbb{D}$ e $\mathbb{C} \setminus \mathbb{D}$, $\widetilde{F}$ and $\widetilde{G}$ respectively, satisfy

$$\widetilde{F}(\phi(t)) = \widetilde{G}(t)$$

for all $t \in \partial\mathbb{D}$. A simple normalization condition and well-known results ensure that the sewing problem associated to ϕ has a unique solution (F, G) and we denote by $(\boldsymbol{F}[\cdot], \boldsymbol{G}[\cdot])$ the pair of operators which maps ϕ to the trace on $\partial\mathbb{D}$ of such solution. The regularity properties of the operators $\boldsymbol{F}[\phi]$ and $\boldsymbol{G}[\phi]$ in spaces of regular functions can be used in the study of Teichmüller spaces, which constitute an important subject in geometric function theory (see Nag (1996)). Our aim is to find natural Banach spaces of regular functions where to obtain the analyticity of $\boldsymbol{F}[\cdot]$ and $\boldsymbol{G}[\cdot]$. First we study the regularity of such operators in Schauder spaces $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$, with $m \geq 1$, $\alpha \in]0, 1[$. By using the classical integral equation approach to the sewing problem, we show that $\boldsymbol{G}[\phi]$ and $\boldsymbol{F}[\phi] = \boldsymbol{G}[\phi] \circ \phi^{(-1)}$ belong to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ when ϕ belongs to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. In this setting, by using the analyticity of the Cauchy singular integral (cf. Ch. II) and by applying Implicit Function Theorem to a suitable integral equation, we show that $\boldsymbol{G}[\cdot]$ extends to a complex analytic operator. Then we prove that this Schauder spaces setting is not sufficient in order to obtain an analytic extension of the operator $\boldsymbol{F}[\cdot]$. Indeed a natural assumption in order to have $\boldsymbol{F}[\cdot]$ analytic, is that ϕ belongs to a space of real analytic functions of $\partial\mathbb{D}$ to $\mathbb{C}$. In Chapter IV we introduce Banach spaces of real analytic functions, namely the Roumieu spaces associated to the differentiation operator. In this setting we show that $\boldsymbol{G}[\cdot]$ and $\boldsymbol{F}[\cdot]$ can be extended to complex analytic operators by employing the regularity results on the composition and on the inversion operator of Lanza (1994 and 1996b).

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CHAPTER 1

Preliminaries and notation

1.1. Notation

Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say that $\mathcal{X}$ is continuously imbedded in $\mathcal{Y}$, provided that $\mathcal{X} \subseteq \mathcal{Y}$ and that the inclusion map is continuous. Let $\mathbb{N}$ be the set of nonnegative integers including 0 and let $n \in \mathbb{N} \setminus \{0\}$. Let $\{\mathcal{X}_i : i = 1, \ldots, n\}$ be normed spaces over $\mathbb{K}$. Unless otherwise specified, we understand that the product $\prod_{i=1}^n \mathcal{X}_i$ is equipped with the sup-norm of the norms of the components. $\mathcal{L}^n_{\mathbb{K}}(\prod_{i=1}^n \mathcal{X}_i, \mathcal{Y})$ (or $\mathcal{L}^n(\prod_{i=1}^n \mathcal{X}_i, \mathcal{Y})$ if there is no ambiguity) denotes the normed space of the continuous $\mathbb{K}$-n-linear maps of $\prod_{i=1}^n \mathcal{X}_i$ into $\mathcal{Y}$ and is equipped with the topology of the uniform convergence on the unit sphere of $\prod_{i=1}^n \mathcal{X}_i$. Let $x_0 \in \mathcal{X}$, $\rho > 0$. $\mathcal{B}(x_0, \rho)$ denotes the open ball of center x_0 and radius ρ of $\mathcal{X}$. To emphasize the finite dimensional case, when $\mathcal{X} = \mathbb{R}^n$ we use the symbol $\mathbb{B}(x_0, \rho)$ with the same meaning of $\mathcal{B}(x_0, \rho)$. To emphasize that the variables of a certain operator $\boldsymbol{F}$ are functions rather than scalars, we write $\boldsymbol{F}[\phi]$ or $\boldsymbol{F}[\phi, f]$ instead $\boldsymbol{F}(\phi)$ or $\boldsymbol{F}(\phi, f)$. For standard definitions of Calculus in normed spaces, we refer *e.g.* to Prodi & Ambrosetti (1973) or to Berger (1977). $[\,\cdot\,]^n$ denotes the diagonal map of $\mathcal{X}$ to $\mathcal{X}^n$ defined by $[v]^n \equiv (v, \ldots, v)$ for all $v \in \mathcal{X}$. A complex normed space can be viewed naturally as a real normed space. Accordingly, we will say that a certain map between complex normed spaces is real differentiable, real analytic or real linear, to indicate that such map is differentiable, analytic or linear respectively as a map between the corresponding underlying real spaces. To emphasize that we are retaining the complex structure, we will say that the map is complex differentiable, complex analytic or complex linear. The inverse function of a function f is denoted $f^{(-1)}$ as opposed to the reciprocal of a complex valued function G or the inverse of a matrix A, which are denoted G^{-1} and A^{-1} respectively. Let $r \in \mathbb{N} \setminus \{0\}$. $M_r(\mathbb{K})$ denotes the set of $r \times r$ matrices with entries in $\mathbb{K}$. A dot '$\cdot$' denotes the matrix product. Throughout the paper, we make no formal distinction between complex numbers and pairs of real numbers, so $\mathbb{D}$ denotes the open unit disk both in $\mathbb{C}$ and in $\mathbb{R}^2$. Let $B \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\, B$ denotes the closure of B and $\mathrm{int}\, B$ denotes the interior of B. If B is an open subset of $\mathbb{R}^2$, $\mathcal{H}(B)$ is the set of all holomorphic functions of B to $\mathbb{C}$. If $G \equiv G_1 + iG_2 \in \mathcal{C}^1(\mathrm{cl}\, B, \mathbb{C})$, we set as usual $\overline{\partial} G \equiv \frac{1}{2}(\partial_{x_1} G + i\partial_{x_2} G) = \frac{1}{2}[(\partial_{x_1} G_1 - \partial_{x_2} G_2) + i(\partial_{x_1} G_2 + \partial_{x_2} G_1)]$.

1.2. Basic properties of Schauder spaces

We now introduce the Schauder spaces on open subsets of $\mathbb{R}^n$. As usual we set $|\eta| \equiv \eta_1 + \cdots + \eta_n$ for all $\eta \in \mathbb{N}^n$. Let $m \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^n$ and let $(\mathcal{Y}, \| \ \|_{\mathcal{Y}})$ be a normed space over $\mathbb{K}$. We denote by $\mathcal{C}^m(\Omega, \mathcal{Y})$ the space of m-times continuously real differentiable functions of Ω to $\mathcal{Y}$ and by $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})$ the subspace of those functions $F \in \mathcal{C}^m(\Omega, \mathcal{Y})$ such that for all $\eta \in \mathbb{N}^n$ with $|\eta| \leq m$, the function $D^\eta F \equiv \frac{\partial^{|\eta|} F}{\partial_1^{\eta_1} \ldots \partial_n^{\eta_n}}$ can be extended with continuity to $\mathrm{cl}\,\Omega$. In particular if n is even the elements of $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})$ or of $\mathcal{C}^m(\Omega, \mathcal{Y})$ are not necessarily holomorphic (*i.e.* complex analytic) in Ω even when $m > 0$. If Ω is a bounded open subset of $\mathbb{R}^n$, we equip $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})$ with the norm $\|F\|_{\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})} \equiv \sum_{|\eta| \leq m} \sup_{\mathrm{cl}\,\Omega} \|D^\eta F\|_{\mathcal{Y}}$. The subspace of $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathcal{Y})$. If $F \in \mathcal{C}^{0,\alpha}(\mathrm{cl}\,\Omega, \mathbb{C})$, then its Hölder quotient $|F : \Omega|_\alpha$ or more simply $|F|_\alpha$, is defined as

$$\sup \left\{ \frac{\|F(\xi_1) - F(\xi_2)\|_{\mathcal{Y}}}{|\xi_1 - \xi_2|^\alpha} : \xi_1, \xi_2 \in \mathrm{cl}\,\Omega,\ \xi_1 \neq \xi_2 \right\} .$$

] The space $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathcal{Y})$ is equipped with its usual norm

$$\|F\|_{\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathcal{Y})} \equiv \|F\|_{\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})} + \sum_{|\eta| = m} |D^\eta F|_\alpha .$$

It is well known that $(\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y}), \| \ \|_{\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y})})$ and $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathcal{Y}), \| \ \|_{\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathcal{Y})})$ are Banach spaces over $\mathbb{K}$. If $B \subseteq \mathcal{Y}$, $\mathcal{C}^m(\mathrm{cl}\,\Omega, B)$ denotes the set $\{F \in \mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y}) : F(\mathrm{cl}\,\Omega) \subseteq B\}$. Similarly we define $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, B)$. Let $r \in \mathbb{N} \setminus \{0\}$. As usual, if $F \in \mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^r)$ and $x_0 \in \mathrm{cl}\,\Omega$, $(DF)(x_0)$ denotes the $r \times n$ Jacobian matrix of F at x_0. Standard norm inequalities imply that the normed spaces $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{K}^r)$, $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{K}^r)$ and $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbb{K}))$ are isomorphic to $(\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{K}))^r$, $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{K}))^r$ and $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{K}))^{r^2}$, respectively. In particular $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{K}^r)$, $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{K}^r)$ and $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbb{K}))$ are Banach spaces. In accordance with our definitions, the real Banach spaces $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{C})$ and $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^2)$ coincide algebraically and have equivalent norms. Let K be a bounded connected subset of $\mathbb{R}^n$ and let $x, y \in K$. Let $\lambda(x, y)$ be the infimum of the lengths of piecewise smooth curves γ of $[0,1]$ to K (*i.e.* there exist $r \in \mathbb{N}$ and an increasing finite sequence $\{a_i \in [0,1] : i = 0, \ldots, r\}$ with $a_0 = 0$, $a_r = 1$ such that $f_{[a_i, a_{i+1}]} \in \mathcal{C}^1([a_i, a_{i+1}], K)$ for all $i < r$) with $\gamma(0) = x, \gamma(1) = y$. We set

$$c[K] \equiv \sup \left\{ \frac{\lambda(x, y)}{|x - y|} : x, y \in K,\ x \neq y \right\} . \tag{1.2.1}$$

The subset K of $\mathbb{R}^n$ is said to be regular in the sense of Whitney if K is bounded, connected and if $c[K] < +\infty$. It is well known that if Ω is a bounded, connected, open subset of $\mathbb{R}^n$ of class $\mathcal{C}^1$, then $c[\Omega] < +\infty$ (cf. *e.g.* Jones (1981, p. 73)).

We now state two abstract results that we need to prove some technical facts on the composition and on the reciprocal operator in Schauder spaces. The validity of the following has been pointed out in Lanza (1996, Prop. 3.11).

LEMMA 1.2.2. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be normed spaces. Let $\mathcal{A} \subseteq \mathcal{X}$. Let $\boldsymbol{S}$ be a map of $\mathcal{Y} \times \mathcal{A}$ to $\mathcal{Z}$ such that for all $x \in \mathcal{A}$, the map $\boldsymbol{S}[\cdot, x]$ is linear from $\mathcal{Y}$ to $\mathcal{Z}$ (i.e. $\boldsymbol{S}$ is linear in its first variable). Then the following statements are equivalent.*

(i) *$\boldsymbol{S}$ maps bounded sets of $\mathcal{Y} \times \mathcal{A}$ to bounded sets of $\mathcal{Z}$.*
(ii) *There exists an increasing function ψ (i.e. $\psi(\rho_1) \leq \psi(\rho_2)$ whenever $\rho_1 \leq \rho_2$) of $[0, \infty[$ to itself such that $\|\boldsymbol{S}[y, x]\|_{\mathcal{Z}} \leq \|y\|_{\mathcal{Y}} \, \psi(\|x\|_{\mathcal{X}})$ for all $(y, x) \in \mathcal{Y} \times \mathcal{A}$.*

Proof. Statement (ii) follows by statement (i) by setting $\psi(r) \equiv \sup\{\|\boldsymbol{S}[y, x]\|_{\mathcal{Z}} : (y, x) \in \mathcal{Y} \times \mathcal{A}, \|y\|_{\mathcal{Y}} = 1, \|x\|_{\mathcal{X}} \leq r\}$ where $\sup \emptyset \equiv 0$. Statement (i) is an obvious consequence of statement (ii). □

The validity of the following abstract Proposition concerning the regularity of the reciprocal map is well-known and can be verified by a standard argument (cf. *e.g.* Hille & Phillips (1957, Thm. 4.3.2 and Thm. 4.3.4)).

PROPOSITION 1.2.3. *Let $\mathcal{X}$ be a real or complex Banach algebra with unity (possibly noncommutative). Let $\mathcal{I}$ be the subset of the elements of $\mathcal{X}$ which are invertible with respect to the product of $\mathcal{X}$. Then $\mathcal{I}$ is open and the reciprocal map, which takes an element x of $\mathcal{I}$ to its reciprocal with respect to the product of $\mathcal{X}$, is analytic.*

Then we have the following.

LEMMA 1.2.4. *Let $m, n, r, h \in \mathbb{N}$, $n, r, h \geq 1$, $\alpha, \beta \in]0, 1]$. Let Ω be an open subset of $\mathbb{R}^n$, regular in the sense of Whitney. Then we have the following.*

(i) *$\mathcal{C}^{m+1}(\mathrm{cl}\, \Omega, \mathbb{C})$ is continuously imbedded in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{C})$.*
(ii) *The pointwise product in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{R})$ is a continuous bilinear map of $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{R}))^2$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{R})$. In particular $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{R})$ with this product is a commutative Banach algebra with unity.*
(iii) *The pointwise matrix product in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R}))$ is a continuous bilinear map of $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R})))^2$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R}))$. In particular $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R}))$ with this product is a noncommutative Banach algebra with unity.*
(iv) *The reciprocal map in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R}))$, which maps an invertible matrix of functions M to its inverse matrix M^{-1}, is real analytic from the open subset $\{M \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R})) : \det(M(x)) \neq 0, \forall x \in \mathrm{cl}\, \Omega\}$ of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\, \Omega, M_r(\mathbb{R}))$ to itself.*

(v) *Let* Ω_1 *be an open subset of* $\mathbb{R}^r$*, regular in the sense of Whitney. If* $F \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbb{R}^h)$ *and if* $G \in \mathcal{C}^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *then* $F \circ G \in \mathcal{C}^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\Omega, \mathbb{R}^h)$*, with* $\gamma_0(\alpha,\beta) = \alpha\beta$ *and* $\gamma_m(\alpha,\beta) = \min\{\alpha,\beta\}$ *if* $m > 0$*. Furthermore, there exists an increasing function* ψ *of* $[0,+\infty[$ *to itself such that*

$$\|F \circ G\|_{\mathcal{C}^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\Omega,\mathbb{R}^h)} \leq \|F\|_{\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega_1,\mathbb{R}^h)} \psi(\|G\|_{\mathcal{C}^{m,\beta}(\mathrm{cl}\,\Omega,\mathbb{R}^r)}),$$

for all $(F,G) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbb{R}^h) \times \mathcal{C}^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$*.*

(vi) *Let* $m \geq 1$*. If* $G \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^n)$ *is injective and satisfies the condition* $\det DG(x) \neq 0$ *for all* x *in* $\mathrm{cl}\,\Omega$ *then* $G(\Omega)$ *is a bounded connected open subset of* $\mathbb{R}^n$*, and* $G(\mathrm{cl}\,\Omega) = \mathrm{cl}\,G(\Omega)$*, and* $c[G(\Omega)] < +\infty$*, and* $G^{(-1)} \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,G(\Omega), \mathrm{cl}\,\Omega)$*.*

Proof. Statement (i) is an obvious consequence of the inclusion $\mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{C}) \subseteq \mathcal{C}^{0,\alpha}(\mathrm{cl}\,\Omega, \mathbb{C})$, which holds because Ω is regular in the sense of Whitney. Statement (ii) is well-known (cf. *e.g.* Lanza (1991, Lemma 2.4 (v))); we can prove statement (iii) by using (ii) and by a simple computation. By Proposition 1.2.3 and by statement (iii) we obtain statement (iv). The first part of statement (v) can be proved by induction on m, by using the chain rule and by statement (i) and (ii) (see also Lanza (1991, Lemma 4.20)). We can prove the second part of (v) by Lemma 1.2.2 and by showing that the composition operator of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbb{R}^h) \times \mathcal{C}^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ to $\mathcal{C}^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\Omega, \mathbb{R}^h)$ maps bounded sequences to bounded sequences, a fact which easily follows by induction on m, by the chain rule and by statement (ii). To prove statement (vi), we note that $G(\Omega)$ is open by the Inverse Function Theorem. Since $\mathrm{cl}\,\Omega$ is compact, G is a homeomorphism of $\mathrm{cl}\,\Omega$ onto $G(\mathrm{cl}\,\Omega)$ and then we have $G(\mathrm{cl}\,\Omega) = \mathrm{cl}\,G(\Omega)$. Inequality $c[G(\Omega)] < +\infty$ follows for example from Lanza (1991, Lemma 4.26). Then by induction on m, by exploiting statement (v) and equality $DG^{(-1)}(y) = (DG(G^{(-1)}(y)))^{-1}$ for all $y \in \mathrm{cl}\,G(\Omega)$, we obtain $G^{(-1)} \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,G(\Omega), \mathrm{cl}\,\Omega)$. □

As we shall see later, we parametrize Jordan domains by one to one functions defined on the unit disk. Thus we will employ the following (cf. Lanza (1991, Cor. 4.24, Prop. 4.29)).

LEMMA 1.2.5. *Let* $h \in \mathbb{N} \setminus \{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^h$*, regular in the sense of Whitney. Let* $\Psi \in \mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h)$*. Let*

$$\boldsymbol{l}_\Omega[\Psi] \equiv \inf\left\{\frac{|\Psi(x) - \Psi(y)|}{|x-y|} : x, y \in \mathrm{cl}\,\Omega, x \neq y\right\},$$

$$\mathcal{A}_\Omega \equiv \{\Psi \in \mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h) : \boldsymbol{l}_\Omega[\Psi] > 0\}.$$

Then the following statements hold.

(i) $\boldsymbol{l}_\Omega[\Psi] > 0$ *if and only if* Ψ *is injective and* $\det D\Psi(x) \neq 0$ *for all* x *in* $\mathrm{cl}\,\Omega$*.*

(ii) *The function of* $\mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h)$ *to* $\mathbb{R}$ *which maps* Ψ *to* $\boldsymbol{l}_\Omega[\Psi]$ *is continuous; in particular* $\mathcal{A}_\Omega$ *is open in* $\mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h)$.

(iii) *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1]$. *Then* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^h) \cap \mathcal{A}_\Omega$ *is open in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^h)$.

Proof. Statement (i) follows by Lanza (1991, Cor. 4.24). Since Ω is regular in the sense of Whitney, $\mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h)$ is continuously imbedded in $\mathcal{C}^{0,1}(\mathrm{cl}\,\Omega, \mathbb{R}^h)$. Then Lanza (1991, Prop. 4.29) implies statement (ii). Since $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^h)$ is continuously imbedded in $\mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^h)$, statement (ii) yields statements (iii). $\square$

We now want to define the Schauder spaces on plane Jordan curves, which are particular compact subsets of $\mathbb{C}$ with no isolated points. With somewhat more generality, we define the Schauder spaces on a general compact subset K of $\mathbb{C}$ with no isolated points. We say that a function f of K to $\mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if $\lim_{K \ni z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists finite. We denote such limit by $f'(z_0)$. As usual the higher order derivatives, if they exist, are defined inductively. Let $m \in \mathbb{N}$. We denote by $\mathcal{C}^m_*(K, \mathbb{C})$ the complex normed space of m-times continuously complex differentiable functions f of K to $\mathbb{C}$ equipped with the norm $\|f\|_{\mathcal{C}^m_*(K,\mathbb{C})} = \sum_{i=0}^m \sup_K |f^{(i)}|$. We say that f is Hölder continuous on K with exponent $\alpha \in]0,1]$ provided that $|f : K|_\alpha \equiv \sup\left\{\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha} : z_1, z_2 \in K,\ z_1 \neq z_2\right\}$ is finite. We denote by $\mathcal{C}^{m,\alpha}_*(K, \mathbb{C})$ the subspace of $\mathcal{C}^m_*(K, \mathbb{C})$ of functions having α-Hölder continuous m-th order derivatives. We equip $\mathcal{C}^{m,\alpha}_*(K, \mathbb{C})$ with the norm $\|f\|_{\mathcal{C}^{m,\alpha}_*(K,\mathbb{C})} \equiv \|f\|_{\mathcal{C}^m_*(K,\mathbb{C})} + |f^{(m)} : K|_\alpha$. If $B \subseteq \mathbb{C}$, we set $\mathcal{C}^{m,\alpha}_*(K, B) \equiv \{f \in \mathcal{C}^{m,\alpha}_*(K, \mathbb{C}) : f(K) \subseteq B\}$. Then the following variant of Lanza (1991, Cor. 4.24, Prop. 4.29) holds.

Lemma 1.2.6. *Let* K *be a compact subset of* $\mathbb{C}$ *with no isolated points. Let* $\phi \in \mathcal{C}^1_*(K, \mathbb{C})$. *Let*

$$\boldsymbol{l}_K[\phi] \equiv \inf\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|} : x, y \in K, x \neq y\right\}$$
$$\mathcal{A}_K \equiv \{\phi \in \mathcal{C}^1_*(K, \mathbb{C}) : \boldsymbol{l}_K[\phi] > 0\}\ .$$

Then the following statements hold.

(i) *Assume that for all* $\phi \in \mathcal{C}^1_*(K, \mathbb{C})$ *and for all* $\overline{x} \in K$, *the limit*

$$\lim_{\{(\xi,\eta)\in K^2, \xi\neq\eta\} \ni (x,y) \to (\overline{x},\overline{x})} \frac{\phi(x)-\phi(y)}{x-y}$$

exists and equals $\phi'(\overline{x})$. *Then* $\boldsymbol{l}_K[\phi] > 0$ *if and only if* ϕ *is injective and* $\phi'(\xi) \neq 0$ *for all* ξ *in* K.

(ii) *If* K *is such that* $\mathcal{C}^1_*(K, \mathbb{C})$ *is continuously imbedded in* $\mathcal{C}^{0,1}_*(K, \mathbb{C})$, *then the function of* $\mathcal{C}^1_*(K, \mathbb{C})$ *to* $\mathbb{R}$ *which maps* ϕ *to* $\boldsymbol{l}_K[\phi]$ *is continuous, and in particular* $\mathcal{A}_K$ *is open in* $\mathcal{C}^1_*(K, \mathbb{C})$.

(iii) *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1]$. *Then* $\mathcal{C}_*^{m,\alpha}(K,\mathbb{C}) \cap \mathcal{A}_K$ *is open in* $\mathcal{C}_*^{m,\alpha}(K,\mathbb{C})$.

Proof. The necessity of the condition of statement (i) is obvious. We now show the sufficiency by a contradiction argument (cf. Lanza & Antman (1991, Lemma 4.11)). If $\boldsymbol{l}_K[\phi] = 0$, then by the compactness of K, there exist two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in K with $x_n \neq y_n$ for all n, which converge to $\overline{x}$ and $\overline{y}$ respectively, and such that

$$\lim_{n\to+\infty} \frac{|\phi(x_n) - \phi(y_n)|}{|x_n - y_n|} = 0 \, .$$

If $\overline{x} \neq \overline{y}$ then $\phi(\overline{x}) = \phi(\overline{y})$, a contradiction. If $\overline{x} = \overline{y}$, then, by the assumption on K, we must have $\phi'(\overline{x}) = 0$, a contradiction. Statement (ii) can be shown by following the proof of the corresponding statement for $\phi \in \mathcal{C}^1(\mathrm{cl}\,\Omega, \mathbb{R}^2)$, with Ω open and bounded in $\mathbb{R}^2$ (cf. Lanza (1991, Prop. 4.29)). Since $\mathcal{C}_*^{m,\alpha}(K,\mathbb{C})$ is continuously imbedded in $\mathcal{C}_*^1(K,\mathbb{C})$, statement (ii) implies statement (iii). $\square$

Remark 1.2.7. It can be easily verified that $K = \partial\mathbb{D}$ satisfies the assumptions on K of conditions (i), (ii) of Lemma 1.2.6 and that accordingly the conclusions of Lemma 1.2.6 (i), (ii) hold for $K = \partial\mathbb{D}$.

We are now ready to state the following, which collects a few facts which we need on the spaces $\mathcal{C}_*^{m,\alpha}(K,\mathbb{C})$.

Lemma 1.2.8. *Let* $m \in \mathbb{N}$, $\alpha, \beta \in]0,1]$, $\phi \in \mathcal{A}_{\partial\mathbb{D}}$, $L = \phi(\partial\mathbb{D})$. *Then the following statements hold.*

(i) *There exists a positive constant* c_ϕ *depending only on* ϕ *such that for all* $f \in \mathcal{C}_*^1(L,\mathbb{C})$ *and for all* $z_1, z_2 \in L$

$$|f(z_1) - f(z_2)| \leq c_\phi (\sup_L |f'|) |z_1 - z_2| \, .$$

(ii) $\mathcal{C}_*^{m+1}(L,\mathbb{C})$ *is continuously imbedded in* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$.

(iii) $\mathcal{C}_*^{m}(L,\mathbb{C})$ *and* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$ *are complex Banach spaces.*

(iv) *The pointwise product is a continuous bilinear map of* $(\mathcal{C}_*^{m,\alpha}(L,\mathbb{C}))^2$ *to* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$.

(v) *The reciprocal map in* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$, *which maps a nonvanishing function* f *to its reciprocal, is complex analytic from the open subset* $\{f \in \mathcal{C}_*^{m,\alpha}(L,\mathbb{C}) : f(\xi) \neq 0, \forall \xi \in L\}$ *of* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$ *to itself.*

(vi) *Let* $\phi_1 \in \mathcal{A}_{\partial\mathbb{D}}$, $L_1 = \phi_1(\partial\mathbb{D})$. *If* $f \in \mathcal{C}_*^{m,\alpha}(L_1,\mathbb{C})$ *and if* $g \in \mathcal{C}_*^{m,\beta}(L,L_1)$, *then* $f \circ g \in \mathcal{C}_*^{m,\gamma_m(\alpha,\beta)}(L,\mathbb{C})$ *with* $\gamma_0(\alpha,\beta) = \alpha\beta$ *and* $\gamma_m(\alpha,\beta) = \min\{\alpha,\beta\}$ *if* $m > 0$. *Furthermore there exists an increasing function* ψ *of* $[0,+\infty[$ *to itself such that*

$$\|f \circ g\|_{\mathcal{C}_*^{m,\gamma_m(\alpha,\beta)}(L,\mathbb{C})} \leq \|f\|_{\mathcal{C}_*^{m,\alpha}(L_1,\mathbb{C})} \psi(\|g\|_{\mathcal{C}_*^{m,\beta}(L,\mathbb{C})}),$$
$$\forall (f,g) \in \mathcal{C}_*^{m,\alpha}(L_1,\mathbb{C}) \times \mathcal{C}_*^{m,\beta}(L,L_1) \, .$$

(vii) *Let $m \geq 1$. If $g \in \mathcal{C}^{m,\alpha}_*(L,\mathbb{C})$ is injective and satisfies condition $g'(\xi) \neq 0$, for all $\xi \in L$, then $g^{(-1)} \in \mathcal{C}^{m,\alpha}_*(g(L),L)$.*

(viii) *Let $m \geq 1$ and $\phi \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. Then the map $\boldsymbol{T}_\phi$ defined by $\boldsymbol{T}_\phi[f] \equiv f \circ \phi$ is a complex linear homeomorphism of $\mathcal{C}^{m,\alpha}_*(L,\mathbb{C})$ onto $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$.*

Proof. We prove (i). Let $j = 1,2$, $\theta_j \in [0,2\pi]$, $s_j = e^{i\theta_j}$, $z_j = \phi(s_j)$,

$$\sigma(\theta_1,\theta_2) \equiv \min\left\{|t_1 - t_2| : t_l \in \mathbb{R}, e^{it_l} = e^{i\theta_l},\ l = 1,2\right\},$$

$$\eta \equiv \inf\left\{\frac{|e^{i\theta_1} - e^{i\theta_2}|}{\sigma(\theta_1,\theta_2)} :\ \theta_1,\theta_2 \in [0,2\pi],\ \sigma(\theta_1,\theta_2) \neq 0\right\}.$$

As shown in Lanza & Antman (1991, Lemma 4.11), $\eta > 0$. Since $f \circ \phi(e^{it}) \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$, we have

$$\begin{aligned}|f(z_1) - f(z_2)| &\leq \Big(\sup_{\theta\in[0,2\pi]} |f'(\phi(e^{i\theta}))\phi'(e^{i\theta})|\Big)\sigma(\theta_1,\theta_2)\\ &\leq (\sup_L |f'|)(\sup_{[0,2\pi]} |\phi'|)\eta^{-1}(l_{\partial\mathbb{D}}[\phi])^{-1}|z_1 - z_2|\,.\end{aligned}$$

Then statement (i) follows by setting $c_\phi \equiv (\sup_{[0,2\pi]} |\phi'|)\eta^{-1}(l_{\partial\mathbb{D}}[\phi])^{-1}$. Statement (ii) is an immediate consequence of (i). We now prove statement (iii). It clearly suffices to show that $\mathcal{C}^m_*(L,\mathbb{C})$ is complete. We proceed by induction on m. Case $m = 0$ is well-known. Case $m = 1$ can be shown by observing that if $f \in \mathcal{C}^1_*(L,\mathbb{C})$, then $f(\phi(e^{it})) \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$ and by using a standard argument. Case $m+1$ can be deduced by case m and by applying case $m = 1$. Statement (iv) can be proved by a standard inductive argument (cf. *e.g.* Lanza (1991, Lemma 2.4 (v))) and by using statement (ii). Statement (v) is an immediate consequence of (iii), (iv) and of Proposition 1.2.3. We can prove statement (vii) and the first part of (vi) by induction on m, by using the chain rule, the rule of differentiation of the inverses and statements (ii) and (iv). By statements (ii) and (iv) and by induction on m, it can be easily shown that the composition operator maps bounded sequences of $\mathcal{C}^{m,\alpha}_*(L_1,\mathbb{C}) \times \mathcal{C}^{m,\beta}_*(L,L_1)$ to bounded sequences of $\mathcal{C}^{m,\gamma_m(\alpha,\beta)}_*(L,\mathbb{C})$. Then by Lemma 1.2.2, we conclude the existence of ψ as in the second part of statement (vi). Statement (viii) is an immediate consequence of statements (vi) and (vii). □

Now let $\phi \in \mathcal{A}_{\partial\mathbb{D}}$. By the Jordan Theorem (cf. *e.g.* Godbillon (Cor. 4.4 p. 214)), $\mathbb{C}\setminus\phi(\partial\mathbb{D})$ consists of two open connected components. We denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and the unbounded component of $\mathbb{C} \setminus \phi(\partial\mathbb{D})$, respectively. We collect in the following Lemma some properties of $\mathbb{I}[\phi]$, of $\mathbb{E}[\phi]$ and of the trace of a function of class $\mathcal{C}^{m,\alpha}$ in $\mathrm{cl}\,\mathbb{I}[\phi]$.

LEMMA 1.2.9. *The following statements hold.*

(i) *If $\phi \in \mathcal{A}_{\partial\mathbb{D}}$, then $\partial\mathbb{I}[\phi] = \partial\mathbb{E}[\phi] = \phi(\partial\mathbb{D})$ and $c[\mathbb{I}[\phi]] < +\infty$.*

(ii) *Let* $m \in \mathbb{N}$, $\alpha \in]0,1]$, $\phi \in \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. *Then the trace operator* $\boldsymbol{R}$ *from* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})$ *to* $\mathcal{C}^{m,\alpha}_{*}(\phi(\partial\mathbb{D}),\mathbb{C})$ *defined by* $\boldsymbol{R}[F] = F_{/\phi(\partial\mathbb{D})}$ *is complex linear and continuous.*

Proof. Conditions $\phi(e^{it}) \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$, $\phi'(e^{it}) \neq 0$ for all $t \in \mathbb{R}$, $l_{\partial\mathbb{D}}[\phi] > 0$ imply that $\phi(\partial\mathbb{D})$ is a real connected submanifold of class $\mathcal{C}^1$ and of codimension one of $\mathbb{C}$. It follows that the boundary of $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ is $\phi(\partial\mathbb{D})$ and that $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ are open subsets of $\mathbb{C}$ of class $\mathcal{C}^1$. Then $c[\mathbb{I}[\phi]] < +\infty$. We now prove statement (ii). Let ϕ_1 and ϕ_2 be the real and the imaginary part, respectively, of ϕ. A simple induction on m shows that ϕ_1 and ϕ_2 belong to $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$. We now fix m and prove by induction on $j = 0,\dots,m$ that $\boldsymbol{R}$ is continuous from $\mathcal{C}^{j,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})$ to $\mathcal{C}^{j,\alpha}_{*}(\phi(\partial\mathbb{D}),\mathbb{C})$. Case $j = 0$ is obvious. We now assume that the statement holds for an arbitrary but fixed $j \in \{0,\dots,m-1\}$, and prove it for $j+1$. Let $F \in \mathcal{C}^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})$. By Lemma 1.2.4 (v), we have $F(\phi(e^{it})) \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$. Then F' exists. By the chain rule applied to the function $F(\phi(e^{it}))$, we deduce that

$$F'(z) = \left[\frac{\partial F}{\partial x}(z)\phi_1'(\phi^{(-1)}(z)) + \frac{\partial F}{\partial y}(z)\phi_2'(\phi^{(-1)}(z))\right](\phi'(\phi^{(-1)}(z)))^{-1}$$

for all $z \in \phi(\partial\mathbb{D})$. By inductive assumption and by Lemma 1.2.8 (iv), (v), (vi), (vii), there exists a constant $c > 0$ such that $\|F'\|_{\mathcal{C}^{j,\alpha}_{*}(\phi(\partial\mathbb{D}),\mathbb{C})} \leq c\|F\|_{\mathcal{C}^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})}$. Then statement (ii) follows immediately. □

We now show that our representation of a Jordan domain depends analytically on the curve which parametrizes the boundary of the Jordan domain. To do so we need the following which is a restatement of a corresponding lemma of Lanza (1997, Lemma 2.13).

LEMMA 1.2.10. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\phi_0 \in \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$, $z_0 \in \mathbb{I}[\phi_0]$. *Then the following hold.*

(i) *There exists at least an element* $\Psi_0 \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}$ *such that* $\Psi_{0/\partial\mathbb{D}} = \phi_0$ *and that* $\Psi_0(0) = z_0$.

(ii) *There exists a continuous complex linear extension map* $\boldsymbol{E}$ *of* $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ *such that the affine map between the same spaces defined by*

$$\boldsymbol{E}_{\phi_0}[\phi] \equiv \Psi_0 + \boldsymbol{E}[\phi - \phi_0]\,, \tag{1.2.11}$$

maps an open neighborhood $\mathcal{U}_{\phi_0}$ *of* ϕ_0 *contained in* $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$ *into* $\{\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A} : \Psi(0) = z_0\}$ *and satisfies* $\boldsymbol{E}_{\phi_0}[\phi]_{/\partial\mathbb{D}} = \phi$ *for all* $\phi \in \mathcal{U}_{\phi_0}$.

Proof. Clearly, the function $\phi_0(e^{it})$ is a simple closed curve of class $\mathcal{C}^{m,\alpha}$ defined on $[0,2\pi]$ with $\frac{d}{dt}\{\phi_0(e^{it})\} \neq 0$ for all $t \in [0,2\pi]$. Then by Lanza (1997, Lemmas 2.7, 2.13 (i)) statement (i) holds. To prove statement (ii), we take $k \in$

$\mathcal{C}^\infty([0,1],[0,1])$ such that $k([0,\frac{1}{3}]) = \{0\}$, $k([\frac{2}{3},1]) = \{1\}$ and we set $\boldsymbol{E}[h](x) \equiv h(\frac{x}{|x|})k(|x|)$, for all $x \in \mathrm{cl}\,\mathbb{D}$, $h \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. By exploiting Lemma 1.2.4 (ii) and (v), it can be verified that $\boldsymbol{E}[\,\cdot\,]$ is a complex linear and continuous operator of $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ and that $\boldsymbol{E}[h]_{/\partial\mathbb{D}} = h$ for all $h \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Then by using Lemmas 1.2.5 (ii) and 1.2.6 (ii), it is easy to check that the affine map defined in (1.2.11) satisfies the required properties. $\square$

CHAPTER 2

Complex analyticity of the Cauchy singular integral in Schauder spaces

2.1. Introduction

In this chapter, which contains the results of Lanza & Preciso (1998), we analyze the analytic dependence of the Cauchy singular integral

$$\boldsymbol{C}[\phi, f](\cdot) \equiv \frac{1}{2\pi i}\,\text{p.v.}\int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t)-\phi(\cdot)}\,dt \tag{2.1.1}$$

upon the oriented simple closed curve ϕ and the density function f, both defined on the counterclockwise oriented boundary $\partial\mathbb{D}$ of the plane unit disk $\mathbb{D}$. The Cauchy singular integral is involved in the functional equation associated to the conformal sewing problem. Then our aim is to apply the results of this chapter to a perturbation analysis of the conformal sewing problem.

We assume that both ϕ and f belong to a Schauder space, say $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$, of complex valued functions of class $\mathcal{C}^{m,\alpha}$ on $\partial\mathbb{D}$, with m a positive natural number and $\alpha \in]0,1[$. (The '$*$' subscript just means that we are taking the derivatives with respect to the variable on $\partial\mathbb{D}$.) As it is well-known, under such conditions on ϕ and f, the function $\boldsymbol{C}[\phi, f](\cdot)$ is also of class $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$, and we consider $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ as the target space of $\boldsymbol{C}[\phi, f]$. Although the linear operator $\boldsymbol{C}[\phi, \cdot]$ for a fixed ϕ has been studied extensively during the last century and a considerable amount of work has been done on the numerical computation of $\boldsymbol{C}[\phi, f]$ (cf. *e.g.* Muskhelishvili (1953), Gakhov (1966) and Wegert (1992)), especially in view of the several applications to integral equations and to boundary value problems, the analysis of the nonlinear functional dependence of $\boldsymbol{C}[\phi, f]$ upon both of its arguments, and in particular on ϕ, seems to be a subject analyzed only more recently. We mention the contribution of Calderón, Coifman, Meyer, McIntosh, David, whose work implies the analyticity of singular integral operators strictly related to $\boldsymbol{C}$. Calderón (1977, Thm. 1) has shown that if ϕ is the graph of a Lipschitz function ψ, *i.e.* if $\phi(x) = x + i\psi(x)$ with $\psi' \in L^\infty(\mathbb{R})$, and if $\|\psi'\|_{L^\infty(\mathbb{R})} < \epsilon$ for some $\epsilon > 0$, then the linear integral operator with singular kernel $\frac{\phi'(y)}{\phi(y)-\phi(x)}$ is an element of the space $\mathcal{L}_{\mathbb{C}}\left(L^2(\mathbb{R}, \mathbb{C}), L^2(\mathbb{R}, \mathbb{C})\right)$ of the linear and continuous operators of $L^2(\mathbb{R}, \mathbb{C})$ to itself. Then by using a standard argument of truncated kernels, one can deduce the analytic dependence of the operator with kernel $\frac{\phi'(y)}{\phi(y)-\phi(x)}$ upon ψ', when $\|\psi'\|_{L^\infty(\mathbb{R})} < \min\{1, \epsilon\}$ (cf. *e.g.* Meyer & Coifman (1991, p. 438)). Later Coifman, McIntosh & Meyer (1982, Thm. 1) and, by different methods, David (1984b, p. 178) have extended the validity of the same analyticity result to the case in which $\|\psi'\|_{L^\infty(\mathbb{R})} < 1$. Coifman & Meyer (1983b) have considered the dependence of the Cauchy singular integral upon an arc-length parametrized curve ϕ defined on $\mathbb{R}$, with values in the plane and determined by a function, say θ, which represents the direction of ϕ', and have shown that the Cauchy singular operator of $\mathcal{L}_{\mathbb{C}}\left(L^2(\mathbb{R},\mathbb{C}), L^2(\mathbb{R},\mathbb{C})\right)$ with kernel $\frac{\phi'(y)}{\phi(y)-\phi(x)}$ depends analytically on θ, if θ ranges in a suitable open subset of the John-Nirenberg space BMO of functions with bounded mean oscillation (cf. Coifman & Meyer (1983b, p. 10)). Later Wu (1993, p. 1310), under the advice of Coifman, has extended the analyticity result of Coifman & Meyer (1983b) on the Cauchy singular integral to arc-length parametrized simple closed curves.

In our work, we consider simple closed curves ϕ, which are not necessarily arc-length parametrized, but which are more regular than those considered by Coifman & Meyer (1983b). Correspondingly, the Cauchy singular operator $\boldsymbol{C}[\phi,\cdot]$ acts in $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$, as opposed to that of Coifman & Meyer (1983b) or Wu (1993), which acts in $L^2(\mathbb{R},\mathbb{C})$ or in $L^2(\partial\mathbb{D},\mathbb{C})$, respectively. Although our curves are more regular, our analyticity results cannot be deduced by the work of the authors mentioned above, and do not seem to follow by an immediate modification of their methods. An advantage of this approach , is that the Cauchy singular operator $\boldsymbol{C}[\phi, f]$ is defined for ϕ in an open subset of $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$.

We present an alternative approach to the study of the regularity of the Cauchy singular operator. As in Lanza (1997), we represent a Jordan domain by an injective and differentiable function, which we denote by Ψ, of $\mathrm{cl}\,\mathbb{D}$ to $\mathbb{R}^2$, rather than by the more traditional curve ϕ parametrizing the boundary of the Jordan domain. Then we observe that a Cauchy singular integral on a contour is uniquely determined by the pair, say (S^+, S^-), of "sectionally holomorphic" functions, which are associated to the Cauchy singular integral, which are defined in the interior and in the exterior of the contour respectively and which satisfy a certain boundary value problem. Then we transform such boundary value problem into a nonlinear boundary value problem of elliptic nature defined on the unit disk, which we now write in the form of an abstract nonlinear operator equation as

$$\boldsymbol{\Lambda}[\Psi, f, T^+, \widetilde{T}] = 0, \tag{2.1.2}$$

where $(T^+, \widetilde{T})$ is a pair of functions which is associated to the pair (S^+, S^-). Then we show that we can apply to equation (2.1.2) the Implicit Function Theorem and we deduce that the solution set of (2.1.2) is the graph of a real analytic operator depending on (Ψ, f). By this result, it follows easily the real analytic dependence of $\boldsymbol{C}[\phi, f]$ on (ϕ, f). An advantage of such approach is that in equation (2.1.2) there are no singular integrals and that the operator Λ of equation (2.1.2) is easily seen to be analytic. Although to apply the Implicit Function Theorem we still have to prove an isomorphism theorem for the linearized problem associated to (2.1.2), the difficulties we encounter in doing so are only of linear type. In principle, it seems that our method could be employed even with weaker regularity assumptions on the curve ϕ and on the density f. Once the real analyticity of $\boldsymbol{C}$ is established, we compute all order derivatives of $\boldsymbol{C}$ and we show that $\boldsymbol{C}$ is actually complex analytic. The statement concerning the real analyticity of $\boldsymbol{C}[\phi, f]$ as a function of (ϕ, f) we prove in this chapter, finds application in problems of nonlinear integral equations, and in particular in those of perturbation nature (cf. Lanza & Rogosin (1997)).

2.2. Introduction of a modified problem and real analyticity of the Cauchy singular integral

We now turn our attention to the dependence of the Cauchy singular integral of (2.1.1) upon ϕ, f. We understand that all line integrals on $\partial\mathbb{D}$ are computed with respect to the parametrization $\theta \mapsto e^{i\theta}$, $\theta \in [0, 2\pi]$ and that all line integrals on $\phi \in \mathcal{C}^1_*(\partial\mathbb{D}, \mathbb{C})$ are computed with respect to the parametrization $\theta \mapsto \phi(e^{i\theta})$. Let $\phi \in \mathcal{C}^1_*(\partial\mathbb{D}, \mathbb{C})$. We denote by $\mathrm{ind}[\phi]$ the index of the curve $\theta \mapsto \phi(e^{i\theta})$, $\theta \in [0, 2\pi]$ with respect to any of the points of $\mathbb{I}[\phi]$:

$$\mathrm{ind}[\phi] \equiv \frac{1}{2\pi i} \int_\phi \frac{d\xi}{\xi - z}, \; z \in \mathbb{I}[\phi] \, . \tag{2.2.1}$$

The map $\mathrm{ind}\,[\cdot]$ is obviously constant on the open connected components of $\mathcal{A}_{\partial\mathbb{D}}$ in $\mathcal{C}^1_*(\partial\mathbb{D}, \mathbb{C})$. Now it is well known that the Cauchy singular integral

$$\frac{1}{2\pi i} \int_\phi \frac{f \circ \phi^{(-1)}(\xi)}{\xi - z} \, d\xi$$

determines a so-called "sectionally holomorphic function" which vanishes at infinity and which jumps across the contour of integration, as shown by the Plemelj formula. Also the jump condition and the condition at infinity determine the "sectionally holomorphic function". We formulate such known facts in the following statement.

Theorem 2.2.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in \,]0, 1[$. Let $\phi \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$, $f \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. Then there exists one and only one pair of functions*

$$(S^+, S^-) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi], \mathbb{C}) \times \big(\mathcal{C}^1(\mathrm{cl}\,\mathbb{E}[\phi], \mathbb{C}) \cap \mathcal{C}^{m,\alpha}_*(\phi(\partial\mathbb{D}), \mathbb{C})\big)$$

which satisfies the following boundary value problem associated to (ϕ, f)

$$
\begin{cases}
\overline{\partial} S^+ = 0 & \text{in } \mathbb{I}[\phi], \\
\overline{\partial} S^- = 0 & \text{in } \mathbb{E}[\phi], \\
S^+ - S^- = f \circ \phi^{(-1)} & \text{on } \phi(\partial\mathbb{D}), \\
S^-(\infty) \equiv \lim_{z\to\infty} S^-(z) = 0\,.
\end{cases} \tag{2.2.3}
$$

We denote such unique solution (S^+, S^-) *by* $(\boldsymbol{S}^+[\phi, f], \boldsymbol{S}^-[\phi, f])$*. The functions* $\boldsymbol{S}^+[\phi, f]$ *and* $\boldsymbol{S}^-[\phi, f]$ *can be written explicitly as follows*

$$
\boldsymbol{S}^+[\phi, f](z) = \frac{\mathrm{ind}[\phi]}{2\pi i} \int_\phi \frac{f \circ \phi^{(-1)}(\xi)}{\xi - z}\, d\xi \qquad \forall z \in \mathbb{I}[\phi],
$$

$$
\boldsymbol{S}^-[\phi, f](z) = \frac{\mathrm{ind}[\phi]}{2\pi i} \int_\phi \frac{f \circ \phi^{(-1)}(\xi)}{\xi - z}\, d\xi \qquad \forall z \in \mathbb{E}[\phi],
$$

and the following Plemelj Formula holds

$$
\boldsymbol{S}^\pm[\phi, f](z) = \pm\frac{1}{2} f \circ \phi^{(-1)}(z) + \frac{\mathrm{ind}[\phi]}{2\pi i}\, \mathrm{p.\,v.} \int_\phi \frac{f \circ \phi^{(-1)}(\xi)}{\xi - z}\, d\xi \quad \forall z \in \phi(\partial\mathbb{D})\,. \tag{2.2.4}
$$

Proof. We first consider the uniqueness. Assume that (S_j^+, S_j^-), $j = 1, 2$ are solutions of (2.2.3), then

$$
\begin{cases}
\overline{\partial}[S_1^+ - S_2^+] = 0 & \text{in } \mathbb{I}[\phi], \\
\overline{\partial}[S_1^- - S_2^-] = 0 & \text{in } \mathbb{E}[\phi], \\
(S_1^+ - S_2^+) - (S_1^- - S_2^-) = 0 & \text{on } \phi(\partial\mathbb{D}), \\
(S_1^- - S_2^-)(\infty) = 0.
\end{cases}
$$

We observe that by Lemma 1.2.9 (i), we have $\partial\mathbb{I}[\phi] = \partial\mathbb{E}[\phi] = \phi(\partial\mathbb{D})$. Thus the function

$$
G(z) = \begin{cases}
(S_1^+ - S_2^+)(z) & \text{if } z \in \mathrm{cl}\,\mathbb{I}[\phi], \\
(S_1^- - S_2^-)(z) & \text{if } z \in \mathbb{C} \setminus \mathrm{cl}\,\mathbb{I}[\phi],
\end{cases}
$$

is holomorphic in $\mathbb{C} \setminus \phi(\partial\mathbb{D})$ and continuous on $\mathbb{C}$. Then a well known result (cf. *e.g.* Muskhelishvili (1953, p. 36)) implies that G is holomorphic in $\mathbb{C}$. Since $G(\infty) = 0$, Liouville's Theorem implies that $G = 0$. By Lemma 1.2.8 (vi) and (vii), we have $f \circ \phi^{(-1)} \in \mathcal{C}_*^{m,\alpha}(\phi(\partial\mathbb{D}), \mathbb{C})$ and thus, by the well-known properties of the Cauchy singular integral (see *e.g.* Lu (1993, Thm. 2.5.1 p. 23, Thm. 3.1.1 p. 28 and Corollary 3.2.2 p. 36)) and by Lemma 1.2.9 (ii), we deduce the existence of (S^+, S^-) and equation (2.2.4). □

Then by Lemma 1.2.9 (ii), we deduce the following corollary.

COROLLARY 2.2.5. *Let $m \in \mathbb{N} \setminus \{0\}$ and let $\alpha \in]0,1[$. The integral defined by*

$$\boldsymbol{C}[\phi, f](\cdot) \equiv \frac{1}{2\pi i} \text{p. v.} \int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t) - \phi(\cdot)}\, dt = \frac{1}{2\pi i} \text{p. v.} \int_{\phi} \frac{f \circ \phi^{(-1)}(\xi)}{\xi - \phi(\cdot)}\, d\xi$$

for all $(\phi, f) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$, belongs to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Thus $\boldsymbol{C}$ defines a nonlinear operator of $(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$.

REMARK 2.2.6. By (2.2.4), by Lemma 1.2.9 (ii) and by the constancy of ind $[\cdot]$ on the open connected components of $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$, the study of the regularity of the operator $\boldsymbol{C}[\phi, f]$ is equivalent to that of the operator from $(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ which maps a pair (ϕ, f) to $\boldsymbol{S}^+[\phi, f] \circ \phi$.

To enable the application of our methods, we now represent the contour of integration of our Cauchy singular integrals by the restriction to $\partial\mathbb{D}$ of a function Ψ defined on $\mathrm{cl}\,\mathbb{D}$. Let $\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$, $m \geq 1$, $\phi \equiv \Psi_{/\partial\mathbb{D}}$. By Brouwer's Theorem on the invariance of the domain (cf. *e.g.* Hurewicz & Wallman (1948, p. 95)) and by a simple topological argument (cf. *e.g.* Lanza (1997, Lemma 2.2)), we have $\Psi(\mathbb{D}) = \mathbb{I}[\phi]$ and $\mathbb{C} \setminus \Psi(\mathrm{cl}\,\mathbb{D}) = \mathbb{E}[\phi]$. Now our aim is to prove that the nonlinear operator defined by

$$\boldsymbol{T}^+[\Psi, f] \equiv \boldsymbol{S}^+[\Psi_{/\partial\mathbb{D}}, f] \circ \Psi \tag{2.2.7}$$

is real analytic from $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$. Clearly, the function $\boldsymbol{S}^+[\phi, f] \circ \phi$ is the restriction to $\partial\mathbb{D}$ of $\boldsymbol{T}^+[\Psi, f]$. By Lemmas 1.2.9 (ii) and 1.2.10, the real analyticity of $\boldsymbol{T}^+$ implies the real analyticity of the operator $\boldsymbol{C}$.

We note that problem (2.2.3) has been formulated in part on the unbounded domain $\mathbb{C} \setminus \Psi(\mathrm{cl}\,\mathbb{D})$. Since we find more convenient to work on a bounded domain, we now transform the problem in $\mathbb{C} \setminus \Psi(\mathrm{cl}\,\mathbb{D})$ into a problem defined in a bounded domain, by means of the following.

PROPOSITION 2.2.8. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$, $f \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Let γ be the function of $\partial\mathbb{D}$ to $\mathbb{C}$ defined by $\gamma(t) \equiv \frac{1}{\Psi(t) - \Psi(0)}$. Then we have the following.*

(i) $\gamma \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C} \setminus \{0\}) \cap \mathcal{A}_{\partial\mathbb{D}}$.

(ii) *Let $(S^+, S^-) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi(\mathbb{D}),\mathbb{C}) \times \big(\mathcal{C}^1(\mathrm{cl}(\mathbb{C} \setminus \Psi(\mathrm{cl}\,\mathbb{D})),\mathbb{C}) \cap \mathcal{C}^{m,\alpha}_*(\Psi(\partial\mathbb{D}),\mathbb{C})\big)$. Let the limit $S^-(\infty) \equiv \lim_{z\to\infty} S^-(z)$ exist in $\mathbb{C}$. Let $\widetilde{S}$ be the map of $\mathrm{cl}\,\mathbb{I}[\gamma]$ to $\mathbb{C}$ defined by*

$$\widetilde{S}(w) \equiv \begin{cases} S^-(\Psi(0) + \frac{1}{w}) & \textit{if } w \in \mathrm{cl}\,\mathbb{I}[\gamma] \setminus \{0\}, \\ S^-(\infty) & \textit{if } w = 0\,. \end{cases}$$

Then $(S^+, S^-) = (\boldsymbol{S}^+[\Psi_{/\partial\mathbb{D}}, f], \boldsymbol{S}^-[\Psi_{/\partial\mathbb{D}}, f])$ *holds if and only if the pair* $(S^+, \widetilde{S}) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\gamma], \mathbb{C})$ *satisfies the following boundary value problem*

$$
\begin{cases}
\overline{\partial} S^+ = 0 & \text{in } \Psi(\mathbb{D}), \\
\overline{\partial} \widetilde{S} = 0 & \text{in } \mathbb{I}[\gamma], \\
S^+(z) - \widetilde{S}(\frac{1}{z-\Psi(0)}) = f \circ \Psi^{(-1)}(z) & \forall z \in \Psi(\partial\mathbb{D}), \\
\widetilde{S}(0) = 0\,.
\end{cases}
\tag{2.2.9}
$$

(iii) *There exists a unique solution* $(S^+, \widetilde{S})$ *in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\gamma], \mathbb{C})$ *of problem* (2.2.9)*.*

Proof. We first observe that the function $H(z) = \frac{1}{z-\Psi(0)}$ is a one to one map of $\mathbb{C} \setminus \{\Psi(0)\}$ onto $\mathbb{C} \setminus \{0\}$ and that H is holomorphic with its inverse map $G(w) = \Psi(0) + \frac{1}{w}$. Furthermore H is a one to one map of $\mathrm{cl}(\mathbb{C} \setminus \Psi(\mathrm{cl}\,\mathbb{D}))$ onto $\mathrm{cl}\,\mathbb{I}[\gamma] \setminus \{0\}$. Since $\mathcal{C}^{m+1}_*(\Psi(\partial\mathbb{D}), \mathbb{C})$ is continuously imbedded in $\mathcal{C}^{m,\alpha}_*(\Psi(\partial\mathbb{D}), \mathbb{C})$ (cf. Lemma 1.2.8 (ii)), Lemmas 1.2.8 (vi) and 1.2.6 (i) imply that $\gamma \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C} \setminus \{0\}) \cap \mathcal{A}_{\partial\mathbb{D}}$. We now prove statement (ii). If $(S^+, S^-) = (\boldsymbol{S}^+[\Psi_{/\partial\mathbb{D}}, f], \boldsymbol{S}^-[\Psi_{/\partial\mathbb{D}}, f])$, then an easy computation shows that

$$
\widetilde{S}(w) = -w \cdot \boldsymbol{S}^+\left[\gamma, \frac{f}{\gamma}\right](w)
$$

for all $w \in \mathrm{cl}\,\mathbb{I}[\gamma]$. By Theorem 2.2.2, $\boldsymbol{S}^+[\phi, f] \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi], \mathbb{C})$ for all $(\phi, f) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. Then by using Lemmas 1.2.8 (iv), (v) and 1.2.4 (i), (ii), it is easy to check that $(S^+, \widetilde{S})$ has the required regularity and satisfies (2.2.9). The converse follows by an easy computation. The existence in statement (iii) is a consequence of statement (ii) and of Theorem 2.2.2. We now show the uniqueness of problem (2.2.9). Let $(S^+_j, \widetilde{S}_j) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\gamma], \mathbb{C})$, $j = 1, 2$ be solutions of (2.2.9). By Lemma 1.2.9 (ii), we have $\widetilde{S}_{j/\partial\mathbb{I}[\gamma]} \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{I}[\gamma], \mathbb{C})$. Then by Lemma 1.2.8 (ii) and (vi), by chain rule and by simple computations, $(S^+_j, S^-_j \equiv \widetilde{S}_j \circ H)$ has the required regularity properties and satisfies (2.2.3) associated to $(\Psi_{/\partial\mathbb{D}}, f)$. Then Theorem 2.2.2 yields the conclusion. □

With the same notations of Proposition 2.2.8, we wish to represent the closure of $\mathbb{I}[\frac{1}{\Psi_{/\partial\mathbb{D}}(\cdot)-\Psi(0)}]$ as the image of some regular function $\boldsymbol{G}[\Psi]$ of $\mathrm{cl}\,\mathbb{D}$ to $\mathbb{C}$ with $\boldsymbol{G}[\Psi](t) = \frac{1}{\Psi(t)-\Psi(0)}$ for all $t \in \partial\mathbb{D}$. We do so by means of the following.

Lemma 2.2.10. *Let* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in\,]0,1[$*. Let* $\Psi_0 \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$*. Then there exists an open neighborhood* $\mathcal{W}_{\Psi_0}$ *of* Ψ_0 *in the open subset* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ *of* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ *and a complex analytic map* $\boldsymbol{G}$ *of* $\mathcal{W}_{\Psi_0}$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ *such* *that*

$$
\begin{aligned}
&\boldsymbol{G}[\Psi](t) = \frac{1}{\Psi(t) - \Psi(0)} \qquad \forall t \in \partial\mathbb{D}, \\
&\boldsymbol{G}[\Psi](0) = 0,
\end{aligned}
$$

for all $\Psi \in \mathcal{W}_{\Psi_0}$.

Proof. By Lemma 1.2.9 (ii), the map of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ to $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C})$, which takes Ψ to $(\Psi_{/\partial\mathbb{D}} - \Psi(0))$, is complex linear and continuous and thus complex analytic. Moreover, $(\Psi_{/\partial\mathbb{D}} - \Psi(0)) \in \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C}\setminus\{0\}) \cap \mathcal{A}_{\partial\mathbb{D}}$ for all $\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$. By Lemma 1.2.8 (v), the reciprocal map is complex analytic from $\mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C}\setminus\{0\}) \cap \mathcal{A}_{\partial\mathbb{D}}$ to itself. Then we can conclude the proof by applying Lemma 1.2.10, with $\phi_0(\cdot) \equiv \frac{1}{\Psi_0(\cdot) - \Psi_0(0)}$ in $\partial\mathbb{D}$ and with $z_0 = 0$. $\square$

We now reformulate the boundary value problem (2.2.9) as a boundary value problem on the fixed domain $\mathrm{cl}\,\mathbb{D}$. To do so, we note that by Lemma 1.2.4 (v), (vi), Lemma 1.2.8 (vi), (vii) and by immediate computations, it can be easily verified that the following holds.

PROPOSITION 2.2.11. *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. *Let* $\Psi_0 \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$. *Let* $\mathcal{W}_{\Psi_0}$ *be the neighborhood of* Ψ_0 *in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ *of Lemma* 2.2.10. *Let* $(\Psi, f) \in \mathcal{W}_{\Psi_0} \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C})$. *Let* γ *be the function of* $\partial\mathbb{D}$ *to* $\mathbb{C}$ *defined by* $\gamma(t) = \frac{1}{\Psi(t)-\Psi(0)}$. *The pair of functions* $(S^+, \widetilde{S}) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\gamma], \mathbb{C})$ *satisfies* (2.2.9) *if and only if the pair of functions* $(T^+, \widetilde{T})$ *defined by*

$$
\begin{cases}
T^+ \equiv S^+ \circ \Psi, \\
\widetilde{T} \equiv \widetilde{S} \circ \boldsymbol{G}[\Psi],
\end{cases}
$$

belongs to $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$ *and satisfies the following boundary value problem*

$$
\begin{cases}
\overline{\partial}[T^+ \circ \Psi^{(-1)}] = 0 & \text{in } \Psi(\mathbb{D}), \\
\overline{\partial}[\widetilde{T} \circ (\boldsymbol{G}[\Psi])^{(-1)}] = 0 & \text{in } \boldsymbol{G}[\Psi](\mathbb{D}), \\
T^+ - \widetilde{T} = f & \text{on } \partial\mathbb{D}, \\
\widetilde{T}(0) = 0\,.
\end{cases}
\tag{2.2.12}
$$

In particular, problem (2.2.12) *has a unique solution in* $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$.

To proceed further, we wish to rewrite the equations of (2.2.12) in a way suitable to the application of our methods. To do so we introduce the following Lemma, whose proof is of immediate verification (cf. Lanza (1997, Lemma 3.1)).

LEMMA 2.2.13. *Let* $m \in \mathbb{N}$, $\alpha \in]0,1]$. *Let* Ω *be an open subset of* $\mathbb{R}^2$. *Let* $\boldsymbol{L}$ *be the linear and continuous map of* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, M_2(\mathbb{R}))$ *to itself defined by*

$$
\boldsymbol{L}[F] = \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix} \qquad \forall F \equiv \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, M_2(\mathbb{R})),
$$

and let $\boldsymbol{I}$ be the identity map in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, M_2(\mathbb{R}))$. Then the following hold.

(i) $\boldsymbol{L} \circ \boldsymbol{L} = \boldsymbol{I}$ *and, in particular,* $(\boldsymbol{I} + \boldsymbol{L}) \circ (\boldsymbol{I} - \boldsymbol{L}) = 0$.

(ii) $(\boldsymbol{I} - \boldsymbol{L})[F] = 0$ *if and only if* $F_{11} = F_{22}$, $F_{12} = -F_{21}$.

Remark 2.2.14. With the same notation of Lemma 2.2.13, let $G \equiv G_1 + iG_2 \in \mathcal{C}^{m+1,\alpha}(\mathrm{cl}\,\Omega, \mathbb{C})$. Then both the first row and the first column of the 2×2 matrix

$$\frac{1}{2}(\boldsymbol{I} - \boldsymbol{L})[DG]$$

equal $(\mathrm{Re}\,\overline{\partial}G, \mathrm{Im}\,\overline{\partial}G)$. Furthermore, we have $(\boldsymbol{I} - \boldsymbol{L})[DG] = 0$ in Ω if and only if $G_1 + iG_2$ is holomorphic in Ω.

Thus we have the following.

Proposition 2.2.15. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, $\Psi_0 \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$. Let $\mathcal{W}_{\Psi_0}$ be the neighborhood of Ψ_0 in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ of Lemma 2.2.10. Let $(\Psi, f) \in \mathcal{W}_{\Psi_0} \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D}, \mathbb{C})$. The pair $(T^+, \widetilde{T}) \in (\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$ satisfies the boundary value problem (2.2.12) if and only if the same pair satisfies the following boundary value problem*

$$\begin{cases} (\boldsymbol{I} - \boldsymbol{L})[DT^+(\cdot) \cdot (D\Psi(\cdot))^{-1}] = 0 & \textit{in } \mathbb{D}, \\ (\boldsymbol{I} - \boldsymbol{L})[D\widetilde{T}(\cdot) \cdot (D\boldsymbol{G}[\Psi](\cdot))^{-1}] = 0 & \textit{in } \mathbb{D}, \\ T^+ - \widetilde{T} = f & \textit{on } \partial\mathbb{D}, \\ \widetilde{T}(0) = 0\,. \end{cases} \tag{2.2.16}$$

In particular, problem (2.2.16) associated to (Ψ, f) has a unique solution in $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$, which we denote by $(\boldsymbol{T}^+_{\Psi_0}[\Psi, f], \widetilde{\boldsymbol{T}}_{\Psi_0}[\Psi, f])$. Finally we have that $\boldsymbol{T}^+_{\Psi_0}[\Psi, f] = \boldsymbol{S}^+[\Psi_{/\partial\mathbb{D}}, f] \circ \Psi$.

Proof. By Remark 2.2.14, condition $\overline{\partial}[T^+ \circ \Psi^{(-1)}] = 0$ can be rewritten as

$$(\boldsymbol{I} - \boldsymbol{L})[D(T^+ \circ \Psi^{(-1)})] = 0 \qquad \text{in } \Psi(\mathbb{D})\,. \tag{2.2.17}$$

By taking the composition of both hand-sides of (2.2.17) with Ψ, one obtains the first equation of (2.2.16). The second equation can be obtained similarly. Then we conclude by Propositions 2.2.11 and 2.2.8. $\square$

Our strategy is now to recast (2.2.16) in a form suitable for the application of the Implicit Function Theorem. We note that the application of the Implicit Function Theorem normally involves difficulties of two types. The first type of difficulty is concerned with showing the regularity of the nonlinear operators involved and with this respect we know that all the operators appearing in (2.2.16) are easily seen to be real analytic (cf. Lemmas 1.2.4 (iii), (iv) and 2.2.10). The second type of difficulty is inherent with the unique solvability of the linearized problem. Although the latter type of difficulty still remains, we note that our approach has completely annihilated all the difficulties of "nonlinear type". A direct approach to show the analyticity of $(\phi, f) \to \int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t)-\phi(\cdot)}\,dt$ would, instead, have to deal with difficulties of nonlinear type aggravated by the presence of a singular integral.

THEOREM 2.2.18. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in \,]0,1[$. Then the nonlinear operator defined by*

$$\boldsymbol{T}^{+}[\Psi, f] \equiv \boldsymbol{S}^{+}[\Psi_{/\partial\mathbb{D}}, f] \circ \Psi \tag{2.2.19}$$

is real analytic from $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$.

Proof. Let $(\Psi_0, f_0) \in (\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$. Let $\mathcal{W}_{\Psi_0}$ be the neighborhood of Ψ_0 of Lemma 2.2.10. By Proposition 2.2.15, we have $\boldsymbol{T}^{+}[\Psi, f] = \boldsymbol{T}^{+}_{\Psi_0}[\Psi, f]$, for all $(\Psi, f) \in \mathcal{W}_{\Psi_0} \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$. Thus by the arbitrariness of (Ψ_0, f_0), it suffices to show that the map

$$(\Psi, f) \longmapsto \boldsymbol{T}^{+}_{\Psi_0}[\Psi, f]$$

is real analytic in a open neighborhood of (Ψ_0, f_0) contained in $\mathcal{W}_{\Psi_0} \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C})$. With the notation of Lemma 2.2.13, we set

$$\mathcal{V}^{r,\alpha} \equiv \{V \in \mathcal{C}^{r,\alpha}(\mathrm{cl}\,\mathbb{D}, M_2(\mathbb{R})) : (\boldsymbol{I} + \boldsymbol{L})[V] = 0 \text{ in } \mathrm{cl}\,\mathbb{D}\},$$

for all $r \in \mathbb{N}$. Then $\mathcal{V}^{r,\alpha}$ is a closed subspace of the Banach space $\mathcal{C}^{r,\alpha}(\mathrm{cl}\,\mathbb{D}, M_2(\mathbb{R}))$. To recast problem (2.2.16) in the form of a nonlinear operator equation, we define the operator $\boldsymbol{\Lambda}$ of $\mathcal{W}_{\Psi_0} \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C}) \times (\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}))^2$ to $(\mathcal{V}^{m-1,\alpha})^2 \times \mathcal{C}^{m,\alpha}_{*}(\partial\mathbb{D},\mathbb{C}) \times \mathbb{C}$ by means of the following equality

$$\begin{aligned}\boldsymbol{\Lambda}[\Psi, f, T^{+}, \widetilde{T}] \equiv \Big(&(\boldsymbol{I} - \boldsymbol{L})[DT^{+}(\cdot) \cdot (D\Psi(\cdot))^{-1}],\\ &(\boldsymbol{I} - \boldsymbol{L})[D\widetilde{T}(\cdot) \cdot (D\boldsymbol{G}[\Psi](\cdot))^{-1}], T^{+} - \widetilde{T} - f, \widetilde{T}(0)\Big) .\end{aligned}$$

The membership of the first two components of $\boldsymbol{\Lambda}[\Psi, f, T^{+}, \widetilde{T}]$ to $(\mathcal{V}^{m-1,\alpha})^2$ is an easy consequence of Lemma 2.2.13 (i). By Proposition 2.2.15 we have

$$\boldsymbol{\Lambda}[\Psi, f, T^{+}, \widetilde{T}] = 0 \text{ if and only if } (T^{+}, \widetilde{T}) = (\boldsymbol{T}^{+}_{\Psi_0}[\Psi, f], \widetilde{\boldsymbol{T}}_{\Psi_0}[\Psi, f]) . \tag{2.2.20}$$

We now apply the Implicit Function Theorem (cf. Prodi & Ambrosetti (1973, Thm. 11.6) or Berger (1977, p. 134)) to the operator equation (2.2.20). By the real analyticity of the real multi-linear continuous operators and by Lemmas 1.2.4 (iii), (iv) and 2.2.10, the operator $\boldsymbol{\Lambda}$ is real analytic. Furthermore $\boldsymbol{\Lambda}$ is defined between an open subset of a Banach space and a Banach space. Thus all we have to show is that the differential

$$\boldsymbol{H} \equiv d_{(T^{+},\widetilde{T})}\boldsymbol{\Lambda}[\Psi_0, f_0, \boldsymbol{T}^{+}_{\Psi_0}[\Psi_0, f_0], \widetilde{\boldsymbol{T}}_{\Psi_0}[\Psi_0, f_0]]$$

of the affine map $(T^+, \widetilde{T}) \to \boldsymbol{\Lambda}[\Psi_0, f_0, T^+, \widetilde{T}]$ is a real linear homeomorphism of $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$ onto $(\mathcal{V}^{m-1,\alpha})^2 \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \times \mathbb{C}$. Since $\boldsymbol{H}$ is real linear and continuous, by the Open Mapping Theorem it suffices to show that for all $(V^+_*, \widetilde{V}_*, g, c) \in (\mathcal{V}^{m-1,\alpha})^2 \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \times \mathbb{C}$ there exists a unique pair $(W^+_*, \widetilde{W}_*) \in (\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$ satisfying

$$\begin{cases} (\boldsymbol{I} - \boldsymbol{L})[DW^+_*(\cdot) \cdot (D\Psi_0(\cdot))^{-1}] = V^+_* & \text{in } \mathbb{D}, \\ (\boldsymbol{I} - \boldsymbol{L})[D\widetilde{W}_*(\cdot) \cdot (D\boldsymbol{G}[\Psi_0](\cdot))^{-1}] = \widetilde{V}_* & \text{in } \mathbb{D}, \\ W^+_* - \widetilde{W}_* = g & \text{on } \partial\mathbb{D}, \\ \widetilde{W}_*(0) = c\,. \end{cases} \tag{2.2.21}$$

By composing the first and the third equation of (2.2.21) with $\Psi_0^{(-1)}$ and the second with $\boldsymbol{G}[\Psi_0]^{(-1)}$, system (2.2.21) can be rewritten as

$$\begin{cases} (\boldsymbol{I} - \boldsymbol{L})[D(W^+_* \circ \Psi_0^{(-1)})] = V^+_* \circ \Psi_0^{(-1)} & \text{in } \Psi_0(\mathbb{D}), \\ (\boldsymbol{I} - \boldsymbol{L})[D(\widetilde{W}_* \circ \boldsymbol{G}[\Psi_0]^{(-1)})] = \widetilde{V}_* \circ \boldsymbol{G}[\Psi_0]^{(-1)} & \text{in } \boldsymbol{G}[\Psi_0](\mathbb{D}), \\ W^+_* \circ \Psi_0^{(-1)} - \widetilde{W}_* \circ \Psi_0^{(-1)} = g \circ \Psi_0^{(-1)} & \text{on } \Psi_0(\partial\mathbb{D}), \\ \widetilde{W}_*(0) = c\,. \end{cases} \tag{2.2.22}$$

Clearly

$$\Psi_0^{(-1)}(z) = \boldsymbol{G}[\Psi_0]^{(-1)}\Big(\frac{1}{z - \Psi_0(0)}\Big) \qquad \forall z \in \Psi_0(\partial\mathbb{D})\,.$$

Now we set

$$\begin{aligned} W^+ &\equiv W^+_* \circ \Psi_0^{(-1)}, \\ \widetilde{W} &\equiv \widetilde{W}_* \circ \boldsymbol{G}[\Psi_0]^{(-1)}, \\ V^+ &\equiv \text{first row of } \frac{1}{2} V^+_* \circ \Psi_0^{(-1)}, \\ \widetilde{V} &\equiv \text{first row of } \frac{1}{2} \widetilde{V}_* \circ \boldsymbol{G}[\Psi_0]^{(-1)}\,. \end{aligned}$$

Then in view of Remark 2.2.14 and of Lemma 1.2.4 (v) and (vi), the existence and unique solvability in $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}))^2$ of problem (2.2.21) for all $(V^+_*, \widetilde{V}_*, g, c) \in (\mathcal{V}^{m-1,\alpha})^2 \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \times \mathbb{C}$ is equivalent to existence and unique solvability in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi_0(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\boldsymbol{G}[\Psi_0](\mathbb{D}), \mathbb{C})$ of the following linear boundary value problem

$$\begin{cases} \bar\partial W^+ = V^+ & \text{in } \Psi_0(\mathbb{D}), \\ \bar\partial \widetilde{W} = \widetilde{V} & \text{in } \boldsymbol{G}[\Psi_0](\mathbb{D}), \\ W^+(z) - \widetilde{W}(\frac{1}{z - \Psi_0(0)}) = g \circ \Psi_0^{(-1)}(z) & \forall z \in \Psi_0(\partial\mathbb{D}), \\ \widetilde{W}(0) = c, \end{cases} \tag{2.2.23}$$

for all $(V^+, \widetilde{V}, g, c) \in \mathcal{C}^{m-1,\alpha}(\mathrm{cl}\,\Psi_0(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m-1,\alpha}(\mathrm{cl}\,\boldsymbol{G}[\Psi_0](\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \times \mathbb{C}$. By Lemma 1.2.9 (ii) and by Proposition 2.2.8 (iii), system (2.2.23) has at most one solution. We now consider the existence. Let $(V^+, \widetilde{V}, g, c) \in \mathcal{C}^{m-1,\alpha}(\mathrm{cl}\,\Psi_0(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m-1,\alpha}(\mathrm{cl}\,\boldsymbol{G}[\Psi_0](\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \times \mathbb{C}$. It is well known (cf. Vekua (1963, p. 56)) that there exist $U^+ \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi_0(\mathbb{D}), \mathbb{C})$ and $\widetilde{U} \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\boldsymbol{G}[\Psi_0](\mathbb{D}), \mathbb{C})$ such that

$$\begin{aligned} \overline{\partial} U^+ &= V^+ \qquad && \text{in } \Psi_0(\mathbb{D}), \\ \overline{\partial} \widetilde{U} &= \widetilde{V} \qquad && \text{in } \boldsymbol{G}[\Psi_0](\mathbb{D}) \,. \end{aligned}$$

By possibly subtracting a constant to $\widetilde{U}$, we can assume that $\widetilde{U}(0) = c$. Thus all we have to show is the existence of a pair of functions $(S^+, \widetilde{S}) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Psi_0(\mathbb{D}), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\boldsymbol{G}[\Psi_0](\mathbb{D}), \mathbb{C})$ such that

(2.2.24)
$$\begin{cases} \overline{\partial} S^+ = 0 & \text{in } \Psi_0(\mathbb{D}), \\ \overline{\partial} \widetilde{S} = 0 & \text{in } \boldsymbol{G}[\Psi_0](\mathbb{D}), \\ S^+(z) - \widetilde{S}(\frac{1}{z-\Psi_0(0)}) = g \circ \Psi_0^{(-1)}(z) - U^+(z) + \widetilde{U}(\frac{1}{z-\Psi_0(0)}) & \forall z \in \Psi_0(\partial\mathbb{D}), \\ \widetilde{S}(0) = 0 \,. \end{cases}$$

Let h be the function of $\Psi_0(\partial\mathbb{D})$ to $\mathbb{C}$ be defined by

$$h(z) \equiv g \circ \Psi_0^{(-1)}(z) - U^+(z) + \widetilde{U}\left(\frac{1}{z - \Psi_0(0)}\right) \,.$$

By Lemmas 1.2.9 (ii), 1.2.8 (vi) and by Proposition 2.2.8, we have $h \circ \Psi_{0/\partial\mathbb{D}} \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. Then Proposition 2.2.8 (iii) ensures the existence of a pair of solutions $(S^+, \widetilde{S})$ of problem (2.2.24). $\square$

By the previous Theorem, by Lemma 1.2.10 and by Remark 2.2.6, we immediately deduce the validity of the following.

THEOREM 2.2.25. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. The nonlinear operator $\boldsymbol{C}$ from $(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ defined by*

$$\boldsymbol{C}[\phi, f](\cdot) = \frac{1}{2\pi i}\, \mathrm{p.v.} \int_{\partial\mathbb{D}} \frac{f(t)\phi'(t)}{\phi(t) - \phi(\cdot)}\, dt$$

is real analytic in all its domain.

2.3. Complex analyticity of the Cauchy singular integral

Let $\boldsymbol{C}$ be the Cauchy singular integral as in Theorem 2.2.25. We now compute all the differentials $\boldsymbol{C}^{(n)}$ of $\boldsymbol{C}$ and show that $\boldsymbol{C}$ is complex analytic in its domain.

PROPOSITION 2.3.1. *Let $m, n \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $\boldsymbol{C}$ be the nonlinear operator of Theorem 2.2.25. Let $(\phi_0, f_0) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$.*

Then the following formulas for the real partial differentials of $\boldsymbol{C}$ hold.

(i) $$\frac{\partial^n \boldsymbol{C}}{(\partial\phi)^n}[\phi_0, f_0][h_1, \ldots, h_n](\cdot) = $$
$$= \frac{(-1)^n (n-1)!}{2\pi i} \int_{\partial\mathbb{D}} f_0'(t) \prod_{i=1}^{n} \Big(\frac{h_i(t) - h_i(\cdot)}{\phi_0(t) - \phi_0(\cdot)} \Big)\, dt,$$
for all $(h_1, \ldots, h_n) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}))^n$.

(ii) $$\frac{\partial^n \boldsymbol{C}}{(\partial\phi)^{n-1}\partial f}[\phi_0, f_0][h_1, \ldots, h_{n-1}, k_n](\cdot) =$$
$$= \begin{cases} \boldsymbol{C}[\phi_0, k_n](\cdot) & \textit{if } n = 1, \\ \frac{(-1)^{n-1}(n-2)!}{2\pi i} \int_{\partial\mathbb{D}} k_n'(t) \prod_{i=1}^{n-1} \Big(\frac{h_i(t) - h_i(\cdot)}{\phi_0(t) - \phi_0(\cdot)} \Big)\, dt & \text{if } n \geq 2, \end{cases}$$
for all $(h_1, \ldots, h_{n-1}, k_n) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}))^n$.

(iii) $$\boldsymbol{C}^{(n)}[\phi_0, f_0][(h_1, k_1), \ldots, (h_n, k_n)] =$$
$$= \frac{\partial^n \boldsymbol{C}}{(\partial\phi)^n}[\phi_0, f_0][h_1, \ldots, h_n] + \sum_{i=1}^{n} \frac{\partial^n \boldsymbol{C}}{(\partial\phi)^{(n-1)}\partial f}[\phi_0, f_0][h_1, \ldots, \widehat{h_i}, \ldots, h_n, k_i],$$

for all $(h_1, k_1, \ldots, h_n, k_n) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C}))^{2n}$. *The '$\widehat{\ }$' symbol on a term denotes that such term must be omitted. In particular,* $\boldsymbol{C}$ *is complex analytic in its domain.*

Proof. It clearly suffices to consider the case in which $\mathrm{ind}[\phi_0] = 1$. Let $\boldsymbol{R}$ be the trace operator of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. Let $z_0 \in \mathbb{I}[\phi_0]$ and let $\boldsymbol{E}$, $\boldsymbol{E}_{\phi_0}$, Ψ_0 and $\mathcal{U}_{\phi_0}$ be as in Lemma 1.2.10. Clearly, we can assume that $\mathrm{ind}[\phi] = 1$ for all $\phi \in \mathcal{U}_{\phi_0}$. The operator $\boldsymbol{E}_{\phi_0}$ is complex differentiable at all points of its domain, with differential given by the operator $\boldsymbol{E}$ which satisfies $\boldsymbol{E}[h]_{/\partial\mathbb{D}} = h$ for all $h \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$. We now compute the Taylor expansion of the real analytic operator $\boldsymbol{C}[\,\cdot, f_0]$ at ϕ_0. By the definition of $\boldsymbol{T}^+$ (cf. (2.2.19)) and of $\boldsymbol{C}$, we have

$$\boldsymbol{C}[\phi, f_0] = -\frac{1}{2} f_0 + \boldsymbol{R}[\boldsymbol{T}^+[\boldsymbol{E}_{\phi_0}[\phi], f_0]], \tag{2.3.2}$$

for all $\phi \in \mathcal{U}_{\phi_0}$. Since $\boldsymbol{R}$ is linear, it suffices to find the Taylor expansion at $\Psi_0 \equiv \boldsymbol{E}_{\phi_0}[\phi_0]$ of the operator $\boldsymbol{T}^+[\cdot, f_0]$ of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$. Since for all $\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ $\boldsymbol{T}^+[\Psi, f_0]$ is a Cauchy type integral and in particular is not singular, we try to reach the boundary values of $\boldsymbol{T}^+[\Psi, f_0]$ by extending the disks of a smaller radius. We set

$$\mathbb{D}_\rho = \{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2)| < \rho\} \qquad \forall \rho \in \,]0, 1[$$

and we denote by $\boldsymbol{R}_\rho$ the restriction operator of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}_\rho,\mathbb{C})$. Then we have that

$$\boldsymbol{R}_\rho \circ \boldsymbol{T}^+[\Psi, f_0](w) = \frac{\mathrm{ind}[\Psi_{/\partial\mathbb{D}}]}{2\pi i}\int_{\partial\mathbb{D}} \frac{f_0(t)\Psi'(t)}{\Psi(t)-\Psi(w)}\,dt \quad \forall w \in \mathrm{cl}\,\mathbb{D}_\rho, \tag{2.3.3}$$

for all $\Psi \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ and for all $\rho \in \,]0,1[$. Now we note that, by Theorem 2.2.18, $\boldsymbol{R}_\rho \circ \boldsymbol{T}^+[\,\cdot, f_0]$ is real analytic on $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})\cap\mathcal{A}_{\mathbb{D}}$ and that by standard calculus

$$\begin{aligned}\boldsymbol{R}_\rho \circ \Big(\frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0, f_0]\Big)[[U]^n] &= \frac{\partial^n}{(\partial\Psi)^n}(\boldsymbol{R}_\rho\circ\boldsymbol{T}^+)[\Psi_0,f_0][[U]^n] \\ &= \frac{d^n}{(d\varepsilon)^n|_{\varepsilon=0}}\{\boldsymbol{R}_\rho\circ\boldsymbol{T}^+[\Psi_0+\varepsilon U, f_0]\}\end{aligned} \tag{2.3.4}$$

for all $U \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$. Since the integral in (2.3.3) is not singular when $w \in \mathrm{cl}\,\mathbb{D}_\rho$ and since $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ is continuously embedded in $\mathcal{C}^0(\mathrm{cl}\,\mathbb{D},\mathbb{C})$, then by a standard result on the differentiation of integrals depending on a parameter and by a straightforward computation, we obtain that

$$\begin{aligned}&\frac{d^n}{(d\varepsilon)^n|_{\varepsilon=0}}\{\boldsymbol{R}_\rho\circ\boldsymbol{T}^+[\Psi_0+\varepsilon U, f_0]\}(w) = \\ &\quad= \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{d^n}{(d\varepsilon)^n|\varepsilon=0}\left\{\frac{f_0(t)(\Psi_0'+\varepsilon U')(t)}{(\Psi_0+\varepsilon U)(t)-(\Psi_0+\varepsilon U)(w)}\right\}dt \\ &\quad= \frac{(-1)^{n-1}n!}{2\pi i}\int_{\partial\mathbb{D}}\Big(\frac{U(t)-U(w)}{\Psi_0(t)-\Psi_0(w)}\Big)^{n-1}\Big\{-\frac{f_0(t)\Psi_0'(t)(U(t)-U(w))}{(\Psi_0(t)-\Psi_0(w))^2} \\ &\qquad + \frac{f_0(t)U'(t)}{\Psi_0(t)-\Psi_0(w)}\Big\}\,dt \\ &\quad= \frac{(-1)^{n-1}(n-1)!}{2\pi i}\int_{\partial\mathbb{D}} f_0(t)\frac{d}{dt}\left\{\Big(\frac{U(t)-U(w)}{\Psi_0(t)-\Psi_0(w)}\Big)^n\right\}dt \\ &\quad= \frac{(-1)^n(n-1)!}{2\pi i}\int_{\partial\mathbb{D}} f_0'(t)\Big(\frac{U(t)-U(w)}{\Psi_0(t)-\Psi_0(w)}\Big)^n dt\end{aligned}$$

for all $w \in \mathrm{cl}\,\mathbb{D}_\rho$, where the last equality follows by integration by parts. Then if $w \in \mathbb{D}$, we can choose ρ such that $|w| < \rho < 1$ and by (2.3.4), we obtain

$$\frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0, f_0][[U]^n](w) = \frac{(-1)^n(n-1)!}{2\pi i}\int_{\partial\mathbb{D}} f_0'(t)\Big(\frac{U(t)-U(w)}{\Psi_0(t)-\Psi_0(w)}\Big)^n dt\,. \tag{2.3.5}$$

Since $U \equiv U_1 + iU_2$ belongs to $\mathcal{C}^1(\mathrm{cl}\,\mathbb{D},\mathbb{C})$, by Lemma 1.2.5 (i) and by the mean value inequality, we obtain

$$\Big|\frac{U(t)-U(w)}{\Psi_0(t)-\Psi_0(w)}\Big| \le \frac{|U(t)-U(w)|}{\boldsymbol{l}_{\mathbb{D}}[\Psi_0]|t-w|} \le \frac{\sum_{1\le h,j\le 2}\sup_{\mathrm{cl}\,\mathbb{D}}\Big|\frac{\partial U_h}{\partial x_j}\Big|}{\boldsymbol{l}_{\mathbb{D}}[\Psi_0]}, \tag{2.3.6}$$

for all $(t,w) \in \partial\mathbb{D}\times\mathrm{cl}\,\mathbb{D}$ with $t \neq w$. Then by the Theorem of continuity of integrals depending on a parameter, the right hand-side of (2.3.5) depends continuously on $w \in \mathrm{cl}\,\mathbb{D}$ and then equality (2.3.5) holds for all $w \in \mathrm{cl}\,\mathbb{D}$. By

(2.3.6), the equality

$$\boldsymbol{H}[U_1,\dots,U_n](w) \equiv \frac{(-1)^n(n-1)!}{2\pi i}\int_{\partial\mathbb{D}} f_0'(t)\prod_{i=1}^{n}\Big(\frac{U_i(t)-U_i(w)}{\Psi_0(t)-\Psi_0(w)}\Big)\,dt,\ \forall w\in \mathrm{cl}\,\mathbb{D}$$

for all $(U_1,\dots,U_n)\in(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}))^n$, defines a complex n-linear symmetric map of $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}))^n$ to $\mathcal{C}^0(\mathrm{cl}\,\mathbb{D},\mathbb{C})$. Clearly

$$\boldsymbol{H}[[U]^n] = \frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0,f_0][[U]^n]$$

for all $U\in\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}))$. Since $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ is continuously embedded in $\mathcal{C}^0(\mathrm{cl}\,\mathbb{D},\mathbb{C})$ and both $\boldsymbol{H}$ and $\frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0,f_0]$ are real n-linear symmetric maps which coincide on the diagonal of $(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{C}))^n$, we must have

$$\boldsymbol{H} = \frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0,f_0]$$

and accordingly $\frac{\partial^n \boldsymbol{T}^+}{(\partial\Psi)^n}[\Psi_0,f_0]$ is a complex n-linear map. By using the chain rule combined with the properties of the map $\boldsymbol{E}_{\phi_0}$, we obtain statement (i). The linearity of $\boldsymbol{C}$ in the variable f implies the validity of statement (ii) and of equality

$$\frac{\partial^{j+2}\boldsymbol{C}}{(\partial\phi)^j(\partial f)^2}[\phi,f]=0 \tag{2.3.7}$$

for all (ϕ,f) in the domain of $\boldsymbol{C}$ and for all $j\in\mathbb{N}$. Since the n-th differential of $\boldsymbol{C}$ is the sum of the partial differentials of $\boldsymbol{C}$ of order n, we obtain statement (iii) by statements (i), (ii) and (2.3.7). Since for all $r\in\mathbb{N}\setminus\{0\}$, $\boldsymbol{C}^{(r)}[\phi_0,f_0]$ is a complex r-linear symmetric map from $(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}))^r$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$, Theorem 2.2.25 implies that $\boldsymbol{C}$ is a complex analytic operator. □

We now show that the formal expansion of the Cauchy kernel with respect to the curve ϕ studied by Tran-Oberlé (1989) for graph curves in $\mathbb{C}$ around the inclusion map of $\mathbb{R}$ in $\mathbb{C}$, gives, in our setting, the Taylor series of the Cauchy singular operator as a map of the contour ϕ. As a consequence, we deduce the validity of a result of Coifman and Meyer (1983b, p. 10) in our Schauder space setting and for curves which are not necessarily arc-length parametrized.

Corollary 2.3.8. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $\boldsymbol{C}$ be the nonlinear operator of Theorem 2.2.25. The nonlinear operator $\widetilde{\boldsymbol{C}}$ from $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})\cap\mathcal{A}_{\partial\mathbb{D}}$ to $\mathcal{L}_{\mathbb{C}}\big(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}),\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})\big)$ defined by*

$$\widetilde{\boldsymbol{C}}[\phi]\equiv\boldsymbol{C}[\phi,\cdot]$$

is complex analytic. Furthermore, if $\phi_0\in\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})\cap\mathcal{A}_{\partial\mathbb{D}}$ and if we denote the integral operators $f\mapsto\int_{\partial\mathbb{D}}f(t)k(\cdot,t)dt$ and $f\mapsto$ p.v. $\int_{\partial\mathbb{D}}f(t)k(\cdot,t)dt$ associated to a given complex-valued function $k(\cdot,\cdot)$ defined on $\{(s,t)\in(\partial\mathbb{D})^2:\ s\neq t\}$

by $k(\cdot,t)dt$ *and* p.v.$k(\cdot,t)dt$ *respectively, we have the following Taylor expansion which has radius of convergence greater or equal to* $\widetilde{r} \equiv \sup\{r > 0 : \mathrm{cl}_{\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})}\mathcal{B}(\phi_0, r) \subseteq \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}\}$*:*

$$\begin{aligned}
&\frac{1}{2\pi i}\,\mathrm{p.\,v.}\,\frac{\phi_0'(t)+h'(t)}{\phi_0(t)-\phi_0(\cdot)+h(t)-h(\cdot)}\,dt \\
&\qquad = \frac{1}{2\pi i}\,\mathrm{p.\,v.}\,\frac{\phi_0'(t)}{\phi_0(t)-\phi_0(\cdot)}\,dt + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2\pi i n}\frac{d}{dt}\left\{\Big(\frac{h(t)-h(\cdot)}{\phi_0(t)-\phi_0(\cdot)}\Big)^n\right\}dt\,.
\end{aligned}$$

for all $h \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$ *such that* $\|h\|_{\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})} < r$*.*

Proof. We first observe that $\boldsymbol{C}$ is linear in its second variable. Accordingly, if f_0 is an arbitrary but fixed element of $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$, we have

$$\widetilde{\boldsymbol{C}}[\phi] = \frac{\partial \boldsymbol{C}}{\partial f}[\phi, f_0]$$

for all $\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. Then the analyticity of $\widetilde{\boldsymbol{C}}$ follows from that of $\boldsymbol{C}$. A simple computation based on the Hölder continuity of ϕ_0', h' and on Lemmas 1.2.6 (i), 1.2.8 (i), shows that there exists a constant $c > 0$ depending only on h, ϕ_0, such that $\left|\frac{d}{dt}\Big(\frac{h(t)-h(t_0)}{\phi_0(t)-\phi_0(t_0)}\Big)\right| \leq \frac{c}{|t-t_0|^{1-\alpha}}$, for all $t, t_0 \in \partial\mathbb{D}$, $t \neq t_0$ (see also Lu (1993, Ex. 5 p. 20)) and that accordingly the integral $\int_{\partial\mathbb{D}} f(t)\frac{d}{dt}\Big(\frac{h(t)-h(t_0)}{\phi_0(t)-\phi_0(t_0)}\Big)dt$ exists in the sense of Lebesgue for all $f \in \mathcal{C}_*^0(\partial\mathbb{D},\mathbb{C})$ and for all $t_0 \in \partial\mathbb{D}$. By Proposition 2.3.1 (ii) and by integration by parts, we obtain the validity of the Taylor expansion of the statement in a ball $\mathcal{B}(\phi_0, r_\prime)$ of sufficiently small radius $r_\prime > 0$. Let $r > 0$ be such that $\overline{\mathcal{B}} \equiv \mathrm{cl}_{\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})}\mathcal{B}(\phi_0, r)$ is contained in $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. To complete the proof it suffices to show that $\widetilde{\boldsymbol{C}}[\cdot]$ is bounded on $\overline{\mathcal{B}}$. By a standard application of the Ascoli-Arzela Theorem, the set $\overline{\mathcal{B}}$ is a compact subset of $\mathcal{C}_*^1(\partial\mathbb{D},\mathbb{C})$. Then Lemma 1.2.6 (ii) implies that the map $\boldsymbol{l}_{\partial\mathbb{D}}[\cdot]$ has a strictly positive minimum on $\overline{\mathcal{B}}$. Then by Privalov Theorem (cf. *e.g.* Lu (1993, Thm. 3.1.1)) and by standard properties of the Cauchy integral, it follows the boundedness of $\widetilde{\boldsymbol{C}}[\cdot]$ on $\overline{\mathcal{B}}$.

□

Now we restate Theorem 2.2.25 by using a domain of integration more general than $\partial\mathbb{D}$.

Corollary 2.3.9. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$, $L = \phi(\partial\mathbb{D})$. *The set of* $g \in \mathcal{C}_*^1(L,\mathbb{C})$ *which are injective and satisfy condition* $g'(\xi) \neq 0$*, for all* $\xi \in L$ *coincides with* $\mathcal{A}_L$ *and is open. The nonlinear operator of* $(\mathcal{C}_*^{m,\alpha}(L,\mathbb{C}) \cap \mathcal{A}_L) \times \mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$ *to* $\mathcal{C}_*^{m,\alpha}(L,\mathbb{C})$ *defined by*

$$\boldsymbol{C}_L[\gamma, f] \equiv \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_L \frac{f(t)\gamma'(t)}{\gamma(t)-\gamma(\cdot)}\,dt,$$

where we understand that the line integral is computed with respect to the parametrization $\phi(e^{i\theta})$, $\theta \in [0, 2\pi]$, *is complex analytic. The partial differentials of* $\boldsymbol{C}_L$ *can be obtained by those of* $\boldsymbol{C}_{\partial\mathbb{D}} = \boldsymbol{C}$ *by replacing the integration on* $\partial\mathbb{D}$ *with an integration on* L.

Proof. By simple computations and by Lemma 1.2.8 (ii), it follows that $K = \phi(\partial\mathbb{D})$ satisfies the assumptions on K of conditions (i), (ii) of Lemma 1.2.6. Accordingly the conclusions of Lemma 1.2.6 (i), (ii) hold for $K = \phi(\partial\mathbb{D})$. Let $\boldsymbol{T}_\phi$ be the complex linear homeomorphism of $\mathcal{C}_*^{m,\alpha}(L, \mathbb{C})$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ of Lemma 1.2.8 (viii). Clearly

$$\boldsymbol{C}_L[\gamma, f] = \boldsymbol{T}_\phi^{(-1)}\left[\boldsymbol{C}[\boldsymbol{T}_\phi[\gamma], \boldsymbol{T}_\phi[f]]\right]$$

for $(\gamma, f) \in (\mathcal{C}_*^{m,\alpha}(L, \mathbb{C}) \cap \mathcal{A}_L) \times \mathcal{C}_*^{m,\alpha}(L, \mathbb{C})$. Then the chain rule yields the conclusion. $\square$

CHAPTER 3

Perturbation analysis of the conformal sewing problem in Schauder spaces

3.1. Introduction

In this chapter we do a perturbation analysis of a classical boundary value problem with shift, namely the conformal sewing problem. Although the boundary value problems with shift have been extensively studed (cf. *e.g.* Litvinchuk & Zwerovich (1968), Aizenshtadt, Karlovich & Litvinchuk (1992) and Kravchenko & Litvinchuk (1994)), the perturbation analysis of such problems seems to be a subject less analyzed. We first introduce the geometric interpretation of the sewing problem to sketch the setting of our work and its geometric motivations.

Let $\widehat{\mathbb{C}} \equiv \mathbb{C} \cup \{ \infty \}$ and let $\mathbb{J}$ be a Jordan domain of $\widehat{\mathbb{C}}$, *i.e.* an open subset of $\widehat{\mathbb{C}}$ which boundary is the image of a Jordan curve of $\widehat{\mathbb{C}}$, namely continuous and injective function g of $\partial\mathbb{D}$ to $\widehat{\mathbb{C}}$. Clearly $\mathbb{J}$ is one of the two connected components of $\widehat{\mathbb{C}} \setminus g(\partial\mathbb{D})$. We want to define a map, say welding map, which associates to a normalized Jordan domain a homeomorphism of $\partial\mathbb{D}$ to itself. Let t_1, t_2, t_3 be three distinct points of $\partial\mathbb{D}$ such that the orientation induced by these points in $\partial\mathbb{D}$ is counterclockwise and let z_1, z_2, z_3 be three distinct points of $\widehat{\mathbb{C}}$. Let $\varDelta_\prime$ be the set of the Jordan domains $\mathbb{J}$ such that $\partial\mathbb{J}$ contains z_1, z_2, z_3 and has $\mathbb{J}$ to the left when $\partial\mathbb{J}$ has the orientation induced by z_1, z_2, z_3. Let F_1 and G_1 be conformal maps (*i.e.* meromorphic and injective and then with at most one simple pole) of $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \mathrm{cl}\,\mathbb{D}$, respectively, such that $F_1(\mathbb{D}) = \mathbb{J}$ and $G_1(\widehat{\mathbb{C}} \setminus \mathrm{cl}\,\mathbb{D}) = \widehat{\mathbb{C}} \setminus \mathrm{cl}\,\mathbb{J}$. As it is well-known the maps F_1 and G_1 can be extended by continuity to $\mathrm{cl}\,\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \mathbb{D}$, respectively, and such extensions are homeomorphisms in the respective images. The pair (F_1, G_1) is uniquely fixed by the normalization conditions $F_1(t_i) = z_i$, $G_1(t_i) = z_i$, $i = 1, 2, 3$. Let W_1 be the welding map from $\varDelta_\prime$ to the subset of $\mathcal{C}^0_*(\partial\mathbb{D}, \mathbb{C})$ of the homeomorphisms of $\partial\mathbb{D}$ to itself which fix t_i, $i = 1, 2, 3$, defined by

$$W_1[\mathbb{J}] = F^{(-1)}_{1/\partial\mathbb{D}} \circ G_{1/\partial\mathbb{D}} \tag{3.1.1}$$

for all $\mathbb{J} \in \varDelta_\prime$. Clearly with a different choice of the points t_i and z_i, $i = 1, 2, 3$, we obtain a map which differs from the first only in a composition and in a conjugation by Möbius transformations. If (F_1, G_1) is a pair of maps as above, there is a unique Möbius transformation H such that the pair $(H \circ F_1, H \circ G_1)$

satisfies the normalization condition $\lim_{z\to\infty} H(G_1(z)) - z = 0$. Then there is a natural one-to-one correspondence between $\Delta_\prime$ and the set Δ of all pairs (F, G) of continuous and injective functions, defined in $\mathrm{cl}\,\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \mathbb{D}$, holomorphic in $\mathbb{D}$ and $\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}$, respectively, and satisfying the normalization condition

$$\lim_{z\to\infty} G(z) - z = 0 \, . \tag{3.1.2}$$

The conformal sewing problem consists in finding a sort of right inverse of the welding map W_1. Indeed, let ϕ be a homeomorphism of $\partial\mathbb{D}$ to itself, more simply a shift of $\partial\mathbb{D}$. A solution of the conformal sewing problem associated to ϕ is a pair (F, G) in Δ such that the following condition holds

$$F_{/\partial\mathbb{D}} \circ \phi = G_{/\partial\mathbb{D}} \, . \tag{3.1.3}$$

It is easy to check that the sewing problem associated to a shift ϕ consists in finding the conformal structures of the topological space, homeomorphic to $\widehat{\mathbb{C}}$, $\mathrm{cl}\,\mathbb{D} \cup_\phi \widehat{\mathbb{C}} \setminus \mathbb{D}$ (obtained by identifying t with $\phi(t)$, for all $t \in \partial\mathbb{D}$), which extend the natural conformal structures of $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \mathrm{cl}\,\mathbb{D}$ (see Nag (1990, Prop. II.1)).

For a general shift ϕ, the sewing problem can admit no solutions (as observed by Lehto (1987, p. 100)) or admit infinitely many solutions (see Nag (1990, Part II)). By the theory of the quasiconformal maps in $\mathbb{C}$ (cf. *e.g.* Lehto (1987, Ch. 1 and p. 96–101)), W_1 restricts to a one-to-one map from the subset $\Delta_{\prime\prime}$ of $\Delta_\prime$, consisting of the quasidisks (*i.e.* the image of $\mathbb{D}$ by a quasiconformal map) in $\Delta_\prime$, and the subset Υ of the quasisymmetric shifts (see *e.g.* Pommerenke (1992, p. 95)) which fix t_1, t_2, t_3. $\Delta_{\prime\prime}$ and Υ are models of the universal Teichmüller space which is an important subject in geometric functions theory. Under the assumption that ϕ is quasisymmetric, we define the pair of operators $(\boldsymbol{F}[\cdot], \boldsymbol{G}[\cdot])$ which maps the shift ϕ to the trace on $\partial\mathbb{D}$ of the solution (F, G) of the sewing problem (3.1.3) (by composing by Möbius transformations, the uniqueness follows also for quasisymmetric shifts which do not fix t_1, t_2, t_3). Clearly $\boldsymbol{F}[\phi]$ and $\boldsymbol{G}[\phi]$ determine uniquely the conformal maps F and G. Nag (1996, sec. 1) illustrates the importance to obtaining $\boldsymbol{F}[\phi]$ and $\boldsymbol{G}[\phi]$ from ϕ in the theory of Teichmüller spaces. Nag (1996) considers a real analytic family of shifts ω_t, $t \in]-\varepsilon, \varepsilon[$, $\varepsilon > 0$ with ω_0 real analytic. By assuming the real analyticity on t of $\boldsymbol{F}[\omega_t]$ and $\boldsymbol{G}[\omega_t]$ and the applicability of the Plemelj formula, he obtains a recursive method to determine all the variations of both $\boldsymbol{F}[\omega_t]$ and $\boldsymbol{G}[\omega_t]$ in terms of the variations of ω_t. Our aims for these chapters is to show that in a suitable Banach space setting, the solution of the conformal sewing problem, $(\boldsymbol{F}[\phi], \boldsymbol{G}[\phi])$, depends real analytically upon ϕ. As a corollary of our result, we can deduce that if ϕ depends analytically on a real parameter t then $(\boldsymbol{F}[\phi], \boldsymbol{G}[\phi])$ depends analytically on the same parameter. Thus in particular, Nag's perturbation scheme can be applied. In the literature there are some results about the continuous dependence of $(\boldsymbol{F}[\phi], \boldsymbol{G}[\phi])$ on ϕ (cf. Monakhov (1983, p. 363) and Huber & Kühnau (1994, p. 319)). In this chapter we study the regularity of the operators $\boldsymbol{F}[\cdot]$ and $\boldsymbol{G}[\cdot]$ in a setting of Schauder spaces of regular functions and we show that this setting is not sufficient to obtain the analyticity of $\boldsymbol{F}[\cdot]$. To pursue our goals, we proceed independently on the quasiconformal maps theory and we employ the classical results about boundary value problems with shift (cf. *e.g.* Lu (1993, p. 416)) and the Implicit Function Theorem for maps between Banach spaces.

Let $m \in \mathbb{N} \setminus \{0\}$, $0 < \alpha < 1$. We first assume that (F, G) is a solution for the conformal sewing problem associated to the shift ϕ, which we assume to belong to the space $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ of m-times continuously differentiable functions, with α-Hölder continuous m-th order derivative, and to be one-to-one, index preserving, and with everywhere nonvanishing tangent vector. Though a shift ϕ belonging to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ is quasisymmetric and then there is a unique solution for the sewing problem (3.1.3), we will show the unique solvability of problem (3.1.3) by different tools. Then we derive (see section 2) an integral equation involving $(\phi, g \equiv G_{/\partial\mathbb{D}})$, which we can write as

$$\boldsymbol{\Gamma}[\phi, g] = 1_{\partial\mathbb{D}} \tag{3.1.4}$$

where $1_{\partial\mathbb{D}}$ is the identity map of $\partial\mathbb{D}$. Since ultimately, we intend to employ the Implicit Function Theorem to equation (3.1.4), and since the Implicit Function Theorem requires that the domain of $\boldsymbol{\Gamma}$ be open, while the set of functions which map $\partial\mathbb{D}$ to $\partial\mathbb{D}$ is not open, we make sure that the operator $\boldsymbol{\Gamma}$ in equation (3.1.4) makes sense and is regular for ϕ's in a neighborhood $\mathcal{U}$ in $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ of the functions which map $\partial\mathbb{D}$ to itself. As a next step, we need to show that equation (3.1.4) has a unique solution g of class $\mathcal{C}^{m,\alpha}_*$ for all ϕ belonging to $\mathcal{U}$. We do so by a two steps argument. We first show that for ϕ in $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$, equation (3.1.4) has a unique solution which is of class $\mathcal{C}^{0,\alpha}_*$. Then we show that if (ϕ, g) solves (3.1.4), ϕ of class $\mathcal{C}^{m,\alpha}_*$, g of class $\mathcal{C}^{0,\alpha}_*$, then g must belong to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D}, \mathbb{C})$ (cf. section 2). Then we show that if (ϕ, g) is of class $\mathcal{C}^{m,\alpha}_*$, and solves (3.1.4), then $(g \circ \phi^{(-1)}, g)$ is the trace of the solution for the conformal sewing problem. Next we apply the Implicit Function Theorem to equation (3.1.4) for complex analytic operator by exploiting the complex analyticity of the dependence of a Cauchy singular integral upon its contour (cf. Lanza & Preciso (1998, Prop. 4.1)) and we deduce that the set of zeros of equation (3.1.4) is the graph of a holomorphic operator, which for shifts ϕ's satisfying $\phi(\partial\mathbb{D}) = \partial\mathbb{D}$ coincides with the map which takes ϕ to $\boldsymbol{G}[\phi]$. In section 4, we study the differentiability properties of the operator $\boldsymbol{F}$ defined by

$$\boldsymbol{F}[\phi] \equiv \boldsymbol{G}[\phi] \circ \phi^{(-1)}$$

for all shift ϕ of class $\mathcal{C}_*^{m,\alpha}$. Let $k \in \mathbb{N} \setminus \{0\}$. By exploiting the result of Lanza (1997), we prove that F has an extension to an open neighborhood $\mathcal{U}_{\prime}$ in the subspace $\mathcal{C}_*^{m+k,\alpha,p}(\partial\mathbb{D},\mathbb{C})$, defined as the closure in $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$ of the smooth functions and this extension is of class $\mathcal{C}^k$ from $\mathcal{U}_{\prime}$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$. Then we show that the introduction of $\mathcal{C}_*^{m+k,\alpha,p}(\partial\mathbb{D},\mathbb{C})$ was actually necessary. Namely, we prove that if $r \in \mathbb{N} \setminus \{0\}$ and $\boldsymbol{F}$ were to have an extension of class $\mathcal{C}^r$ from an open neighborhood of the admissible ϕ's in $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$, then the ϕ's in the domain of $\boldsymbol{F}$ must belong to $\mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D},\mathbb{C})$.

3.2. The integral equation associated to the sewing problem. Analyticity of the operator $\boldsymbol{G}$

In this section we present the classical integral equation approach to show the unique solvability of the sewing problem in a Schauder space setting. In view of a perturbation analysis in a Banach space and of a consequent application of the Implicit Function Theorem, we need to deal with an open set of shifts. Let $\mathcal{A}_{\partial\mathbb{D}}$ be the open subset of $\mathcal{C}_*^1(\partial\mathbb{D},\mathbb{C})$ of the injective maps with nonvanishing derivative (cf. Lemma 1.2.6). We set

$$\mathcal{A}_{\partial\mathbb{D}}^* \equiv \{ \phi \in \mathcal{A}_{\partial\mathbb{D}} : \operatorname{ind}[\phi] = 1 \} .$$

Analogously we define $\mathcal{A}_L^*$ for $L = \gamma(\partial\mathbb{D})$ and $\gamma \in \mathcal{A}_{\partial\mathbb{D}}$. By the constancy of the index under small perturbations in $\mathcal{C}_*^1(\partial\mathbb{D},\mathbb{C})$, the set $\mathcal{A}_{\partial\mathbb{D}}^*$ is open in $\mathcal{C}_*^1(\partial\mathbb{D},\mathbb{C})$. Let $m \in \mathbb{N} \setminus \{0\}$, $0 < \alpha < 1$. Then we consider the open neighborhood $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ of $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\partial\mathbb{D}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ and we call the elements of $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ "generalized shifts". As usual, if Ω is an open subset of $\mathbb{C}$, $\mathcal{H}(\Omega)$ will denote the holomorphic functions of Ω.

Definition 3.2.1. Let $0 < \alpha < 1$, $\gamma \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$, $L \equiv \gamma(\partial\mathbb{D})$. Let $\phi \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$. We say that the pair of complex-valued functions (F,G) defined in $\operatorname{cl}\mathbb{I}[\phi\circ\gamma]$ and $\operatorname{cl}\mathbb{E}[\gamma]$, holomorphic in $\mathbb{I}[\phi\circ\gamma]$ and $\mathbb{E}[\gamma]$, respectively, is a solution of the boundary value problem with generalized shift ϕ on L (or $(BVP\phi)$ on L) if $(F_{/\phi(L)}, G_{/L}) \in \mathcal{C}_*^{0,\beta}(\phi(L),\mathbb{C}) \times \mathcal{C}_*^{0,\beta}(L,\mathbb{C})$, for some $\beta \in]0,1[$, and the following condition hold

$$F(\phi(t)) = G(t) \tag{3.2.2}$$

for all $t \in L$.

Now we state an integral equation satisfied by a solution of problem $(BVP\phi)$.

Proposition 3.2.3 (cf. Lu (1993, p. 417)). *Let* $0 < \alpha < 1$, $\gamma \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$, $L \equiv \gamma(\partial\mathbb{D})$. *Let* $\phi \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$. *Let* (F,G) *be a solution of problem* $(BVP\phi)$ *and let* $P(\cdot)$ *be a polynomial function of* $\mathbb{C}$ *such that* $\lim_{z\to\infty} G(z) - P(z) = 0$. *Let* $\boldsymbol{C}$ *be the Cauchy singular integral defined in Corollary* 2.2.5. *Let*

$k_\phi(\cdot,\cdot)$ be the complex-valued function of $L^2 \setminus \{ (t,\xi) \in L^2 : t \neq \xi \}$ defined by

$$k_\phi(t,\xi) \equiv \frac{1}{\xi - t} - \frac{\phi'(\xi)}{\phi(\xi)-\phi(t)} \tag{3.2.4}$$

for all $(t,\xi) \in L^2$, $t \neq \xi$. Then (F,G) satisfies the following integral equation

$$\begin{aligned} G(t) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_L k_\phi(t,\xi)G(\xi)\,d\xi \\ = G(t) + \boldsymbol{C}[1_{\partial\mathbb{D}}, G_{/L}](t) - \boldsymbol{C}[\phi, G_{/L}](t) = P(t) \end{aligned} \tag{3.2.5}$$

for all $t \in L$. We intend that the line integral on L are computed with respect the parametrization $\gamma(e^{i\theta})$, $\theta \in [0,2\pi]$.

Proof. Since ϕ is a positively oriented closed curve of class $\mathcal{C}^{1,\alpha}_*$, by the Cauchy formula and the Plemelj formula we have

$$F(z) = \frac{1}{2\pi i}\int_{\phi(L)} \frac{F(\eta)}{\eta - z}\,d\eta, \qquad z \in \mathbb{I}[\phi\circ\gamma],$$

$$G(w) - P(w) = -\frac{1}{2\pi i}\int_L \frac{G(\xi)-P(\xi)}{\xi - w}\,d\xi, \qquad w \in \mathbb{E}[\gamma],$$

and

$$F(t_1) = \lim_{z\to t_1} F(z) = \frac{1}{2}F(t_1) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\phi(L)} \frac{F(\eta)}{\eta - t_1}\,d\eta \tag{3.2.6}$$

$$\begin{aligned} G(t) - P(t) &= \lim_{w\to t} G(w) - P(w) \\ &= \frac{1}{2}(G(t)-P(t)) - \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_L \frac{G(\xi)-P(\xi)}{\xi - t}\,d\xi \end{aligned} \tag{3.2.7}$$

for all $t_1 \in \phi(L)$ and for all $t \in L$. By setting $t_1 = \phi(t)$, $\eta = \phi(\xi)$ and by exploiting equation (3.2.2), (3.2.6) becomes

$$G(t) = \frac{1}{2}G(t) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_L \frac{G(\xi)\phi'(\xi)}{\phi(\xi)-\phi(t)}\,d\xi \tag{3.2.8}$$

for all $t \in L$. By using that $P(\cdot)$ is holomorphic in $\mathbb{C}$ and by adding (3.2.7) with (3.2.8), we obtain (3.2.5). □

In order to study the integral equation (3.2.5), we show this lemma which is a variant of a classical result about integral operators (cf. *e.g.* Kantorovich & Akilov (1964, p. 363 Thm. 4 and p. 365 Rem. 2)).

LEMMA 3.2.9. *Let $0 < \alpha < 1$ and let $k(\cdot,\cdot)$ be a complex-valued function of $(\partial\mathbb{D})^2 \setminus \{ (t,\xi) \in (\partial\mathbb{D})^2, t \neq \xi \}$. We assume that for each fixed $\xi \in \partial\mathbb{D}$, the derivative with respect t exists and is continuous in all compact subsets of $\partial\mathbb{D} \setminus \{\xi\}$. Let M_1 and M_2 be positive constants such that the following conditions hold*

$$|k(t,\xi)| \leq \frac{M_1}{|\xi - t|^{1-\alpha}} \tag{3.2.10}$$

$$\left|\frac{\partial k}{\partial t}(t,\xi)\right| \le \frac{M_2}{|\xi - t|^{2-\alpha}} \tag{3.2.11}$$

for all $(t,\xi) \in (\partial\mathbb{D})^2$, $t \neq \xi$. *Then the integral operator* $\boldsymbol{U}$ *with kernel* $k(\cdot,\cdot)$ *maps continuously* $L^\infty(\partial\mathbb{D},\mathbb{C})$ *in* $\mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$.

Proof. Let $t_1, t_2 \in \partial\mathbb{D}$ and let

$$\sigma(t_1,t_2) \equiv \inf\{\, |\theta_1 - \theta_2| : e^{i\theta_1} = t_1, e^{i\theta_2} = t_2 \,\}. \tag{3.2.12}$$

It is easy to check that

$$|\xi - t| \le \sigma(\xi,t) \le \frac{\pi}{2}|\xi - t| \tag{3.2.13}$$

for all $(\xi,t) \in (\partial\mathbb{D})^2$. Let $|\cdot|_0$ the usual norm of $L^\infty(\partial\mathbb{D},\mathbb{C})$. We observe that

$$\begin{aligned} |\boldsymbol{U}[g](t)| &= \left|\int_{\partial\mathbb{D}} k(t,\xi) g(\xi)\, d\xi\right| \le \int_{\partial\mathbb{D}} \frac{M_1 |g|_0}{|\xi - t|^{1-\alpha}}\, |d\xi| \\ &\le \frac{2^{1-\alpha} M_1 |g|_0}{\pi^{1-\alpha}} \int_{\partial\mathbb{D}} \frac{1}{\sigma(\xi,t)^{1-\alpha}}\, |d\xi| \\ &= \frac{2^{1-\alpha} M_1 |g|_0}{\pi^{1-\alpha}} \left(\int_0^\pi \frac{1}{\theta^{1-\alpha}}\, d\theta + \int_\pi^{2\pi} \frac{1}{(2\pi - \theta)^{1-\alpha}}\, d\theta \right) \\ &= 2^{2-\alpha} \pi^{2\alpha - 1} M_1 |g|_0 = C_1 |g|_0 \end{aligned} \tag{3.2.14}$$

for all $g \in L^\infty(\partial\mathbb{D},\mathbb{C})$ and for all $t \in \partial\mathbb{D}$ by setting $C_1 \equiv 2^{2-\alpha}\pi^{2\alpha-1}M_1$ ($|d\xi|$ is the usual arc-measure of $\partial\mathbb{D}$). Then by using the theorem of continuity of an integral depending on a parameter, $\boldsymbol{U}$ is continuous from $L^\infty(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})$. Now we have to estimate the quotient

$$\frac{|\boldsymbol{U}[g](t+h) - \boldsymbol{U}[g](t)|}{|t|^\alpha} \tag{3.2.15}$$

for all $g \in L^\infty(\partial\mathbb{D},\mathbb{C})$ and for all $(t, t+h) \in (\partial\mathbb{D})^2$, $h \neq 0$. First we assume $|h| \ge 1/2$. Then it easy to check that quotient (3.2.15) is less or equal to $2^{\alpha+1}|\boldsymbol{U}[g]|_0 \le 2^{\alpha+1} C_1 |g|_0$. Let $(t, t+h) \in \partial\mathbb{D}^2$ and let $|h| \le 1/2$. We set

$$\mathbb{I}(t,h) = \{\, \xi \in \partial\mathbb{D} : |\xi - t| < 2|h| \,\}, \qquad \mathbb{E}(t,h) = \{\, \xi \in \mathbb{D} : |\xi - t| \ge 2|h| \,\}\,.$$

Then we have

$$\begin{aligned} |\boldsymbol{U}[g](t+h) - \boldsymbol{U}[g](t)| &\le \left|\int_{\mathbb{E}(t,h)} (k(t+h,\xi) - k(t,\xi)) g(\xi)\, d\xi\right| \\ &\quad + \left|\int_{\mathbb{I}(t,h)} (k(t+h,\xi) - k(t,\xi)) g(\xi)\, d\xi\right| \\ &\le \Big(\int_{\mathbb{E}(t,h)} |k(t+h,\xi) - k(t,\xi)|\, |d\xi| \\ &\quad + \int_{\mathbb{I}(t,h)} |k(t+h,\xi)|\, |d\xi| + \int_{\mathbb{I}(t,h)} |k(t,\xi)|\, |d\xi| \Big) |g|_0\,. \end{aligned} \tag{3.2.16}$$

We now choose two points $a_{t,h}$, $b_{t,h}$ on $\partial\mathbb{D}$ in such away that the triple $(b_{t,h}, t, a_{t,h})$ induces a counterclockwise orientation on $\partial\mathbb{D}$ and in such away that $\{ a_{t,h}, b_{t,h} \} = \{ \eta \in \partial\mathbb{D} : |\eta - t| = 2|h| \}$. Let $\widehat{t}(\cdot)$ be the parametrization of $\partial\mathbb{D}$ defined by $\widehat{t}(\theta) = te^{i\theta}$, $\theta \in [0, 2\pi[$. Let $\theta_h \in]0, 2\pi[$ such that $\widehat{t}(\theta_h) = t+h$ and let $[t, t+h]$ be the smaller arc of $\partial\mathbb{D}$ between t and $t+h$. Clearly $|\eta - t| \leq |h|$ for all $\eta \in [t, t+h]$. Then we have that

$$\begin{aligned} |\eta - \xi| &\geq \Big| |\eta - t| - |t - \xi| \Big| \geq |t - \xi| - |\eta - t| \\ &\geq |t - \xi| - |h| \geq \frac{|t - \xi|}{2} \end{aligned} \tag{3.2.17}$$

for all $\xi \in \mathbb{E}(t, h)$ and for all $\eta \in [t, t+h]$. Then by (3.2.11) and (3.2.17) we can write

$$\begin{aligned} |k(t+h, \xi) - k(t, \xi)| &= |k(\widehat{t}(\theta_h), \xi) - k(\widehat{t}(0), \xi)| \\ &\leq \left(\sup_{[0,\theta_h]} \left\{ \left| \frac{d}{d\theta} k(\widehat{t}(\theta), \xi) \right| \right\} \right) \theta_h \\ &\leq \left(\sup_{\eta \in [t,t+h]} \left\{ \left| \frac{\partial k}{\partial t}(\eta, \xi) \right| \right\} \right) \frac{\pi}{2} |h| \\ &\leq \left(\sup_{\eta \in [t,t+h]} \left\{ \frac{M_2}{|\eta - \xi|^{2-\alpha}} \right\} \right) \frac{\pi}{2} |h| \leq \frac{2^{1-\alpha}\pi}{|\xi - t|^{2-\alpha}} |h| \end{aligned}$$

for all $\xi \in \mathbb{E}(t, h)$. Let $\theta_a, \theta_b \in]0, 2\pi[$ such that $\widehat{t}(\theta_a) = a_{t,h}$, $\widehat{t}(\theta_b) = b_{t,h}$. By (3.2.13), we have that

$$\begin{aligned} \int_{\mathbb{E}(t,h)} \frac{1}{|\xi - t|^{2-\alpha}} \, |d\xi| &\leq \left(\frac{\pi}{2} \right)^{2-\alpha} \int_{\mathbb{E}(t,h)} \frac{1}{\sigma(\xi, t)} \, |d\xi| \\ &\leq \left(\frac{\pi}{2} \right)^{2-\alpha} \left[\int_{\theta_a}^{\pi} \frac{1}{\theta^{2-\alpha}} \, d\theta + \int_{\pi}^{\theta_b} \frac{1}{(2\pi - \theta)^{2-\alpha}} \, d\theta \right] \\ &= \frac{\pi^{2-\alpha}}{2^{2-\alpha}(1-\alpha)} \left[\theta_a^{\alpha-1} + (2\pi - \theta_b)^{\alpha-1} - 2\pi^{\alpha-1} \right] \\ &\leq \frac{\pi^{2-\alpha}}{2^{2-\alpha}(1-\alpha)} \left[2^{\alpha-1}|h|^{\alpha-1} + 2^{\alpha-1}|h|^{\alpha-1} \right] \\ &= \frac{\pi^{2-\alpha}}{2^{2-2\alpha}(1-\alpha)} |h|^{\alpha-1} . \end{aligned}$$

As a consequence, the following inequality holds

$$\int_{\mathbb{E}(t,h)} |k(t+h, \xi) - k(t, \xi)| \, |d\xi| \leq C_2 |h|^{\alpha} \tag{3.2.18}$$

where $C_2 \equiv 2^{\alpha-1}\pi^{3-\alpha}(1-\alpha)^{-1}$. Let $\eta \in \mathbb{I}(t, h)$ and let $\widehat{\gamma}$ the parametrization of $\partial\mathbb{D}$ defined by $\widehat{\gamma}(s) = b_{t,h}e^{is}$, $s \in [0, 2\pi[$. Let $s_t, s_\eta, s_a \in]0, 2\pi[$ such that

$\widehat{\gamma}(s_t) = t$, $\widehat{\gamma}(s_\eta) = \eta$, $\widehat{\gamma}(s_a) = a_{t,h}$. By (3.2.10), the following inequalities hold

(3.2.19)

$$\begin{aligned}
\int_{\mathbb{I}(t,h)} |k(\eta,\xi)|\,|d\xi| &\leq \int_{\mathbb{I}(t,h)} \frac{M_1}{|\eta-\xi|^{1-\alpha}}\,|d\xi| \\
&\leq \left(\frac{\pi}{2}\right)^{1-\alpha} \int_{\mathbb{I}(t,h)} \frac{M_1}{\sigma(\xi,\eta)^{1-\alpha}}\,|d\xi| \\
&= \frac{\pi^{1-\alpha}M_1}{2^{1-\alpha}}\left(\int_0^{s_\eta} \frac{1}{(s_\eta - s)^{1-\alpha}}\,ds + \int_{s_\eta}^{s_a} \frac{1}{(s-s_\eta)^{1-\alpha}}\,ds\right) \\
&= \frac{\pi^{1-\alpha}M_1}{2^{1-\alpha}}\left(s_\eta^\alpha + (s_a - s_\eta)^\alpha\right) \\
&\leq 2^\alpha \pi^{1-\alpha} M_1 s_a^\alpha \leq \pi M_1 |a_{t,h} - b_{t,h}|^\alpha \leq 2^{2\alpha} M_1 |h|^\alpha\,.
\end{aligned}$$

By (3.2.14), (3.2.18) and (3.2.19) it follows that

$$|\boldsymbol{U}[g]|_{\mathcal{C}_*^{0,\alpha}(\partial\mathbb{D},\mathbb{C})} \leq ((1+2^\alpha)C_1 + C_2 + 2^{2\alpha+1}M_1)|g|_0$$

for all $g \in L^\infty(\partial\mathbb{D},\mathbb{C})$. This yields the conclusion. □

In the hypothesis of Proposition 3.2.3, we show that $(BVP\phi)$ admits at most one solution when the polynomial $P(\cdot)$ is fixed.

Proposition 3.2.20. *Let $0 < \alpha < 1$, $\gamma \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$, $L = \gamma(\partial\mathbb{D})$. Let $\phi \in \mathcal{C}_*^{1,\alpha}(L,\mathbb{C}) \cap \mathcal{A}^*_L$. The following statements hold.*

(i) *Let $k_\phi(\cdot,\cdot)$ be the complex-valued function of $L^2 \setminus \{\,(t,\xi) \in L^2 : t \neq \xi\,\}$ defined in* (3.2.4). *Then the function $k(\cdot,\cdot)$ from $\partial\mathbb{D}^2 \setminus \{\,(s,\eta) \in \partial\mathbb{D}^2 : s \neq \eta\,\}$ to $\mathbb{C}$ defined by*

$$k(s,\eta) \equiv k_\phi(\gamma(s),\gamma(\eta))\gamma'(\eta)$$

for all $(s,\eta) \in \partial\mathbb{D}^2, s \neq \eta$, satisfies conditions (3.2.10) *and* (3.2.11) *of Lemma* 3.2.9.

(ii) *If (F,G) is a solution of $(BVP\phi)$ on L satisfying $G(\infty) \equiv \lim_{z\to\infty} G(z) = 0$, then $F = 0$ and $G = 0$.*

Proof. By arguing as in Lu (1993, p. 418), we have that there exists $c_\phi > 0$ such that

$$\left|\frac{\phi(\xi)-\phi(t)}{\xi - t} - \phi'(\xi)\right| \leq c_\phi |\xi - t|^\alpha \tag{3.2.21}$$

for all $(t,\xi) \in L^2$, $t \neq \xi$. Then we obtain that

$$\begin{aligned}
|k_\phi(t,\xi)| &= \frac{1}{|\xi - t|} \cdot \frac{|\xi - t|}{|\phi(\xi)-\phi(t)|} \cdot \left(\left|\frac{\phi(\xi)-\phi(t)}{\xi - t} - \phi'(\xi)\right|\right) \\
&\leq \frac{c_\phi}{\boldsymbol{l}_L[\phi]} \cdot \frac{1}{|\xi - t|^{1-\alpha}}
\end{aligned}$$

(see Lemma 1.2.6 for the definition of $\boldsymbol{l}_L[\cdot]$). Then bu substituting $t = \gamma(s)$, $\xi = \gamma(\eta)$ and by using $\boldsymbol{l}_{\partial\mathbb{D}}[\gamma] > 0$, it follows that $k(\cdot,\cdot)$ satisfies condition (3.2.10). A computation based on (3.2.21), $\boldsymbol{l}_L[\phi] > 0$ and $\boldsymbol{l}_{\partial\mathbb{D}}[\gamma] > 0$, shows that $k(\cdot,\cdot)$ satisfies condition (3.2.11). Now we prove statement (ii). Let (F,G) be a solution of $(BVP\phi)$ on L such that $G(\infty) = 0$. We can assume that $G_{/L} \in \mathcal{C}^{0,\beta}_*(L,\mathbb{C})$ with $0 < \beta \leq \alpha$. Let $\boldsymbol{U}_\phi$ be the integral operator with kernel $k_\phi(\cdot,\cdot)$. We observe that the right composition by γ is a complex linear homeomorphism from $L^\infty(L,\mathbb{C})$ to $L^\infty(\partial\mathbb{D},\mathbb{C})$ and its restriction to $\mathcal{C}^{0,\alpha}_*(L,\mathbb{C})$ is a complex linear homeomorphism onto $\mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Then by statement (i) and by Lemma 3.2.9, $\boldsymbol{U}_\phi$ maps continuously $L^\infty(L,\mathbb{C})$ to $\mathcal{C}^{0,\alpha}_*(L,\mathbb{C})$. Since $\beta \leq \alpha$ and $\mathcal{C}^{0,\beta}(L,\mathbb{C})$ is compactly imbedded in $L^\infty(L,\mathbb{C})$, $\boldsymbol{U}_\phi$ is a compact operator from $\mathcal{C}^{0,\beta}_*(L,\mathbb{C})$ to itself. Let $n \in \mathbb{N} \setminus \{0\}$. By Proposition 3.2.3, $(G_{/L})^n$ belongs to the kernel of the operator $\boldsymbol{I} + \boldsymbol{U}_\phi$ (where $\boldsymbol{I}$ is the identity operator of $\mathcal{C}^{0,\beta}_*(L,\mathbb{C})$). We now assume by contradiction that $(F,G) \neq (0,0)$. By the holomorphic extendability of $G_{/L}$ to $\mathbb{C} \setminus \mathrm{cl}\,\mathbb{I}[\gamma]$ it is easy to check that $\{ (G_{/L})^n : n \in \mathbb{N} \setminus \{0\} \}$ is a linearly independent subset of $\mathcal{C}^{0,\beta}_*(L,\mathbb{C})$. Since the kernel of $\boldsymbol{I} + \boldsymbol{U}_\phi$ is a subspace of finite dimension of $\mathcal{C}^{0,\beta}_*(L,\mathbb{C})$, we have a contradiction. Then $(F,G) = (0,0)$. □

Definition 3.2.22. Let $0 < \alpha < 1$, $\phi \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. The pair of injective functions (F,G) defined in $\mathbb{I}[\phi]$ and $\mathbb{C}\setminus\mathbb{D}$, respectively, is a normalized solution of the generalized conformal sewing problem associated to ϕ (or $(GSP\phi)$) if (F,G) satisfies $(BVP\phi)$ on $\partial\mathbb{D}$ and the following normalization condition holds

$$\lim_{z\to\infty} G(z) - z = 0\,.$$

Remark 3.2.23. Let (F,G) be a solution of $(GSP\phi)$. It is easy to check that $F(\mathbb{I}[\phi])$ and $G(\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D})$ are the two open connected components of $\mathbb{C} \setminus G(\partial\mathbb{D})$. Clearly if $\phi(\partial\mathbb{D}) = \partial\mathbb{D}$, (F,G) is a normalized solution for the conformal sewing problem (cf. (3.1.3)).

By Proposition 3.2.20 (ii), the problem $(GSP\phi)$ has at most one solution. In the sequel we show the existence of a solution for problem $(GSP\phi)$ which has the same regularity of ϕ.

Lemma 3.2.24 (cf. Lu (1993, p. 419–420)). *Let $0 < \alpha < 1$, $\phi \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. Let $\boldsymbol{I}$ be the identity operator of $\mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Let $k_\phi(\cdot,\cdot)$ be the map defined in (3.2.4) and let $\boldsymbol{U}_\phi$ be the operator from $\mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to itself defined by*

$$\boldsymbol{U}_\phi[g](t) \equiv \frac{1}{2\pi i}\int_{\partial\mathbb{D}} k_\phi(t,\xi) g(\xi)\, d\xi = \boldsymbol{C}[1_{\partial\mathbb{D}}, g](t) - \boldsymbol{C}[\phi, g](t) \tag{3.2.25}$$

for all $g \in \mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ and for all $t \in \partial\mathbb{D}$. Then $\boldsymbol{I} + \boldsymbol{U}_\phi$ is a complex linear homeomorphism of $\mathcal{C}^{0,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to itself.

Proof. By Proposition 3.2.20 and by Lemma 3.2.9, $\boldsymbol{U}_\phi$ is a compact operator. To conclude it suffices to show that $\boldsymbol{I}+\boldsymbol{U}_\phi$ is injective. Let g_0 be such that $(\boldsymbol{I}+\boldsymbol{U}_\phi)[g_0]=0$. We consider the pair of functions (Φ_0^+,Φ_0^-) defined by

$$\Phi_0^+(z)\equiv\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{g_0(\xi)}{\xi-z}\,d\xi \tag{3.2.26}$$

$$\Phi_0^-(w)\equiv\frac{1}{2\pi i}\int_{\phi(\partial\mathbb{D})}\frac{g_0\circ\phi^{(-1)}(\eta)}{\eta-w}\,d\eta \tag{3.2.27}$$

for all $z\in\mathbb{D}$ and for all $w\in\mathbb{E}[\phi]$. By well-known properties of a Cauchy integral Φ_0^+ and Φ_0^- have continuous extension to $\operatorname{cl}\mathbb{D}$ and $\operatorname{cl}\mathbb{E}[\phi]=\mathbb{E}[\phi]\cup\phi(\partial\mathbb{D})$ which are of class $\mathcal{C}_*^{0,\alpha}$ in $\partial\mathbb{D}$ and $\phi(\partial\mathbb{D})$, respectively. By the Plemelj formula

$$\Phi_0^+(t)=\frac{1}{2}g_0(t)+\frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\partial\mathbb{D}}\frac{g_0(\xi)}{\xi-t}\,d\xi \tag{3.2.28}$$

$$\Phi_0^-(t_1)=-\frac{1}{2}g_0\circ\phi^{(-1)}(t_1)+\frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\phi(\partial\mathbb{D})}\frac{g_0\circ\phi^{(-1)}(\eta)}{\eta-t_1}\,d\eta \tag{3.2.29}$$

for all $t\in\partial\mathbb{D}$ and for all $t_1\in\phi(\partial\mathbb{D})$. We set $\eta=\phi(\xi)$ and $t_1=\phi(t)$ in (3.2.29) and we observe that

$$\Phi_0^+(t)-\Phi_0^-(\phi(t))=g_0(\xi)+\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\left[\frac{1}{\xi-t}-\frac{\phi'(\xi)}{\phi(\xi)-\phi(t)}\right]g_0(\xi)\,d\xi=0$$

for all $t\in\partial\mathbb{D}$. Then $\Phi^+_{0/\partial\mathbb{D}}\circ\phi^{(-1)}=\Phi^-_{0/\phi(\partial\mathbb{D})}$ and $\lim_{z\to\infty}\Phi_0^-(z)=0$. By Proposition 3.2.20 (ii), $\Phi_0^+=0$ and $\Phi_0^-=0$. By (3.2.28) and (3.2.29), we obtain that

$$g_0(t)=-\frac{1}{\pi i}\,\mathrm{p.\,v.}\int_{\partial\mathbb{D}}\frac{g_0(\xi)}{\xi-t}\,d\xi$$
$$g_0\circ\phi^{(-1)}(t_1)=\frac{1}{\pi i}\,\mathrm{p.\,v.}\int_{\phi(\partial\mathbb{D})}\frac{g_0\circ\phi^{(-1)}(\eta)}{\eta-t_1}\,d\eta$$

for all $t\in\partial\mathbb{D}$ and for all $t_1\in\phi(\partial\mathbb{D})$. By the conditions of holomorphic extension, there exist two maps Ψ_0^+ and Ψ_0^- defined in $\operatorname{cl}\mathbb{I}[\phi]$ and $\mathbb{C}\setminus\mathbb{D}$ and holomorphic in $\mathbb{I}[\phi]$ and $\mathbb{C}\setminus\operatorname{cl}\mathbb{D}$, respectively, such that

$$\Psi_0^+(t_1)=g_0\circ\phi^{(-1)}(t_1)$$
$$\Psi_0^-(t)=g_0(t)$$

for all $t_1\in\phi(\partial\mathbb{D})$, $t\in\partial\mathbb{D}$ and $\lim_{z\to\infty}\Psi_0^-(z)=0$. By the equality $\Psi^+_{0/\phi(\partial\mathbb{D})}\circ\phi=\Psi^-_{0/\partial\mathbb{D}}$ and by Proposition 3.2.20 (ii), it follows that $g_0=0$. □

In order to obtain a regularity result about the solution of the integral equation (3.2.5), we need the following Lemma.

LEMMA 3.2.30. *Let* $m \in \mathbb{N} \setminus \{0\}$, $0 < \alpha < 1$, $\beta < \alpha$. *Let* $\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. *We assume that the integral operator* $\boldsymbol{C}_*$ *defined by*

$$\boldsymbol{C}_*[g] \equiv \boldsymbol{C}[1_{\partial\mathbb{D}}, g] - \boldsymbol{C}\Big[\phi, \frac{g}{(\phi')^m}\Big](\phi')^m$$

is continuous from $\mathcal{C}_*^{0,\beta}(\partial\mathbb{D}, \mathbb{C})$ *to* $\mathcal{C}_*^{0,\alpha}(\partial\mathbb{D}, \mathbb{C})$. *Then the integral operator* $\boldsymbol{U}_\phi$ *defined in* (3.2.25) *maps continuously* $\mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$ *to* $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$.

Proof. It suffices to show that the operator which maps g to $\boldsymbol{U}_\phi[g]^{(h)}$ is continuous from $\mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$ to $\mathcal{C}^{0,\alpha}(\partial\mathbb{D}, \mathbb{C})$ for all $h = 0, \dots, m$. We will prove by induction that for all $h = 0, \dots, m$ there exists a continuous operator $\boldsymbol{H}_h$ from $\mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$ to $\mathcal{C}_*^{m-h,\alpha}(\partial\mathbb{D}, \mathbb{C})$ such that

$$\boldsymbol{U}_\phi[g]^{(h)} = \boldsymbol{C}[1_{\partial\mathbb{D}}, g^{(h)}] - \boldsymbol{C}\Big[\phi, \frac{g^{(h)}}{(\phi')^h}\Big](\phi')^h + \boldsymbol{H}_h[g] \tag{3.2.31}$$

for all $g \in \mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$. Then conclusion will follow by using well-known properties of continuity of the operator $\boldsymbol{C}[\gamma, \cdot]$ for all $\gamma \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$ and Lemma 1.2.8 whenever $0 \le h < m$ and by using the hypothesis whenever $h = m$. Case $h = 0$ holds with $\boldsymbol{H}_0 = 0$. We assume that property (3.2.31) holds for $0 \le h < m$ and we show that holds for $h+1$. We observe that, by formula of the derivative of a Cauchy singular integral, the following equality holds,

$$\boldsymbol{C}[\gamma, f]'(t) = \boldsymbol{C}\Big[\gamma, \frac{f'}{\gamma'}\Big](t)\gamma'(t) \tag{3.2.32}$$

for all $(\gamma, f) \in (\mathcal{C}^{1,\beta}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}) \times \mathcal{C}_*^{1,\beta}(\partial\mathbb{D}, \mathbb{C})$ and for all $t \in \partial\mathbb{D}$. Then if $h = 0$, case $h+1$ holds by setting $\boldsymbol{H}_1 \equiv 0$. If $h \ge 1$, by using (3.2.32), induction and Lemma 1.2.8, it easy to check that the operator $\boldsymbol{H}_{h+1}$ defined by

$$\begin{aligned}\boldsymbol{H}_{h+1}[g](\cdot) \equiv h\boldsymbol{C}\Big[\phi, \frac{g^{(h)}\phi^{(2)}}{(\phi')^{h+2}}\Big](\cdot)(\phi'(\cdot))^{h+1}\\ - h\boldsymbol{C}\Big[\phi, \frac{g^{(h)}}{(\phi')^h}\Big](\cdot)(\phi'(\cdot))^{h-1}\phi^{(2)}(\cdot) + \boldsymbol{H}_h[g]'(\cdot)\end{aligned}$$

for all $g \in \mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$ is continuous from $\mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$ to $\mathcal{C}_*^{m-(h+1),\alpha}(\partial\mathbb{D}, \mathbb{C})$. □

Then we have the following Proposition which shows the existence of a unique solution for the problem $(GSP\phi)$ with the same regularity of ϕ (cf. Lu (1993, p. 424 Thm. 2.2.1)).

PROPOSITION 3.2.33. *Let* $\alpha \in \,]0,1[$. *Let* $\widehat{\boldsymbol{G}}$ *be the operator from* $\mathcal{C}_*^{1,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ *to* $\mathcal{C}_*^{0,\alpha}(\partial\mathbb{D}, \mathbb{C})$ *which maps* γ *to the unique Hölderian solution of the integral equation*

$$g + \boldsymbol{C}[1_{\partial\mathbb{D}}, g] - \boldsymbol{C}[\gamma, g] = 1_{\partial\mathbb{D}}$$

for all $\gamma \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. *Then the following statements hold.*

(i) *Let* $m \in \mathbb{N} \setminus \{0\}$*. Let* $\boldsymbol{I}$ *and* $\boldsymbol{U}_\phi$ *be the operators of Lemma* 3.2.24*. We assume that* $\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$*. Then* $\boldsymbol{I} + \boldsymbol{U}_\phi$ *is a complex linear homeomorphism of* $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ *to itself. In particular,* $\widehat{\boldsymbol{G}}[\phi] \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$*.*

(ii) *Let* $\phi \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$*. Then* $\widehat{\boldsymbol{G}}[\phi] \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ *and the pair of functions* (F, G) *defined by*

$$F(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\phi(\partial\mathbb{D})} \dfrac{(\widehat{\boldsymbol{G}}[\phi] \circ \phi^{(-1)})(\eta)}{\eta - z}\, d\eta & \text{if } z \in \mathbb{I}[\phi] \\ (\widehat{\boldsymbol{G}}[\phi] \circ \phi^{(-1)})(z) & \text{if } z \in \phi(\partial\mathbb{D}) \end{cases} \tag{3.2.34}$$

$$G(w) = \begin{cases} -\dfrac{1}{2\pi i} \displaystyle\int_{\partial\mathbb{D}} \dfrac{\widehat{\boldsymbol{G}}[\phi](\xi)}{\xi - w}\, d\xi + w & \text{if } w \in \mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}, \\ \widehat{\boldsymbol{G}}[\phi](w) & \text{if } w \in \partial\mathbb{D}, \end{cases} \tag{3.2.35}$$

for all $z \in \mathrm{cl}\,\mathbb{I}[\phi]$*,* $w \in \mathbb{C} \setminus \mathbb{D}$*, is the unique normalized solution of problem* $(GSP\phi)$*. Furthermore, for all* $m \in \mathbb{N} \setminus \{0\}$ $(\widehat{\boldsymbol{G}}[\phi] \circ \phi^{(-1)}, \boldsymbol{G}[\phi]) \in (\mathcal{C}_*^{m,\alpha}(\phi(\partial\mathbb{D}), \mathbb{C}) \cap \mathcal{A}^*_{\phi(\partial\mathbb{D})}) \times (\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}})$ *whenever* $\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$*.*

Proof. We prove statement (i). By Lemma 3.2.24, it suffices to show that $\boldsymbol{U}_\phi$ is a compact operator of $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ to itself. Let $0 < \beta < \alpha$. Clearly, the kernel $k(\cdot, \cdot)$ of the integral operator $\boldsymbol{C}_*$ of Lemma 3.2.30 is given by the formula

$$\begin{aligned} k(t, \xi) &= \frac{1}{\xi - t} - \Big(\frac{\phi'(\xi)}{\phi(\xi) - \phi(t)}\Big)\Big(\frac{\phi'(t)}{\phi'(\xi)}\Big)^m \\ &= k_\phi(t, \xi) - \frac{\phi'(\xi)}{\phi(\xi) - \phi(t)}\Big(\frac{\phi'(t)^m - \phi'(\xi)^m}{\phi'(\xi)^m}\Big) \end{aligned}$$

for all $(t, \xi) \in \partial\mathbb{D}^2$, $t \neq \xi$. A computation based on the α-Hölder continuity of ϕ' and on $\boldsymbol{l}_{\partial\mathbb{D}}[\phi] > 0$, shows that the function defined by

$$(t, \xi) \longmapsto \frac{\phi'(\xi)}{\phi(\xi) - \phi(t)}\Big(\frac{\phi'(t)^m - \phi'(\xi)^m}{\phi'(\xi)^m}\Big)$$

for all $(t, \xi) \in (\partial\mathbb{D})^2$, $t \neq \xi$, satisfies conditions (3.2.10), (3.2.11) of Lemma 3.2.9. Then by Proposition 3.2.20 (i), $k(\cdot, \cdot)$ satisfies conditions (3.2.10), (3.2.11) of Lemma 3.2.9. By Lemma 3.2.9 and since $\mathcal{C}_*^{0,\beta}(\partial\mathbb{D}, \mathbb{C})$ is continuously embedded in $L^\infty(\partial\mathbb{D}, \mathbb{C})$, it follows that $\boldsymbol{C}_*$ is continuous from $\mathcal{C}_*^{0,\beta}(\partial\mathbb{D}, \mathbb{C})$ to $\mathcal{C}_*^{0,\alpha}(\partial\mathbb{D}, \mathbb{C})$. Since $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ is compactly embedded in $\mathcal{C}_*^{m,\beta}(\partial\mathbb{D}, \mathbb{C})$, Lemma 3.2.30 yields statement (i). Statement (ii) follows by adapting standard argument of theory of boundary value problems with shift to our setting of generalized shifts and by using statement (i). Indeed by *e.g.* Lu (1993, p. 419–420), the maps F and G, defined in 3.2.34 and 3.2.35 respectively, satisfy problem $(BVP\phi)$ on $\partial\mathbb{D}$ (cf. Definition 3.2.1). By statement (i) it follows that $\widehat{\boldsymbol{G}}[\phi] \in \mathcal{C}_*^{1,\alpha}(\partial\mathbb{D}, \mathbb{C})$. By *e.g.*

Lu (1993, p. 425–426 together with p. 421), we obtain that F and G are injective and $\widehat{\boldsymbol{G}}[\phi] \in \mathcal{A}^*_{\partial\mathbb{D}}$. The regularity property of the second part of statement (ii) follows by statement (i). $\square$

Then we have the following.

COROLLARY 3.2.36. *Let $\alpha \in]0,1[$. Let $\widehat{\boldsymbol{G}}$ be the operator defined in Proposition* 3.2.33. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be the operators from $\mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\partial\mathbb{D})\cap\mathcal{A}^*_{\partial\mathbb{D}}$ to $\mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\mathbb{C})\cap \mathcal{A}^*_{\partial\mathbb{D}}$ defined by*

$$(\boldsymbol{F}[\phi],\boldsymbol{G}[\phi]) \equiv (\widehat{\boldsymbol{G}}[\phi]\circ\phi^{(-1)},\widehat{\boldsymbol{G}}[\phi]) \tag{3.2.37}$$

for all $\phi \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. Then $(\boldsymbol{F}[\phi],\boldsymbol{G}[\phi])$ is the trace of the normalized solution of the conformal sewing problem associated to the shift $\phi \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. Furthermore, if $m \in \mathbb{N}\setminus\{0\}$ and $\phi \in \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ then $(\boldsymbol{F}[\phi],\boldsymbol{G}[\phi]) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}})^2$.

REMARK 3.2.38. Let $\alpha \in]0,1[$, $\phi \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\mathbb{C})\cap\mathcal{A}^*_{\partial\mathbb{D}}$. Let $(F,G) \in \mathcal{C}^0(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C}) \times\mathcal{C}^0(\mathbb{C}\setminus\mathbb{D},\mathbb{C})$ be a pair functions, holomorphic in $\mathbb{I}[\phi]$ and $\mathbb{C}\setminus\mathrm{cl}\,\mathbb{D}$, respectively. We assume that (F,G) satisfies the boundary condition

$$F_{/\phi(\partial\mathbb{D})}\circ\phi = G_{/\partial\mathbb{D}}\ .$$

By adapting a standard argument to transfer a problem with shift to a ordinary problem for "sectionally holomorphic" functions and by using Proposition 3.2.33, it follows that $F_{/\phi(\partial\mathbb{D})} \in \mathcal{C}^{1,\alpha}_*(\phi(\partial\mathbb{D}),\mathbb{C})$ and that $G_{/\partial\mathbb{D}} \in \mathcal{C}^{1,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. Then by removing the hypothesis of Hölder continuity for the traces of a solution in the problem $(BVP\phi)$ (cf. Definition 3.2.1), we obtain an equivalent problem.

Now we show the analytic dependence of $\boldsymbol{G}[\phi]$ on ϕ in Schauder spaces.

THEOREM 3.2.39. *Let $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let $\widehat{\boldsymbol{G}}$, $\boldsymbol{G}$ be the operators defined in Proposition* 3.2.33 *and Corollary* 3.2.36*, respectively. Then $\widehat{\boldsymbol{G}}$ extends $\boldsymbol{G}$ and is complex analytic from $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ to itself.*

Proof. By (3.2.37) $\widehat{\boldsymbol{G}}$ coincides with $\boldsymbol{G}$ in $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\partial\mathbb{D})\cap\mathcal{A}^*_{\partial\mathbb{D}}$. We now show the analyticity of $\widehat{\boldsymbol{G}}$. Let $\boldsymbol{\Gamma}$ be the operator of $(\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ defined by

$$\boldsymbol{\Gamma}[\phi,g] \equiv g + \boldsymbol{C}[1_{\partial\mathbb{D}},g] - \boldsymbol{C}[\phi,g]$$

for all $(\phi,g) \in (\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}) \times \mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$. By Proposition 2.3.1, $\boldsymbol{\Gamma}[\cdot,\cdot]$ is complex analytic in its domain. By Lemma 3.2.24, the graph of the operator $\widehat{\boldsymbol{G}}[\cdot]$ from $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})\cap\mathcal{A}^*_{\partial\mathbb{D}}$ to itself is the solution set of the functional equation

$$\boldsymbol{\Gamma}[\phi,g] = 1_{\partial\mathbb{D}}\ .$$

By linearity of $\boldsymbol{\Gamma}[\cdot,\cdot]$ on g and by Proposition 3.2.33 (i), it follows that $\frac{\partial\boldsymbol{\Gamma}}{\partial g}[\phi,\widehat{\boldsymbol{G}}[\phi]] = \boldsymbol{\Gamma}[\phi,\cdot]$ is a complex linear homeomorphism of $\mathcal{C}^{m,\alpha}_*(\partial\mathbb{D},\mathbb{C})$ to itself for all

$\phi \in \mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. Then, Implicit Function Theorem in the form of Prodi & Ambrosetti (1973, Thm. 11.6) or Berger (1977, p. 134), yields the conclusion. □

3.3. Regularity of the operator $\boldsymbol{F}$ associated to the sewing problem

Let $(\boldsymbol{F}[\phi] = \boldsymbol{G}[\phi] \circ \phi^{(-1)}, \boldsymbol{G}[\phi])$ be the trace of the normalized solution (F, G) of the conformal sewing problem associated to the shift ϕ (cf. Corollary 3.2.36). In this section we study the regularity of the dependence of $\boldsymbol{F}[\phi]$ on ϕ in Schauder spaces. In particular, we show that $\boldsymbol{F}$ does not admit an extension of class $\mathcal{C}^\infty$ in Schauder spaces.

Let $m, h, k \in \mathbb{N} \setminus \{0\}$, $h, k \geq 1$, $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^h$, regular in the sense of Whitney. As in Lanza (1994, Def. 2.18), $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ denotes the closure in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ of the set of polynomial functions. By Stone-Weierstrass Theorem and well-known properties of the open subsets regular in the sense of Whitney, $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ coincides with the closure in $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ of $\mathcal{C}^{m+1}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ and, in particular, of $\mathcal{C}^\infty(\mathrm{cl}\,\Omega, \mathbb{R}^k) \equiv \bigcap_{r=0}^{+\infty} \mathcal{C}^r(\mathrm{cl}\,\Omega, \mathbb{R}^k)$. In this Lemma we introduce the subset of function of $\partial\mathbb{D}$ to $\mathbb{C}$ of class $\mathcal{C}^{m,\alpha,p}$.

LEMMA 3.3.1. *Let $\alpha \in]0,1[$, $m \in \mathbb{N}$. We set*

$$\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D},\mathbb{C}) = \mathrm{cl}_{\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})}(\mathcal{C}_*^{m+1}(\partial\mathbb{D},\mathbb{C}))\ .$$

Then the following statements hold.

(i) *$\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D},\mathbb{C})$ coincides with the closure in $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$ of $\mathcal{C}_*^\infty(\partial\mathbb{D},\mathbb{C}) \equiv \bigcap_{r=0}^{+\infty} \mathcal{C}_*^r(\partial\mathbb{D},\mathbb{C})$.*

(ii) *Let $\phi_0 \in \mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$. Then there exist an open neighborhood $\mathcal{U}_{\phi_0}$ of ϕ_0 in $\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}$ and a complex analytic extension operator $\boldsymbol{E}_{\phi_0}$ from $\mathcal{U}_{\phi_0}$ to $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\mathbb{D},\mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ such that*

$$\boldsymbol{E}_{\phi_0}[\phi]_{/\partial\mathbb{D}} = \phi$$

for all $\phi \in \mathcal{U}_{\phi_0}$.

Proof. Clearly, the operator which maps f to $f(e^{i\theta})$, $\theta \in [0, 2\pi]$ is a linear homeomorphism from $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D},\mathbb{C})$ to the subspace

$$\{g \in \mathcal{C}^{m,\alpha}([0,2\pi],\mathbb{C}) : g^{(i)}(0) = g^{(i)}(2\pi), i = 0, \ldots, m\}\ .$$

Then statement (i) follows by the equality

$$\begin{aligned}\mathcal{C}^{m,\alpha,p}([0,2\pi],\mathbb{C}) &= \mathrm{cl}_{\mathcal{C}^{m,\alpha}([0,2\pi],\mathbb{C})}(\mathcal{C}^{m+1}([0,2\pi],\mathbb{C})) \\ &= \mathrm{cl}_{\mathcal{C}^{m,\alpha}([0,2\pi],\mathbb{C})}(\mathcal{C}^{\infty}([0,2\pi],\mathbb{C}))\ .\end{aligned}$$

To prove statement (ii), it suffices to consider the operator defined in the proof of Lemma 1.2.10 and to apply statement (i). □

Then we have the following.

PROPOSITION 3.3.2. *Let* $\alpha \in]0,1[$*,* $m, r \in \mathbb{N}$*,* $m \geq 1$*. Let* $\boldsymbol{F}$ *be the operator defined in Corollary* 3.2.36 *and let* $\phi_0 \in \mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$*. Then there exist an open neighborhood* $\mathcal{U}_{\phi_0}$ *of* ϕ_0 *in* $\mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ *and an operator* $\widehat{\boldsymbol{F}}_{\phi_0}$ *of class* $\mathcal{C}^r$ *from* $\mathcal{U}_{\phi_0}$ *to* $\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D}, \mathbb{C})$ *such that*

$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] = \boldsymbol{F}[\phi]$$

for all $\phi \in \mathcal{U}_{\phi_0}$*.*

Proof. Let $\widehat{\boldsymbol{G}}$ be the operator from $\mathcal{C}_*^{m+r,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ to itself defined in Proposition 3.2.33. By Proposition 3.2.33 (i), $\widehat{\boldsymbol{G}}[\gamma] \in \mathcal{C}_*^{\infty}(\partial\mathbb{D}, \mathbb{C})$ whenever $\gamma \in \mathcal{C}_*^{\infty}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$. Then by the continuity of $\widehat{\boldsymbol{G}}$ and by Lemma 3.3.1 (i), $\widehat{\boldsymbol{G}}$ maps $\mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ to itself. Let $g_0 \equiv \widehat{\boldsymbol{G}}[\phi_0]$. Let $\mathcal{U}_{g_0}$ and $\boldsymbol{E}_{g_0}[\cdot]$ be the open neighborhood of g_0 in $\mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ and the extension operator from $\mathcal{U}_{g_0}$ to $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$, respectively, of Lemma 3.3.1 (ii). Analogously, let $\mathcal{U}_{\phi_0}$ and $\boldsymbol{E}_{\phi_0}[\cdot]$ be the open neighborhood of ϕ_0 in $\mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\partial\mathbb{D}}^*$ and the extension operator from $\mathcal{U}_{g_0}$ to $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$, respectively, of Lemma 3.3.1 (ii). Since $\mathbb{D}$ is convex, by Lanza (1994, Lemma 2.10 and Lemma 2.20 (iv)), there exist $R > 1$ and an extension operator $\boldsymbol{E}[\cdot]$ from $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ to $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{B}(0,R), \mathbb{C})$ such that

$$\boldsymbol{E}[F]_{/\,\mathrm{cl}\,\mathbb{D}} = F \tag{3.3.3}$$

for all $F \in \mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$. By Lanza (1994, Thm. 5.9 together with Lemma 2.20 (iv)), there exist a neighborhood $\mathcal{W}_{\Phi_0}$ of Φ_0 in $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C}) \cap \mathcal{A}_{\mathbb{D}}$ and an operator $\widehat{\boldsymbol{J}}_{\Phi_0}[\cdot]$ of class $\mathcal{C}^r$ from $\mathcal{W}_{\Phi_0}$ to $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{B}(0,R))$ such that

$$\widehat{\boldsymbol{J}}_{\Phi_0}[\Phi] = \Phi^{(-1)} \tag{3.3.4}$$

for all $\Phi \in \mathcal{W}_{\Phi_0} \cap \mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathrm{cl}\,\mathbb{D})$. Let $\boldsymbol{R}$ be the trace map from $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$. By Lemma 1.2.9 (ii) and Lemma 3.3.1 (ii), $\boldsymbol{R}$ is linear and continuous from $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$ to $\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D}, \mathbb{C})$. Let $\boldsymbol{T}$ be the composition operator from $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{B}(0,R), \mathbb{C}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{B}(0,R))$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$. By Lanza (1994, Thm. 4.2), $\boldsymbol{T}$ is of class $\mathcal{C}^r$ and maps $\mathcal{C}^{m+r,\alpha,p}(\mathrm{cl}\,\mathbb{B}(0,R), \mathbb{C}) \times \mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{B}(0,R))$ into $\mathcal{C}^{m,\alpha,p}(\mathrm{cl}\,\mathbb{D}, \mathbb{C})$. By possibly shrinking $\mathcal{U}_{\phi_0}$, we define the operator $\widehat{\boldsymbol{F}}_{\phi_0}$ by setting

$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] \equiv \boldsymbol{R}\Big[\boldsymbol{T}\Big[\boldsymbol{E}\Big[\boldsymbol{E}_{g_0}\Big[\boldsymbol{G}[\phi]\Big]\Big], \widehat{\boldsymbol{J}}_{\Phi_0}\Big[\boldsymbol{E}_{\phi_0}[\phi]\Big]\Big]\Big]$$

for all $\phi \in \mathcal{U}_{\phi_0}$. Then $\widehat{\boldsymbol{F}}_{\phi_0}$ is of class $\mathcal{C}^r$ from $\mathcal{U}_{\phi_0}$ to $\mathcal{C}_*^{m,\alpha,p}(\partial\mathbb{D}, \mathbb{C})$. By (3.3.3) and (3.3.4), we have that

$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] = \boldsymbol{F}[\phi]$$

for all $\phi \in \mathcal{U}_{\phi_0} \cap \mathcal{C}_*^{m+r,\alpha,p}(\partial\mathbb{D}, \partial\mathbb{D})$. □

In this Proposition we show that the regularity result of Proposition 3.3.2 is sharp and, in particular we clarify the introduction of spaces $\mathcal{C}^{m,\alpha,p}$. It will follow that $\boldsymbol{F}$ does not admit an extension of class $\mathcal{C}^{\infty}$ in Schauder spaces.

PROPOSITION 3.3.5. *Let $\alpha \in \,]0,1[$, $m, r \in \mathbb{N}$, $m \geq 1$. Let $\phi_0 \in \mathcal{C}_*^{m+r,\alpha}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. We assume that there exists an operator $\widehat{\boldsymbol{F}}_{\phi_0}$ of class $\mathcal{C}^{r+1}$ from an open neighborhood $\mathcal{U}_{\phi_0}$ of ϕ_0 in $\mathcal{C}_*^{m+r,\alpha}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$ such that*

$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] = \boldsymbol{F}[\phi] \tag{3.3.6}$$

for all $\phi \in \mathcal{U}_{\phi_0} \cap \mathcal{C}_*^{m+r,\alpha}(\partial\mathbb{D}, \partial\mathbb{D})$. Then $\phi_0 \in \mathcal{C}_*^{m+r+1,\alpha,p}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$.

Proof. Let $\varepsilon > 0$. We consider the $\mathcal{C}^{\infty}$, one-parametric family of functions belonging to $\mathcal{C}_*^{m+r,\alpha}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$

$$\phi_\theta(t) \equiv e^{-i\theta}\phi_0(t)$$

for all $\theta \in \,]-\varepsilon, \varepsilon[$, $t \in \partial\mathbb{D}$. Let $f_0 \equiv \boldsymbol{F}[\phi_0]$. By the unique solvability of problem (GSP-ϕ_θ), we have that

$$\boldsymbol{G}[\phi_\theta] = \boldsymbol{G}[\phi_0]$$
$$\boldsymbol{F}[\phi_\theta](t) = f_0(e^{i\theta}t)$$

for all $\theta \in \,]-\varepsilon, \varepsilon[$, $t \in \partial\mathbb{D}$. By the regularity assumption on $\widehat{\boldsymbol{F}}_{\phi_0}$, the family $\Big\{\boldsymbol{F}[\phi_\theta]\Big\}_{\theta \in]-\varepsilon,\varepsilon[}$ is of class $\mathcal{C}^{r+1}$ from $]-\varepsilon, \varepsilon[$ to $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$. It is easy to check that

$$\frac{d^h}{(d\theta)^h}\Big\{\boldsymbol{F}[\phi_\theta]\Big\}(\eta)(t) = \frac{d^h}{(d\theta)^h}\Big\{f_0(e^{i\theta}t)\Big\}(\eta) \tag{3.3.7}$$

for all $\eta \in \,]-\varepsilon, \varepsilon[$, $t \in \partial\mathbb{D}$ and for all $h = 0, \dots, r+1$. It follows that $f_0 \in \mathcal{C}_*^{m+r+1,\alpha}(\partial\mathbb{D}, \mathbb{C})$. Furthermore, by the equality

$$\lim_{\theta \to 0}\Big\| \frac{\boldsymbol{F}[\phi_\theta] - f_0}{\theta} - \frac{d}{d\theta}\Big\{\boldsymbol{F}[\phi_\theta]\Big\}(0)\Big\|_{m,\alpha} = 0$$

and by (3.3.7), we obtain that $f_0 \in \mathcal{C}_*^{m+r+1,\alpha,p}(\partial\mathbb{D}, \mathbb{C})$. Since $\boldsymbol{G}[\phi_0]$ is a Riemann map of $\mathbb{E}[f_0]$ and f_0 is of class $\mathcal{C}_*^{m+r+1,\alpha,p}$, a standard argument implies that $\boldsymbol{G}[\phi_0] \in \mathcal{C}_*^{m+r+1,\alpha,p}(\partial\mathbb{D}, \mathbb{C})$. By adapting standard result (cf. *e. g.* Lanza (1994, Lemma 5.5 (i) and Thm. 3.3 (ii))), it follows that $\phi_0 \in \mathcal{C}_*^{m+r+1,\alpha,p}(\partial\mathbb{D}, \partial\mathbb{D})$. $\square$

CHAPTER 4

Roumieu spaces and Sewing Problem

4.1. Introduction

In section III.3 we have shown a regularity result for the operator $\boldsymbol{F}$ defined by

$$\boldsymbol{F}[\phi] = \boldsymbol{G}[\phi] \circ \phi^{(-1)}$$

for all shifts ϕ of class $\mathcal{C}_*^{1,\alpha}$, $0 < \alpha < 1$, in a setting of Schauder spaces $\mathcal{C}_*^{m,\alpha}(\partial\mathbb{D}, \mathbb{C})$, $m \geq 1$. Clearly $\boldsymbol{F}[\phi]$ is the trace of the first component of the solution (F, G) of the conformal sewing problem associated to the shift ϕ. In section III.3 we have seen that the regularity of the shift ϕ must increase if we require a higher order of differentiability for the operator $\boldsymbol{F}[\cdot]$. Then the assumption that ϕ belongs to a Banach space of real analytic functions of $\partial\mathbb{D}$ to $\mathbb{C}$ appears natural in order to show the existence of an analytic extension of the operator $\boldsymbol{F}[\cdot]$. The operator $\boldsymbol{F}[\cdot]$ can be expressed as a composition of the operator, say $\boldsymbol{J}$, which takes an invertible function into its inverse, of a composition operator and of the operator $\boldsymbol{G}[\cdot]$, which has an analytic extension in Schauder spaces. Thus we will concerned with the problem of finding a proper space of real analytic functions for our ϕ's in order to have such operators analytic. In section 2 we introduce a space which turn to be suitable to this purpose, namely the Roumieu space $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ (see Definition (4.2.11)). Then in this setting we prove an analyticity result for the composition operator, for the inversion operator $\boldsymbol{J}$ and for the operators $\boldsymbol{G}[\cdot]$ and $\boldsymbol{F}[\cdot]$ by adapting the regularity results on the composition and on the inversion operator of Lanza (1994 and 1996b). In doing so we encounter some technical difficulties, among which we mention the construction of an extension operator (cf. Proposition 4.2.10 (ii) and Lemma 4.2.10 (iii)). We want to underline that there are a lot of problems in which the Roumieu spaces have been applied. We mention *e.g.* the contributions of Roumieu (1960), Lions & Magenes (1970) and Nazarov (1990).

4.2. The composition operator in Roumieu spaces associated to the differentiation operator

Let $h, k \in \mathbb{N} \setminus \{ 0 \}$ and let Ω be a bounded open subset of $\mathbb{R}^h$. In this section we will introduce a natural Banach space setting to find an analytic extension of the operator $\boldsymbol{F}[\cdot]$. By Proposition 3.3.5, we need that our Banach spaces are continuously embedded in $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{R}^k)$ for all $m \in \mathbb{N}$. By observing that the composition operator is involved in the definition of $\boldsymbol{F}[\cdot]$, we deduce that the assumption of real analyticity for the shifts ϕ in the domain of $\boldsymbol{F}[\cdot]$ is a natural condition.

PROPOSITION 4.2.1. *Let $m, h, k \in \mathbb{N}$, $h, k \geq 1$. Let Ω and Ω' be bounded open subsets of $\mathbb{R}^h$ and $\mathbb{R}^k$, respectively. Let F be a function of $\mathrm{cl}\,\Omega$ to $\mathbb{R}$. We assume that the operator $\boldsymbol{T}_F$ from $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathrm{cl}\,\Omega)$ to $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathbb{R})$ defined by*

$$\boldsymbol{T}_F[G] \equiv F \circ G$$

for all $G \in \mathcal{C}^m(\mathrm{cl}\,\Omega', \mathrm{cl}\,\Omega)$ has a real analytic extension $\widetilde{\boldsymbol{T}}_F$ defined in an open neighborhood $\mathcal{N}$ of $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathrm{cl}\,\Omega)$ in $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathbb{R}^h)$. Then F extends to a map $\widetilde{F}$ which is real analytic in some open subset $\widetilde{\Omega} \supseteq \mathrm{cl}\,\Omega$.

Proof. Let $\underline{x} \in \mathbb{R}^h$ and let $c(\underline{x})$ be the constant function of $\mathrm{cl}\,\Omega'$ to $\mathbb{R}^h$ with constant value $\underline{x}$. Clearly, the map $c(\cdot)$ is linear and continuous from $\mathbb{R}^h$ to $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathbb{R}^h)$ and

$$\widetilde{\Omega} \equiv \left\{ \underline{x} \in \mathbb{R}^h : c(\underline{x}) \in \mathcal{N} \right\}$$

is an open subset of $\mathbb{R}^h$ which contains $\mathrm{cl}\,\Omega$. Let $\underline{a} \in \mathrm{cl}\,\Omega'$ and let $\boldsymbol{V}$ be the linear and continuous functional of $\mathcal{C}^m(\mathrm{cl}\,\Omega', \mathbb{R})$ defined by $\boldsymbol{V}[H] = H(\underline{a})$ for all $H \in \mathcal{C}^m(\mathrm{cl}\,\Omega', \mathbb{R})$. Then the map $\widetilde{F}$ defined by $\widetilde{F}(\underline{x}) \equiv \boldsymbol{V}[\widetilde{T}_F[c(\underline{x})]]$ for all $\underline{x} \in \widetilde{\Omega}$, yields the required extension of F. □

Before we introduce Banach spaces of real analytic functions, we observe the validity of the following standard lemma.

LEMMA 4.2.2. *Let $h \in \mathbb{N} \setminus \{0\}$ and let Ω, $\widetilde{\Omega}$ be bounded open subsets of $\mathbb{R}^h$ such that $\widetilde{\Omega} \supseteq \mathrm{cl}\,\Omega$. Let $(\mathcal{Y}, \|\ \|_{\mathcal{Y}})$ be a real Banach space and let F be a real analytic function of $\widetilde{\Omega}$ to $\mathcal{Y}$. Let $\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})$ be the space of all n-linear maps of $(\mathbb{R}^h)^n$ to $\mathcal{Y}$ and let $\|\ \|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}$ be its usual norm. Then there exist $\rho' > 0$ and $M_1 > 0$ such that the following Cauchy estimates of analyticity hold*

$$\frac{\rho'^n \|d^n F(\underline{x})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}}{n!} \leq M_1 \tag{4.2.3}$$

for all $\underline{x} \in \mathrm{cl}\,\Omega$ and for all $n \in \mathbb{N}$. ($d^n F(\underline{x})$ is the n-th differential of F at $\underline{x}$). In particular we have that

$$\frac{\rho'^{|\eta|} \|D^\eta F(\underline{x})\|_{\mathcal{Y}}}{|\eta|!} \leq M_1 \tag{4.2.4}$$

for all $\underline{x} \in \mathrm{cl}\,\Omega$ and for all $\eta \in \mathbb{N}^h$.

Proof. By the compactness of $\mathrm{cl}\,\Omega$, it suffices to show that for all $\underline{a} \in \mathrm{cl}\,\Omega$ there exist $\rho' > 0$ and $M_1 > 0$ such that $\mathbb{B}(\underline{a}, \rho') \subseteq \widetilde{\Omega}$ and the estimates (4.2.3) hold for all $\underline{x} \in \mathbb{B}(\underline{a}, \rho_\prime)$. By the analyticity of F there exists $R > 0$ such that $\mathbb{B}(\underline{a}, R) \subseteq \widetilde{\Omega}$ and that

$$\sum_{n \geq 0} \frac{R^n \|d^n F(\underline{a})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}}{n!} < +\infty \tag{4.2.5}$$

and

$$F(\underline{x}) = \sum_{n \geq 0} \frac{d^n f(\underline{a})[\underline{x} - \underline{a}]^n}{n!} \tag{4.2.6}$$

for all $\underline{x} \in B(\underline{a}, R)$. By standard calculus on normed spaces, we obtain that

$$d^k F(\underline{x})[v_1, \ldots, v_k] = \sum_{n \geq 0} \frac{d^{n+k} F(\underline{a})[v_1, \ldots, v_k, [\underline{x} - \underline{a}]^n]}{n!} \tag{4.2.7}$$

for all $k \in \mathbb{N}$, $\underline{x} \in B(\underline{a}, R)$ and for all $(v_1, \ldots, v_k) \in (\mathbb{R}^h)^k$. By (4.2.5) there exists $M > 0$ such that

$$\frac{R^n \|d^n F(\underline{a})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}}{n!} \leq M\ . \tag{4.2.8}$$

Let $\rho_\prime < R/2$. Then by (4.2.7) and (4.2.8), we obtain that for all $k \in \mathbb{N}$ and for all $\underline{x} \in \mathbb{B}(\underline{a}, \rho_\prime)$. the following inequality holds.

$$\begin{aligned}
\frac{\rho_\prime^k \|d^k F(\underline{x})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}}{k!} &\leq \frac{\rho_\prime^k}{k!} \Bigg(\sum_{n \geq 0} \frac{\rho_\prime^n \|d^{n+k} F(\underline{a})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n, \mathcal{Y})}}{n!} \Bigg) \\
&\leq M \Bigg(\sum_{n \geq 0} \left(\frac{\rho_\prime}{R} \right)^{n+k} \frac{(n+k)!}{n!k!} \Bigg) \\
&\leq M \Bigg(\sum_{n \geq 0} \left(\frac{\rho_\prime}{R} \right)^{n+k} 2^{n+k} \Bigg) \\
&\leq M \frac{1}{1 - 2\rho_\prime / R}.
\end{aligned} \tag{4.2.9}$$

This yields the conclusion. □

We now define standard Banach spaces of real analytic functions of a bounded open subset Ω, namely the Roumieu spaces (cf. Roumieu (1960)) constructed on the basis of the differentiation operator for functions of Ω.

Proposition 4.2.10. *Let $\rho > 0$, $k \in \mathbb{N} \setminus \{0\}$ and let Ω be a bounded open subset of $\mathbb{R}^k$. Let $(\mathcal{Y}, \| \|_\mathcal{Y})$ be a real Banach space and let $\| \|_0$ be the usual sup-norm of the space $\mathcal{C}^0(\mathrm{cl}\,\Omega, \mathcal{Y})$. We set*

$$\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega, \mathcal{Y}) \equiv \left\{ F \in \bigcap_{m \in \mathbb{N}} \mathcal{C}^m(\mathrm{cl}\,\Omega, \mathcal{Y}) : \sup_{\eta \in \mathbb{N}^k} \left\{ \frac{\rho^{|\eta|} \|D^\eta F\|_0}{|\eta|!} \right\} < +\infty \right\} \tag{4.2.11}$$

and we endow $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ with the norm

$$\|F\|_{\omega,\rho} \equiv \sup_{\eta\in\mathbb{N}^k}\left\{\frac{\rho^{|\eta|}\|D^\eta F\|_0}{|\eta|!}\right\} \tag{4.2.12}$$

for all $F \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$. If $B\subseteq\mathcal{Y}$, $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,B)$ denotes the subset of $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ of functions F such that $F(\mathrm{cl}\,\Omega)\subseteq B$. The following statements hold.

(i) *$\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ is a Banach space and for all $m\in\mathbb{N}$ $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ is continuously embedded in $\mathcal{C}^m(\mathrm{cl}\,\Omega,\mathcal{Y})$.*

(ii) *We set $\Omega_\rho\equiv\Omega+\mathbb{B}(0,\rho)$ for all $\rho>0$ and we assume that Ω is regular in the sense of Whitney, i.e. $c[\Omega]<+\infty$. Then there exist $\rho\prime>0$, depending only on ρ and $c[\Omega]$, and a continuous extension operator $\boldsymbol{E}_{\Omega,\rho\prime}$ from $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ to $\mathcal{C}^0_{\omega,\rho\prime}(\mathrm{cl}\,\Omega_{\rho\prime},\mathcal{Y})$ such that*

$$\boldsymbol{E}_{\Omega,\rho\prime}[F]_{/\,\mathrm{cl}\,\Omega}=F$$

for all $F\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$.

(iii) *The set $\bigcup_{\rho>0}\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ coincides with the set of all real analytic functions F of Ω to $\mathcal{Y}$ which admit a real analytic extension to an open subset $\Omega_F\supseteq\mathrm{cl}\,\Omega$.*

Proof. Statement (i) follows by standard arguments. We prove statement (ii). Let $F\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathcal{Y})$ and let $\underline{x}\in\Omega$. Since all partial derivatives exist and are continuous in Ω, F is a $\mathcal{C}^\infty$ map. We want to estimate the norm of the n-th differential of F. Let $\beta(n,k_1,\dots,k_h)$ be natural numbers such that

$$(t_1+\cdots+t_h)^n=\sum_{k_1+\cdots+k_h=n}\beta(n,k_1,\dots,k_h)t_1^{k_1}\cdots t_h^{k_h}$$

for all $n\in\mathbb{N}$ and for all $(t_1,\dots,t_h)\in\mathbb{R}^h$. Clearly

$$\sum_{k_1+\cdots+k_h=n}\beta(n,k_1,\dots,k_h)=h^n\ .$$

Let $\{\,e_i : i=1,\dots,h\,\}$ be the canonical basis of $\mathbb{R}^h$ and let $v=\sum_{i=1}^h v_ie_i\in\mathbb{R}^h$ such that $|v|\equiv\sup\{\,|v_i| : i=1,\dots,h\,\}\leq 1$. Then the following inequality holds (4.2.13)

$$\begin{aligned}\|d^nF(\underline{x})([v]^n)\|_{\mathcal{Y}} &= \left\|\sum_{k_1+\cdots+k_h=n}(\beta(n,k_1,\dots,k_n)v_1^{k_1}\cdots v_h^{k_h}D^{(k_1,\dots,k_h)}F(\underline{x}))\right\|_{\mathcal{Y}}\\ &\leq\left(\sum_{k_1+\cdots+k_h=n}\beta(n,k_1,\dots,k_h)\right)\frac{n!\|F\|_{\omega,\rho}}{\rho^n}\\ &=\frac{n!\|F\|_{\omega,\rho}}{(\rho/h)^n}\ .\end{aligned}$$

By standard properties of n-linear symmetric maps (cf. *e.g.* Prodi & Ambrosetti (1973, p. 84–85)), it follows that

$$\|d^n F(\underline{x})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n,\mathcal{Y})} \leq \frac{n^n \|F\|_{\omega,\rho}}{(\rho/h)^n},$$

for all $n \in \mathbb{N}$. By ratio criterion there exists $M < +\infty$ such that

$$\frac{1}{3^n}\frac{n^n}{n!} \leq M$$

for all $n \in \mathbb{N}$. Then we obtain

$$\|d^n F(\underline{x})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n,\mathcal{Y})} \leq \frac{Mn!\|F\|_{\omega,\rho}}{(\rho/3h)^n} \tag{4.2.14}$$

for all $n \in \mathbb{N}$. Clearly the map $F_{\underline{x}}$ from the open ball $\mathbb{B}(\underline{x}, \rho/3h)$ of $\mathbb{R}^h$ to $\mathcal{Y}$, defined by

$$F_{\underline{x}}(\underline{w}) = \sum_{n=0}^{+\infty} \frac{d^n F(\underline{x})([\underline{w} - \underline{x}]^n)}{n!}$$

for all $\underline{w} \in \mathbb{B}(\underline{x}, \rho/3h)$, is real analytic. Let $\rho_{\prime\prime} < \rho/6h$. Then by (4.2.14) and by arguing as in (4.2.8), there exists M_1 independent on F and $\underline{x}$ such that

$$\frac{\rho_{\prime\prime}^n \|d^n F_{\underline{x}}(\underline{w})\|_{\mathcal{L}^{(n)}((\mathbb{R}^h)^n,\mathcal{Y})}}{n!} \leq M_1 \|F\|_{\omega,\rho} \tag{4.2.15}$$

for all $n \in \mathbb{N}$ and for all $\underline{w} \in \mathrm{cl}\,\mathbb{B}(\underline{x}, \rho_{\prime\prime})$. Clearly by (4.2.14) and by a standard argument on Cauchy estimates (cf. *e.g.* Prodi & Ambrosetti (1973, Thm. 10.5)), F is real analytic in Ω and coincides with $F_{\underline{x}}$ in a neighborhood of $\underline{x}$. Let $\rho_{\prime} < \rho_{\prime\prime}/c[\Omega]$. Let $\underline{z} \in \mathbb{B}(\underline{x}, \rho_{\prime}) \cap \Omega$. Since Ω is regular in the sense of Whitney, there exists a piecewise smooth curve γ from $[0,1]$ to Ω such that $\gamma(0) = \underline{x}$, $\gamma(1) = \underline{z}$ and $\ell_\gamma < c[\Omega]\rho_{\prime}$ (ℓ_γ is the length of γ). It follows that $\gamma([0,1]) \subseteq \mathbb{B}(\underline{x}, \rho_{\prime\prime}) \cap \Omega$ and then all points of $\mathbb{B}(\underline{x}, \rho_{\prime}) \cap \Omega$ belong to the same connected component in $\mathbb{B}(\underline{x}, \rho_{\prime\prime}) \cap \Omega$. This implies that F and $F_{\underline{x}}$ coincide in $\mathbb{B}(\underline{x}, \rho_{\prime}) \cap \Omega$. Let $\underline{y} \in \Omega$. We show that $F_{\underline{x}}$ and $F_{\underline{y}}$ coincides in $\mathbb{S} \equiv \mathbb{B}(\underline{x}, \rho_{\prime}) \cap \mathbb{B}(\underline{y}, \rho_1)$. If $\mathbb{S}$ is different from $\emptyset$, there exists a piecewise smooth curve γ from $[0,1]$ to Ω such that $\gamma(0) = \underline{x}$, $\gamma(1) = \underline{y}$ and $\ell_\gamma < 2c[\Omega]\rho_{\prime}$. Let $\underline{t} \in \gamma([0,1])$ be the point with arc-length $\ell_\gamma/2$. Since $\underline{x}, \underline{y} \in \mathbb{B}(\underline{t}, \rho_{\prime\prime})$, by considering $F_{\underline{t}}$ we obtain that $F_{\underline{x}}$ and $F_{\underline{y}}$ coincide in $\mathbb{B}(\underline{x}, \rho_{\prime}) \cap \mathbb{B}(\underline{y}, \rho_{\prime})$. Then we can define an extension $\boldsymbol{E}_{\Omega,\rho_{\prime}}[F]$ of F in $\mathrm{cl}(\Omega + \mathbb{B}(0, \rho_{\prime}))$ and by (4.2.15), it follows statement (ii). Statement (iii) is an easy consequence of Lemma 4.2.2 and of statement (ii). □

Now we show an analyticity result about the composition operator (cf. Lanza (1996a, Prop. 2.17)).

PROPOSITION 4.2.16. *Let $m, h, k \in \mathbb{N}$, $h, k \geq 1$. Let $\alpha \in]0,1[$, $\rho > 0$. Let Ω, $\Omega_{\prime}$ be bounded open subsets of $\mathbb{R}^h$, $\mathbb{R}^k$, respectively, regular in the sense of Whitney.*

Then the operator $\boldsymbol{T}$ *defined by*

$$\boldsymbol{T}[F,G] \equiv F \circ G$$

for all $(F,G) \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$ *is real analytic from the open subset* $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$ *of* $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R}^h)$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R})$.

Proof. Let $(F_0,G_0) \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$. By a well known argument on Cauchy estimates (cf. *e.g.* Prodi & Ambrosetti (1973, Thm. 10.5)), it suffices to show that there exist $\rho_0 > 0$, $M_0 > 0$ and a neighborhood $\mathcal{U}$ of (F_0,G_0) in $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$ such that

$$\frac{\rho_0^s \|(d^s\boldsymbol{T})[F,G]\|_{\mathcal{L}^{(s)}}}{s!} \leq M_0 \tag{4.2.17}$$

for all $(F,G) \in \mathcal{U}$ and for all $s \in \mathbb{N}$. For the sake of brevity, we set $d^0\boldsymbol{T} \equiv \boldsymbol{T}$ and

$$\mathcal{L}^{(s)} \equiv \mathcal{L}^{(s)}((\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R}^h))^s, \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R})).$$

Let $s \in \mathbb{N}\setminus\{0\}$. Since $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R})$ is continuously embedded in $\mathcal{C}^{m+s,\alpha,p}(\mathrm{cl}\,\Omega,\mathbb{R})$ (cf. Proposition 4.2.10 (i) and Lemma 1.2.4 (i)), Lanza (1996b, Thm. 4.1) (cf. also Lanza (1994, Thm. 4.2)) implies that the operator $\boldsymbol{T}$ is of class $\mathcal{C}^s$ and that the s-th differential of $\boldsymbol{T}$ at (F,G) is delivered by the formula

$$\begin{aligned}(d^s\boldsymbol{T}[F,G])&[(V_{[1]},W_{[1]}),\dots,(V_{[s]},W_{[s]})] \\ &= \sum_{j=1}^{s} \sum_{\ell_1,\dots,\widehat{\ell_j},\dots,\ell_s=1}^{s} \boldsymbol{T}[D_{\ell_1}\cdots D_{\ell_j}\cdots D_{\ell_s}V_{[j]},G](W_{1,\ell_1}\cdots \widehat{W}_{j,\ell_j}\cdots W_{s,\ell_s}) \\ &\qquad + \sum_{\ell_1,\dots,\ell_s=1}^{h} \boldsymbol{T}[D_{\ell_1}\cdots D_{\ell_s}F,G](W_{1,\ell_1}\cdots W_{s,\ell_s})\end{aligned} \tag{4.2.18}$$

for all $(V_{[j]}, W_{[j]} \equiv (W_{j,1},\dots,W_{j,n})) \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R}^h)$, $j = 1,\dots,s$. (We understand that the first summation of the right hand side of (4.2.18) is $\boldsymbol{T}[V_{[1]},G]$ if $s = 0$; the "ˆ" symbol on a term denotes that such term must be omitted). Let $r \in \mathbb{N}$ and let $\|\ \|_r$ and $\|\ \|_{r,\alpha}$ be the usual norms of the spaces of functions of class $\mathcal{C}^r$ and $\mathcal{C}^{r,\alpha}$, respectively, in the closure of an open subset. Since $c[\Omega] < +\infty$, there exists $C_1 > 0$ such that $\|H\|_{0,\alpha} \leq C_1\|H\|_1$ for all $H \in \mathcal{C}^1(\mathrm{cl}\,\Omega,\mathbb{R})$. Let $C_2 > 0$ such that $\|W\prime W\prime\prime\|_{m,\alpha} \leq C_2\|W\prime\|_{m,\alpha}\|W\prime\prime\|_{m,\alpha}$ for all $W\prime$, $W\prime\prime \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\mathbb{R})$. By Lemma 1.2.4 (v), there exists an increasing function of $[0,+\infty[$ to itself such that

$$\|V \circ \widetilde{W}\|_{m,\alpha} \leq \|V\|_{m,\alpha}\Psi(\|\widetilde{W}\|_{m,\alpha})$$

for all $(V,\widetilde{W}) \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$. By (4.2.18), we have that for all $s \in \mathbb{N}\setminus\{0\}$ and for all $((V_{[1]},W_{[1]}),\dots,(V_{[s]},W_{[s]})) \in (\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega))^s$ with $\|V_{[j]}\|_{\omega,\rho} \leq 1$, $\|W_{j,\rho}\|_{m,\alpha} \leq 1$, $j = 1,\dots,s$, $\ell = 1,\dots,h$, the following inequality hold.

$$
\begin{aligned}
(4.2.19)\quad & \|(d^s\boldsymbol{T}[F,G])[(V_{[1]},W_{[1]}),\dots,(V_{[s]},W_{[s]})]\|_{m,\alpha} \\
&\leq \sum_{j=1}^{s} \sum_{\ell_1,\dots,\widehat{\ell_j},\dots,\ell_s=1}^{h} (C_1C_2^{s-1}\|D_{\ell_1}\cdots\widehat{D}_{\ell_j}\cdots D_{\ell_s}V_{[j]}\|_{m+1}\Psi(\|G\|_{m,\alpha})) \\
&\qquad + \sum_{\ell_1,\dots,\ell_s=1}^{h} C_1C_2^s\|D_{\ell_1}\cdots D_{\ell_s}F\|_{m+1}\Psi(\|G\|_{m,\alpha}) \,.
\end{aligned}
$$

If $\widetilde{V} \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R})$, then for all $\eta \in \mathbb{N}^h$ we have that

$$
\begin{aligned}
\|D^\eta\widetilde{V}\|_{m+1} &\leq \left(\sum_{p=0}^{m+1} \frac{h^p[(|\eta|+p)!]}{\rho^{|\eta|+p}}\right)\|\widetilde{V}\|_{\omega,\rho} \\
&\leq \frac{(|\eta|+m+1)!}{\rho^{|\eta|}}\left(\sum_{p=0}^{m+1}\left(\frac{h}{\rho}\right)^p\right)\|\widetilde{V}\|_{\omega,\rho} \qquad (4.2.20)\\
&= \frac{(|\eta|+m+1)!}{\rho^{|\eta|}}C_3\|\widetilde{V}\|_{\omega,\rho}
\end{aligned}
$$

with C_3 positive constant depending only on m, h, ρ and independent on $|\eta|$. By (4.2.19) and (4.2.20), it follows that for all $s \in \mathbb{N}$

$$
\begin{aligned}
\|(d^s\boldsymbol{T})[F,G]\|_{\mathcal{L}^{(s)}} &\leq \frac{sh^{s-1}C_1C_2^{s-1}C_3((s+m)!)}{\rho^{s-1}}\Psi(\|G\|_{m,\alpha}) \\
&\quad + \frac{h^sC_1C_2^sC_3((s+m+1)!)}{\rho^s}\|F\|_{\omega,\rho}\Psi(\|G\|_{m,\alpha}) \qquad (4.2.21)\\
&= a_s\frac{\rho C_1C_3\Psi(\|G\|_{m,\alpha})}{hC_2} + b_sC_1C_3\|F\|_{\omega,\rho}\Psi(\|G\|_{m,\alpha})
\end{aligned}
$$

with

$$
(a_s,b_s) \equiv \left(\frac{(s+m)!(hC_2)^s s}{\rho^s}, \frac{(s+m+1)!(hC_2)^s}{\rho^s}\right) .
$$

Let $0 < \rho_1 < 1$ and let $\rho_2 \equiv (\rho_1\rho)/(hC_2)$. By ratio criterion, it can be easily to checked that there exists $M > 0$ such that

$$
\frac{\rho_2^s a_s}{s!} \leq M, \qquad \frac{\rho_2^s b_s}{s!} \leq M
$$

for all $s \in \mathbb{N}$. Let $\mathcal{U}$ be a bounded open neighborhood of (F_0,G_0) in $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}) \times \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega\prime,\Omega)$. By setting $\rho_0 \equiv \rho_2$, it is easy to find $M_0 > 0$ such that the estimates (4.2.17) hold for all $(F,G) \in \mathcal{U}$. □

Then we have the following analyticity result on the inversion operator (cf. Lanza (1994, Thm. 5.9)).

Corollary 4.2.22. *Let $m, h \in \mathbb{N}$, $h \geq 1$. Let $0 < \alpha < 1$, $\rho > 0$. Let Ω, $\Omega\prime$ be bounded open subsets of $\mathbb{R}^h$, regular in the sense of Whitney. Let $\mathcal{A}_\Omega$ be the open* *subset of* $\mathcal{C}^1(\mathrm{cl}\,\Omega,\mathbb{R}^h)$ *of all injective functions of* $\mathrm{cl}\,\Omega$ *to* $\mathbb{R}^h$ *with nonvanishing jacobian in* $\mathrm{cl}\,\Omega$ *(cf. Lemma 1.2.5). Let* $F_0 \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega')\cap\mathcal{A}_\Omega$ *and let* $\boldsymbol{J}$ *be the inversion operator from* $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega')\cap\mathcal{A}_\Omega$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\mathbb{R}^h)$ *defined by* $\boldsymbol{J}[F] = F^{(-1)}$ *for all* $F \in \mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega')\cap\mathcal{A}_\Omega$*. Then there exists an open neighborhood* $\mathcal{W}_{F_0}$ *of* F_0 *in* $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)\cap\mathcal{A}_\Omega$ *and a real analytic operator* $\widehat{\boldsymbol{J}}_{F_0}$ *from* $\mathcal{W}_{F_0}$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\mathbb{R}^h)$ *such that*

$$\widehat{\boldsymbol{J}}_{F_0}[F] = \boldsymbol{J}[F] \tag{4.2.23}$$

for all $F\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega')\cap\mathcal{W}_{F_0}$*. The differential of* $\widehat{\boldsymbol{J}}_{F_0}$ *at* $F\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega')\cap\mathcal{W}_{F_0}$ *is delivered by the formula*

$$(d\widehat{\boldsymbol{J}}_{F_0}[F])[H] = -((DF)^{-1}\circ F^{(-1)})\cdot(H\circ F^{(-1)}) \tag{4.2.24}$$

for all $H\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)$*.*

Proof. Let $\rho' > 0$, $\Omega_{\rho'} \equiv \Omega + \mathbb{B}(0,\rho')$, $\boldsymbol{E}_{\Omega,\rho'}$ be as in Lemma 4.2.10 (ii). By possibly changing ρ', a compactness argument shows that $\boldsymbol{E}_{\Omega,\rho'}[F_0]\in\mathcal{C}^0_{\omega,\rho'}(\mathrm{cl}\,\Omega_{\rho'},\mathbb{R}^h)\cap\mathcal{A}_{\Omega_{\rho'}}$. By a standard argument on partitions of unity, there exists an open subset Ω'' of $\mathbb{R}^h$ of class $\mathcal{C}^\infty$ (in particular regular in the sense of Whitney) such that $\mathrm{cl}\,\Omega\subseteq\Omega''\subseteq\Omega_{\rho'}$. Since $\mathcal{C}^0_{\omega,\rho'}(\mathrm{cl}\,\Omega'',\mathbb{R}^h)$ is continuously embedded in $\mathcal{C}^1(\mathrm{cl}\,\Omega'',\mathbb{R}^h)$, by Lemma 1.2.5 there exists a neighborhood $\widetilde{\mathcal{W}}_{F_0}$ of F_0 in $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)\cap\mathcal{A}_\Omega$ such that

$$\boldsymbol{E}_{\Omega,\rho'}[F]_{\mathrm{cl}\,\Omega''}\in\mathcal{A}_{\Omega''}$$

for all $F\in\widetilde{\mathcal{W}}_{F_0}$. Let $\widetilde{\boldsymbol{T}}$ be the operator defined by

$$\widetilde{\boldsymbol{T}}[F,G]\equiv\boldsymbol{E}_{\Omega,\rho'}[F]\circ G$$

for all $(F,G)\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)\times\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\Omega'')$. By Proposition 4.2.16, $\widetilde{\boldsymbol{T}}$ is real analytic from $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)\times\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\Omega'')$ to $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\mathbb{R}^h)$. Furthermore, if $F\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega')\cap\widetilde{\mathcal{W}}_{F_0}$, the equation

$$\widetilde{\boldsymbol{T}}[F,G] = \mathrm{id}_{\mathrm{cl}\,\Omega'} \tag{4.2.25}$$

has a unique solution $G\equiv F_0^{(-1)}$. We now apply Implicit Function Theorem to equation (4.2.25) around the pair $(F_0,F_0^{(-1)})$. By formula (4.2.18), the partial differential of $\widetilde{\boldsymbol{T}}$ at $(F_0,F_0^{(-1)})$ with respect to the variable G, is delivered by

$$\frac{\partial\widetilde{\boldsymbol{T}}}{\partial G}[F_0,F_0^{(-1)}][H'] = (DF_0\circ F_0^{(-1)})\cdot H'$$

for all $H'\in\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\mathbb{R}^h)$. Since $F_0\in\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^h)\cap\mathcal{A}_\Omega$, by Lemma 1.2.4 (iv) it follows that $\frac{\partial\widetilde{\boldsymbol{T}}}{\partial G}[F_0,F_0^{(-1)}]$ is a linear homeomorphism of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega',\mathbb{R}^h)$. Then Implicit Function Theorem yields the existence of a real analytic operator $\widehat{\boldsymbol{J}}_{F_0}$ which has property (4.2.23) by the unique solvability of equation (4.2.25). An easy computation, based on formula (4.2.18), yields formula (4.2.24). $\square$

4.3. Analyticity of the operators associated to the sewing problem in Roumieu spaces

In this section we introduce some natural Banach spaces of complex differentiable functions of a compact subset K of $\mathbb{R}^2$ to $\mathbb{C}$. When $K = \partial\mathbb{D}$, these spaces constitute the natural setting in order to find an analytic extension of the operators $\boldsymbol{F}$ and $\boldsymbol{G}$ associated to the sewing problem.

In this Proposition, we define the space of complex-valued functions of class $\mathcal{C}^0_{*,\omega,\rho}$ on a compact subset K with no isolated points: in this setting by using the methods of the last section, we state an analyticity result for the composition and the inversion operator.

PROPOSITION 4.3.1. *Let $\rho > 0$ and let K be a compact subset of $\mathbb{C}$ with no isolated points. Let $\| \ \|_0$ be the usual sup-norm of the Banach space $\mathcal{C}^0_*(K, \mathbb{C})$ (cf. Ch. II). We set*

$$\mathcal{C}^0_{*,\omega,\rho}(K,\mathbb{C}) \equiv \left\{ f \in \bigcap_{n=0}^{+\infty} \mathcal{C}^n_*(K,\mathbb{C}) : \sup_{n\in\mathbb{N}} \left\{ \frac{\rho^n \|f^{(n)}\|_0}{n!} \right\} < +\infty \right\} \tag{4.3.2}$$

and we endow $\mathcal{C}^0_{*,\omega,\rho}(K,\mathbb{C})$ with the norm

$$\|f\|_{\omega,\rho} \equiv \sup_{n\in\mathbb{N}} \left\{ \frac{\rho^n \|f^{(n)}\|_0}{n!} \right\} \tag{4.3.3}$$

for all $f \in \mathcal{C}^0_{*,\omega,\rho}(K,\mathbb{C})$. If $B \subseteq \mathbb{C}$, $\mathcal{C}^0_{*,\omega,\rho}(K,B)$ denotes the subset of $\mathcal{C}^0_{*,\omega,\rho}(K,\mathbb{C})$ of functions f such that $f(K) \subseteq B$. The following statements hold.

- (i) *$\mathcal{C}^0_{*,\omega,\rho}(K,\mathbb{C})$ is continuously imbedded in $\mathcal{C}^m_*(K,\mathbb{C})$ for all $m \in \mathbb{N}$.*
- (ii) *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$, $\rho > 0$. Let Ω be a bounded open subset of $\mathbb{C}$ regular in the sense of Whitney. Then the space $\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega,\mathbb{C})$ and the subspace $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C}) \cap \mathcal{H}(\Omega)$ of $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$ coincide algebraically and have equivalent norms. Furthermore, the space $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})$ and the subspace $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^2) \cap \mathcal{H}(\Omega)$ of $\mathcal{C}^0_{\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{R}^2)$ coincide algebraically and have the same norms. In particular $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})$ is a Banach space.*
- (iii) *We set $\mathbb{A}_\ell \equiv \{ z \in \mathbb{C} : \mathrm{dist}(z,\partial\mathbb{D}) < \ell \}$ for all $\ell \in]0,1[$. Then there exist $\rho_\prime > 0$ and a complex linear and continuous extension operator $\boldsymbol{E}_{\partial\mathbb{D},\rho_\prime}$ from $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^0_{*,\omega,\rho_\prime}(\mathrm{cl}\,\mathbb{A}_{\rho_\prime},\mathbb{C})$ such that*

  $$\boldsymbol{E}_{\partial\mathbb{D},\rho_\prime}[f]_{/\partial\mathbb{D}} = f$$

  for all $f \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$. In particular $\boldsymbol{E}_{\partial\mathbb{D},\rho_\prime}[\cdot]$ is a complex linear homeomorphism of $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$ onto $\mathcal{C}^0_{*,\omega,\rho_\prime}(\mathrm{cl}\,\mathbb{A}_{\rho_\prime},\mathbb{C})$.

(iv) *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$, $\rho > 0$. Let Ω and Ω' be bounded open subsets of $\mathbb{C}$, regular in the sense of Whitney. Then the operator $\boldsymbol{T}$ defined by*

$$\boldsymbol{T}[F,G] \equiv F \circ G$$

for all $(F,G) \in \mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C}) \times \mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\Omega)$ is complex analytic from the open subset $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C}) \times \mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\Omega)$ of $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C}) \times \mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\mathbb{C})$.

(v) *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$, $\rho > 0$. Let Ω, Ω' be bounded open subsets of $\mathbb{C}$ regular in the sense of Whitney. Let $F_0 \in \mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C}) \cap \mathcal{A}_\Omega$. We assume that $\mathrm{cl}\,\Omega' \subseteq F_0(\Omega)$. Then there exist an open neighborhood $\mathcal{W}_{F_0}$ of F_0 in $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C}) \cap \mathcal{A}_\Omega$ and a complex analytic operator $\boldsymbol{J}_{F_0,\Omega'}[\cdot]$ from $\mathcal{W}_{F_0}$ to $\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\mathbb{C})$ such that*

$$\boldsymbol{J}_{F_0,\Omega'}[F] = F^{(-1)}_{/\,\mathrm{cl}\,\Omega'} \tag{4.3.4}$$

for all $F \in \mathcal{W}_{F_0}$.

Proof. Statement (i) follows by an immediate estimate. By using the Cauchy-Riemann equations and the well-known properties of differentiability at the boundary for functions of class $\mathcal{C}^1$ in the closure of an open subset regular in the sense of Whitney, we obtain statement (ii). We prove statement (iii). Let $\rho_0 < \rho$ and let $f \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$. It is easy to check that the power series

$$S_t(z) \equiv \sum_{i=0}^{+\infty} \frac{f^{(i)}(t)}{i!}(z-t)^i$$

defines an holomorphic function on $\mathbb{B}(t,\rho_0)$. Let $t,z \in \partial\mathbb{D}$ and let $\underline{\ell} = \sigma(t,z)$ (cf. (3.2.12)) be the length of the shorter arc between t and z. Let γ be an injective curve of class $\mathcal{C}^1$ from $[0,\underline{\ell}]$ to $\partial\mathbb{D}$ such that $\gamma(0) = t$ and $\gamma(\underline{\ell}) = z$. Let $n \in \mathbb{N} \setminus \{0\}$. By integrating by parts the line integral

$$\frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1} f^{(n)}(\eta)\,d\eta \tag{4.3.5}$$

we obtain the following Taylor formula for f which holds for $t,z \in \partial\mathbb{D}$.

$$f(z) = \sum_{i=0}^{n-1} \frac{f^{(i)}(t)}{i!}(z-t)^i + \frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1} f^{(n)}(\eta)\,d\eta\,. \tag{4.3.6}$$

We now estimate the line integral (4.3.5). We observe that

$$\left|\frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1} f^{(n)}(\eta)\,d\eta\right| \le \frac{\|f^{(n)}\|_0}{(n-1)!}\int_0^{\underline{\ell}} (\underline{\ell}-x)^{n-1}\,dx = \frac{\underline{\ell}^n \|f^{(n)}\|_0}{n!}\,.$$

Since $\underline{\ell} = \sigma(t,z) \le \frac{\pi}{2}|z-t|$, by (4.3.6) it follows that f coincides with $S_t(z)$ in $\mathbb{B}(t,\frac{2}{\pi}\rho_0) \cap \partial\mathbb{D}$. Let $\rho' \equiv \rho_0/5$ and let $\boldsymbol{E}_{\partial\mathbb{D},\rho'}[f]$ be the function of $\mathbb{A}_{\rho'}$ to $\mathbb{C}$ defined by

$$\boldsymbol{E}_{\partial\mathbb{D},\rho'}[f](z) \equiv S_t(z) \tag{4.3.7}$$

for all $(t,z) \in \partial\mathbb{D}\times\mathbb{A}_{\rho'}$ such that $z \in \mathbb{B}(t,\rho_0/4)$. Although there are more choices of t in the definition (4.3.7), the value $S_t(z)$ does not depend on t. Indeed let $\underline{z} \in \mathbb{C}$ and let $t_1, t_2 \in \partial\mathbb{D}$ be such that $\underline{z} \in \mathbb{B}(t_1,\rho_0/4)\cap\mathbb{B}(t_2,\rho_0/4)$. Let $\underline{\eta} \in \partial\mathbb{D}$ be such that $|\underline{z}-\underline{\eta}| = \min\{|\underline{z}-t| : t \in \partial\mathbb{D}\}$. Then $\underline{\eta} \in \mathbb{B}(t_1,\frac{2}{\pi}\rho_0)\cap\mathbb{B}(t_2,\frac{2}{\pi}\rho_0)$ and $f(\eta) = S_{t_1}(\eta) = S_{t_2}(\eta)$ for all $\eta \in \mathbb{B}(t_1,\frac{2}{\pi}\rho_0)\cap\mathbb{B}(t_2,\frac{2}{\pi}\rho_0)\cap\partial\mathbb{D}$. By the Identity Principle for holomorphic functions, it follows that $S_{t_1}(\underline{z}) = S_{t_2}(\underline{z})$. Now we prove that $\boldsymbol{E}_{\partial\mathbb{D},\rho'}[f] \in \mathcal{C}^0_{*,\omega,\rho'}(\mathrm{cl}\,\mathbb{A}_{\rho'},\mathbb{C})$ for all $f \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$ and that the operator $\boldsymbol{E}_{\partial\mathbb{D},\rho'}[\cdot]$ is continuous from $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^0_{*,\omega,\rho'}(\mathrm{cl}\,\mathbb{A}_{\rho'},\mathbb{C})$. Let $z \in \mathbb{A}_{\rho'}$ and $t \in \partial\mathbb{D}$ such that $z \in \mathbb{B}(t,\rho')$. By (4.3.7) and by standard differentiation properties of a power series, we have that

$$\begin{aligned}
\frac{\rho'^k|\boldsymbol{E}_{\partial\mathbb{D},\rho'}[f]^{(k)}(z)|}{k!} &= \frac{\rho_0^k|S_t^{(k)}(z)|}{5^k(k!)} \\
&\le \frac{\rho_0^k}{2^k(k!)}\left(\sum_{n=0}^{+\infty}\frac{|f^{(n+k)}(t)|}{n!}|z-t|^n\right) \\
&\le \frac{\rho_0^k\|f\|_{\omega,\rho}}{2^k}\left(\sum_{n=0}^{+\infty}\frac{(n+k)!}{n!k!}\frac{\rho_0^n}{\rho^{n+k}2^n}\right) \\
&\le \|f\|_{\omega,\rho}\left(\sum_{n=0}^{+\infty}\binom{n+k}{k}\frac{1}{2^{n+k}}\left(\frac{\rho_0}{\rho}\right)^{n+k}\right) \\
&\le \frac{\|f\|_{\omega,\rho}}{1-\rho_0/\rho}\,.
\end{aligned}$$

This yields statement (iii). Statement (iv) follows by Proposition 4.2.16, formula (4.2.18) and by observing that the differential of $\boldsymbol{T}$, which is delivered by the formula

$$(d\boldsymbol{T}[F,G])[V,W] = V\circ G + (F'\circ G)W$$

for all $(F,G) \in \mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})\times\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\Omega)$ and for all $(V,W) \in \mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})\times\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\mathbb{C})$, is complex linear. We consider statement (v). By Corollary 4.2.22, there exist a neighborhood $\mathcal{W}_{F_0}$ of F_0 in $\mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})\cap\mathcal{A}_\Omega$ and a real analytic operator $\boldsymbol{J}_{F_0,\Omega'}$ from $\mathcal{W}_{F_0}$ to $\mathcal{C}^{m,\alpha}_*(\mathrm{cl}\,\Omega',\mathbb{C})$ satisfying property (4.3.4). By (4.2.24), we have that

$$(d\boldsymbol{J}_{F_0,\Omega'}[F])[H] = -(F'\circ F^{(-1)}_{/\mathrm{cl}\,\Omega'})(H\circ F^{(-1)}_{/\mathrm{cl}\,\Omega'})$$

for all $F \in \mathcal{W}_{F_0}$ and for all $H \in \mathcal{C}^0_{*,\omega,\rho}(\mathrm{cl}\,\Omega,\mathbb{C})$. Since $d\boldsymbol{J}_{F_0,\Omega'}[F]$ is complex linear, it follows the complex analyticity of $\boldsymbol{J}_{F_0,\Omega'}[\cdot]$. □

Now we can prove our analyticity result about the operator $\boldsymbol{F}[\cdot]$ and $\boldsymbol{G}[\cdot]$ associated to the sewing problem.

THEOREM 4.3.8. *Let* $\rho > 0$. *Let* $(\boldsymbol{F}[\cdot], \boldsymbol{G}[\cdot])$ *be the pair of operators which maps the shift* $\phi \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ *to the trace of the solution of the sewing problem* (3.2.22) *(cf. Cor.* 3.2.36*). Let* $\phi_0 \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$*. Then there exist* $\rho_\prime > 0$*, a neighborhood* $\mathcal{W}_{\phi_0}$ *of* ϕ_0 *in* $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ *and two complex analytic operators* $\widehat{\boldsymbol{G}}_{\phi_0}[\cdot]$*,* $\widehat{\boldsymbol{F}}_{\phi_0}[\cdot]$ *from* $\mathcal{W}_{\phi_0}$ *to* $\mathcal{C}^0_{*,\omega,\rho_\prime}(\partial\mathbb{D}, \mathbb{C})$ *such that*

$$\widehat{\boldsymbol{G}}_{\phi_0}[\phi] = \boldsymbol{G}[\phi]$$
$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] = \boldsymbol{F}[\phi]$$

for all $\phi \in \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \partial\mathbb{D}) \cap \mathcal{W}_{\phi_0}$.

Proof. We first consider the operator $\boldsymbol{G}[\cdot]$. Let $m \in \mathbb{N}$. By Proposition 3.2.39 and Lemma 4.3.1 (i), $\boldsymbol{G}[\cdot]$ has a complex analytic extension $\widehat{\boldsymbol{G}}[\cdot]$ from $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ to $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$. Let $\varepsilon > 0$ such that $\mathcal{B}(\phi_0, 2\varepsilon) \subseteq \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ and such that the map $\widehat{\boldsymbol{G}}[\cdot]$ from $\mathcal{B}(\phi_0, 2\varepsilon)$ to $\mathcal{C}^0_*(\partial\mathbb{D}, \mathbb{C})$ is bounded. Let $\ell > 0$ and let $\|\ \|^\ell_m$ be the norm of $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$ defined by

$$\|f\|^\ell_m \equiv \sup_{h=0,\dots,m} \frac{\ell^h \|f^{(h)}\|_{\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})}}{h!}$$

for all $f \in \mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$. Clearly $\|\ \|^\ell_m$ is equivalent to the usual norm of $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$. Let

$$C(m, \ell, \eta) \equiv \sup_{\phi \in \mathcal{B}(\phi_0, \eta)} \left\{ \|\widehat{\boldsymbol{G}}[\phi]\|^\ell_m \right\}$$

for all $m \in \mathbb{N}$, $0 < \ell \leq \rho$, $0 < \eta \leq \varepsilon$ (possibly $+\infty$). Let $m \in \mathbb{N}$. Since $\widehat{\boldsymbol{G}}[\cdot]$ is complex analytic from $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$ to $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$ and $\mathcal{B}(\phi_0, 2\varepsilon) \subseteq \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}) \cap \mathcal{A}^*_{\partial\mathbb{D}}$, the following Cauchy estimates hold

$$\frac{(\eta/6)^k \|d^k \widehat{\boldsymbol{G}}[\phi]\|^\ell_{m,\mathcal{L}^{(k)}}}{k!} \leq C(m, \ell, \eta) \tag{4.3.9}$$

for all $k \in \mathbb{N}$, $0 < \ell \leq \rho$, $0 < \eta \leq \varepsilon$ and for all $\phi \in \mathcal{B}(\phi_0, \eta/2)$ (see *e.g.* Berger (1977, p. 88) together with Prodi & Ambrosetti (1973, p. 85); $\mathcal{L}^{(k)} \equiv \mathcal{L}^{(k)}_{\mathbb{C}}((\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}))^k, \mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C}))$ for the sake of brevity and $\|\ \|^\ell_{m,\mathcal{L}^{(k)}}$ is the usual norm on $\mathcal{L}^{(k)}$ associated to the sup norm on $((\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \mathbb{C}))^k$ and $\|\ \|^\ell_m$ on $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$). We will show that there exist $0 < \rho_\prime < \rho$ and $0 < \varepsilon_\prime \leq \varepsilon$ such that

$$\sup_{m \in \mathbb{N}} \{ C(m, \rho_\prime, \varepsilon_\prime) \} < +\infty\ . \tag{4.3.10}$$

Then , by (4.3.9), (4.3.10) and by a standard argument on Cauchy estimates, it follows that $\widehat{\boldsymbol{G}}[\cdot]$ is a complex analytic operator from $\mathcal{B}(\phi_0, \varepsilon_\prime/6)$ to $\mathcal{C}^0_{*,\omega,\rho_\prime}(\partial\mathbb{D}, \mathbb{C})$. We prove (4.3.10). Let $0 < \rho_0 < 1$ and let $\boldsymbol{E}_{\partial\mathbb{D},\rho_0}$ be the extension operator of Proposition 4.3.1 (iii). A standard compactness argument shows that, by possibly shrinking ρ_0, we can assume $\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi_0] \in \mathcal{A}_{\mathbb{A}_{\rho_0}}$. Since $\mathcal{C}^0_{*,\omega,\rho_0}(\mathrm{cl}\,\mathbb{A}_{\rho_0}, \mathbb{C}) \cap$

$\mathcal{A}_{\mathbb{A}_{\rho_0}}$ is open in $\mathcal{C}^0_{*,\omega,\rho_0}(\mathrm{cl}\,\mathbb{A}_{\rho_0},\mathbb{C})$, there exists a neighborhood $\mathcal{W}_{\phi_0}$ of ϕ_0 such that

$$\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi] \in \mathcal{A}_{\mathbb{A}_{\rho_0}},$$
$$\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi](\mathbb{A}_{\rho_0}\cap\mathbb{D}) \subseteq \mathbb{I}[\phi],$$
$$\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi](\mathbb{A}_{\rho_0}\cap(\mathbb{C}\setminus\mathrm{cl}\,\mathbb{D})) \subseteq \mathbb{E}[\phi],$$

for all $\phi \in \mathcal{W}_{\phi_0}$. Let F_ϕ be the holomorphic extension to $\mathbb{I}[\phi]$ of the function $\widehat{\boldsymbol{G}}[\phi]\circ\phi^{(-1)}$ of $\phi(\partial\mathbb{D})$ to $\mathbb{C}$ and let G_ϕ be the holomorphic extension to $\mathbb{C}\setminus\mathrm{cl}\,\mathbb{D}$ of $\widehat{\boldsymbol{G}}[\phi]$ (cf. Proposition 3.2.33 (ii)). We set

$$S_\phi(z) \equiv \begin{cases} G_\phi(z) & \text{if } z \in \mathbb{A}_{\rho_0}\cap(\mathbb{C}\setminus\mathbb{D}) \\ F_\phi(\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi](z)) & \text{if } z \in \mathbb{A}_{\rho_0}\cap\mathbb{D}\,. \end{cases}$$

Since $S_\phi(\cdot)$ is holomorphic in $\mathbb{A}_{\rho_0}\setminus\partial\mathbb{D}$ and continuous in $\mathbb{A}_{\rho_0}$, it follows that $S_\phi(\cdot)$ is holomorphic in $\mathbb{A}_{\rho_0}$. Furthermore the Maximum Principle applied to the functions F_ϕ and $G_\phi(1/w)-1/w$ belonging to $\mathcal{C}^0(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})\cap\mathcal{H}(\mathbb{I}[\phi])$ and $\mathcal{C}^0(\mathrm{cl}\,\mathbb{D},\mathbb{C})\cap\mathcal{H}(\mathbb{D})$ respectively, shows that

$$\|F_\phi\|_{\mathcal{C}^0(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})} \le \|F\|_{\mathcal{C}^0_*(\phi(\partial\mathbb{D}),\mathbb{C})} = \|\widehat{\boldsymbol{G}}[\phi]\|_{\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})}$$
$$\|G_\phi\|_{\mathcal{C}^0(\mathrm{cl}(\mathbb{A}_{\rho_0}\cap(\mathbb{C}\setminus\mathbb{D})),\mathbb{C})} \le \|\widehat{\boldsymbol{G}}[\phi]\|_{\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})}$$

for all $\phi \in \mathcal{W}_{\phi_0}$. Let $0<\varepsilon_\prime<\varepsilon$ such that $\mathcal{B}(\phi_0,\varepsilon_\prime)\subseteq\mathcal{W}_{\phi_0}$ and let

$$M \equiv \sup\{\, \|\widehat{\boldsymbol{G}}[\phi]\|_{\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})} : \phi\in\mathcal{B}(\phi_0,\varepsilon_\prime)\,\}\,.$$

By the assumption on ε, $M<+\infty$. Let $\rho_\prime\equiv\rho_0/3$. Then the Cauchy estimates of the function $S_\phi(\cdot)$ (cf. Berger (1977, p. 88)) yield

$$\frac{(\rho_0/3)^h|\widehat{\boldsymbol{G}}[\phi]^{(h)}(t)|}{h!} \le M \tag{4.3.11}$$

for all $h\in\mathbb{N}$, $t\in\partial\mathbb{D}$ and for all $\phi\in\mathcal{B}(\phi_0,\varepsilon_\prime)$. It follows that $C(m,\rho_\prime,\varepsilon_\prime)\le M$ for all $m\in\mathbb{N}$. This completes the proof for the operator $\boldsymbol{G}[\cdot]$. We now consider the operator $\boldsymbol{F}[\cdot]$. We first construct an extension $\widehat{\boldsymbol{F}}_{\phi_0}$ of $\boldsymbol{F}[\cdot]$ in a neighborhood of ϕ_0 in $\mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D},\mathbb{C})\cap\mathcal{A}^*_{\partial\mathbb{D}}$ and then we show the analyticity of $\widehat{\boldsymbol{F}}_{\phi_0}$. Let $0<\rho_2<1$. Let $\boldsymbol{E}_{\partial\mathbb{D},\rho_2}$ be the extension operator from $\mathcal{C}^0_{*,\omega,\rho_\prime}(\partial\mathbb{D},\mathbb{C})$ to $\mathcal{C}^0_{*,\omega,\rho_2}(\mathrm{cl}\,\mathbb{A}_{\rho_2},\mathbb{C})$ of Proposition 4.3.1 (iii). Clearly, $\boldsymbol{E}_{\partial\mathbb{D},\rho_2}\circ\widehat{\boldsymbol{G}}$ is a complex analytic operator from $\mathcal{B}(\phi_0,\varepsilon_\prime)$ to $\mathcal{C}^0_{*,\omega,\rho_2}(\mathrm{cl}\,\mathbb{A}_{\rho_2},\mathbb{C})$. Let $0<\rho_0<\rho_2$. Let $\boldsymbol{E}_{\partial\mathbb{D},\rho_0}$ be the extension operator of Proposition 4.3.1 (iii) considered above. Let $\Phi_0\equiv\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi_0]$. Since $\Phi_0(\mathbb{A}_{\rho_0})$ is an open subset of $\mathbb{C}$ and contains $\partial\mathbb{D}$, there exists $0<\rho_3<1$ such that $\mathrm{cl}\,\mathbb{A}_{\rho_3}\subseteq\Phi_0(\mathbb{A}_{\rho_0})$. Let $0<\alpha<1$. Let $\mathcal{W}_{\Phi_0}$ and $\boldsymbol{J}\equiv\boldsymbol{J}_{\Phi_0,\mathbb{A}_{\rho_3}}$ be the open neighborhood of Φ_0 in $\mathcal{C}^0_{*,\omega,\rho_0}(\mathrm{cl}\,\mathbb{A}_{\rho_0},\mathbb{C})\cap\mathcal{A}_{\mathbb{A}_{\rho_0}}$ and the complex analytic operator from $\mathcal{W}_{\Phi_0}$ to $\mathcal{C}^{1,\alpha}_*(\mathrm{cl}\,\mathbb{A}_{\rho_3},\mathbb{A}_{\rho_0})$ of Proposition 4.3.1 (v). Let $m\in\mathbb{N}$. By applying Proposition 4.3.1 (v) to each $\Phi\in\mathcal{W}_{\Phi_0}$, we obtain that $\boldsymbol{J}$ is complex analytic from $\mathcal{W}_{\Phi_0}$ to $\mathcal{C}^m_*(\mathrm{cl}\,\mathbb{A}_{\rho_3},\mathbb{A}_{\rho_0})$. By possibly shrinking $\varepsilon_\prime$, we can assume that $\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi] \in \mathcal{W}_{\Phi_0}$ for all $\phi \in \mathcal{B}(\phi_0, 2\varepsilon_\prime)$. By Proposition 4.3.1 (iv), (v), the operator $\widetilde{\boldsymbol{F}}_{\phi_0}$ defined by

$$\widetilde{\boldsymbol{F}}_{\phi_0}[\phi] \equiv \boldsymbol{E}_{\partial\mathbb{D},\rho_2}[\widehat{\boldsymbol{G}}[\phi]] \circ \boldsymbol{J}[\boldsymbol{E}_{\partial\mathbb{D},\rho_0}[\phi]]$$

for all $\phi \in \mathcal{B}(\phi_0, 2\varepsilon_\prime)$, is complex analytic from $\mathcal{B}(\phi_0, 2\varepsilon_\prime)$ to $\mathcal{C}^m_*(\mathrm{cl}\,\mathbb{A}_{\rho_3}, \mathbb{C})$ for all $m \in \mathbb{N}$. Let $\boldsymbol{R}$ be the operator from $\mathcal{C}^m_*(\mathrm{cl}\,\mathbb{A}_{\rho_3}, \mathbb{C})$ to $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$ which maps F to $F_{/\partial\mathbb{D}}$. Clearly, the operator $\widehat{\boldsymbol{F}}_{\phi_0} \equiv \boldsymbol{R} \circ \widetilde{\boldsymbol{F}}_{\phi_0}$ satisfies

$$\widehat{\boldsymbol{F}}_{\phi_0}[\phi] = \boldsymbol{F}[\phi]$$

for all $\phi \in \mathcal{B}(\phi_0, 2\varepsilon_\prime) \cap \mathcal{C}^0_{*,\omega,\rho}(\partial\mathbb{D}, \partial\mathbb{D})$. Let $\rho_4 \equiv \rho_3/3$ and let

$$M_1 \equiv \sup\{\, \|\widetilde{\boldsymbol{F}}_{\phi_0}[\phi]\|_{\mathcal{C}^0(\mathrm{cl}\,\mathbb{A}_{\rho_3}, \mathbb{C})} : \phi \in \mathcal{B}(\phi_0, \varepsilon_\prime) \,\} \,.$$

By the continuity of $\boldsymbol{E}_{\partial\mathbb{D},\rho_2}$ and by (4.3.11), $M_1 < +\infty$. The Cauchy estimates, applied to the holomorphic function $\widetilde{\boldsymbol{F}}_{\phi_0}[\phi]$ and to the complex analytic operator $\widehat{\boldsymbol{F}}_{\phi_0}[\,\cdot\,]$ from $\mathcal{B}(\phi_0, 2\varepsilon_\prime)$ to $\mathcal{C}^m_*(\partial\mathbb{D}, \mathbb{C})$, imply that

$$\begin{aligned}\frac{(\varepsilon_\prime/6)^k \|d^k \widehat{\boldsymbol{F}}_{\phi_0}[\phi]\|^{\rho_4}_{m,\mathcal{L}^{(k)}}}{k!} &\leq \sup_{h=0,\dots,m} \frac{\rho_4^h \|\widehat{\boldsymbol{F}}_{\phi_0}[\phi]^{(h)}\|_{\mathcal{C}^0_*(\partial\mathbb{D},\mathbb{C})}}{h!} \\ &\leq \|\widetilde{\boldsymbol{F}}_{\phi_0}[\phi]_{/\,\mathrm{cl}\,\mathbb{A}_{2\rho_4}}\|_{\mathcal{C}^0(\mathrm{cl}\,\mathbb{A}_{2\rho_4}, \mathbb{C})} \leq M_1\end{aligned}$$

for all $k \in \mathbb{N}$ and for all $\phi \in \mathcal{B}(\phi_0, \varepsilon_\prime/2)$. Since M_1 does not depend on $m \in \mathbb{N}$, it follows the complex analyticity of $\widehat{\boldsymbol{F}}_{\phi_0}[\,\cdot\,]$ from $\mathcal{B}(\phi_0, \varepsilon_\prime/6)$ to $\mathcal{C}^0_{*,\omega,\rho_4}(\partial\mathbb{D}, \mathbb{C})$. $\square$

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