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ON THE GLOBAL CONTROLLABILITY OF SCALAR CONSERVATION LAWS WITH BOUNDARY AND SOURCE CONTROLS*

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Abstract. We provide local and global controllability results for hyperbolic conservation laws on a bounded domain, with a general (not necessarily convex) flux and a time dependent source term acting as a control. The results are achieved for possibly critical states, both continuously differentiable states and BV states. The proofs are based on a combination of the return method and on the analysis of the Riccati equation for the space derivative of the solution.

Key words. conservation laws, source control, global exact controllability, return method

AMS subject classifications. 35L65, 35Q93, 93B05, 93C20

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1. Introduction.

We are concerned with the problem of controllability of a one space-dimensional scalar conservation law on a bounded domain

$$\partial_t u + \partial_x f(u) = 0\,, \qquad t > 0,\ x \in [a, b]\,, \tag{1.1}$$

where $u = u(t, x)$ is the state variable and the flux function $f : I \mapsto \mathbb{R}$ is a smooth map defined on some open interval $I \subseteq \mathbb{R}$. Most of the literature concerning the controllability of hyperbolic partial differential equations analyzes the states $\psi \doteq u(T, \cdot)$ that can be reached at a fixed time $T > 0$, through the influence of boundary controls acting at the endpoints $\{a, b\}$, when an initial condition is given:

$$u(0, x) \;=\; \overline{u}(x)\,, \qquad x \in [a, b]\,. \tag{1.2}$$

In the case of conservation laws (1.1) with a strictly convex flux f, Ancona and Marson [4, 5] and Adimurthi, Ghoshal, and Gowda [1] established a characterization of the reachable states with boundary controls. A similar characterization of approximately reachable states for the Burgers equation was provided by Horsin [26]. From these results it follows that, if we start with a general initial data $\overline{u} \in \mathbf{L}^\infty([a, b])$, the profiles ψ that are attainable at a time $T > 0$ with boundary controls at $x = a$ and $x = b$ are only those which satisfy suitable Oleinik-type inequalities, provided that

$$T \geq \overline{T}(\psi) \doteq \max\left\{ \sup_{x\in(a,b)} \frac{x-a}{\lfloor f' \circ \psi(x)\rfloor_+},\ \sup_{x\in(a,b)} \frac{b-x}{\lfloor f' \circ \psi(x)\rfloor_-} \right\}, \tag{1.3}$$

where $\lfloor a\rfloor_- \doteq \max\{-a, 0\}$ and $\lfloor a\rfloor_+ \doteq \max\{a, 0\}$ denote the negative and positive parts, respectively, of $a \in \mathbb{R}$.

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For conservation laws with general nonconvex flux, Léautaud [29] proved the attainability in finite time of constant states, employing boundary controls, while Andreianov, Donadello, and Marson [8] derived sufficient conditions for the reachability of (nonconstant) states in the case of a nonconvex flux with a single inflection point, where one regards as controls the initial data. All these results show, in particular, that conservation laws are not exactly controllable in finite time to *critical states* (with vanishing characteristic speed).

Here, in the same spirit of Chapouly [12] and Perrollaz [35], we wish to investigate how the effect of a control acting through a time dependent source term on the right-hand side of (1.1), in combination with the boundary controls, allows us to establish (optimal) local and global controllability results, achieve the reachability of a broader class of states (including critical states), and realize the exact controllability to such states in a shorter time than the one required when employing only boundary controls. Namely, we shall investigate the exact controllability problem for a balance law,

$$\partial_t u + \partial_x f(u) = h(t)\,, \qquad t > 0,\ x \in [a,b]\,, \tag{1.4}$$

where we regard as controls both the boundary data acting at the endpoints $\{a,b\}$ of the domain and the source term h depending only on time. We recall that there are two possible settings within which to study this problem. The first possibility is to consider classical solutions (i.e., Lipschitz continuous functions that satisfy the equation almost everywhere), assuming that the source and the boundary controls are regular functions as well. The other possibility is to consider weak (distributional) solutions which satisfy an entropy admissibility criterion, which are natural in this framework since in general classical solutions of (1.1) develop discontinuities in finite time because of the nonlinearity of the equation.

In the first setting Chapouly [12] showed that, when $f(u) = u^2/2$, for every $T > 0$ one can drive in time T any preassigned continuously differentiable initial data $\overline{u}$ to any continuously differentiable target state ψ with a classical solution of (1.4), using suitable source $h(t)$ and boundary controls at $x = a$, $x = b$. In the same setting, for quasilinear hyperbolic systems, local [30, 31, 32, 39, 40] and global (in the linearly degenerate case) [38] controllability results for C^1 states were established employing boundary and distributed controls on the source that depend on both (t,x) variables. In the second setting and for general strictly convex flux f, Perrollaz [35] provided sufficient conditions for the reachability (in arbitrarily small time) of a state $\psi \in BV([a,b])$ with boundary and source controls, through entropy weak solutions of (1.4). In a related result Corghi and Marson [14] established a characterization of the attainable set for scalar strictly convex balance laws evolving on the whole real line, with the source term (depending on both space and time) regarded as a control.

In this paper we will first establish the local and global controllability of continuously differentiable states for a conservation law with a general smooth flux, employing time dependent source and boundary controls. Namely, when the flux has a bounded derivative we will show that, for a preassigned possibly large time, one can steer the system between any two smooth states provided that their C^1-norm is bounded by a constant depending on the first and second derivatives of the flux.

THEOREM 1.1 (see Theorem 2.3). *Let $f : I \to \mathbb{R}$ be a twice continuously differentiable flux on an open interval $I \subseteq \mathbb{R}$, with bounded C^1-norm and C^2-norm. Assume that for two subintervals $I_1, I_2 \subseteq I$ one has*

$$[|f|]_{I_i} \doteq \sup_{\{k\ \mid\ I_i + k \subseteq I\}} \inf_{u \in I_i} \left| \frac{f(u+k) - f(u)}{k} \right| > 0, \qquad i = 1, 2\,.$$

Then, for any

$$T > (b-a)\left(\frac{1}{[|f|]_{I_1}} + \frac{1}{[|f|]_{I_2}}\right)$$

and for every $\overline{u}, \psi \in C^1([a,b])$*, with* $\mathrm{Im}(\overline{u}) \subsetneq I_1, \mathrm{Im}(\psi) \subsetneq I_2$*, such that*

$$\|\bar{u}'\|_{C_0([a,b])} \;<\; \frac{[|f|]_{I_1}}{(b-a)\cdot\|f''\|_{C_0(I)}}, \qquad |\psi'\|_{C_0([a,b])} \;<\; \frac{[|f|]_{I_2}}{(b-a)\cdot\|f''\|_{C_0(I)}},$$

the Cauchy problem (1.4), (1.2) *admits a classical solution* $u \in C^1([0,T]\times[a,b])$ *that satisfies the terminal condition*

$$u(T,x) \;=\; \psi(x)\,, \qquad x \in [a,b]\,, \tag{1.5}$$

for some $h \in C^0([0,T])$*.*

Instead, in the case of fluxes with unbounded derivatives considered in Theorem 2.5, we will obtain a global controllability result, i.e., we will show that, for any fixed time $T > 0$ and for every smooth initial state $\overline{u}$ and target state ψ, one can choose boundary and source controls that drive $\overline{u}$ to ψ at time T. Next, in the case of convex (not necessarily strictly convex) conservation laws, Theorem 2.7 will provide a priori bounds on the total variation of the source control and of the solution connecting an initial data $\overline{u}$ to a terminal state ψ, in terms of the positive variation of ψ and of the negative variation of $\overline{u}$. Finally, relying on such BV bounds, we will show in Theorem 2.11 and in Theorem 2.12 the reachability in finite time of states $\psi \in BV([a,b])$ that satisfy one-sided Lipschitz estimates similar to those stated in [35]. In particular, as a consequence of Theorem 2.12, we obtain the following.

THEOREM 1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex and twice continuously differentiable map satisfying either*

$$\lim_{u\to\infty} |f'(u)| \;=\; \lim_{u\to\infty} \frac{|f'(u)|}{\sup\limits_{v\le u} |f''(v)|} \;=\; +\infty$$

or

$$\lim_{u\to-\infty} |f'(u)| \;=\; \lim_{u\to-\infty} \frac{|f'(u)|}{\sup\limits_{v\ge u} |f''(v)|} \;=\; +\infty.$$

Then, for every $T > 0$ *and* $\overline{u}, \psi \in BV([a,b])$ *with*

$$\bar{u}(x+z)-\bar{u}(x) \;\ge\; -Cz \qquad \text{and} \qquad \psi(x+z)-\psi(x) \;\le\; Cz \qquad \textit{for all } \; x, x+z \in [a,b],$$

for some constant $C > 0$*, there exists* $h \in BV([0,T])$ *so that the Cauchy problem* (1.4), (1.2) *admits an entropy weak solution on* $[0,T]\times[a,b])$ *that satisfies* (1.5)*.*

The advantage of this construction is that we obtain the source control and the corresponding solution as limits of regular solutions which are easier to handle than the piecewise constant front tracking solutions employed in [35]. In fact, we rely on the approach developed in the present paper to address similar problems of global controllability for diagonal systems of conservation laws in the forthcoming paper [6].

We stress the fact that the main feature which distinguishes the results of the present paper from other controllability results available in the literature is their applicability to conservation laws with

- nonconvex flux, for which very limited, partial results have been obtained so far, due to the quite complex structure of solutions (see [8, 29]);
- flux defined on semilines or bounded intervals (that naturally arise in most applications), for which, as far as we know, no result has been previously established.

The examples of applications discussed in section 5 involve conservation laws that share both of these properties.

Notice that, while the choice of the additional source control is certainly needed to achieve controllability of a larger class of states including the critical states (which cannot be reached employing only boundary control; see Remark 2.4), it would be interesting to investigate whether such states can still be reached with the use of only a pair of controls acting through the source and a single boundary. Another promising research direction which seems natural to investigate is to analyze the controllability of viscous conservation laws in the same setting of the present paper, as the viscosity parameter vanishes.

Control problems for conservation laws arise in many different applications, including vehicular traffic models [2, 3, 13, 17], oil reservoir simulation and sedimentation models [8], supply chain [25, 28], and gas dynamics [22]. In practice a time dependent source control can be viewed as a control parameter acting on the flux function of the conservation law letting vary its flux capacity. We refer to [35] for a discussion of various models where source controls naturally appear to govern the dynamics of the corresponding balance law.

Since we are assuming to have full control on both endpoints $\{a, b\}$ of the domain, and because boundary conditions for nonlinear hyperbolic equations are quite involved (e.g., see [9, 37]), it will be simpler to reformulate the controllability problem in an undetermined form (where the boundary data are not explicitly prescribed). Therefore, given an initial state $\overline{u}$ and a terminal state ψ, we will rephrase the problem of steering (1.1) from $\overline{u}$ to ψ via boundary and source controls into the equivalent one of determining a time dependent source $h = h(t)$ and a solution of (1.4) that satisfies (1.2) together with the terminal condition (1.5). The corresponding boundary controls can be recovered afterwards by taking the traces of u at $x = a$ and $x = b$. Because of the time reversibility of isentropic smooth solutions, we will also reduce the problem of exact controllability of continuously differentiable states to the problem of null controllability with C^1 initial states.

The general strategy adopted in section 3 to establish the main results of the paper is basically an application of the so-called *return method* introduced by Coron (see [15]) in combination with the analysis of the Riccati equation governing the evolution of the space derivative of the solutions. In fact, exploiting the a priori bounds on the solutions of the Riccati equation, we construct a source control which steers in a minimal time the initial data $\overline{u}$ to some constant state, say w_1, that can be quite far from the initial and terminal states $\overline{u}$, ψ. Similarly, one can produce a source control that steers in minimal time some constant states, say w_2 (far away from $\overline{u}$, ψ), to the terminal state ψ. Then, it's straightforward to see that we can connect w_1 and w_2 in an arbitarly small time τ, taking h so that $\int_0^\tau h(t)dt = w_2 - w_1$. In the case of convex flux f, the explicit construction of the source control allows us to provide a priori estimates on the control and on the solution of (1.4) in terms of the $\mathbf{L}^\infty$-norm of $\overline{u}$, ψ, of the negative variation of $\overline{u}$, and of the positive variation of ψ. In turn, such a priori bounds are crucial to establish in section 4 the corresponding controllability results in the BV setting. Some exemplifying applications for traffic flow and sedimentation models are illustrated in section 5.

2. Statement of main results. Before stating the main results, we recall the definition of entropy admissible weak solutions. An entropy/entropy flux pair for (1.4) is a couple of continuously differentiable maps $(\eta, q) : I \to \mathbb{R}$ that satisfy $D\eta(u) \cdot Df(u) = Dq(u)$ for all $u \in I$. Observe that, in particular, $(\eta, q) = (\pm Id, \pm f(u))$ provide two entropy/entropy flux pairs. Then we shall adopt the following definition.

DEFINITION 2.1. *A function $u : [0,T] \times [a,b] \to I$ is called an* entropic weak solution *of* (1.4), (1.2) *on* $[0,T] \times [a,b]$ *if it is a continuous function from* $[0,T]$ *into* $\mathbf{L}^1([a,b]; I)$*, which assumes almost everywhere the initial data* (1.2)*, and that is an entropy admissible distributional solution of* (1.4) *on* $(0,T) \times (a,b)$*, i.e., such that for any entropy/entropy flux pair* (η, q)*, with* η *convex, there holds*

$$\int_0^T \int_a^b \Big\{ \eta(u(t,x))\partial_t \varphi(t,x) + q(u(t,x))\partial_x \varphi(t,x) + \eta'(u(t,x))h(t) \cdot \varphi(t,x) \Big\}\, dx\, dt \geq 0$$

for all test functions $\varphi \in C_c^1$, $\varphi \geq 0$, *with compact support in* $]0,T[\times]a,b[$.

Our first results concern the global controllability of continuously differentiable states. Throughout the paper, for any continuously differentiable map $\varphi : J \to \mathbb{R}$, defined on some interval $J \subset \mathbb{R}$, we shall adopt the notation

$$\|\varphi\|_{C(J)} \doteq \sup\{|\varphi(x)| : x \in J\}. \tag{2.1}$$

Moreover, to estimate the maximal speed of the characteristics with which can travel an initial data taking values in a given set $J' \subseteq J$, we introduce the quantities

$$[|\varphi|]_{J'} \doteq \sup_{\{k \,\mid\, J'+k \subseteq J\}} \inf_{u \in J'} \big|\Delta\varphi(u;k)\big| \tag{2.2}$$

with

$$\Delta\varphi(u;k) \doteq \frac{\varphi(u+k) - \varphi(u)}{k} = \frac{\displaystyle\int_0^k \varphi'(u+v)\, dv}{k},$$

and, for every $\varepsilon > 0$,

$$\arg\sup[|\varphi|]_{J',\varepsilon} \doteq \begin{cases} \inf\Big\{k \geq 0 \,|\, J'+k \subseteq J, \big|\Delta\varphi(u;k)\big| > [|\varphi|]_{J'} - \varepsilon \text{ for all } u \in J'\Big\} \\ \qquad\qquad \text{if} \quad [|\varphi|]_{J'} = \sup\limits_{\{k \geq 0 \,\mid\, J'+k \subseteq J\}} \inf_{u \in J'} \big|\Delta\varphi(u;k)\big|, \\ \sup\Big\{k \leq 0 | J'+k \subseteq J, \big|\Delta\varphi(u;k)\big| > [|\varphi|]_{J'} - \varepsilon \text{ for all } u \in J'\Big\} \\ \qquad\qquad \text{if} \quad [|\varphi|]_{J'} = \sup\limits_{\{k \leq 0 | J'+k \subseteq J\}} \inf_{u \in J'} \big|\Delta\varphi(u;k)\big|. \end{cases} \tag{2.3}$$

We will also use the notation Tot.Var.$\{\varphi;\, J'\}$ for the total variation of $\varphi \in BV(J)$ on an interval $J' \subseteq J$ (e.g., see [21]).

We make the following standing assumptions on the flux function f:

(H1) $f : I = (i_-, i_+) \to \mathbb{R}$ is a twice continuously differentiable map;

(H2) one of the following three conditions holds:

(i) $\lim_{u \to i_\pm} |f'(u)| < +\infty$ and $\lim_{u \to i_\pm} |f''(u)| < +\infty$;

(ii) $i_+ = +\infty$, $\lim_{u \to +\infty} |f'(u)| = +\infty$, and $\lim_{u \to +\infty} \frac{|f'(u)|}{\sup_{z \in (i_-,\, u)} |f''(z)|} = +\infty$;

(iii) $i_- = -\infty$, $\lim_{u\to-\infty} |f'(u)| = +\infty$, and $\lim_{u\to-\infty} \frac{|f'(u)|}{\sup_{z\in(u,\, i_+)}|f''(z)|} = +\infty$.

Remark 2.2. Any function with a polynomial growth satisfies conditions (ii) and (iii). Notice that if the flux f satisties condition (ii) of (H2), then it follows that f satisfies also condition

$$\text{(ii)}' \quad \mathrm{i}_+ = +\infty, \qquad \lim_{\mathrm{u}\to+\infty} |\mathrm{f}'(\mathrm{u})| \;=\; \lim_{\mathrm{u}\to+\infty} \left|\frac{\mathrm{f}'(\mathrm{u})}{\mathrm{f}''(\mathrm{u})}\right| \;=\; +\infty\,.$$

Of course, condition (iii) of (H2) implies a similar condition on f. On the other hand, condition (ii) is stronger that condition (ii)$'$ since (ii) prevents the possibility of too many oscillations in the second derivative of f. In fact there exist functions f that satisfy (ii)$'$ but which do not satisfy (ii). To produce a function of this type, consider, for example, any sequence of twice continuously differentiable maps $g_n : [0, 2n^2] \to [0, \infty)$, $n \geq 1$, such that

$$\begin{cases} g_n(0) = g_n'(0) = g_n'(2n^2) = 0, \quad g_n(n^2) = n, \quad g_n(2n^2) = 1, \\[2ex] g_n'(n^2) = \dfrac{1}{n}, \qquad \left|g_n'(u)\right| \leq \dfrac{2}{n} \quad \text{for all } u \in [0, 2n^2]. \end{cases}$$

Set $a_0 = 0$, $a_n = \sum_{k=1}^{n} 2k^2$ for all $n \geq 1$, and define the continuously differentiable map

$$g(u) = n + g_n(u - a_{n-1}) \qquad \text{if} \qquad u \in [a_{n-1}, a_n[\,,\; n \geq 2\,.$$

Then, the function

$$f(u) \;=\; \int_0^u e^{g(s)} ds, \qquad u \in [0, \infty),$$

is twice continuously differentiable and satisfies

$$\lim_{u\to\infty} f'(u) \;=\; \lim_{u\to\infty} e^{g(u)} \;=\; +\infty\,,$$

$$\lim_{u\to\infty} \left|\frac{f'(u)}{f''(u)}\right| \;=\; \lim_{u\to\infty} \frac{1}{|g'(u)|} \;\geq\; \lim_{n\to\infty} \frac{1}{\sup\limits_{u\in[0,n^2]} |g_n'(u)|} \;=\; +\infty\,.$$

However, the function f does not satisfy (ii). Indeed, for every $n \geq 2$, we compute

$$\begin{aligned} \frac{|f'(a_n)|}{\sup_{z\in[0,a_n)} |f''(z)|} \leq \frac{|f'\,(a_n)|}{|f''(a_{n-1} + n^2)|} \;&=\; \frac{e^{(g(a_n) - g(a_{n-1} + n^2))}}{|g'(a_{n-1} + n^2)|} \\ &= \frac{e^{((n+1+g_n(0)) - (n+g_n(n^2)))}}{|g_n'(n^2)|} \;=\; ne^{(1-n)}, \end{aligned}$$

and this implies

$$\limsup_{n\to\infty} \frac{|f'(a_n)|}{\sup_{z\in[0,a_n)} |f''(z)|} \;=\; 0,$$

which in turn is in contrast with

$$\lim_{u\to+\infty} \frac{|f'(u)|}{\sup\limits_{z\in[0,\,u)} |f''(z)|} = +\infty\,,$$

since $a_n \to +\infty$.

Throughout the paper we fix the endpoint $a < b$ of the bounded interval where the space variable takes values.

THEOREM 2.3. *Let f be a flux satisfying the assumptions* (H1), (H2)(i)*, and assume that $[|f|]_{I_1'} > 0$, $[|f|]_{I_2'} > 0$ for intervals $I_1', I_2' \subseteq I$. Then, for every $\overline{u}, \psi \in C^1([a,b])$ with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$ and $\mathrm{Im}(\psi) \subsetneq I_2'$ such that*

$$\|\overline{u}'\|_{C^0([a,b])} < \frac{[|f|]_{I_1'}}{(b-a)\cdot\|f''\|_{C^0(I)}}, \qquad \|\psi'\|_{C^0([a,b])} < \frac{[|f|]_{I_2'}}{(b-a)\cdot\|f''\|_{C^0(I)}}, \tag{2.4}$$

and for any

$$T > T^* := T_1^* + T_2^* \qquad \text{with} \qquad T_1^* \doteq \frac{(b-a)}{[|f|]_{I_1'}}, \qquad T_2^* \doteq \frac{(b-a)}{[|f|]_{I_2'}}, \tag{2.5}$$

there exists $h \in C^0([0,T])$ so that the Cauchy problem (1.4), (1.2) *admits a classical solution $u \in C^1([0,T]\times[a,b])$ that satisfies* (1.5).

Remark 2.4. Notice that T_1^* is the controllability time needed to steer the initial data $\bar{u}$ to 0 while T_2^* is the controllability time needed to steer 0 to the final state ψ. The controllability time T^* in (2.5) is in general much smaller than the boundary controllability time $\overline{T}$ in (1.3). In particular, we observe that $T_2^* \approx \frac{1}{\sup_{u\in I}|f'(u)|}$, whereas $\overline{T}(\psi) \approx \frac{1}{\inf_{u\in \mathrm{Im}(\psi)}|f'(u)|}$. Therefore, whenever the target state ψ is close to a critical state, i.e., $\inf_{u\in\mathrm{Im}(\psi)}|f'(u)| \approx 0$, we have $\overline{T}(\psi) \approx \frac{1}{\inf_{u\in\mathrm{Im}(\psi)}|f'(u)|} = +\infty$, while this is not the case for T^*. For example, assume that there exists $u^c \in \mathrm{Im}(\psi) \subsetneq I_2' = (i', i'')$ with $f'(u^c) = 0$. Then, clearly one cannot reach ψ at any time $T > 0$, starting from any constant state different from u^c and employing only boundary controls, since $\overline{T}(\psi) = +\infty$. On the other hand, it is sufficient to find k such that $I_2' + k \subset I$, with $\inf_{u\in I_2'}\left|\frac{f(u+k)-f(u)}{k}\right| > 0$, to deduce that $[|f|]_{I_2'} > 0$ and hence $T_2^* < +\infty$. Thus, in this case, Theorem 2.3 guarantees the reachability of ψ at any time $T > T^*$, starting from an initial data $\overline{u}$, with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$, provided that they satisfy (2.4) and that also $[|f|]_{I_1'} > 0$.

THEOREM 2.5. *Let f be a flux satisfying the assumptions* (H1) *and* (H2)(ii) *or* (H2)(iii)*. Then, for every $T > 0$ and $\overline{u}, \psi \in C^1([a,b])$, there exists $h \in C^0([0,T])$ so that the Cauchy problem* (1.4), (1.2) *admits a classical solution $u \in C^1([0,T]\times[a,b])$ that satisfies* (1.5).

Remark 2.6. Clearly the flux $f(u) = \frac{u^2}{2}$ satisifies the assumptions (H1) and (H2)(ii). Thus, as a particular case, we recover from Theorem 2.5 the global controllability result established in [12] for the Burgers equation (by quite a different proof). We observe that we achieve the global controllability stated in Theorem 2.5 whenever the derivative of the flux is unbounded and satisfies the assumption (H2)(ii) or (H2)(iii) because in these cases, for any fixed time $T > 0$, one can use the source control h to bend the characteristics of (1.4) with an arbitrary large slope. Thanks to this property, any smooth initial data $\overline{u}$ and terminal state ψ can travel out of the domain (a,b) and exit from the boundaries $x = a, x = b$ in a time smaller than T, keeping bounded the derivative of the solution. Instead, in the case of a flux having a bounded derivative satisfying the assumption (H2)(i), we obtain only the local controllability result given in Theorem 2.3 since the characteristics need at least a time $T > \frac{2(b-a)}{\|f'\|_{C^0(I)}}$ to cross the interval (a,b), no matter which source control h we choose. In the particular case where f is affine, one has $[|f|]_J = \|f'\|_{C^0(I)}$ for any $J \subseteq I$, and thus we find that the controllability time in (2.5) is $T^* = \frac{2(b-a)}{\|f'\|_{C^0(I)}}$. But for nonaffine fluxes f, in general we have $T^* > \frac{2(b-a)}{\|f'\|_{C^0(I)}}$. In fact, by the proof of Proposition 3.1 it follows that T^* is almost optimal as time needed to steer the initial state $\overline{u}$ to the terminal state ψ. Similarly, the upper bounds (2.4) on the C^1-norm of $\overline{u}, \psi$ are almost optimal to guarantee that one can drive $\overline{u}$ to ψ with a smooth solution.

THEOREM 2.7. *Let f be a convex map satisfying the assumptions* (H1), (H2)(i), *and assume that* $[|f|]_{I'_1} > 0$, $[|f|]_{I'_2} > 0$ *for intervals* $I'_1, I'_2 \subseteq I$. *Then, given any* $\rho > 0$ *and* $T > T^*$ *with* $T^* \geq 0$ *as in* (2.5), *there exists* $C_1 > 0$ *depending on* $b-a$, T, T^*, $\arg\sup[|f|]_{I'_i,\, c_1}$, $i = 1,2$ *(c_1 being a constant depending on $\rho, T - T^*$), so that the following hold. For every* $\overline{u}, \psi \in C^1([a,b])$ *with* $\mathrm{Im}(\overline{u}) \subsetneq I'_1$ *and* $\mathrm{Im}(\psi) \subsetneq I'_2$ *such that*

(2.6)
$$\sup_{x\in[a,b]} \lfloor \overline{u}'(x) \rfloor_- \leq \frac{[|f|]_{I'_1}}{(b-a)\cdot \|f''\|_{C^0(I)}} - \rho\,, \qquad \sup_{x\in[a,b]} \lfloor \psi'(x) \rfloor_+ \leq \frac{[|f|]_{I'_2}}{(b-a)\cdot \|f''\|_{C^0(I)}} - \rho\,,$$

there exists $h \in C^0([0,T])$, *with*

$$\|h\|_{C^0([0,T])} + \mathrm{Tot.Var.}\{h;\, [0,T]\} \leq C_1 \cdot \Big(1 + \|\overline{u}\|_{C^0([a,b])} + \|\psi\|_{C^0([a,b])}\Big), \tag{2.7}$$

so that the Cauchy problem (1.4), (1.2) *admits a classical solution* $u \in C^1([0,T]\times[a,b])$ *that satisfies* (1.5) *and*

$$\begin{aligned} &\|u(t,\cdot)\|_{C^0([a,b])} + \mathrm{Tot.Var.}\{u(t,\cdot);\, [a,b]\} \\ &\qquad \leq C_1 \cdot \left(\|\overline{u}\|_{C^0([a,b])} + \|\psi\|_{C^0([a,b])} + \sup_{x\in[a,b]} \lfloor \overline{u}'(x) \rfloor_- + \sup_{x\in[a,b]} \lfloor \psi'(x) \rfloor_+ \right) \end{aligned} \tag{2.8}$$

for all $t \in (0,T)$.

Remark 2.8. In the case f is a convex map satisfying the assumptions (H1), (H2)(i), $i_+ = +\infty$, $\lim_{\rho\to 0} \arg\sup[|f|]_{I',\rho} = +\infty$ (or $i_- = -\infty$, $\lim_{\rho\to 0} \arg\sup[|f|]_{I',\rho} = -\infty$), the constants $C_1, c_1 > 0$ provided by Theorem 2.7 have the following property: If either $T \to T^*$ or $\rho \to 0$, then $c_1 \to 0$ and $C_1 \to +\infty$.

Remark 2.9. If f is a concave map satisfying the assumptions (H1), (H2)(i) and $[|f|]_{I'_1} > 0$, $[|f|]_{I'_2} > 0$ for $I'_1, I'_2 \subseteq I$, then the same conclusions of Theorem 2.7 hold, replacing $\lfloor \overline{u}'(x) \rfloor_-$ with $\lfloor \overline{u}'(x) \rfloor_+$ and $\lfloor \psi'(x) \rfloor_+$ with $\lfloor \psi'(x) \rfloor_-$ in the inequalities (2.6), (2.8).

Remark 2.10. If $f : I = (i_-, +\infty) \to \mathbb{R}$ is a map satisfying the assumptions (H1) and (H2)(ii), and $I'_1, I'_2 \subset I$ are bounded intervals, then setting

$$[|f|]_{I',\,u} \doteq \sup_{\{k \;|\; I'+k \subseteq (i_-,u)\}} \; \inf_{v\in I'} \big|\Delta f(v;k)\big|\,, \tag{2.9}$$

one finds

$$\lim_{u\to+\infty} \frac{[|f|]_{I',\,u}}{|f'(u)|} = 1\,. \tag{2.10}$$

Hence, taking the limit as $u \to \infty$ in (2.4), (2.5), (2.6) with $[|f|]_{I'_i,\,u}$ in place of $[|f|]_{I'_i}$ and $\|f''\|_{C^0((i_-,u))}$ in place of $\|f''\|_{C^0(I)}$, the controllability time T^* in (2.5) becomes zero and the upper bounds in (2.4), (2.6) become $+\infty$. Therefore, at least formally, one can deduce the conclusions of Theorem 2.5 from Theorem 2.3. Similar formal deductions can be carried out in the case f satisfies the assumptions (H1) and (H2)(iii).

Relying on Theorem 2.7 we then establish a global controllability result for BV states that satisfy one-sided Lipschitz inequalities expressed in terms of Dini derivatives. We recall that

$$D^-\omega(x) = \liminf_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h}, \qquad D^+\omega(x) = \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} \tag{2.11}$$

denote, respectively, the lower and the upper Dini derivative of a function ω at x.

THEOREM 2.11. *Under the same assumptions of Theorem* 2.7*, given any $\rho > 0$ and $T > T^*$ with $T^* \geq 0$ as in* (2.5)*, there exists $C_2 > 0$ depending on $b-a$, T, T^*, $\arg\sup[|f|]_{I_i',c_2}$, $i = 1,2$ (c_2 being a constant depending on $\rho, T-T^*$), so that the following hold. For every $\overline{u}, \psi \in BV([a,b])$ with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$ and $\mathrm{Im}(\psi) \subsetneq I_2'$ such that*

$$\begin{cases} d^- \doteq \sup_{x\in[a,b]} \lfloor D^-\overline{u}(x)\rfloor_- < \frac{[|f|]_{I_1'}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho\,, \\ d^+ \doteq \sup_{x\in[a,b]} \lfloor D^+\psi(x)\rfloor_+ < \frac{[|f|]_{I_2'}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho\,, \end{cases} \tag{2.12}$$

there exists $h \in BV([0,T])$, with

$$\|h\|_{\mathbf{L}^\infty([0,T])} + \mathrm{Tot.Var.}\{h;\,[0,T]\} \leq C_2 \cdot \Big(1 + \|\overline{u}\|_{\mathbf{L}^\infty([a,b])} + \|\psi\|_{\mathbf{L}^\infty([a,b])}\Big), \tag{2.13}$$

so that the Cauchy problem (1.4)*,* (1.2) *admits an entropy weak solution on $[0,T]\times[a,b])$ that satisfies* (1.5) *and*

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty([a,b])} + \mathrm{Tot.Var.}\{u(t,\cdot);\,[a,b]\} \leq C_2 \cdot \Big(\|\overline{u}\|_{\mathbf{L}^\infty([a,b])} + \|\psi\|_{\mathbf{L}^\infty([a,b])} + d^- + d^+\Big) \tag{2.14}$$

for all $t \in (0,T)$

THEOREM 2.12. *Let f be a convex map satisfying the assumptions* (H1) *and* (H2)(ii) *or* (H2)(iii)*. Then, for every $T > 0$ and $\overline{u}, \psi \in BV([a,b])$ with*

$$d^- \doteq \sup_{x\in[a,b]} \big\lfloor D^-\overline{u}(x)\big\rfloor_- < +\infty \qquad \text{and} \qquad d^+ \doteq \sup_{x\in[a,b]} \big\lfloor D^+\psi(x)\big\rfloor_+ < +\infty\,, \tag{2.15}$$

there exists $h \in BV([0,T])$ so that the Cauchy problem (1.4)*,* (1.2) *admits an entropy weak solution on $[0,T]\times[a,b])$ that satisfies* (1.5)*.*

Remark 2.13. By the proofs of Theorem 2.11 it follows that, in its setting, we obtain an approximate controllability result for classical solutions. Namely, if $\overline{u}$, ψ are BV states that satisfy conditions (2.12), then for any $T > T^*$ and for every fixed $\varepsilon > 0$, there exist $h \in C^0([0,T])$ and a classical solution $u \in C^1([0,T]\times[a,b])$ of (1.4) that satisfies

$$\big\|u(0,\cdot) - \overline{u}\big\|_{\mathbf{L^1}([a,b])} < \varepsilon\,, \qquad \big\|u(T,\cdot) - \psi\big\|_{\mathbf{L^1}([a,b])} < \varepsilon\,. \tag{2.16}$$

On the other hand, the same type of approximate controllability result holds also in the setting of Theorem 1.2. In fact, in this case we can approximate any pair of initial and terminal data $\overline{u}, \psi \in BV([a,b])$ with $\overline{u}_\varepsilon,\ \psi_\varepsilon \in C^1([a,b])$ so that $\|\overline{u} - \overline{u}_\varepsilon\|_{\mathbf{L^1}} < \varepsilon$, $\|\psi - \psi_\varepsilon\|_{\mathbf{L^1}} < \varepsilon$. Then, for any $T > 0$ and for every fixed $\varepsilon > 0$, applying Theorem 2.5 we deduce the existence of $h \in C^0([0,T])$ and of a classical solution $u \in C^1([0,T]\times[a,b])$ of (1.4) that satisfies (2.16).

Remark 2.14. If f is a concave map satisfying the assumptions (H1), (H2)(i) and $[|f|]_{I'} > 0$ for $I' \subseteq I$, or (H1) and (H2)(ii), or (H1) and (H2)(iii), then the same conclusions of Theorem 2.11 and of Theorem 1.2 hold, replacing $\lfloor D^- \overline{u}(x) \rfloor_-$ with $\lfloor D^+ \overline{u}(x) \rfloor_+$ and $\lfloor D^+ \psi(x) \rfloor_+$ with $\lfloor D^- \psi(x) \rfloor_-$ in the inequalities (2.6), (2.8).

Remark 2.15. Theorem 1.2 shows that, for conservation laws with convex or concave fluxes satisfying the assumptions (H1) and (H2)(ii) or (H2)(iii), by choosing a suitable source term h in (1.4) we can steer, in any arbitrarily small time $T > 0$, every initial BV state $\overline{u}$ which does not admit shock discontinuities to every BV target state ψ which does not admit discontinuities generating a rarefaction wave. This result is included in the ones established in [35], but here we obtain the solution u as limit of smooth solutions, which are easier to handle both for numeric schemes and for treating similar problems in the case of diagonal systems of conservation laws.

3. Global controllability of C^1 states.

3.1. Reduction to null controllability. Since classical solutions of (1.4) are time reversible, we can recover the global controllability of C^1 states provided by Theorems 2.3, 2.5, and 2.7 from the null controllability of (1.4). Thus, it will be sufficient to prove the following.

PROPOSITION 3.1. *In the same setting and with the same assumptions of Theorem 2.3, for any $T > T_1^*$ with $T_1^* \geq 0$ as in (2.5), and for every $\overline{u} \in C^1([a,b])$ with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$, and satisfying*

$$\|\overline{u}'\|_{C^0([a,b])} < \frac{[|f|]_{I_1'}}{(b-a) \cdot \|f''\|_{C^0(I)}}\,, \tag{3.1}$$

there exists $h \in C^0([0,T])$ vanishing at $t = 0, T$ so that the Cauchy problem (1.4), (1.2) admits a classical solution $u \in C^1([0,T] \times [a,b])$ that satisfies

$$u(T,x) = 0, \qquad x \in [a,b]\,. \tag{3.2}$$

PROPOSITION 3.2. *In the same setting and with the same assumptions of Theorem 2.5, for any $T > 0$, and for every $\overline{u} \in C^1([a,b])$, there exists $h \in C^0([0,T])$ vanishing at $t = 0, T$ so that the Cauchy problem (1.4), (1.2) admits a classical solution $u \in C^1([0,T] \times [a,b])$ that satisfies (3.2).*

PROPOSITION 3.3. *In the same setting and with the same assumptions of Theorem 2.7, given any $\rho > 0$ and $T > T_1^*$ with $T_1^* \geq 0$ as in (2.5), there exists $C_1 > 0$ depending on $b-a$, T, T_1^*, $\arg sup [|f|]_{I_1', c_1}$ (c_1 being a constant depending on $\rho, T - T_1^*$) so that the following hold. For every $\overline{u} \in C^1([a,b])$, with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$, and satisfying*

$$\sup_{x \in [a,b]} \lfloor \overline{u}'(x) \rfloor_- \leq \frac{[|f|]_{I_1'}}{(b-a) \cdot \|f''\|_{C^0(I)}} - \rho\,, \tag{3.3}$$

there exists $h \in C^0([0,T])$ vanishing at $t = 0, T$, with

$$\|h\|_{C^0([0,T])} + \mathrm{Tot.Var.}\{h;\, [0,T]\} \leq C_1 \cdot \Big(1 + \|\overline{u}\|_{C^0([a,b])}\Big), \tag{3.4}$$

so that the Cauchy problem (1.4), (1.2) admits a classical solution $u \in C^1([0,T] \times [a,b])$ that satisfies (3.2) and

(3.5)

$$\|u(t,\cdot)\|_{C^0([a,b])} + \mathrm{Tot.Var.}\{u(t,\cdot);\, [a,b]\} \leq C_1 \cdot \left(\|\overline{u}\|_{C^0([a,b])} + \sup_{x \in [a,b]} \lfloor \overline{u}'(x) \rfloor_- \right)$$

for all $t \in (0,T)$.

The following lemmas show that Theorems 2.3, 2.5, and 2.7 are indeed a consequence of Propositions 3.1, 3.2, and 3.3.

LEMMA 3.4. *Proposition* 3.1 $\Longrightarrow$ *Theorem* 2.3 *,* *Proposition* 3.2 $\Longrightarrow$ *Theorem* 2.5 *.*

Proof. We provide only a proof of the first implication, the second being entirely similar. Let $T > T^*$, and, given $\overline{u}, \psi \in C^1([a, b])$ with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$ and $\mathrm{Im}(\psi) \subsetneq I_2'$, which satisfy (2.4), set

$$\overline{u}_1(x) \doteq \overline{u}(x) \qquad \text{and} \qquad \overline{u}_2(x) \doteq \psi(a + b - x), \qquad x \in [a, b]\,. \tag{3.6}$$

Observe that $\overline{u}_1, \overline{u}_2$ satisfy the assumptions (3.1) (with I_2' in place of I_1' for $\overline{u}_2$). Hence, by Proposition 3.1 there exist $h_i \in C^0([0, T_i])$, $T_i > T_i^*$, $i = 1, 2$, vanishing at $t = 0, T_i$, and $u_i \in C^1([0, T_i] \times [a, b])$, $i = 1, 2$, with $T = T_1 + T_2$, that satisfiy

$$\begin{aligned} &\partial_t u_i + \partial_x f(u_i) = h_i(t)\,, \quad t \in [0, T_i], \quad x \in [a, b]\,, \\ &u_i(0, x) = \overline{u}_i(x), \qquad u_i(T_i, x) = 0, \quad x \in [a, b]\,. \end{aligned} \tag{3.7}$$

Consider the function

$$u(t, x) = \begin{cases} u_1(t, x) & \text{if} \quad t \in [0, T_1], \quad x \in [a, b], \\ u_2(T - t, a + b - x) & \text{if} \quad t \in [T_1, T], \quad x \in [a, b], \end{cases} \tag{3.8}$$

and define

$$h(t) = \begin{cases} h_1(t) & \text{if} \quad t \in [0, T_1], \\ -h_2(T - t) & \text{if} \quad t \in [T_1, T]. \end{cases} \tag{3.9}$$

Then, relying on (3.7), by a direct computation it follows that $u(t, x)$ is a solution of (1.4). Moreover, since (3.7) together with $h_1(T_1) = h_2(T_2) = 0$ imply that $u_1(T_1, \cdot) = u_2(T_2, \cdot) = \partial_t u_1(T_1, \cdot) = \partial_t u_2(T_2, \cdot) \equiv 0$, we deduce that u is a continuously differentiable map on $[0, T] \times [a, b]$. Finally, observe that (3.6), (3.7), (3.8) yield $u(0, \cdot) = \overline{u}_1 = \overline{u}$, $u(T, \cdot) = \overline{u}_2(a + b - \cdot) = \psi$, which shows that u is a C^1 classical solution of (1.4) steering the equation from $\overline{u}$ to ψ. ☐

LEMMA 3.5. *Proposition* 3.3 $\Longrightarrow$ *Theorem* 2.7 *.*

Proof. Let $T > T^*$ and, given $\overline{u}, \psi \in C^1([a, b])$ with $\mathrm{Im}(\overline{u}) \subsetneq I_1'$ and $\mathrm{Im}(\psi) \subsetneq I_2'$, which satisfy (2.6), adopting the same setting (3.6) we observe that

$$\begin{aligned} &\|\overline{u}_1\|_{C^0([a,b])} = \|\overline{u}\|_{C^0([a,b])}, \qquad \sup_{x \in [a,b]} \lfloor \overline{u}_1'(x) \rfloor_- = \sup_{x \in [a,b]} \lfloor \overline{u}'(x) \rfloor_-, \\ &\|\overline{u}_2\|_{C^0([a,b])} = \|\psi\|_{C^0([a,b])}, \qquad \sup_{x \in [a,b]} \lfloor \overline{u}_2'(x) \rfloor_- = \sup_{x \in [a,b]} \lfloor \psi'(x) \rfloor_+\,. \end{aligned} \tag{3.10}$$

Hence, relying on Proposition 3.3 and following the same arguments of the proof of Lemma 3.4 we deduce that the function u defined in (3.8) is a C^1 classical solution of (1.4), with h as in (3.9), steering the equation from $\overline{u}$ to ψ. Moreover, by Proposition 3.3 we are assuming that

$$\|h_i\|_{C^0([0,T])} + \mathrm{Tot.Var.}\{h_i;\, [0, T]\} \leq C_1 \cdot \Big(1 + \|\overline{u}_i\|_{C^0([a,b])}\Big), \quad i = 1, 2, \tag{3.11}$$

and

(3.12)
$$\|u_i(t,\cdot)\|_{C^0([a,b])} + \text{Tot.Var.}\{u_i(t,\cdot);\ [a,b]\} \leq C_1 \cdot \left(\|\overline{u}_i\|_{C^0([a,b])} + \sup_{x\in[a,b]} \lfloor \overline{u}_i'(x) \rfloor_- \right)$$

for all $t \in (0, T_i)$, $i = 1, 2$. Observe that, by (3.8)–(3.9), there holds

(3.13)
$$\|h\|_{C^0([0,T])} \leq \max_i \|h_i\|_{C^0([0,T_i])}, \qquad \text{Tot.Var.}\{h;\ [0,T]\} \leq \sum_i \text{Tot.Var.}\{h_i;\ [0,T_i]\},$$

$$\|u(t,\cdot)\|_{C^0([a,b])} \leq \begin{cases} \|u_1(t,\cdot)\|_{C^0([a,b])} & \text{if} \quad t \in [0, T_1], \\ \|u_2(T-t,\cdot)\|_{C^0([a,b])} & \text{if} \quad t \in [T_1, T], \end{cases}$$

$$\text{Tot.Var.}\{u(t,\cdot);\ [a,b]\} \leq \begin{cases} \text{Tot.Var.}\{u_1(t,\cdot);\ [a,b]\} & \text{if} \quad t \in [0, T_1], \\ \text{Tot.Var.}\{u_2(T-t,\cdot);\ [a,b]\} & \text{if} \quad t \in [T_1, T]. \end{cases}$$

Thus, from (3.11)–(3.12) we deduce that the functions h, u defined in (3.9), (3.8), respectively, satisfy the bounds (2.7)–(2.8) stated in Theorem 2.7 (with a constant C_1 different from the one provided by Proposition 3.3). ∎

3.2. Null controllability.

Proof of Proposition 3.1.

Part 1. Given $T > T_1^*$ with $T_1^* \geq 0$ as in (2.5), and $\overline{u} \in C^1([a,b])$ with $\text{Im}(\overline{u}) \subsetneq I_1'$, satisfying (3.1), let $\varepsilon_1 > 0$ be such that

$$T > T_0 \doteq \frac{(b-a)}{[|f|]_{I_1'}} \cdot (1 + 2\,\varepsilon_1)\,, \tag{3.14}$$

$$\|\overline{u}'\|_{C^0([a,b])} < \frac{[|f|]_{I_1'}}{(b-a)\cdot(1+3\,\varepsilon_1)\cdot\|f''\|_{C^0(I)}}\,. \tag{3.15}$$

Then, we extend $\overline{u}$ to a continuously differentiable function on the entire line $\mathbb{R}$, which we still denote $\overline{u}$ (see Figure 1), so that

$$\text{Im}(\overline{u}) \subsetneq I_1', \qquad \|\overline{u}'\|_{C^0(\mathbb{R})} < \frac{[|f|]_{I_1'}}{(b-a)\cdot(1+3\,\varepsilon_1)\cdot\|f''\|_{C^0(I)}}\,, \tag{3.16}$$

$$\overline{u}(x) = \begin{cases} \alpha_- & \text{if} \qquad x \leq a - \varepsilon_1 \cdot (b-a)\,, \\ \alpha_+ & \text{if} \qquad x \geq b + \varepsilon_1 \cdot (b-a)\,, \end{cases} \tag{3.17}$$

for some constants $\alpha_-, \alpha_+ \in \mathbb{R}$.

Observe that, for any $h \in C^0([0,+\infty])$, the Cauchy problem

$$\begin{aligned} \partial_t u + \partial_x f(u) &= h(t)\,, \qquad t > 0,\ x \in \mathbb{R}\,, \\ u(0,x) &= \overline{u}(x)\,, \qquad x \in \mathbb{R}\,, \end{aligned} \tag{3.18}$$

admits a classical solution $u(t,x)$ defined on some maximal interval $\big[0, T^h\big)$. Given any fixed $x_0 \in \mathbb{R}$, let $x(\cdot)$ denote the unique forward characteristics of (3.18) starting from x_0, i.e., the unique solution of

$$\dot{x}(t) \ = \ f'(u(t,x(t))), \quad t \in \big[0, T^h\big), \tag{3.19}$$

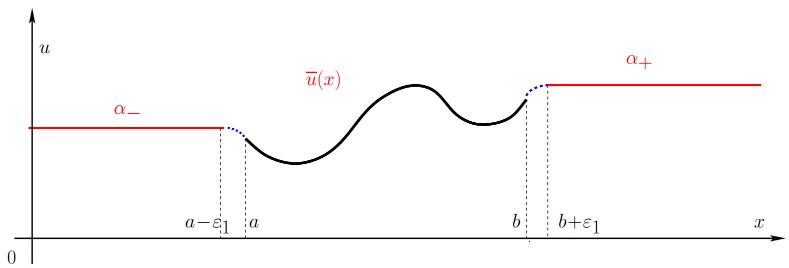

FIG. 1. *Extension of the initial data.*

satisfying $x(0) = x_0$. Then, $z_0(t) \doteq u(t, x(t))$ is a Carathéodory solution of

$$\dot{z_0}(t) \ = \ h(t), \quad t \in [0, T^h), \qquad z_0(0) \ = \ \overline{u}(x_0)\,.$$

On the other hand, observe that the function $w(t,x) = \partial_x u(t,x)$ is a broad solution on $[0, T^h\,) \times \mathbb{R}$ of the semilinear equation

$$\partial_t w(t,x) + f'(u(t,x)) \cdot \partial_x w(t,x) \ = \ - f''(u(t,x)) \cdot w^2(t,x) \tag{3.20}$$

(e.g., see [11, Theorems 3.1 and 3.6]). Hence, relying on (3.20) we deduce that $z_1(t) \doteq \partial_x u(t, x(t))$ is a Carathéodory solution of

$$\dot{z_1}(t) \ = \ - f''(u(t,x(t))) \cdot z_1^2(t), \quad t \in [0, T^h), \qquad z_1(0) \ = \ \overline{u}'(x_0)\,. \tag{3.21}$$

Then, a direct computation yields

$$z_0(t) \ = \ \overline{u}(x_0) + \int_0^t h(\tau) d\tau, \tag{3.22}$$

$$x(t) \ = \ x_0 + \int_0^t f'(z_0(\tau)) d\tau \ = \ x_0 + \int_0^t f' \left(\overline{u}(x_0) + \int_0^\tau h(s) ds \right) d\tau, \tag{3.23}$$

and

$$\frac{1}{z_1(t)} \ = \ \frac{1}{\overline{u}'(x_0)} + \int_0^t f''(z_0(\tau)) d\tau \tag{3.24}$$

for all $t \in [0, T^h)$.

Part 2. Consider the continuous function (see Figure 2)

(3.25)

$$\begin{aligned} h(t) = \ & \frac{t \cdot \overline{h}}{\tau_1} \cdot \chi_{[0,\tau_1]} + \overline{h} \cdot \chi_{[\tau_1, T_0]} + \frac{(T_1 - t) \cdot \overline{h}}{\tau_1} \cdot \chi_{[T_0, T_1]} \\ & - \frac{16(t - T_1) \cdot (\alpha + T_0 \cdot \overline{h}\,)}{3\,(T - T_1)^2} \cdot \chi_{\left[T_1, \,\frac{T + 3T_1}{4}\right]} - \frac{4\,(\alpha + T_0 \cdot \overline{h}\,)}{3\,(T - T_1)} \cdot \chi_{\left[\frac{T + 3T_1}{4}, \,\frac{3T + T_1}{4}\right]} \\ & + \left[- \frac{4\,(\alpha + T_0 \cdot \overline{h}\,)}{3\,(T - T_1)} + \frac{4\,(4t - 3T - T_1) \cdot (\alpha + T_0 \cdot \overline{h}\,)}{3\,(T - T_1)^2} \right] \cdot \chi_{\left[\frac{3T + T_1}{4}, \,T\right]}\,, \end{aligned}$$

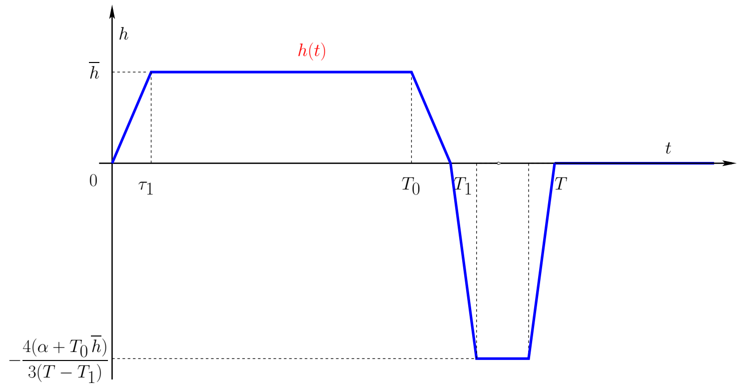

FIG. 2. *The source control.*

where χ_J denotes the characteristic function of an interval $J \subset \mathbb{R}$, T_0 is the time defined in (3.14), and $\alpha \in \{\alpha_-, \alpha_+\}$ and the constants $0 < \tau_1 < T - T_0$, $\overline{h} \in \mathbb{R}$ will be chosen later so that

$$T_1 \doteq T_0 + \tau_1 < T^h \tag{3.26}$$

$\big(T^h$ being the maximal time of existence of a classical solution to (3.18)$\big)$ and such that there holds

$$u(T_1, x) = \alpha + T_0 \cdot \overline{h}, \qquad x \in [a, b]\,. \tag{3.27}$$

Notice that the definition of (3.25), together with (3.27), then implies

$$u(t, x) = \alpha + T_0 \cdot \overline{h} + \int_{T_1}^{t} h(s)\ ds, \qquad t \in [T_1, T]\,, \quad x \in [a, b]\,, \tag{3.28}$$

which in turn, by a direct computation, yields

$$u(T, x) = \alpha + T_0 \cdot \overline{h} - \alpha - T_0 \cdot \overline{h} = 0, \qquad x \in [a, b]\,, \tag{3.29}$$

thus showing that condition (3.2) is verified. Hence, in order to conclude the proof of the theorem we need only to establish (3.26)–(3.27) with $\alpha = \alpha_-$ or $\alpha = \alpha_+$. To this end, relying on (3.22)–(3.24), we find

(3.30)

$$z_0(t) \ = \ \overline{u}(x_0) + \frac{t^2 \cdot \overline{h}}{2\tau_1} \cdot \chi_{[0,\tau_1]} + \Big(t - \frac{\tau_1}{2}\Big) \cdot \bar{h} \cdot \chi_{[\tau_1, T_0]} + \left(T_0 - \frac{(T_1 - t)^2}{2\tau_1}\right) \cdot \overline{h} \cdot \chi_{[T_0, T_1]}$$

for all $t \in [0, T_1]$ and

(3.31)

$$\begin{aligned}
x(T_1) &= x_0 + \int_0^{\tau_1} f'(z_0(s))ds + \int_{\tau_1}^{T_0} f'\Big(\overline{u}(x_0) + \frac{(2s-\tau_1)\cdot \overline{h}}{2}\Big)\ ds + \int_{T_0}^{T_1} f'(z_0(s))ds \\
&= x_0 + \int_0^{\tau_1} f'(z_0(s))ds + \frac{1}{\overline{h}} \cdot \left[f\Big(\overline{u}(x_0) + \Big(T_0 - \frac{\tau_1}{2}\Big)\cdot \overline{h}\Big) - f\left(\overline{u}(x_0) + \frac{\tau_1 \overline{h}}{2}\right)\right] \\
&\quad + \int_{T_0}^{T_1} f'(z_0(s))ds \ = x_0 + \int_0^{\tau_1} f'(z_0(s))ds + T_0 \cdot \Delta f\big(\overline{u}(x_0);\, T_0 \cdot \overline{h}\,\big) \\
&\quad + \frac{\tau_1}{2} \cdot \left[-\Delta f\left(\overline{u}(x_0); \frac{\tau_1 \overline{h}}{2}\right) + \Delta f\left(\overline{u}(x_0) + T_0 \cdot \bar{h}; -\frac{\tau_1 \overline{h}}{2}\right)\right] + \int_{T_0}^{T_1} f'(z_0(s))ds\,.
\end{aligned}$$

Part 3. Since we are assuming that $[|f|]_{I_1'} > 0$, and because $\mathrm{Im}(\overline{u})$ is a closed interval, recalling definition (2.2) and (3.16) there will be some $\overline{k}$ such that either

$$\Delta f(u; \overline{k}\,) > [|f|]_{I_1'} - \frac{\varepsilon_1 \cdot (b-a)}{2\,T_0} \qquad \text{for all} \quad u \in \mathrm{Im}(\overline{u}) \tag{3.32}$$

or

$$\Delta f(u; \overline{k}\,) < -[|f|]_{I_1'} + \frac{\varepsilon_1 \cdot (b-a)}{2\,T_0} \qquad \text{for all} \quad u \in \mathrm{Im}(\overline{u})\,, \tag{3.33}$$

with ε_1 as in (3.14)–(3.15). To fix the ideas, assume that (3.32) holds and that $\overline{k} > 0$. Then, choosing

$$\overline{h} = \frac{\overline{k}}{T_0}, \tag{3.34}$$

we find

$$\Delta f\big(\overline{u}(x_0);\, T_0 \cdot \overline{h}\,\big) > [|f|]_{I_1'} - \frac{\varepsilon_1 \cdot (b-a)}{2\,T_0}\,. \tag{3.35}$$

Hence, if $x(T_1) \in [a, b]$, and we choose

$$\tau_1 < \min\left\{ \frac{\varepsilon_1 \cdot (b-a)}{6 \cdot \|f'\|_{C^0(I)}}\,, T - T_0 \right\}, \tag{3.36}$$

combining (3.31) with (3.35), and recalling (3.14), we derive

$$\begin{aligned} x_0 &\le x(T_1) + 3\tau_1 \cdot \|f'\|_{C^0(I)} - T_0 \cdot \Delta f\big(\overline{u}(x_0);\, T_0 \cdot \overline{h}\,\big) \\ &\le b - T_0 \cdot [|f|]_{I_1'} + 3\tau_1 \cdot \|f'\|_{C^0(I)} + \frac{\varepsilon_1 \cdot (b-a)}{2} \\ &< b - T_0 \cdot [|f|]_{I_1'} + \varepsilon_1 \cdot (b-a) = a - \varepsilon_1 \cdot (b-a)\,. \end{aligned} \tag{3.37}$$

Because of (3.17) and (3.30), the inequality (3.37) implies that $u(T_1, x(T_1)) = \overline{u}(x_0) + T_0 \cdot \overline{h} = \alpha_- + T_0 \cdot \overline{h}$, which proves (3.27), choosing

$$\alpha = \alpha_-\,. \tag{3.38}$$

On the other hand, relying on (3.14), (3.16), (3.24), and taking

$$\tau_1 < \frac{\varepsilon_1 \cdot (b-a)}{2 \cdot [|f|]_{I_1'}}\,, \tag{3.39}$$

we deduce that if $\overline{u}'(x_0) \neq 0$, then

$$\begin{aligned} \frac{1}{|z_1(t)|} &\ge \frac{1}{|\overline{u}'(x_0)|} - \left| \int_0^t f''(z_0(\tau))d\tau \right| \\ &\ge \frac{1}{\|\overline{u}'\|_{C^0(\mathbb{R})}} - t \cdot \|f''\|_{C^0(I)} \\ &> \frac{(b-a)\cdot(1+3\,\varepsilon_1)\cdot\|f''\|_{C^0(I)}}{[|f|]_{I_1'}} - T_1 \cdot \|f''\|_{C^0(I)} \\ &> \frac{(b-a)\cdot \varepsilon_1 \cdot \|f''\|_{C^0(I)}}{2 \cdot [|f|]_{I_1'}} \qquad \text{for all} \quad t \in [0, T_1]\,. \end{aligned} \tag{3.40}$$

Therefore, choosing

$$\tau_1 < \min\left\{ \frac{\varepsilon_1 \cdot (b-a)}{6 \cdot \|f'\|_{C^0(I)}},\ \frac{\varepsilon_1 \cdot (b-a)}{[|f|]_{I_1'}},\ T - T_0 \right\} \tag{3.41}$$

and observing that $z_1(t) \equiv 0$ if $\overline{u}'(x_0) = 0$, we deduce from (3.40) that, for every solution $x(t)$ of (3.19) starting at $x_0 \in \mathbb{R}$, the function $z_1(t) = \partial_x u(t, x(t))$ satisfies

$$|z_1(t)| < +\infty \qquad \text{for all} \quad t \in [0, T_1]\,, \tag{3.42}$$

which yields (3.26). This completes the proof of the theorem with the choice of $\overline{h}$, α, and τ_1 in (3.25) according to (3.34), (3.38), (3.41), respectively. ◻

Proof of Proposition 3.2. To fix the ideas assume that the flux f satisfies the assumptions (H1) and (H2)(ii). Given $\overline{u} \in C^1([a,b])$, set $I_1' \doteq \mathrm{Im}(\overline{u})$ and $I_u \doteq (i_-, u)$. Observe that, because of (H2)(ii) and (2.10), we have

$$\lim_{u \to +\infty} \frac{(b-a)}{[|f|]_{I_1',u}} = \lim_{u \to +\infty} \frac{(b-a)}{|f'(u)|} = 0$$

and

$$\lim_{u \to +\infty} \frac{[|f|]_{I_1',u}}{(b-a) \cdot \|f''\|_{C^0(I_u)}} = \lim_{u \to +\infty} \frac{\|f'\|_{C^0(I_u)}}{(b-a) \cdot \|f''\|_{C^0(I_u)}} = +\infty\,,$$

where $[|f|]_{I_1',u}$ is defined as in (2.9). Then, given any $T > 0$, there will be $u_0 > i_-$ such that

$$T > \frac{(b-a)}{[|f|]_{I_1',u_0}}\,, \qquad \|\overline{u}'\|_{C^0([a,b])} < \frac{[|f|]_{I_1',u_0}}{(b-a) \cdot \|f''\|_{C^0(I_{u_0})}}\,. \tag{3.43}$$

Now, applying Proposition 3.1 to the flux $f : I_{u_0} \to \mathbb{R}$ and to the initial data $\overline{u} \in C^1([a,b])$, which satisfy the assumptions (H1), (H2)(i), $[|f|]_{I_1',u_0} > 0$, and (3.1), respectively, we deduce the conclusion of Proposition 3.2. ◻

Proof of Proposition 3.3.

Part 1. Given $\overline{u} \in C^1([a,b])$ satisfying $\mathrm{Im}(\overline{u}) \subsetneq I_1'$ and (3.3), let $\varepsilon_1 > 0$ (depending on $T - T_1^*$ and ρ) be such that $T > T_0$, with T_0 as in (3.14), and

$$\frac{[|f|]_{I_1'}}{(b-a) \cdot \|f''\|_{C^0(I)}} - \rho < \frac{[|f|]_{I_1'}}{(b-a) \cdot (1 + 3\,\varepsilon_1) \cdot \|f''\|_{C^0(I)}}\,. \tag{3.44}$$

Then, in view of (3.3), (3.44), we extend $\overline{u}$ to a continuously differentiable function on $\mathbb{R}$, which we still denote $\overline{u}$, so that

$$\mathrm{Im}(\overline{u}) \subsetneq I_1', \qquad \sup_{x \in \mathbb{R}} \lfloor \overline{u}'(x) \rfloor_- < \frac{[|f|]_{I_1'}}{(b-a) \cdot (1 + 3\,\varepsilon_1) \cdot \|f''\|_{C^0(I)}}\,, \tag{3.45}$$

$$\|\overline{u}\|_{C^0(\mathbb{R})} \le 2 \cdot \|\overline{u}\|_{C^0([a,b])}\,, \qquad \text{Tot.Var.}\{\overline{u};\, \mathbb{R}\} \le 2 \cdot \text{Tot.Var.}\{\overline{u};\, [a,b]\}\,, \tag{3.46}$$

$$\overline{u}(x) = \begin{cases} \alpha_- & \text{if} \quad x \le a - \varepsilon_1 \cdot (b-a)\,, \\ \alpha_+ & \text{if} \quad x \ge b + \varepsilon_1 \cdot (b-a)\,, \end{cases} \tag{3.47}$$

for some constants

$$\alpha_-, \ \alpha_+ \in \mathrm{Im}(\overline{u})\,. \tag{3.48}$$

Next observe that if we show that the Cauchy problem (3.18), with h defined as in (3.25), admits a classical solution u on $[0,T_1]\times\mathbb{R}$, with T_0 as in (3.14) and $\tau_1>0$ satisfying (3.36), then by the same arguments of the proof of Proposition 3.1 we deduce that (3.27), (3.29) hold. Hence, in order to complete the proof that u is a classical solution of (1.4), (1.2) satisfying (3.2), it remains to prove that (3.26) is also true. To this end notice that, since $f''(u)$ is nonnegative (f being a convex map), by (3.21) it follows that z_1 is a decreasing map on $[0,T^h)$. Moreover, if $\overline{u}'(x_0)>0$ from (3.24) it follows that $z_1(t)>0$ for all $t\in[0,T^h)$. On the other hand, in the case where $\overline{u}'(x_0)<0$, relying on (3.14), (3.24), (3.45), and taking τ_1 as in (3.39), we deduce

$$\begin{aligned}
\frac{1}{z_1(t)} &\leq \frac{1}{\overline{u}'(x_0)} + \left|\int_0^t f''(z_0(\tau))d\tau\right| \\
&\leq \frac{-1}{\sup_{x\in\mathbb{R}}\lfloor\overline{u}'(x)\rfloor_-} + t\cdot\|f''\|_{C^0(I)} \\
&< -\frac{(b-a)\cdot(1+3\,\varepsilon_1)\cdot\|f''\|_{C^0(I)}}{[|f|]_{I_1'}} + T_1\cdot\|f''\|_{C^0(I)} \\
&< -\frac{(b-a)\cdot\varepsilon_1\cdot\|f''\|_{C^0(I)}}{2\cdot[|f|]_{I_1'}} \qquad \text{for all} \ \ t\in[0,T_1]\,.
\end{aligned} \tag{3.49}$$

Thus, choosing τ_1 as in (3.41), we derive

$$-\infty < z_1(t) \leq \overline{u}'(x_0) \qquad \text{for all} \ \ t\in[0,T_1]\,, \quad x_0\in\mathbb{R}\,, \tag{3.50}$$

which shows that (3.26) is verified.

Part 2. By the definition of h in (3.25), and because of (3.48), a direct computation yields

(3.51)

$$\text{Tot.Var.}\{h;\,[0,T]\} = \frac{2\cdot|\overline{k}|}{T_0} + \frac{8\cdot|\alpha_\pm+\overline{k}|}{3\cdot(T-T_1)} \leq \left(\frac{6\,T+2\,T_0-6\,\tau_1}{3\,T_0\cdot(T-T_1)}\right)\cdot|\overline{k}| + \frac{8\cdot\|\overline{u}\|_{C^0([a,b])}}{3(T-T_1)},$$

(3.52)

$$\|h\|_{C^0([0,T])} \leq \frac{\text{Tot.Var.}\{h;\,[0,T]\}}{2} \leq \left(\frac{3\,T+T_0-3\,\tau_1}{3\,T_0\cdot(T-T_1)}\right)\cdot|\overline{k}| + \frac{4\cdot\|\overline{u}\|_{C^0([a,b])}}{3(T-T_1)},$$

$$\left|\int_0^t h(s)ds\right| \leq |\overline{k}| + |\alpha_\pm| \leq |\overline{k}| + \|\overline{u}\|_{C^0([a,b])} \qquad \text{for all} \ \ t\in[0,T], \tag{3.53}$$

where $\overline{k}=T_0\cdot\overline{h}$ is a constant choosen so that (3.32) holds which, recalling (2.3), (3.14), can be taken so that

$$|\overline{k}| \leq \arg\sup[|f|]_{I_1',c_1}+1, \qquad c_1 \leq \frac{\varepsilon_1}{2(1+2\varepsilon_1)}\cdot[|f|]_{I_1'}\,. \tag{3.54}$$

Then, choosing

$$\tau_1 < \min\left\{\frac{\varepsilon_1\cdot(b-a)}{6\cdot\|f'\|_{C^0(I)}},\ \frac{\varepsilon_1\cdot(b-a)}{[|f|]_{I_1'}},\ \frac{T-T_0}{2}\right\}, \tag{3.55}$$

(3.51), (3.52) imply

$$\|h\|_{C^0([0,T])} + \text{Tot.Var.}\{h;\,[0,T]\} \le \max\left\{\frac{6\,T+3\,T_0}{T_0\cdot(T-T_0)},\ \frac{8}{T-T_0}\right\}\cdot\Big(|\overline{k}| + \|\overline{u}\|_{C^0([a,b])}\Big), \tag{3.56}$$

while from (3.22), (3.45), (3.53), we deduce

$$\|u(t,\,\cdot\,)\|_{C^0([a,b])} \le |\overline{k}| + 4\cdot\|\overline{u}\|_{C^0([a,b])}\,. \tag{3.57}$$

Next, observe that, letting $\text{Tot.Var.}^-\{\overline{u};\,[a,b]\}$ denote the negative variation of $\overline{u}$ on $[a,b]$ (e.g., see [21]), one has

$$\text{Tot.Var.}\{\overline{u};\,[a,b]\} \le 2\left(\|\overline{u}\|_{C^0([a,b])} + \text{Tot.Var.}^-\{\overline{u};\,[a,b]\}\right). \tag{3.58}$$

Thus, we have

$$\text{Tot.Var.}\{\overline{u};\,[a,b]\} \le 2\cdot(1+(b-a))\cdot\left(\|\overline{u}\|_{C^0([a,b])} + \sup_{x\in[a,b]}\lfloor\overline{u}'(x)\rfloor_-\right). \tag{3.59}$$

On the other hand, notice that a classical solution of (3.18) is also the unique entropic weak solution of (3.18). Hence, since scalar balance laws as in (3.18), with a source term h only depending on time, admit entropic weak solutions with total variation nonincreasing in time (e.g., obtained by an operator splitting algorithm; see [16]), relying also on (3.46) we derive

$$\begin{aligned} \text{Tot.Var.}\{u(t,\cdot);\,[a,b]\} &\le \text{Tot.Var.}\{u(t,\cdot);\,\mathbb{R}\}\\ &\le \text{Tot.Var.}\{\overline{u};\,\mathbb{R}\}\\ &\le 2\cdot\text{Tot.Var.}\{\overline{u};\,[a,b]\} \qquad \text{for all}\quad t\in[0,T]\,. \end{aligned} \tag{3.60}$$

Then, combinig (3.59), (3.60), we obtain

$$\text{Tot.Var.}\{u(t,\cdot);\,[a,b]\} \le 4\cdot(1+(b-a))\cdot\left(\|\overline{u}\|_{C^0([a,b])} + \sup_{x\in[a,b]}\lfloor\overline{u}'(x)\rfloor_-\right). \tag{3.61}$$

Hence, (3.56), (3.57), (3.61) show that the estimates (3.4), (3.5) are satisfied with

$$C_1 = \max\left\{\frac{(6\,T+3\,T_0)\cdot(1+|\overline{k}|)}{T_0\cdot(T-T_0)},\ \frac{8\,(1+|\overline{k}|)}{T-T_0},\ 4\,(2+(b-a))+|\overline{k}|\right\}, \tag{3.62}$$

where $\overline{k}$ satisfies the bound (3.54). This completes the proof of the theorem. ∎

4. Controllability of BV states.

Proof of Theorem 2.11. Given $\overline{u},\psi\in BV([a,b])$ with $\text{Im}(\overline{u})\subsetneq I_1'$ and $\text{Im}(\psi)\subsetneq I_2'$ such that (2.12) holds, relying on Lemma 6.1 in the appendix there will be sequences $\{\overline{u}_n\}_{n\ge1},\{\psi_n\}_{n\ge1}\subset C^1([a,b])$ with $\text{Im}(\overline{u}_n)\subsetneq I_1'$ and $\text{Im}(\psi_n)\subsetneq I_2'$ such that

$$\overline{u}_n\to\overline{u}\,,\qquad \psi_n\to\psi\qquad \text{in}\quad \mathbf{L^1}([a,b])\,, \tag{4.1}$$

and

$$\sup_{x\in[a,b]} \lfloor \overline{u}_n'(x)\rfloor_- \leq \frac{[|f|]_{I_1'}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho\,, \quad \sup_{x\in[a,b]} \lfloor \psi_n'(x)\rfloor_+ \leq \frac{[|f|]_{I_2'}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho. \tag{4.2}$$

Then, applying Theorem 2.7 for each pair $\overline{u}_n, \psi_n \in C^1([a,b])$, we deduce the existence of $\{h_n\}_{n\geq 1} \subset C^0([0,T])$, with $T > T^*$, and $\{u_n\}_{n\geq 1} \subset C^1([a,b]\times[0,T])$ that are classical solutions of

$$\partial_t u_n + \partial_x f(u_n) = h_n(t)\,, \qquad t\in[0,T],\ x\in[a,b]\,, \tag{4.3}$$

$$u_n(0,x) = \overline{u}_n(x), \qquad x\in[a,b]\,, \tag{4.4}$$

$$u_n(T,x) = \psi_n(x), \qquad x\in[a,b]\,, \tag{4.5}$$

which satisfy the estimates

$$\|h_n\|_{C^0([0,T])} + \text{Tot.Var.}\{h_n;\,[0,T]\} \leq C_1\cdot\Big(1+\|\overline{u}\|_{C^0([a,b])} + \|\psi\|_{C^0([a,b])}\Big) \tag{4.6}$$

and

$$\begin{aligned} &\|u_n(t,\cdot)\|_{C^0([a,b])} + \text{Tot.Var.}\{u_n(t,\cdot);\,[a,b]\} \\ &\quad\leq\ C_1\cdot\left(\|\overline{u}\|_{C^0([a,b])} + \|\psi\|_{C^0([a,b])} + \frac{[|f|]_{I_1'}+[|f|]_{I_2'}}{(b-a)\cdot\|f''\|_{C^0(I)}}\right)\end{aligned} \tag{4.7}$$

for all $n\geq 1$, $t\in(0,T)$.

Observe that each u_n is also a weak entropic solution of (4.3) and that, since (4.7) provides a uniform bound on the total variation of $u_n(t,\cdot)$ for all $t\in[0,T]$, applying [16, Theorem 4.3.1] we deduce that $t\to u_n(t,\cdot)$ is Lipschitz continuous in $\mathbf{L^1}([a,b])$ on $[0,T]$. Moreover, by (4.7), $\{u_n(t,\cdot)\}_{n\geq 1}$ are uniformly bounded for all $t\in[0,T]$. Therefore, invoking a consequence of Helly's compactness theorem (e.g., see [11, Theorem 2.4]), we deduce the existence of a function $u\in\mathbf{L^1}([0,T]\times[a,b];\,I)$, which is Lipschitz continuous from $[0,T]$ into $\mathbf{L^1}([a,b];\,I)$ and such that, up to a subsequence, there holds

$$u_n(t,\cdot)\to u(t,\cdot) \qquad \text{in} \qquad \mathbf{L^1}([a,b]) \qquad \text{for all}\ \ t\in[0,T]\,. \tag{4.8}$$

On the other hand (4.6) provides a uniform bound on $\{h_n\}_n$ and on their total variation. Hence, by Helly's compactness theorem there will be a function $h\in BV([0,T])$ so that, up to a subsequence, there holds

$$h_n\to h \qquad \text{in} \qquad \mathbf{L^1}([0,T])\,. \tag{4.9}$$

Hence, relying on (4.8)–(4.9) and on the fact that each u_n is an entropic weak solution of (4.3), we deduce

$$\begin{aligned} &\int_0^T\int_a^b \Big\{\eta(u)\partial_t\varphi + q(u)\partial_x\varphi + \eta'(u)h(t)\cdot\varphi\Big\}\,dx\,dt \\ &\quad= \lim_{n\to\infty}\int_0^T\int_a^b \Big\{\eta(u_n)\partial_t\varphi + q(u_n)\partial_x\varphi + \eta'(u_n)h_n\cdot\varphi\Big\}\,dx\,dt\ \geq\ 0\end{aligned} \tag{4.10}$$

for every entropy/entropy flux pair (η, q) with η convex. Thus (4.10), together with (4.1), (4.4), (4.8), proves that u is an entropic weak solution of the Cauchy problem (1.4), (1.2), while (4.1), (4.5), (4.8) show that the terminal condition (1.5) is satisfied. Finally, we observe that, by the lower semicontinuity of the total variation with respect to the $\mathbf{L^1}$ convergence, and because of (4.8), (4.9), we recover the estimates (2.13) and (2.14) from (4.6) and (4.7), respectively. This concludes the proof of the theorem. ∎

Proof of Theorem 2.12. To fix the ideas assume that the flux f satisfies the assumptions (H1) and (H2)(ii). Then, given $\overline{u}, \psi \in BV([a,b])$ satisfying (2.15), and $T > 0$, setting $I_1' \doteq \mathrm{Im}(\overline{u})$, $I_2' \doteq \mathrm{Im}(\psi)$, $I_u \doteq (i_-, u)$, by the same arguments and with the same notations of the proof of Theorem 3.2, we deduce that there will be $u_0 > i_-$ such that

$$
\begin{aligned}
&T > (b-a)\cdot\left(\frac{1}{[|f|]_{I_1',u_0}} + \frac{1}{[|f|]_{I_2',u_0}}\right),\\
&\sup_{x\in[a,b]} \big\lfloor D^-\overline{u}(x)\big\rfloor_- < \frac{[|f|]_{I_1',u_0}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho\,,\\
&\sup_{x\in[a,b]} \big\lfloor D^+\overline{u}(x)\big\rfloor_- < \frac{[|f|]_{I_2',u_0}}{(b-a)\cdot\|f''\|_{C^0(I)}} - \rho
\end{aligned}
\tag{4.11}
$$

for some $\rho > 0$. Hence, according to Lemma 6.1 there are sequences $\{\overline{u}_n\}_{n\geq 1}, \{\psi_n\}_{n\geq 1}$ in $C^1([a,b])$ with $\mathrm{Im}(\overline{u}_n) \subsetneq \mathrm{Im}(\overline{u})$ and $\mathrm{Im}(\psi_n) \subsetneq \mathrm{Im}(\psi)$, which satisfy (4.1), (4.2). Now, applying Theorem 2.7 to the flux $f : I_{u_0} \to \mathbb{R}$ which satisfy the assumptions (H1), (H2)(i), $[|f|]_{I_1',u_0} > 0$, $[|f|]_{I_2',u_0} > 0$, and to each pair $\overline{u}_n, \psi_n \in C^1([a,b])$ that satisfy the estimates (2.6), by the same arguments of the proof of Theorem 2.11 we deduce the conclusions of Theorem 1.2. ∎

5. Some applications. In this section we discuss the application of the controllability results established in the paper to some examples of conservation laws describing vehicular traffic and sedimentation processes. Traffic source control can be implemented in a variety of ways so as to modulate the flux capacity of the road, e.g., route recommendation panels, variable speed limit regulation [23], integrated vehicular and roadside sensors [20], and autonomous vehicles [24]. Control strategies adopted in the process of continuous sedimentation taking place in a clarifier-thickener unit, or settler (used, for example, in waste water treatment), usually consist in modulating the inflow and outflow of the settler containing solid particles dispersed in a liquid [18, 19].

5.1. LWR traffic flow models. Consider the Lighthill, Whitham [33], and Richards [36] (LWR) model describing the evolution of unidirectional traffic flow along a stretch of road, say parametrized by $x \in [a,b]$, given by the conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0\,, \tag{5.1}$$

where $\rho(t,x)$ denotes the (normalized) traffic density taking values in the interval $[0,2]$, and where $f(\rho) = \rho\, v(\rho)$ is the flux (the so-called fundamental diagram) depending on the average traffic speed $v(\rho)$. We first assume that, according to Greenshields' relationship, $v(\rho) = 2 - \rho$ which leads to the strictly concave flux

$$f_1(\rho) = \rho\,(2-\rho), \qquad \rho \in [0,2]\,. \tag{5.2}$$

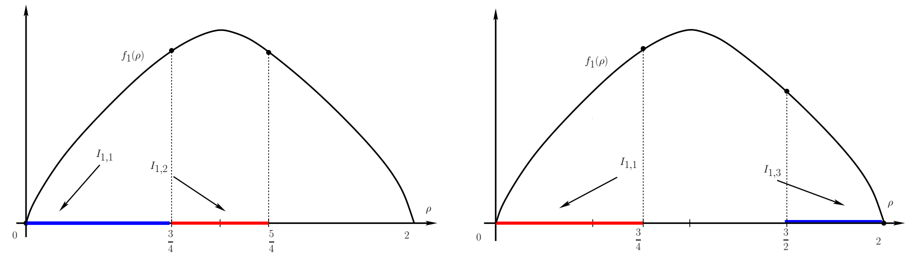

FIG. 3. *Flux* $f_1(\rho) = \rho\,(2-\rho)$.

Then, in connection with the sets $I_{1,1} = [0, \frac{3}{4}]$, $I_{1,2} = [\frac{3}{4}, \frac{5}{4}]$, $I_{1,3} = [\frac{3}{2}, 2]$ (see Figure 3), by a direct computation we find

(5.3)

$$[|f_1|]_{I_{1,1}} = f_1'\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\,, \quad [|f_1|]_{I_{1,2}} = \left|\frac{f_1(\frac{3}{2}) - f_1(\frac{3}{4})}{\frac{3}{4}}\right| = \tfrac{1}{4}\,, \quad [|f_1|]_{I_{1,3}} = \left|f_1'\left(\tfrac{3}{2}\right)\right| = 1\,.$$

On the other hand, we have $f_1''(\rho) = -2$. Hence, invoking Remark 2.9 for (5.1) with $f(\rho) = f_1(\rho)$, we deduce that we can produce a source control $h(t)$ which steers any $\overline{u} \in C^1([a,b])$ to any target profile $\psi \in C^1([a,b])$

- in a time $T > T^*_{1,1} + T^*_{1,2} = 6\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{1,1}$, $\mathrm{Im}(\psi) \subsetneq I_{1,2}$, and

$$\sup_{x\in[a,b]} \lfloor \overline{u}'(x) \rfloor_+ < \frac{1}{4\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor \psi'(x) \rfloor_- < \frac{1}{8\,(b-a)}\,;$$

- in a time $T > T^*_{1,3} + T^*_{1,1} = 3\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{1,3}$, $\mathrm{Im}(\psi) \subsetneq I_{1,1}$, and

$$\sup_{x\in[a,b]} \lfloor \overline{u}'(x) \rfloor_+ < \frac{1}{2\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor \psi'(x) \rfloor_- < \frac{1}{4\,(b-a)}\,.$$

Observe that in the first case we are controlling a state $\overline{u}$ to a target state ψ with possibly vanishing characteristics since $f_1'(1) = 0$ and $1 \in I_{1,2}$. Notice that the choice of the intervals $I_{1,1}, I_{1,2}, I_{1,3}$ is made only to simplify the computation, but one can derive similar results for any pair of intervals $I_1', I_2' \subsetneq [0,2]$ such that $[|f_1|]_{I_1'} > 0$, $[|f_1|]_{I_2'} > 0$ by first solving the optimization problem related to the definition (2.2) of $[|f_1|]_{I_i'}$, $i = 1,2$, and then carrying out similar computations as above. On the other hand, relying on Remark 2.14 we can produce a source control $h(t)$ which steers any $\overline{u} \in BV([a,b])$ to any target profile $\psi \in BV([a,b])$

- in a time $T > T^*_{1,1} + T^*_{1,2} = 6\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{1,1}$, $\mathrm{Im}(\psi) \subsetneq I_{1,2}$, and

$$\sup_{x\in[a,b]} \lfloor D^+\overline{u}(x) \rfloor_+ < \frac{1}{4\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor D^-\psi(x) \rfloor_- < \frac{1}{8\,(b-a)}\,;$$

- in a time $T > T^*_{1,3} + T^*_{1,1} = 3\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{1,3}$, $\mathrm{Im}(\psi) \subsetneq I_{1,1}$, and

$$\sup_{x\in[a,b]} \lfloor D^+\overline{u}(x) \rfloor_+ < \frac{1}{2\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor D^-\psi(x) \rfloor_- < \frac{1}{4\,(b-a)}\,.$$

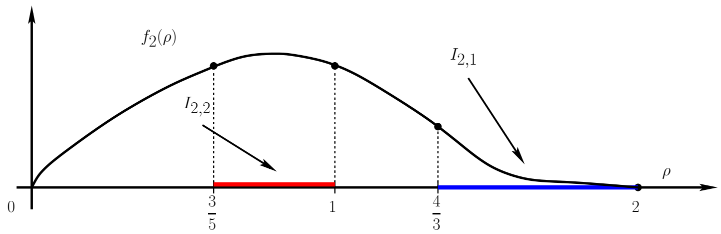

FIG. 4. *Flux* $f_2(\rho) = \rho\, e^{-\dfrac{\rho}{(2-\rho)}}$.

Next, we assume that the traffic speed has the expression $v(\rho) = e^{-\frac{\rho}{(2-\rho)}}$ according to Bonzani and Mussone's model [10], which leads to the (nonconcave) bell-shaped flux

$$f_2(\rho) = \rho\, e^{-\dfrac{\rho}{(2-\rho)}}, \qquad \rho \in [0,2]\,. \tag{5.4}$$

Then, in connection with the set $I_{2,1} = [\frac{4}{3}, 2]$, $I_{2,2} = [\frac{3}{5}, 1]$ (see Figure 4), by a direct computation we find

(5.5)
$$[|f_2|]_{I_{2,1}} \approx \left|\frac{f_2(\frac{4}{3}) - f_2(\frac{4}{3} - 0.717)}{0.717}\right| \approx 0.298 \qquad [|f_2|]_{I_{2,2}} = \left|f_2(\tfrac{8}{5}) - f_2(\tfrac{3}{5})\right| \approx 0.361\,.$$

On the other hand, we have $\|f_2''\|_{C^0([0,2])} = f_2''(\frac{11+\sqrt{13}}{9}) \approx 2.323$ and $\frac{[|f_2|]_{I_{2,1}}}{\|f_2''\|_{C^0([0,2])}} \approx 0.128$, $\frac{[|f_2|]_{I_{2,2}}}{\|f_2''\|_{C^0([0,2])}} \approx 0.155$. Hence, invoking Remark 2.9 for (5.1) with $f(\rho) = f_2(\rho)$, we can produce a source control $h(t)$ which steers any $\overline{u} \in C^1([a,b])$ to any target profile $\psi \in C^1([a,b])$

- in a time $T > T^*_{2,1} + T^*_{2,2} \approx 6.125\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{2,1}$, $\mathrm{Im}(\psi) \subsetneq I_{2,2}$, and

$$\sup_{x\in[a,b]} \lfloor \overline{u}'(x)\rfloor_+ < \frac{0.128}{(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor \psi'(x)\rfloor_- < \frac{0.155}{(b-a)}\,.$$

Similarly, relying on Remark 2.14, we can produce a source control $h(t)$ which steers any $\overline{u} \in BV([a,b])$ to any target profile $\psi \in BV([a,b])$

- in a time $T > T^*_{2,1} + T^*_{2,2} \approx 6.125\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{2,1}$, $\mathrm{Im}(\psi) \subsetneq I_{2,2}$, and

$$\sup_{x\in[a,b]} \lfloor D^+\overline{u}(x)\rfloor_+ < \frac{0.128}{(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor D^-\psi(x)\rfloor_- < \frac{0.155}{(b-a)}\,.$$

Again, we observe that these results guarantee the controllability of possibly critical states since $f_2'(3-\sqrt{5}) = 0$ and $3-\sqrt{5} \in I_{2,2}$.

5.2. Kynch's sedimentation model. According to the solid-flux theory by Kynch [27], the sedimentation of a suspension of small particles dispersed in a viscous fluid can be described by a conservation law,

$$\partial_t u + \partial_x f(u) = 0\,, \tag{5.6}$$

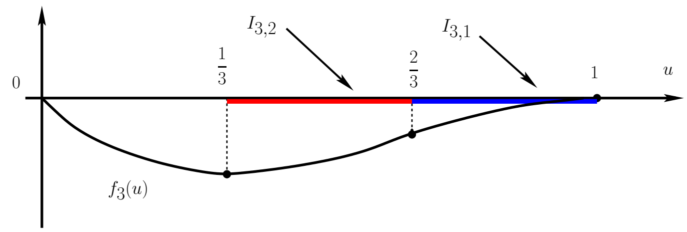

FIG. 5. *Flux* $f_3(u) = -u\,(1-u)^2$.

where $u(t,x)$ denotes the solid fraction, taking values in the interval $[0,1]$, and where the flux function (also called drift-flux) has the same type of expression of the LWR flux, i.e., $f(u) = u\,v(u)$ with $v(u)$ denoting the local settling velocity of the particles. Typically f is a concave-convex map with one inflection point. Here we consider the sedimentation of a solid substance suspended in a cylindrical batch of height L, parametrized so that the bottom is located at $x = 0$ and the top at $x = L$, with the drift-flux function proposed in [34] which, up to normalization, in this case can be written as

$$f_3(u) = -u\,(1-u)^2, \qquad u \in [0,1]\,. \tag{5.7}$$

Then, in connection with the set $I_{3,1} = [\frac{2}{3}, 1]$, $I_{3,2} = [\frac{1}{3}, \frac{2}{3}]$ (see Figure 5), by a direct computation we find

$$[|f_3|]_{I_{3,1}} = [|f_3|]_{I_{3,2}} = \left|\frac{f_3(1) - f_3(\frac{2}{3})}{\frac{1}{3}}\right| = \left|\frac{f_3(\frac{2}{3}) - f_3(\frac{1}{3})}{\frac{1}{3}}\right| = \frac{2}{9} \approx 0.222\,. \tag{5.8}$$

On the other hand, we have $\|f_3''\|_{C^0([0,1])} = |f_3''(0)| = 4$, and thus $\frac{[|f_3|]_{I_{3,1}}}{\|f_3''\|_{C^0([0,1])}} = \frac{[|f_3|]_{I_{3,2}}}{\|f_3''\|_{C^0([0,1])}} \approx 0.055$. Hence, invoking Remark 2.9 for (5.6) with $f(u) = f_3(u)$, we can produce a source control $h(t)$ which steers any $\overline{u} \in C^1([a,b])$ to any target profile $\psi \in C^1([a,b])$

- in a time $T > T^*_{3,1} + T^*_{3,2} = 9\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{3,1}$, $\mathrm{Im}(\psi) \subsetneq I_{3,2}$, and

$$\sup_{x\in[a,b]} \lfloor \overline{u}'(x) \rfloor_+ < \frac{1}{2\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor \psi'(x) \rfloor_- < \frac{1}{2\,(b-a)}\,.$$

Similarly, relying on Remark 2.14, we can produce a source control $h(t)$ which steers any $\overline{u} \in BV([a,b])$ to any target profile $\psi \in BV([a,b])$

- in a time $T > T^*_3 = 9\,(b-a)$, provided that $\mathrm{Im}(\overline{u}) \subsetneq I_{3,1}$, $\mathrm{Im}(\psi) \subsetneq I_{3,2}$, and

$$\sup_{x\in[a,b]} \lfloor D^+\overline{u}(x) \rfloor_+ < \frac{1}{2\,(b-a)}\,, \qquad \sup_{x\in[a,b]} \lfloor D^-\psi(x) \rfloor_- < \frac{1}{2\,(b-a)}\,.$$

Again, we observe that these results guarantee the controllability of possibly critical states since $f_3'(\frac{1}{3}) = 0$ and $\frac{1}{3} \in I_{3,1}$.

6. Appendix. The approximation of the BV function satisfying a one-sided Lipschitz condition in terms of smooth functions satisfying the same Lipschitz condition (used in the proof of Theorem 2.11) is guaranteed by the following lemma. The result is standard, but we provide a proof for completeness.

LEMMA 6.1. *Let* $\varphi \in BV([a,b])$, *with* $Im(\varphi) \subsetneq I$, *satisfy*

$$D^+\varphi(x) < M \qquad \text{for all} \ \ x \in [a,b]\,, \tag{6.1}$$

for some $M > 0$. *Then, there exists* $\{\varphi_n\}_{n\geq 1} \subset C^1([a,b])$ *with* $Im(\varphi_n) \subseteq I$, *for all* n *sufficiently large, and satisfying*

$$\varphi_n'(x) < M \qquad \text{for all} \ \ x \in [a,b]\,, \quad \text{for all} \ \ n \geq 1\,, \tag{6.2}$$

such that

$$\varphi_n \to \varphi \quad \text{in} \quad \mathbf{L^1}([a,b])\,. \tag{6.3}$$

Proof. Observe that, because of (6.1), the map $x \mapsto \psi(x) = \varphi(x) - Mx$ is strictly decreasing on $[a,b]$. Let $\rho_n \in C_c^\infty(\mathbb{R})$, $n > 0$, be a standard mollifier, i.e.,

$$\rho_n \ \geq \ 0, \qquad \sup(\rho_n) \ \subseteq \ (-\tfrac{1}{n}, \tfrac{1}{n}), \qquad \text{and} \qquad \int_{\mathbb{R}} \rho_n(x)dx \ = \ 1.$$

Then, we have that $\psi_n = \rho_n * \psi \in C^\infty([a,b])$, with $Im(\psi_n) \subsetneq I$ for all n sufficiently large, and

$$\psi_n \to \psi \quad \text{in} \quad \mathbf{L^1}([a,b])\,.$$

Moreover, for every $x_1 < x_2$, there holds

$$\psi_n(x_2) - \psi_n(x_1) \ = \ \int [\psi(x_2 - y) - \psi(x_1 - y)] \cdot \rho_n(y)dy \ < \ 0.$$

Thus, one has $D^+\psi_n(x) < 0$ for all n, and the sequence $\varphi_n = \psi_n + Mx$ does the job. ∎

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