Variations of some star selection properties

Cite as: AIP Conference Proceedings **2334**, 020006 (2021); https://doi.org/10.1063/5.0042301
Published Online: 02 March 2021

Ljubiša D. R. Kočinac, Şükran Konca and Sumit Singh

View Online

Export Citation

ARTICLES YOU MAY BE INTERESTED IN

On G-continuity in neutrosophic topological spaces
AIP Conference Proceedings **2334**, 020001 (2021); https://doi.org/10.1063/5.0042369

G-compactness and local G-compactness of topological groups with operations
AIP Conference Proceedings **2334**, 020007 (2021); https://doi.org/10.1063/5.0042236

Preface: Fourth International Conference of Mathematical Sciences (ICMS 2020) Maltepe University, Istanbul-Turkey
AIP Conference Proceedings **2334**, 010001 (2021); https://doi.org/10.1063/12.0003278

© 2021 Author(s).

Variations of Some Star Selection Properties

Ljubiša D.R. Kočinac[a)], Şükran Konca[b)] and Sumit Singh[c)]

[a)] *University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia*
[b)] *Izmir Bakircay University, Faculty of Engineering and Architecture, Department of Fundamental Sciences/Mathematics, 35665 Izmir, Turkey*
[c)] *University of Delhi, Department of Mathematics, 110007 New Delhi, India*

[a)]lkocinac@gmail.com
[b)]sukran.konca@bakircay.edu.tr
[c)]sumitkumar405@gmail.com

Abstract. In this paper we introduce some new types of star covering properties which we call set (strongly) star Menger, set (strongly) star Rothberger, set (strongly) star Hurewicz properties. We consider also weaker versions of these properties, discuss their relationships with some other selective covering properties, and study topological properties of introduced classes of spaces.
Keywords: Star selection principles, set-star Menger, set strongly star Menger, set star Hurewicz, set star Rothberger.
PACS: 54D20, 54A05, 54B05, 54C10.

INTRODUCTION

Let X be a topological space, A a subset of X and $\mathcal{P}$ a collection of subsets of X. Clearly, properties of A or of elements of $\mathcal{P}$ depend on the manner in which A or elements of $\mathcal{P}$ are placed in X. Two natural lines of investigation arise:

(I) To each topological property π assign a relative π of A in X (and say A is π in X);

(II) To each topological property π assign a property $\mathcal{P}$-π which shows how each element $A \in \mathcal{P}$ is located in X.

A.V. Arhangel'skii (with collaborators) was the first who applied this kind of investigation (see [1, 2] in connection with (I) and [3] for (I) and (II)).

The same scenario was applied to selection principles theory, one of the most dynamic areas of research in topology in the last 25-30 years (see, [4, 9, 15] for (I) and [6, 16, 17] for (II)).

In this paper we follow the line (II) of investigation considering some properties related to the Menger [19], Rothberger [20] and Hurewicz [10] covering properties and their star versions.

We use the usual topological notation and terminology as in [8]. Throughout the paper, unless otherwise stated, no separation axioms are assumed. The set on natural numbers is denoted by $\mathbb{N}$, the first uncountable ordinal by ω_1. If $\mathcal{F}$ is a family of subsets of a space X, $A \subset X$, $x \in X$, then

$$\mathrm{St}(A, \mathcal{F}) = \cup\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$$

is the *star* of A with respect to $\mathcal{F}$; $\mathrm{St}(\{x\}, \mathcal{F})$ is denoted by $\mathrm{St}(x, \mathcal{F})$.

Recall the following definitions from [11] (see also [12, 13]).

A space X is said to be:

1. *star Menger* SM (respectively, *star Rothberger* SR) if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(U_n)_{n\in\mathbb{N}}$) such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (respectively, $U_n \in \mathcal{U}_n$) for each $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ (respectively, $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathcal{U}_n)$).

2. *strongly star Menger* SSM (respectively, *strongly star Rothberger* SSR) if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open overs of X there is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of X (respectively, a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X) such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U}_n)$ (respectively, $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(x_n, \mathcal{U}_n)$).

Fourth International Conference of Mathematical Sciences (ICMS 2020)
AIP Conf. Proc. 2334, 020006-1–020006-5; https://doi.org/10.1063/5.0042301
Published by AIP Publishing. 978-0-7354-4078-4/$30.00

3. *star Hurewicz* SH (respectively, *strongly star Hurewicz* SSH) if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(F_n)_{n\in\mathbb{N}}$) such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (respectively, F_n is a finite subset of X) for each $n \in \mathbb{N}$, and each $x \in A$ belongs to all but finitely many sets $\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ (respectively, to all but finitely many sets $\mathrm{St}(F_n, \mathcal{U}_n)$).

More details on star selection principles the reader can find in [5, 11, 21, 22, 24, 25] and the survey papers [13, 14].

In [17], the so-called set versions of these star covering properties (in a very general form) were presented. Here, we continue the investigation of those properties.

New definitions

In this section we introduce new selection properties related to star versions of the Menger, Rothberger and Hurewicz covering properties. As we mention in Introduction the first presentation of these properties was done in [17].

Definition 1 Let $\mathcal{P}$ be a family of nonempty subsets of a space X. We say that X is:

(1) *$\mathcal{P}$-star Menger* (respectively, *weakly $\mathcal{P}$-star Menger*, *almost $\mathcal{P}$-star Menger*, *faintly $\mathcal{P}$-star Menger*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of covers of $\overline{A}$ by sets open in X, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$, and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{\bigcup \mathcal{V}_n}, \mathcal{U}_n)$);

(2) *$\mathcal{P}$-strongly star Menger* (respectively, *weakly $\mathcal{P}$-strongly star Menger*, *almost $\mathcal{P}$-strongly star Menger*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of covers of $\overline{A}$ by sets open in X, there is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of $\overline{A}$ such that $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(F_n, \mathcal{U}_n)}$).

(3) *$\mathcal{P}$-star Rothberger* (respectively, *weakly $\mathcal{P}$-star Rothberger*, *almost $\mathcal{P}$-star Rothberger*, *faintly $\mathcal{P}$-star Rothberger*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \bigcup \mathcal{U}_n$, there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathcal{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(U_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{U_n}, \mathcal{U}_n)$);

(4) *$\mathcal{P}$-strongly star Rothberger* (respectively, *weakly $\mathcal{P}$-strongly star Rothberger*, *almost $\mathcal{P}$-strongly star Rothberger*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \bigcup \mathcal{U}_n$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\overline{A}$ such that $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(x_n, \mathcal{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(x_n, \mathcal{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(x_n, \mathcal{U}_n)}$);

(5) *$\mathcal{P}$-star Hurewicz* (respectively, *almost $\mathcal{P}$-star Hurewicz*, *faintly $\mathcal{P}$-star Hurewicz*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \bigcup \mathcal{U}_n$, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and each $x \in A$ belongs to all but finitely many sets $\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ (respectively, to all but finitely many sets $\overline{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)}$, to all but finitely many $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{\bigcup \mathcal{V}_n}, \mathcal{U}_n)$).

(6) *$\mathcal{P}$-strongly star Hurewicz* (respectively, *almost $\mathcal{P}$-strongly star Hurewicz*) if for each $A \in \mathcal{P}$ and each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \bigcup \mathcal{U}_n$, there is a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of finite subsets of $\overline{A}$ such that each $x \in A$ belongs to all but finitely many sets $\mathrm{St}(F_n, \mathcal{U}_n)$ (respectively, to all but finitely many sets $\overline{\mathrm{St}(F_n, \mathcal{U}_n)}$).

If $\mathcal{P}$ is the family of *all* nonempty subsets of X, then we say that X is *set star Menger* (set-SM) (*set star Rothberger* (set-SR), *set star Hurewicz* (set-SH). Similar terminology is used for other classes defined above.

Results

In this section we mainly consider set versions of star selection principles. Only in the beginning of the section we give a few facts about $\mathcal{P}$-set versions.

Evidently, if a space X belongs to $\mathcal{P}$ and X is $\mathcal{P}$-set (strongly) star Menger, then X is (strongly) star Menger. In fact, we have the following diagram (where set-SC and set SSC are abbreviations for set starcompact and set strongly starcompact [7, 18], respectively) for the Menger-type properties.

Observe, that there are examples showing that none of the implications in this diagram is reversible.

$$\begin{array}{ccccc} & & \text{set-SSC} & \to & \text{set-SC} \\ & & \downarrow & & \downarrow \\ \text{Menger} & \to & \text{set-SSM} & \to & \text{set-SM} \\ & & \downarrow & & \downarrow \\ & & \text{SSM} & \to & \text{SM} \end{array}$$

Diagram 1

For example, the ordinal space $[0, \omega_1)$ with the order topology is a set strongly star Menger space (hence set star Menger) which is not Menger, because it is not Lindelöf [8].

The subspace $Y = \{\alpha + 1 : \alpha \text{ is a limit ordinal }\}$ of the $[0, \omega_1)$. This subspace is not set star Menger (hence not set strongly star Menger). Therefore, the properties set-SM and set-SSM are not hereditary.

Recall that a space X is said to be *extremally disconnected* if the closure of any open set in X is open.

Let $\mathcal{P}_O$ denote the family of all nonempty open subsets of a space X.

Example 1 (1) The Stone-Čech compactification βD of a discrete space D is a $\mathcal{P}_O$-set strongly star Hurewicz (and thus $\mathcal{P}_O$-strongly set star Menger) space.

Indeed, let A be an open subset of βD and $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $\overline{A}$ by sets open in βD. It is well known that βD is an extremally disconnected space, so that $\overline{A}$ is a clopen subset of βD. On the other hand, βD is a strongly star Hurewicz space (being compact). Since the strong star Hurewicz property is hereditary by clopen subspaces (see, for example, [22]), $\overline{A}$ is strongly star Hurewicz. So, there are finite sets $F_1, F_2, F_3, \ldots$ in $\overline{A}$ such that each $a \in \overline{A}$, hence each $a \in A$, belongs to all but finitely many sets $\mathrm{St}(F_n, \mathcal{U}_n)$.

(2) An infinite set X with the cofinite topology (is extremally disconnected and star Menger and thus) is $\mathcal{P}_O$-set star Menger.

Note 1 (1) The following general result is true: Every extremally disconnected (strongly) star Menger space X is $\mathcal{P}_O$-set (strongly) star Menger.

(2) If in the previous example the discrete space D is uncountable, then βD is a $\mathcal{P}_O$-set star Menger space, but its subspace D (which is open in βD) is not.

In [6], quasi Menger and quasi Rothberger (also quasi Hurewicz) have been introduced and studied: a space X is said to be *quasi Menger* (respectively, *quasi Rothnerger*) if for any closed set $A \subset X$ and any sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of open sets in X such that $A \subset \cup\mathcal{U}_n$ for each n, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(U_n)_{n\in\mathbb{N}}$) such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (respectively, $U_n \in \mathcal{U}_n$) and $A \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$).

Natural definitions of star versions of these properties are the following: a space X is said to be *quasi star Menger* q-SM (respectively, *quasi star Rothberger* q-SR) if for any closed set $A \subset X$ and any sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of collections of open sets in X such that $A \subset \cup\mathcal{U}_n$ for each n, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(U_n)_{n\in\mathbb{N}}$) such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (respectively, $U_n \in \mathcal{U}_n$) and $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)}$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathcal{U}_n)}$).

The following proposition follows directly from definitions.

Proposition 1 *Set star Mengerness implies quasi star Mengerness, and quasi star Mengerness implies weak set star Mengerness.*

However, weak set star Mengerness implies quasi star Mengerness. Indeed, let F be a closed subset of a weakly set star Menger space X and $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $\overline{F}$ by sets open in X. Since X is weakly set star Menger there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$, and $F \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)}$, i.e. X is quasi star Menger.

Similarly for other considered classes of spaces. Therefore, we have the following diagram for set star properties.

$$
\begin{tikzcd}
\text{set–SH} \arrow[r] \arrow[d] & \text{SH} \arrow[rr] \arrow[d] & & \text{a–set–SH} \arrow[d] \\
\text{set–SM} \arrow[r] & \text{SM} \arrow[r] & \text{w–set–SM} & \text{a–set–SM} \arrow[l] \\
\text{set–SR} \arrow[r] \arrow[u] & \text{SR} \arrow[r] \arrow[u] & \text{w–set–SR} \arrow[u] & \text{a–set–SR} \arrow[l] \arrow[u]
\end{tikzcd}
$$

Diagram 2

In some classes of spaces, properties that we consider here may coincide.

Theorem 1 *The following hold:*

1. *Let X be a regular space. If X is faintly set star Menger* (*faintly set star Rothberger*)*, then X is set star Menger* (*set star Rothberger*)*.*
2. *Every set strongly star Menger hereditarily metacompact space X is a Menger space.*
3. *If a P-space X is weakly set star Menger, then X is almost set star Menger.*
4. *In paracompact spaces set star Mengerness, set strong star Mengerness and Mengerness are equivalent.*

Some properties of spaces which are the subject of this article are listed below.

Theorem 2 (1) *If (A, τ_A) is a clopen subset of a weakly set star Menger* (*weakly set star Rothberger*) *space (X, τ), then (A, τ_A) is also weakly set star Menger* (*weakly set star Rothberger*)*.*

(2) *A continuous image of a set star Menger* (*set star Rothberger, set strongly star Menger*) *space is also set star Menger* (*set star Rothberger, set strongly star Menger*)*.*

(3) *A continuous image of a weakly* (*almost, faintly*) *set star Menger space is also weakly* (*almost, faintly*) *set star Menger.*

Example 2 1. There are set star Menger spaces X and Y such that the product $X \times Y$ is not set star Menger.

2. There are a set star Menger space X and a Lindelöf space Y whose product is not set star Menger.

REFERENCES

[1] A.V. Arhangelskii and H.M.M. Genedi, Beginnings of the theory of relative topological properties, in: *General Topology. Spaces and Mappings* (MGU, Moscow, 3–48 (1989) (in Russian).

[2] A.V. Arhangelskii, Relative topological properties and relative topological spaces, *Topology Appl.* 70:2-3, 87–99 (1996).

[3] A.V. Arhangel'skii, A generic theorem in the theory of cardinal invariants of topological spaces, *Comment. Math. Univ. Carolinae* 36:2, 3056–327 (1995).

[4] L. Babinkostova, Lj.D.R. Kočinac and M. Scheepers, Combinatorics of open covers (VIII), *Topology Appl.* 140:1, 15–32 (2004).

[5] J. Casas-de la Rosa, S.A. Garcia-Balan and P.J. Szeptycki, Some star and strongly star selection principles, *Topology Appl.* 258, 572–587 (2019).

[6] G. Di Maio, Lj.D.R. Kočinac, A note on quasi-Menger and similar spaces, *Topology Appl.* 179, 148–155 (2015).

[7] E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree, Star covering properties, *Topology Appl.* 39:1, 71–103 (1991).

[8] R. Engelking, General Topology, Revised and completed ed., Heldermann Verlag, Berlin, 1989.

[9] C. Guido and Lj.D.R. Kočinac, Relative covering properties, *Quest. Answ. Gen. Topol.* 19:1, 107–114 (2001).

[10] W. Hurewicz, Uber eine Verallgemeinerung des Borelschen Theorems, *Math. Z.* 24, 401–421 (1925).

[11] Lj.D.R. Kočinac, Star Menger and related spaces, *Publ. Math. Debrecen* 55:3-4, 421–431 (1999).

[12] Lj.D.R. Kočinac, Star Menger and related spaces II, *Filomat* 13, 129–140 (1999).

[13] Lj.D.R. Kočinac, Star selection principles: A survey, *Khayyam J. Math.* 1:1, 82–106 (2015).

[14] Lj.D.R. Kočinac, Variations of classical selelction principles: an overview, *Quaest. Math.* (2020), in press.

[15] Lj.D.R. Kočinac, C. Guido and L. Babinkostova, On relative γ-sets, *East West J. Math.* 2:2, 195–199 (2000).

[16] Lj.D.R. Kočinac and Ş. Konca, Set-Menger and related properties, *Topology Appl.* 275, Art. No. 106996 (2020).

[17] Ş. Konca and Lj.D.R. Kočinac, Set-star Menger and related spaces, *Abstract Book VI ICRAPAM* (Istanbul, Turkey, June 12–15), p. 49 (2019).
[18] M.V. Matveev, A survey on star covering properties, *Topology Atlas*, Preprint No. 330 (1998).
[19] K. Menger, Einige Überdeckungssätze der Punktmengenlehre, *Sitzungsberischte Abt. 2a, Math., Astron., Phys., Meteor. Mech.* (Wiener Akademie, Wien) 133, 421–444 (1924).
[20] F. Rothberger, Eine Verscharfung der Eigenschaft C, *Fund. Math.* 30, 50–55 (1938).
[21] Y.K. Song, Remarks on star-Hurewicz spaces, *Bull. Polish Asad. Sci. Math* 61:3-4, 247–255 (2013).
[22] Y.K. Song, Remarks on strongly star-Hurewicz spaces, *Filomat* 27:6, 1127–1131 (2013).
[23] Y.K. Song, R. Li, On almost Hurewicz spaces, *Quest. Answ. Gen. Topol.* 31, 131–136 (2013).
[24] Y.-K. Song, Remarks on star-Menger spaces, *Houston J. Math.* 40:3, 917–925 (2014).
[25] Y.-K. Song, Remarks on star-Menger spaces II, *Houston J. Math.* 41:1, 355–364 (2015).