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SHARP L^p-L^q ESTIMATE FOR THE SPECTRAL PROJECTION ASSOCIATED WITH THE TWISTED LAPLACIAN

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Abstract: In this note we are concerned with estimates for the spectral projection operator $\mathcal{P}_\mu$ associated with the twisted Laplacian L. We completely characterize the optimal bounds on the operator norm of $\mathcal{P}_\mu$ from L^p to L^q when $1 \le p \le 2 \le q \le \infty$. As an application, we obtain a uniform resolvent estimate for L.

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1. Introduction

We consider the twisted Laplacian L on $\mathbb{C}^d \cong \mathbb{R}^{2d}$, $d \ge 1$, which is defined by

$$L = -\sum_{j=1}^{d} \Big(\Big(\frac{\partial}{\partial x_j} - \frac{1}{2} i y_j\Big)^2 + \Big(\frac{\partial}{\partial y_j} + \frac{1}{2} i x_j\Big)^2 \Big).$$

It is well known that L has a discrete spectrum which consists of the points $2k + d$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any multi-index $\alpha, \beta \in \mathbb{N}_0^d$, the special Hermite function $\Phi_{\alpha,\beta}$ is given by

$$\Phi_{\alpha,\beta}(z) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \Phi_\alpha\Big(\xi + \frac{1}{2}y\Big) \Phi_\beta\Big(\xi - \frac{1}{2}y\Big)\, d\xi, \quad z = x + iy,$$

which are the Fourier–Wigner transform of the Hermite functions Φ_α and Φ_β on $\mathbb{R}^d$. It is easy to see that $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^d\}$ forms a complete orthonormal system in $L^2(\mathbb{C}^d)$. Also, $L\Phi_{\alpha,\beta} = (2|\beta| + d)\Phi_{\alpha,\beta}$, which means that $\Phi_{\alpha,\beta}$ is an eigenfunction of L with eigenvalue $2|\beta| + d$, hence the eigenspace of L is infinite-dimensional. Here $|\beta| = \beta_1 + \cdots + \beta_d$. A simple calculation shows that $\Phi_{\alpha,\beta}$ is also an eigenfunction of the Hermite operator $-\Delta_z + \frac{1}{4}|z|^2$ with eigenvalue $|\alpha| + |\beta| + d$. So, the functions $\Phi_{\alpha,\beta}$ are called the special Hermite functions. For more about the twisted Laplacian and the special Hermite functions, we refer the reader to the monograph by Thangavelu [**23**].

The spectral projection operator $\mathcal{P}_\mu$ onto the eigenspace of L associated with the eigenvalue $\mu = 2k + d \in 2\mathbb{N}_0 + d$ is given by

$$\mathcal{P}_\mu f = \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta : 2|\beta| + d = \mu} \langle f, \Phi_{\alpha,\beta} \rangle \Phi_{\alpha,\beta}, \quad f \in \mathcal{S}(\mathbb{R}^{2d}). \tag{1.1}$$

Thus, it follows that $f = \sum_{\mu \in 2\mathbb{N}_0 + d} \mathcal{P}_\mu f$. It is known ([**23**]) that $\mathcal{P}_\mu$ is also expressed by the twisted convolution

$$\mathcal{P}_\mu f = (2\pi)^{-d} f \times \varsigma_k, \quad \mu = 2k + d, \tag{1.2}$$

where $\varsigma_k(z) = \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z) = L_k^{d-1}\big(\frac{1}{2}|z|^2\big) e^{-\frac{1}{4}|z|^2}$ and L_k^α is the Laguerre polynomial of type α. Here, the twisted convolution $f \times g$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^d} f(z-w) g(w) e^{i\frac{1}{2}\operatorname{Im} z \cdot \overline{w}}\, dw,$$

where $z \cdot w = z_1 w_1 + \cdots + z_d w_d$ for any $z, w \in \mathbb{C}^d$.

The estimates for $\mathcal{P}_\mu$ have been of interest in relation to L^p convergence of the Bochner–Riesz means $S_R^\alpha(L)$ associated with the special Hermite expansion, which is given by $S_R^\alpha(L) f := \sum_{\mu \le R} (1 - \mu/R)^\alpha \mathcal{P}_\mu f$ (see, for example, [**23**]). In particular, L^2-L^q estimates for $\mathcal{P}_\mu$ (equivalently, $L^{q'}$-L^2 estimates for $\mathcal{P}_\mu$) were studied by Thangavelu [**22, 23, 24**], Ratnakumar, Rawat, and Thangavelu [**19**], Stempak and Zienkiewicz [**21**], and Koch and Ricci [**13**]. The sharp L^2-L^q bound for $\mathcal{P}_\mu$ is now well understood. More precisely, for $2 \le q \le \infty$,

$$\|\mathcal{P}_\mu\|_{2 \to q} \le C_q \mu^{\varrho(q)} \tag{1.3}$$

holds with the exponent $\varrho(q)$ given by

$$\varrho(q) = \begin{cases} -\frac{1}{2}\big(\frac{1}{2} - \frac{1}{q}\big) & \text{if } 2 \le q \le \frac{2(2d+1)}{2d-1}, \\ \frac{d-1}{2} - \frac{d}{q} & \text{if } \frac{2(2d+1)}{2d-1} \le q \le \infty, \end{cases}$$

and the estimate (1.3) is optimal in that the exponent $\varrho(q)$ cannot be taken to be a smaller one. Here, $\|T\|_{p \to q}$ denotes the usual operator norm from L^p to L^q of a linear operator T defined by

$$\|T\|_{p \to q} = \sup_{f \in \mathcal{S},\, f \ne 0} \|Tf\|_q / \|f\|_p.$$

The estimate (1.3) was shown by Thangavelu [**22, 23**] and, subsequently, Ratnakumar, Rawat, and Thangavelu [**19**] for $q \ge q_\circ$ with some $q_\circ > 2(2d+1)/(2d-1)$. Afterward, Stempak and Zienkiewicz ([**21**]) proved (1.3) for all $q \ge 2$ except $q = 2(2d+1)/(2d-1)$. The remaining endpoint case $q = 2(2d+1)/(2d-1)$ was settled by Koch and Ricci [**13**]; moreover, they showed that the estimate (1.3) is sharp. A local version of the endpoint estimate was obtained earlier by Thangavelu [**24**].

The purpose of this paper is to establish the optimal L^p-L^q estimate for $\mathcal{P}_\mu$ when $1 \le p \le 2 \le q \le \infty$. Our result was inspired by the recent work [**9**] of the authors regarding L^p-L^q estimates for the Hermite spectral projection Π_μ, which is the orthogonal projection onto the eigenspace of the Hermite operator $H = |x|^2 - \Delta$ associated with the eigenvalue μ. In [**9**], a systematic study concerning the bound on $\|\Pi_\mu\|_{p\to q}$ was carried out. The main ingredients were a representation formula for Π_μ and a modification of the TT^*-argument. In particular, the representation formula was obtained by making use of the Schrödinger propagator e^{itH} and the fact that the eigenvalues of H are in $2\mathbb{N}_0 + d$. See [**9**, Section 2]. It turns out that a similar approach works even more efficiently for the projection operator $\mathcal{P}_\mu$ and we obtain a complete characterization of the L^p-L^q bound on $\|\mathcal{P}_\mu\|_{p\to q}$ in terms of μ.

In Theorem 1.2 below, we show that the boundedness property of the spectral projection $\mathcal{P}_\mu$ is similar to that of the operator

$$\wp_k f(x,y) = \frac{1}{(2\pi)^{2d}} \int_{k-1\le|\xi|^2<k} e^{i(x,y)\cdot\xi} \widehat{f}(\xi)\, d\xi, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

which is the spectral projection operator associated with the Laplacian $-\Delta$ in $\mathbb{R}^{2d}$. For a discussion regarding the sharp L^p-L^q bounds for the operator $\wp_k$, we refer to [**9**, Section 3.3]. Compared with the Hermite spectral projection Π_μ, the sharp exponent $\varrho(p,q)$ exhibits less involved behavior and we do not have to appeal to the heavy machinery used in [**9**]. Consequently, we obtain the sharp estimates much more easily.

Before stating our result, we need to introduce some notations. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F} \in [1/2,1] \times [0,1/2]$ be the points defined by

$$\mathfrak{A} = \Big(\frac{2d+3}{2(2d+1)}, \frac{1}{2}\Big),\ \mathfrak{B} = \Big(\frac{(2d)^2+8d-1}{4d(2d+1)}, \frac{2d-1}{4d}\Big),\ \mathfrak{C} = \Big(1, \frac{2d-1}{4d}\Big),$$

$$\mathfrak{D} = \Big(\frac{d+1}{2d}, \frac{1}{2}\Big),\ \mathfrak{F} = \Big(\frac{(2d)^2+4d-4}{4d(2d-1)}, \frac{d-1}{2d-1}\Big),\ \mathfrak{G} = \Big(\frac{2d+3}{4d}, \frac{2d-1}{4d}\Big).$$

For a point $(x,y) \in [1/2,1] \times [0,1/2]$, let $(x,y)' = (1-y, 1-x)$ and, similarly, for a set $S \subset [1/2,1] \times [0,1/2]$ we put $S' = \{(x,y)' : (x,y) \in S\}$. Then, we define the set $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3 \subset [1/2,1] \times [0,1/2]$ as follows.

Definition 1.1. Let $\mathcal{R}_1$ denote the closed pentagon with vertices $\big(\frac{1}{2}, \frac{1}{2}\big)$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{A}'$, from which two points $\mathfrak{B}$ and $\mathfrak{B}'$ are removed. Let $\mathcal{R}_2$ be the closed trapezoid with vertices $\mathfrak{A}$, $\big(1, \frac{1}{2}\big)$, $\mathfrak{C}$, $\mathfrak{B}$, from which the closed line segment $[\mathfrak{B},\mathfrak{C}]$ is removed, and $\mathcal{R}_3$ denote the closed pentagon with vertices $\mathfrak{B}$, $\mathfrak{C}$, $(1,0)$, $\mathfrak{C}'$, and $\mathfrak{B}'$, from which the closed line segments $[\mathfrak{B},\mathfrak{C}]$ and $[\mathfrak{B}',\mathfrak{C}']$ are removed. (See Figure 1.)

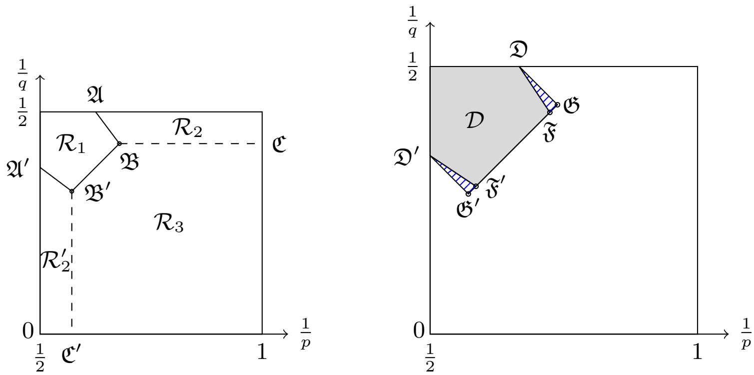

FIGURE 1. The points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{F}$, $\mathfrak{G}$ and the regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{D}$. The point $\mathfrak{B}$ lies on the line segment $[\mathfrak{A},(1,0)]$ connecting $\mathfrak{A}$ and $(1,0)$. Also the midpoint of $[\mathfrak{B},\mathfrak{B}']$ with $\left(\frac{1}{2},\frac{1}{2}\right)$, $\mathfrak{A}$ and $\mathfrak{A}'$ together forms a square.

For $(p,q)\in[1,2]\times[2,\infty]$, we define the exponent $\varrho(p,q)$ by setting

$$\varrho(p,q)=\begin{cases}-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right), & \left(\frac{1}{p},\frac{1}{q}\right)\in\mathcal{R}_1,\\ d\left(\frac{1}{p}+\frac{1}{q}\right)-\frac{2d+1}{2}, & \left(\frac{1}{p},\frac{1}{q}\right)\in\mathcal{R}_2,\\ \frac{2d-1}{2}-d\left(\frac{1}{p}+\frac{1}{q}\right), & \left(\frac{1}{p},\frac{1}{q}\right)\in\mathcal{R}_2',\\ d\left(\frac{1}{p}-\frac{1}{q}\right)-1, & \left(\frac{1}{p},\frac{1}{q}\right)\in\overline{\mathcal{R}_3}.\end{cases}\tag{1.4}$$

We are now ready to state our main result.

Theorem 1.2. *Let $d\ge 1$ and $1\le p\le 2\le q\le\infty$. We have the estimate*

$$\|\mathcal{P}_\mu\|_{p\to q}\le C_{p,q}\mu^{\varrho(p,q)}\tag{1.5}$$

if and only if $(1/p,1/q)\notin[\mathfrak{B},\mathfrak{C}]\cup[\mathfrak{B}',\mathfrak{C}']$. The bounds are sharp in that the exponents $\varrho(p,q)$ cannot be improved. Additionally, we have

(i) *If $(1/p,1/q)\in(\mathfrak{B},\mathfrak{C}]$, we have $\|\mathcal{P}_\mu\|_{L^p\to L^{q,\infty}}\lesssim\mu^{\varrho(p,q)}$.*

(ii) *If $(1/p,1/q)=\mathfrak{B}$ or $\mathfrak{B}'$, we have $\|\mathcal{P}_\mu\|_{L^{p,1}\to L^{q,\infty}}\lesssim\mu^{\varrho(p,q)}$.*

Here $\|\mathcal{P}_\mu\|_{L^{p,r}\to L^{q,s}}$ means the operator norm of $\mathcal{P}_\mu$ from the Lorentz space $L^{p,r}$ to $L^{q,s}$.

The twisted Laplacian is related to the Heisenberg sub-Laplacian [**13, 23, 25**]. The reduced Heisenberg group h_d is the set $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ with group law

$$(x, y, e^{it})(x', y,' e^{it'}) = (x + x', y + y', e^{i(t+t'+\frac{1}{2}(x'\cdot y - x\cdot y'))}),$$

and the sub-Laplacian $\mathcal{L}$ on h_d is defined by

$$\mathcal{L} = -\sum_{j=1}^{d}\Big(\Big(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\Big)^2 + \Big(\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}\Big)^2\Big).$$

The estimate (1.5) can be used to generate the spectral projection estimates for the differential operators acting on a special class of functions on h_d. Especially, on the class of functions of the form $g(x, y, t) = e^{imt}f(x, y)$, $m \in \mathbb{Z}$, we have $\mathcal{L}(e^{imt}f) = e^{imt}L_m f$, where

$$L_m = -\sum_{j=1}^{d}\Big(\Big(\frac{\partial}{\partial x_j} - \frac{m}{2}iy_j\Big)^2 + \Big(\frac{\partial}{\partial y_j} + \frac{m}{2}ix_j\Big)^2\Big).$$

By scaling, it is easy to see that, for each nonzero $m \in \mathbb{Z}$, the numbers $(2k + d)|m|$, $k \in \mathbb{N}_0$, are eigenvalues of L_m with the corresponding eigenfunctions

$$\Phi^m_{\alpha,\beta}(x, y) = |m|^{d/2}\Phi_{\alpha,\beta}(|m|^{1/2}x, \mathrm{sgn}(m)|m|^{1/2}y),$$

which form an orthonormal basis of $L^2(\mathbb{R}^{2d})$. So, the pairs $(|m|(2k + d), m)$, $m \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{N}_0$, give the discrete joint spectrum of $\mathcal{L}$ and $-i\partial_t$. Let $\mathcal{P}_{m,k}$ be the projection onto the joint eigenspace corresponding to the eigenvalue $(|m|(2k + d), m)$ (see [**23, 13**] for further details). Then the spectral projection estimate (1.5) yields

$$\|\mathcal{P}_{m,k}u\|_{L^q(h_d)} \lesssim (2k + d)^{\varrho(p,q)}|m|^{d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(h_d)}.$$

We now consider the estimate for the resolvent of L, which takes the form

$$\|(L - z)^{-1}\|_{p\to q} \le C_{p,q,z}, \quad z \in \mathbb{C} \setminus (2\mathbb{N}_0 + d), \tag{1.6}$$

where $(L - z)^{-1}$ is defined by

$$(L - z)^{-1}f = \sum_{\mu}(\mu - z)^{-1}\mathcal{P}_\mu f. \tag{1.7}$$

Estimates for resolvents have a wide range of applications. In particular, uniform resolvent estimates for partial differential operators which hold with C_z independent of the spectral parameter have been studied in relation to Carleman estimates and strong/weak unique continuation properties (for example, see [**12, 5, 14, 8, 9**] and references therein).

For the closely related Hermite operator H, it was shown in [**5, 14, 9**] that

$$\|(H-z)^{-1}\|_{p\to q} \le C_{p,q} \tag{1.8}$$

under the spectral gap condition

$$\operatorname{dist}(z, 2\mathbb{N}_0 + d) \ge c_\circ \tag{1.9}$$

for some $c_\circ > 0$. Escauriaza and Vega proved (1.8) for $\frac{2d}{d+2} \le p \le 2 \le q \le \frac{2d}{d-2}$ and showed the strong unique continuation property for the differential inequality $|\partial_t u + \Delta u| \le |Vu|$ with $V \in L^\infty_t L^{d/2}_x$. Also, in [**9**], the authors extended the range of p, q for (1.8) to an interval on $1/p - 1/q = 2/d$ and obtained the strong unique continuation property with $V \in L^\infty_t L^{d/2,\infty}_x$.

As an application of the spectral projection estimate we obtain the following uniform resolvent estimate for the twisted Laplacian.

Theorem 1.3. *Let $d \ge 2$ and $c_\circ > 0$. Suppose that $(1/p, 1/q)$ is in the closed pentagon $\mathcal{D}$ with vertices $(1/2, 1/2)$, $\mathfrak{D}$, $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{D}'$, from which $\mathfrak{F}$ and $\mathfrak{F}'$ are removed (the gray region in Figure 1). Then there is a constant $C = C(c_\circ) > 0$ such that*

$$\|(L-z)^{-1}\|_{p\to q} \le C \tag{1.10}$$

provided that $z \in \mathbb{C}$ satisfies (1.9). Furthermore, if $\left(\frac{1}{p}, \frac{1}{q}\right) = \mathfrak{F}$ or $\mathfrak{F}'$, we have the restricted-weak type estimate $\|(L-z)^{-1} f\|_{q,\infty} \le C\|f\|_{p,1}$ provided that (1.9) holds.

For the 1-dimensional case the uniform resolvent estimate (1.10) holds for all $1 < p \le 2 \le q < \infty$ (see Remark 1).

It is natural to expect that, as an application of the uniform resolvent estimates, one may be able to show the strong unique continuation property for the heat equation associated with L as in the previous works but we do not intend to pursue the matter here. When $p = q'$, the estimate (1.10) was previously obtained by Cuenin [**4**, Proposition 2.2] to show clustering estimates for eigenvalues of the twisted Laplacian with L^p potentials. In fact, he obtained the resolvent estimate (1.6) with the bound $(1 + |\operatorname{Re} z|)^{\varrho(q',q)}(1 + \delta(z)^{-1})$, where $\delta(z) := \operatorname{dist}(z, 2\mathbb{N}_0 + d)$.

As in the case of the Hermite resolvent estimate, the gap condition (1.9) is necessary for the uniform estimate (1.10) because the twisted Laplacian has discrete eigenvalues. Indeed, $\|(L-z)^{-1}\|_{p\to q} \ge |\mu - z|^{-1}\|f\|_q/\|f\|_p$ if f is in the eigenspace corresponding to $\mu \in 2\mathbb{N}_0 + d$. Thus, the operator norm goes to infinity as z goes towards μ.

By making use of Theorem 5.1, one can easily show that (1.10) holds only for $1/p - 1/q \leq 1/d$. Thus, $1/p - 1/q = 1/d$ is the critical case for the estimate (1.10), and it is more difficult to obtain the uniform estimate (1.10) for p, q satisfying $1/p - 1/q = 1/d$. As for the critical case, we establish (1.10) for $(1/p, 1/q) \in (\mathfrak{F}, \mathfrak{F}')$ in Theorem 1.3. However, we could not obtain the result in the expected range. More precisely, from (1.7) and the estimate for the fractional twisted Laplacian operator (Theorem 5.1), one may expect that the uniform resolvent estimate (1.10) holds for any p, q for which the uniform spectral projection estimate $\|\mathcal{P}_\mu\|_{p\to q} \lesssim 1$ holds. Let $\widetilde{\mathcal{D}}$ be the pentagon with vertices $(1/2, 1/2)$, $\mathfrak{D}$, $\mathfrak{G}$, $\mathfrak{G}'$, and $\mathfrak{D}'$ from which two points $\mathfrak{G}$, $\mathfrak{G}'$ are removed. Then, the operator $\mathcal{P}_\mu$ is bounded uniformly in μ if $(1/p, 1/q) \in \widetilde{\mathcal{D}}$. The region $\widetilde{\mathcal{D}}$ is the union of the hatched region in Figure 1 and $\mathcal{D}$, which contains $\mathcal{R}_1$.

The rest of the paper is organized as follows. In Section 2 we provide a representation formula for $\mathcal{P}_\mu$ which will be useful to show the sharp L^p-L^q estimate for $\mathcal{P}_\mu$. We separately prove the sufficiency and the necessary parts of Theorem 1.2 in Sections 3 and 4. We provide the proof of the uniform resolvent estimate for L in Section 5.

2. Preliminaries

2.1. Representation formula for $\boldsymbol{\mathcal{P}_\mu}$. The Schrödinger propagator e^{itL} associated with L can be expressed by using the spectral decomposition of L, that is to say,

$$e^{itL} f = \sum_\mu e^{it\mu} \mathcal{P}_\mu f, \quad t \in \mathbb{R}.$$

So, we clearly have

$$\|e^{itL} f\|_2 = \|f\|_2, \quad t \in \mathbb{R}. \tag{2.1}$$

Since the eigenvalues of L are in $2\mathbb{N}_0 + d$, the difference between two eigenvalues μ, μ' of L is in $2\mathbb{Z}$, i.e., $\mu - \mu' \in 2\mathbb{Z}$. As in the case of the Hermite spectral projection, $\mathcal{P}_\mu$ is also written as follows:

$$\mathcal{P}_\mu f(z) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{it\mu} e^{-itL} f(z)\, dt, \quad f \in \mathcal{S}(\mathbb{R}^{2d}). \tag{2.2}$$

Set $z = x + iy$ and $z' = x' + iy' \in \mathbb{C}^d \cong \mathbb{R}^{2d}$. The same idea of exploiting the specific form of the eigenvalues was already used in [**9**]. We note that the Schrödinger propagator e^{-itL} also has the following kernel representation:

$$e^{-itL} f(z) = C_d (\sin t)^{-d} \int e^{i(\frac{|z-z'|^2}{4} \cot t + \frac{1}{2} \operatorname{Im} z \cdot \overline{z}')} f(z')\, dz', \tag{2.3}$$

where C_d is a constant depending only on d. This can be easily deduced from the corresponding kernel formula for the heat operator e^{-tL} by replacing t with it (see [**23**, p. 37]). Since $\sum_\mu \mathcal{P}_\mu f$ converges absolutely and uniformly for $f \in \mathcal{S}(\mathbb{R}^{2d})$ (see (1.1)), we now get

$$\mathcal{P}_\mu f(z) = C_d \int_{\mathbb{C}^d} \int_{-\pi/2}^{\pi/2} (\sin t)^{-d} e^{i(t\mu + \frac{|z-z'|^2}{4} \cot t + \frac{1}{2} \operatorname{Im} z\cdot \bar{z}')} f(z')\, dt\, dz' \tag{2.4}$$

for any $f \in \mathcal{S}(\mathbb{R}^{2d})$.

Since the kernel of $e^{it(\mu-L)}$ has a singularity at $t = 0$, we need to decompose it away from the singularity. For any function $\eta \in C^\infty(\mathbb{R})$, let us define $\mathcal{P}_\mu[\eta]$ by

$$\mathcal{P}_\mu[\eta] f := \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \eta(t) e^{it\mu} e^{-itL} f\, dt. \tag{2.5}$$

Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth function such that ψ is supported in $\left[-\frac{\pi}{16}, \frac{\pi}{16}\right]$ and equals 1 on $\left[-\frac{\pi}{32}, \frac{\pi}{32}\right]$, and set $\psi_j = \psi(2^{j-1}t) - \psi(2^j t)$. Then we have $\sum_{j=1}^\infty \psi_j(t) = \psi(t)$ except $t = 0$. Moreover, we define periodic functions φ^0 and φ_k of period π by setting

$$\varphi_k(t) = \psi_k(t - \pi/2) \quad \text{and} \quad \varphi^0(t) = 1 - \psi(t) - \psi(2^2(t - \pi/2)) \tag{2.6}$$

for $t \in [-\pi/4, 3\pi/4]$. Clearly, φ^0, φ_k are smooth and

$$\sum_{j=1}^\infty \psi_j(t) + \sum_{k=3}^\infty \varphi_k(t) + \varphi^0(t) = 1 \tag{2.7}$$

for $t \in (-\pi/2, \pi/2) \setminus \{0\}$. Here, we break it away from $\pm\pi/2$ as well as 0, because the second derivative of the phase function vanishes at $\pm\pi/2$. Using the partition of unity, we decompose the projection operator as follows:

$$\mathcal{P}_\mu = \sum_{j\ge 1} \mathcal{P}_\mu[\psi_j] + \sum_{k\ge 3} \mathcal{P}_\mu[\varphi_k] + \mathcal{P}_\mu[\varphi^0]. \tag{2.8}$$

Since the eigenvalues of L are in $2\mathbb{N}_0 + d$, as before it is clear from the spectral decomposition that

$$e^{it(L-\mu)} = e^{i(\pi+t)(L-\mu)}, \quad \mu \in 2\mathbb{N}_0 + d. \tag{2.9}$$

Thus, by periodicity it follows that $\mathcal{P}_\mu[\eta] f = \frac{1}{\pi} \int_0^\pi \eta(t) e^{it(L-\mu)} f\, dt$ for any π periodic η. In particular, we have

$$\mathcal{P}_\mu[\varphi_k] = \frac{1}{\pi} \int_0^\pi \varphi_k(t) e^{it(L-\mu)} f\, dt, \quad k = 3, 4, \dots \tag{2.10}$$

2.2. Estimate for oscillatory integral. In this subsection, we obtain some elementary oscillatory integral estimates to obtain the L^1-L^∞ bound for $\mathcal{P}_\mu[\eta]$.

Let the phase function ϕ be defined by

$$\phi(t) := \phi(z, z', t) := t + \frac{|z - z'|^2}{4} \cot t + \frac{1}{2} \operatorname{Im}(z \cdot \overline{z}'), \quad z, z' \in \mathbb{C}^d.$$

By (2.3), (2.5), and scaling, we see that

$$\begin{aligned} \mathcal{P}_\mu[\eta] f(\sqrt{\mu} z) = & \frac{C_d}{\pi} \\ & \times \int_{\mathbb{C}^d} \Big(\int_{-\pi/2}^{\pi/2} (\sin t)^{-d} \eta(t) e^{i\mu\phi(z,z',t)}\, dt \Big) f(\sqrt{\mu} z') \mu^d \, dz'. \end{aligned} \tag{2.11}$$

Since $\|f(\sqrt{\mu}\cdot)\mu^d\|_1 = \|f\|_1$, to obtain the L^1-L^∞ bound for $\mathcal{P}_\mu[\eta]$ we need only to consider the kernel $\int_{-\pi/2}^{\pi/2} (\sin t)^{-d} \eta(t) e^{i\mu\phi(z,z',t)}\, dt$ in the above. We define the oscillatory integrals $\mathcal{I}_j$, $\mathcal{J}_k$, and $\mathcal{J}^0$ by

$$\begin{aligned} \mathcal{I}_j(\mu) &:= \mathcal{I}_j(z, z', \mu) := \int \eta(2^j t) e^{i\mu\phi(z,z',t)}\, dt, \\ \mathcal{J}_k(\mu) &:= \mathcal{J}_k(z, z', \mu) := \int \eta(2^k(t - \pi/2)) e^{i\mu\phi(z,z',t)}\, dt, \\ \mathcal{J}^0(\mu) &:= \mathcal{J}^0(z, z', \mu) := \int_0^\pi \varphi^0(t) e^{i\mu\phi(z,z',t)}\, dt, \end{aligned}$$

for $j, k \in \mathbb{Z}$, $\mu \in \mathbb{R}$, and $z, z' \in \mathbb{C}^d$, where η is a function supported in $[-\pi/8, -\pi/32] \cup [\pi/32, \pi/8]$.

In what follows we show the estimates for $\mathcal{I}_j(\mu)$, $\mathcal{J}_k(\mu)$, and $\mathcal{J}^0(\mu)$, which are crucial for obtaining the sharp estimates for $\mathcal{P}_\mu$.

Lemma 2.1. *Let $d \geq 1$, $j, k \geq 1$. Let η be a C^1-function supported in $[-\pi/8, -\pi/32] \cup [\pi/32, \pi/8]$. Then we have*

$$|\mathcal{I}_j(z, z', \mu)| \leq C\mu^{-1/2} 2^{-j/2} \|\eta\|_{C^1}, \tag{2.12}$$

$$|\mathcal{J}_k(z, z', \mu)| \leq C\mu^{-1/2} 2^{k/2} \|\eta\|_{C^1}, \tag{2.13}$$

$$|\mathcal{J}^0(z, z', \mu)| \leq C\mu^{-1/2}, \tag{2.14}$$

with C independent of $z, z' \in \mathbb{C}^d$, j, k, and $\mu > 1$.

Proof: To show (2.12)–(2.14), we make use of the well-known van der Corput lemma (see for example [**20**, p. 334]). We consider the time derivative of the phase function ϕ of the integrals $\mathcal{I}_j$, $\mathcal{J}_k$, and $\mathcal{J}^0$. A simple computation shows that

$$\phi'(t) = \frac{4 \sin^2 t - |z - z'|^2}{4 \sin^2 t}. \tag{2.15}$$

We first show (2.12) for $j \ge 1$. If $|z-z'| \ge 2^{-j+1}$ or $|z-z'| \le 2^{-j-3}$, there is no critical point of ϕ on $\operatorname{supp} \eta(2^j \cdot)$ because η is supported in $[-\pi/2, -\pi/8] \cup [\pi/8, \pi/2]$. So, it is easy to see that

$$|\phi'(t)| \gtrsim 2^{2j} |(2\sin t - |z-z'|)(2\sin t + |z-z'|)| \gtrsim 1$$

on the support of $\eta(2^j \cdot)$. Thus, applying van der Corput's lemma yields

$$|\mathcal{I}_j(\mu)| \lesssim \min\{\mu^{-1}, 2^{-j}\} \lesssim \mu^{-1/2} 2^{-j/2}.$$

To complete the proof of (2.12) we only need to consider the case

$$|z-z'| \sim 2^{-j}.$$

Let us note that

$$\phi''(t) = \cos t |z-z'|^2 / 2\sin^3 t \tag{2.16}$$

and $|\phi''| \gtrsim 2^j$ on the support of $\eta(2^j \cdot)$. Applying van der Corput's lemma again, we have (2.12). This completes the proof of (2.12).

Next we show the estimate (2.13) for $k \ge 1$. If $|z-z'| \le 1/2$, we have $|\phi'(t)| \gtrsim 1$ on the support of $\eta(2^k(\cdot - \pi/2))$ because $2|\sin t| \ge 1$. Thus $|\mathcal{J}_k(\mu)| \lesssim \mu^{-1} \le \mu^{-1/2} 2^{k/2}$. We may now assume $|z-z'| > 1/2$. Using (2.16), we have

$$|\phi''(t)| \gtrsim |\cos t| = |\cos t - \cos(\pi/2)| \gtrsim 2^{-k}$$

on $\operatorname{supp} \eta(2^k(\cdot - \pi/2))$. Thus, van der Corput's lemma gives the desired result (2.13).

Finally, noting that $\operatorname{dist}(\operatorname{supp} \varphi^0, \{0, \pi/2, \pi\}) \ge c$ for some $c > 0$ because of (2.6), we see that $|\phi'(t)| \gtrsim 1$ if $|z-z'| \le 1/2$ and $|\phi''(t)| \gtrsim 1$ if $|z-z'| \ge 1/2$ on the support of φ^0. Hence, the estimate (2.14) follows from the van der Corput lemma. □

We frequently make use of the following summation trick to handle the endpoint cases [**1, 3**].

Lemma 2.2 ([**9**, Lemma 2.4])**.** *Let $1 \le p_l, q_l \le \infty$ and $\epsilon_l > 0$ for $l = 0, 1$, and set $\theta = \frac{\epsilon_0}{\epsilon_0 + \epsilon_1}$, $\frac{1}{p_*} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$, and $\frac{1}{q_*} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0}$. Suppose that T_j, $j \in \mathbb{Z}$, are sublinear operators defined from $L^{p_l} \to L^{q_l}$ with*

$$\|T_j\|_{p_l \to q_l} \le B_l 2^{j(-1)^l \epsilon_l}, \quad l = 0, 1.$$

Then we have the following.

(i) *If $p_0 = p_1 = p$ and $q_0 \ne q_1$, then $\big\|\sum_j T_j f\big\|_{L^{q_*,\infty}} \lesssim B_0^{1-\theta} B_1^\theta \|f\|_p$.*
(ii) *If $q_0 = q_1 = q$ and $p_0 \ne p_1$, then $\big\|\sum_j T_j f\big\|_{L^q} \lesssim B_0^{1-\theta} B_1^\theta \|f\|_{p_*,1}$.*
(iii) *If $p_0 \ne p_1$ and $q_0 \ne q_1$, then $\big\|\sum_j T_j f\big\|_{L^{q_*,\infty}} \lesssim B_0^{1-\theta} B_1^\theta \|f\|_{p_*,1}$.*

We close this section with some sharp L^1-L^2, L^1-L^∞ estimates for $\mathcal{P}_\mu[\eta_k]$, which are useful in showing the weak type estimate for $\mathcal{P}_\mu$ at $(1/p,1/q) \in (\mathfrak{B},\mathfrak{C}]$.

Lemma 2.3. *Let $d \ge 1$, $\mu \in 2\mathbb{N}_0 + d$, and $k \in \mathbb{N}_0$. Suppose $\eta_k \in C_c^\infty((-\pi/2,\pi/2))$ such that $\operatorname{supp}\eta_k$ is contained in an interval of length $\sim 2^{-k}$ and satisfies $\left|\frac{d^l}{dt^l}\eta_k(t)\right| \le C_l 2^{kl}$ for all $l \in \mathbb{N}_0$. If $2^k \lesssim \mu$, then we have*

$$\|\mathcal{P}_\mu[\eta_k]\|_{1\to2} \lesssim 2^{-k/2}\mu^{\frac{d-1}{2}}, \tag{2.17}$$

$$\|\mathcal{P}_\mu[\eta_k]\|_{1\to\infty} \lesssim \mu^{d-1}. \tag{2.18}$$

Proof: We prove (2.17) and (2.18) by combining with the known L^1-L^2 estimate for $\mathcal{P}_\mu$ (see (1.3)) and the spectral decomposition. Note that

$$\mathcal{P}_\mu[\eta_k]f = \sum_{\mu'}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\eta_k(t)e^{it(\mu-\mu')}P_{\mu'}f\,dt = \frac{1}{\pi}\sum_{\mu'}\widehat{\eta}_k(\mu'-\mu)P_{\mu'}f. \tag{2.19}$$

By orthogonality and the estimate $\|\mathcal{P}_\mu\|_{1\to2} \lesssim \mu^{\frac{d-1}{2}}$ (see (1.3)) we see that

$$\|\mathcal{P}_\mu[\eta_k]f\|_2^2 \lesssim \sum_{\mu'}|\widehat{\eta}_k(\mu'-\mu)|^2\|P_{\mu'}f\|_2^2 \le C\sum_{\mu'}|\widehat{\eta}_k(\mu'-\mu)|^2(\mu')^{d-1}\|f\|_1^2.$$

Since $|\widehat{\eta}_k(t)| \le C_N 2^{-k}(1+2^{-k}|t|)^{-N}$ for any N with C_N independent of k and since $2^k \lesssim \mu$, we have

$$\|\mathcal{P}_\mu[\eta_k]f\|_2^2 \lesssim \sum_{\mu'}2^{-2k}(1+2^{-k}|\mu'-\mu|)^{-2N}(\mu')^{d-1}\|f\|_1^2 \lesssim 2^{-k}\mu^{d-1}\|f\|_1^2,$$

which yields (2.17). The estimate (2.18) can be shown in the same manner using (2.19) since we have $\|P_\mu f\|_\infty \lesssim \mu^{d-1}\|f\|_1$ by (1.3) and duality. We omit the detail. □

3. Proof of Theorem 1.2: Sufficiency part

In this section, we show (1.5) for p, q satisfying $1 \le p \le 2 \le q \le \infty$ and $(1/p,1/q) \notin [\mathfrak{B},\mathfrak{C}] \cup [\mathfrak{B}',\mathfrak{C}']$ and obtain the weak/restricted-weak type estimates for $\mathcal{P}_\mu$ for $(1/p,1/q) \in [\mathfrak{B},\mathfrak{C}] \cup [\mathfrak{B}',\mathfrak{C}']$. Our argument here is similar to the one used in the proof of the local estimate for the Hermite spectral projection (Theorem 1.5 of [**9**]).

From the known L^2-L^q bound (1.3) for $\mathcal{P}_\mu$ and duality, we already have (1.5) when $p = 2$, $q = 2$, or $p = q'$. Thus, by duality and interpolation, it suffices to show (1.5) for $(1/p,1/q) \in \mathcal{R}_1$, the weak type estimate $\|\mathcal{P}_\mu\|_{L^p\to L^{q,\infty}} \lesssim \mu^{\varrho(p,q)}$ for $(1/p,1/q) \in (\mathfrak{B},\mathfrak{C}]$ (the assertion (i)), and the restricted-weak type estimate $\|\mathcal{P}_\mu\|_{L^{p,1}\to L^{q,\infty}} \lesssim \mu^{\varrho(p,q)}$ at $(1/p,1/q) = \mathfrak{B}$ (the assertion (ii)).

Strong type estimate for $\mathcal{P}_\mu$ when $(1/p, 1/q) \in \mathcal{R}_1$. We first prove (1.5) for $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{R}_1$. In view of (2.8) and Lemma 2.2, it is enough to show that for $j \geq 1$ and $k \geq 3$

$$\|\mathcal{P}_\mu[\psi_j]\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} 2^{j(-1+\frac{2d+1}{2}(\frac{1}{p}-\frac{1}{q}))}, \tag{3.1}$$

$$\|\mathcal{P}_\mu[\varphi_k]\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} 2^{k(-1+\frac{2d+1}{2}(\frac{1}{p}-\frac{1}{q}))}, \tag{3.2}$$

$$\|\mathcal{P}_\mu[\varphi^0]\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{3.3}$$

whenever $(1/p, 1/q)$ is in the closed quadrangle $\mathcal{Q}(d)$ with vertices $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\mathfrak{A}$, $(1, 0)$, and $\mathfrak{A}'$. Indeed, by (2.8), (3.1), (3.2), and the triangle inequality, we obtain

$$\begin{aligned}\|\mathcal{P}_\mu\|_{p\to q} &\lesssim \sum_{j\geq 1} \|\mathcal{P}_\mu[\psi_j]\|_{p\to q} + \sum_{k\geq 3} \|\mathcal{P}_\mu[\varphi_k]\|_{p\to q} + \|\mathcal{P}_\mu[\varphi^0]\|_{p\to q} \\ &\lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\end{aligned}$$

if $(1/p, 1/q) \in \mathcal{Q}(d)$ satisfying $\frac{1}{p} - \frac{1}{q} < \frac{2}{2d+1}$. For $(1/p, 1/q) = \mathfrak{B}$ or $\mathfrak{B}' \in \mathcal{Q}(d)$, which satisfies $\frac{1}{p} - \frac{1}{q} = \frac{2}{2d+1}$, (iii) in Lemma 2.2 implies

$$\|\mathcal{P}_\mu\|_{L^{p,1}\to L^{q,\infty}} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}.$$

Moreover, this shows the assertion (ii) in Theorem 1.2. By real interpolation between the restricted-weak type (p, q) estimates with $(1/p, 1/q) = \mathfrak{B}$ and $\mathfrak{B}'$, we get (1.5) for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$ and, hence, for all $(1/p, 1/q) \in \mathcal{R}_1$.

As will be seen later, better bounds are possible for $\mathcal{P}_\mu[\varphi_k]$ and $\mathcal{P}_\mu[\varphi^0]$, but (3.2) and (3.3) are sufficient for our purpose.

We now show (3.1)–(3.3) for $(1/p, 1/q) \in \mathcal{Q}(d)$. Thanks to (2.1) we clearly have the isometry $\|e^{-itL} f\|_2 = \|f\|_2$. It is clear that the modified TT^*-argument in [**9**, Lemma 2.3] works without modification. From (2.12) and (2.11), we have $\|\mathcal{P}_\mu[\eta_j]\|_{1\to\infty} \lesssim \mu^{-1/2} 2^{j(d-1/2)}$ whenever η_j is a smooth function supported in $[-2^{-j}, -2^{-j-2}] \cup [2^{-j-2}, 2^{-j}]$ and satisfies $\left|\frac{d^l}{dt^l}\eta_j(t)\right| \leq C2^{jl}$ for $l = 0, 1$. Thus, [**9**, Lemma 2.3] gives the estimate (3.1) for $j \geq 1$ and $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{Q}(d)$. Indeed, we apply [**9**, Lemma 2.3] to $\mathcal{P}_\mu[\psi_j^\pm]$ by splitting $\psi_j = \psi_j^+ + \psi_j^-$, where $\psi_j^+(t) = \psi_j(t)$ if $t > 0$, and 0 otherwise.

Next we consider the estimate (3.2) for $\mathcal{P}_\mu[\varphi_k]$. Here, the cutoff function φ_k is supported near $\frac{\pi}{2}$. So, Lemma 2.3 of [**9**] does not apply directly, but a little modification of the argument gives the desired result. Since $|\sin t| \gtrsim 1$ on the support of φ_k, from (2.13), we have $\|\mathcal{P}_\mu[\varphi_k]\|_{1\to\infty} \lesssim$

$\mu^{-1/2}2^{k/2}$ for any $k \geq 3$. Taking interpolation with $\|\mathcal{P}_\mu[\varphi_k]\|_{2\to 2} \lesssim 2^{-k}$, which follows from the isometry (2.1) and Minkowski's inequality, we get

$$\|P_\mu[\varphi_k]\|_{p\to p'} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{p'})}2^{k(-1+\frac{3}{2}(\frac{1}{p}-\frac{1}{p'}))} \leq \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{p'})}2^{k(-1+\frac{2d+1}{2}(\frac{1}{p}-\frac{1}{p'}))}$$

for $1 \leq p \leq 2$. Thus, in order to show (3.2) for $(1/p, 1/q) \in \mathcal{Q}(d)$, by interpolation and duality it suffices to show (3.2) with $(1/p, 1/q) = (1/p_\circ, 1/q_\circ) := \mathfrak{A}$, i.e., $q_\circ = 2$ and $p_\circ = 2(2d+1)/(2d+3)$. Equivalently, we will show that

$$\|\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k]\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}2^{-k} \tag{3.4}$$

with $(p, q) = (p_\circ, p'_\circ)$. By a simple change of variables, we see that

$$\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k]f = \int_0^\pi \varphi_k(s) \int_{-\pi/2}^{\pi/2} \varphi_k(t+s)e^{it(\mu-L)}f\,dt\,ds.$$

Here, we note that the support of $\varphi_k(\cdot+s)$ is contained in $[-2^{-k+1}, 2^{-k+1}]$ for any $s \in \operatorname{supp}\varphi_k$. Let us define $(\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k])_l$ for $l \in \mathbb{Z}$ by

$$(\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k])_l f := \int_0^\pi \varphi_k(s) \int_{-\pi/2}^{\pi/2} \varphi_k(t+s)\psi_l(t)e^{it(\mu-L)}f\,dt\,ds$$

and we may write

$$\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k] = \sum_{l\geq k-2} (\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k])_l.$$

Note that $l \geq 1$ and the estimate (3.1) is valid with ψ_j replaced by a smooth function η_j supported in $[-2^{-j}, -2^{-j-2}] \cup [2^{-j-2}, 2^{-j}]$ and satisfies $\left|\frac{d^n}{dt^n}\eta_j(t)\right| \leq C2^{jn}$ for $n = 0, 1$. Applying (3.1) to $\mathcal{P}_\mu[\varphi_k(\cdot+s)\psi_l]$, we have

$$\|(\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k])_l\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}2^{-k}2^{l(-1+\frac{2d+1}{2}(\frac{1}{p}-\frac{1}{q}))}$$

for all $(1/p, 1/q) \in \mathcal{Q}(d)$. As before, Lemma 2.2 gives

$$\|\mathcal{P}_\mu[\varphi_k]^*\mathcal{P}_\mu[\varphi_k]\|_{L^{p,1}\to L^{q,\infty}} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}2^{-k}$$

for $(1/p, 1/q) \in \mathcal{Q}(d)$ satisfying $1/p - 1/q = 2/(2d+1)$. Real interpolation yields (3.4) for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$. Since $(1/p_\circ, 1/p'_\circ)$ is the intersection between the line segment $(\mathfrak{B}, \mathfrak{B}')$ and the line of duality, we get the desired estimate (3.4) with $(p, q) = (p_\circ, p'_\circ)$.

It remains to show the estimate for $\mathcal{P}_\mu[\varphi^0]$. As before, $|\sin t| \gtrsim 1$ on the support of φ^0. So, from (2.14) we have $\|\mathcal{P}_\mu[\varphi^0]\|_{1\to\infty} \lesssim \mu^{-1/2}$. Interpolating this with the L^2 estimate derived from the L^2 isometry of e^{itL}, we obtain for $1 \leq p \leq 2$

$$\|\mathcal{P}_\mu[\varphi^0]\|_{p\to p'} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{p'})}. \tag{3.5}$$

Thus, in view of interpolation, it is enough to show (3.3) for $(1/p, 1/q) = \mathfrak{A}$. Equivalently, we will show that

$$\|\mathcal{P}_\mu[\varphi^0]^*\mathcal{P}_\mu[\varphi^0]\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}$$

for $(p,q) = (p_\circ, p_\circ')$ with $p_\circ = 2(2d+1)/(2d+3)$. Actually, it follows from the known bounds (3.1), (3.2), and (3.5). Indeed, from the periodicity of φ^0 and (2.9) it is easy to see that

$$\mathcal{P}_\mu[\varphi^0]^*\mathcal{P}_\mu[\varphi^0]f = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi^0(s) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi^0(t+s)e^{it(\mu-L)} f\, dt\, ds.$$

So, using this and the partition of unity (2.7) we note that $\mathcal{P}_\mu[\varphi^0]^*\mathcal{P}_\mu[\varphi^0]$ equals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi^0(s)\Big\{\sum_{j\ge1} \mathcal{P}_\mu[\varphi^0(\cdot+s)\psi_j] + \sum_{k\ge3} \mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi_k] + \mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi^0]\Big\}\, ds.$$

We have already had estimates for $\mathcal{P}_\mu[\varphi^0(\cdot+s)\psi_j^\pm]$ and $\mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi_k]$ (see (3.1), (3.2)). Thus, Lemma 2.2 and the real interpolation imply

$$\Big\|\sum_{j\ge1} \mathcal{P}_\mu[\varphi^0(\cdot+s)\psi_j] + \sum_{k\ge3} \mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi_k]\Big\|_{p\to q} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}$$

for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$. Since $(1/p_\circ, 1/p_\circ')$ is in $(\mathfrak{B}, \mathfrak{B}')$, this particularly yields

$$\Big\|\int_{-\pi/2}^{\pi/2} \varphi^0(s)\Big(\sum_{j\ge1} \mathcal{P}_\mu[\varphi^0(\cdot+s)\psi_j] + \sum_{k\ge3} \mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi_k]\Big)\, ds\Big\|_{p_\circ\to p_\circ'} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p_\circ}-\frac{1}{p_\circ'})}.$$

Moreover, from (3.5), we also have

$$\Big\|\int_{-\pi/2}^{\pi/2} \varphi^0(s)\mathcal{P}_\mu[\varphi^0(\cdot+s)\varphi^0]\, ds\Big\|_{p_\circ\to p_\circ'} \lesssim \mu^{-\frac{1}{2}(\frac{1}{p_\circ}-\frac{1}{p_\circ'})}.$$

Combining these two estimates, we obtain the desired estimate.

Weak type estimate for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{C}]$. Recalling (2.8), we first handle $\sum_{j\ge1} \mathcal{P}_\mu[\psi_j]$. To obtain the weak type (p,q) estimate for $\mathcal{P}_\mu$, we prove

$$\Big\|\sum_{j\ge1} \mathcal{P}_\mu[\psi_j]\Big\|_{L^p\to L^{q,\infty}} \lesssim \mu^{d(\frac{1}{p}+\frac{1}{q})-\frac{2d+1}{2}} \tag{3.6}$$

for $(1/p,1/q)\in(\mathfrak{B},\mathfrak{C}]$. Note that $\sum_{2^{-j}\lesssim\mu^{-1}}\psi_j=\zeta_\mu$ for some $\zeta_\mu\in C_c^\infty((-c\mu^{-1},c\mu^{-1}))$ satisfying $|\zeta_\mu^{(l)}(t)|\le C_l\mu^l$ for all $l\in\mathbb{N}_0$. So, we write

$$\sum_{j\ge1}\mathcal{P}_\mu[\psi_j]=\mathcal{P}_\mu[\zeta_\mu]+\sum_{2^j\lesssim\mu}\mathcal{P}_\mu[\psi_j].$$

From Lemma 2.3, we have $\|\mathcal{P}_\mu[\psi_j]\|_{1\to2}\lesssim2^{-j/2}\mu^{\frac{d-1}{2}}$ for $2^j\lesssim\mu$. By interpolation between this estimate and (3.1) with $(1/p,1/q)=\mathfrak{B}$ and $(1,0)$, we obtain

$$\|\mathcal{P}_\mu[\psi_j]\|_{p\to q}\lesssim2^{j2d(\frac{2d-1}{4d}-\frac1q)}\mu^{d(\frac1p+\frac1q)-\frac{2d+1}{2}}$$

for $2^j\lesssim\mu$ whenever $(1/p,1/q)$ is in the closed triangle $\mathcal{T}(d)$ with vertices $(1,1/2)$, $(1,0)$, and $\mathfrak{B}$. Thus, choosing $q_0<\frac{4d}{2d-1}<q_1$ such that $(1/p,1/q_0)$, $(1/p,1/q_1)\in\mathcal{T}(d)$ and using (i) in Lemma 2.2, we obtain

$$\Big\|\sum_{2^j\lesssim\mu}\mathcal{P}_\mu[\psi_j]\Big\|_{L^p\to L^{q,\infty}}\lesssim\mu^{d(\frac1p+\frac1q)-\frac{2d+1}{2}}\tag{3.7}$$

for any $(1/p,1/q)\in(\mathfrak{B},\mathfrak{C}]$.

We now handle $\mathcal{P}_\mu[\zeta_\mu]$. Since $\sum_{2^{-j}\lesssim\mu^{-1}}\psi_j=\zeta_\mu$, by (3.1) and (iii) of Lemma 2.2 we have the restricted-weak type $(p_\circ,q_\circ)$ estimate for $\mathcal{P}_\mu[\zeta_\mu]$ with $(1/p_\circ,1/q_\circ)=\mathfrak{B}$:

$$\|\mathcal{P}_\mu[\zeta_\mu]\|_{L^{p_\circ,1}\to L^{q_\circ,\infty}}\lesssim\mu^{-\frac12(\frac{1}{p_\circ}-\frac{1}{q_\circ})}=\mu^{-\frac{1}{2d+1}}.$$

Interpolating this and the estimates $\|\mathcal{P}_\mu[\zeta_\mu]\|_{1\to2}\lesssim\mu^{\frac{d-2}{2}}$, $\|\mathcal{P}_\mu[\zeta_\mu]\|_{1\to\infty}\lesssim\mu^{d-1}$ which follow from Lemma 2.3, we obtain

$$\|\mathcal{P}_\mu[\zeta_\mu]\|_{p\to q}\lesssim\mu^{d(\frac1p-\frac1q)-1}$$

whenever $(1/p,1/q)$ is in $\mathcal{T}(d)\setminus\{\mathfrak{B}\}$. Therefore, we get the estimate $\|\mathcal{P}_\mu[\zeta_\mu]\|_{p\to q}\lesssim\mu^{d(\frac1p+\frac1q)-\frac{2d+1}{2}}$ for $(1/p,1/q)\in(\mathfrak{B},\mathfrak{C}]$, since $d\big(\frac1p-\frac1q\big)-1=d\big(\frac1p+\frac1q\big)-\frac{2d+1}{2}$ when $\frac1q=\frac{2d-1}{4d}$. Combining this with (3.7), we obtain (3.6) for $(1/p,1/q)\in(\mathfrak{B},\mathfrak{C}]$.

We now turn to $\sum_{k\ge3}\mathcal{P}_\mu[\varphi_k]$ and $\mathcal{P}_\mu[\varphi^0]$. Applying Lemma 2.3, we have the estimate $\|\mathcal{P}_\mu[\varphi_k]\|_{1\to2}\lesssim2^{-k/2}\mu^{\frac{d-1}{2}}$ for $2^k\lesssim\mu$ and (3.2), especially, with $(1/p,1/q)=\mathfrak{B},(1,0)$. We also have the restricted-weak type (p,q) estimate for $\sum_{2^{-k}\lesssim\mu^{-1}}\mathcal{P}_\mu[\varphi_k]$ at $(p,q)=\mathfrak{B}$. Thus, in the same manner as before we can obtain

$$\Big\|\sum_{k\ge3}\mathcal{P}_\mu[\varphi_k]\Big\|_{L^p\to L^{q,\infty}}\lesssim\mu^{d(\frac1p+\frac1q)-\frac{2d+1}{2}}\tag{3.8}$$

for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{C}]$. Finally, we have $\|\mathcal{P}_\mu[\varphi^0]\|_{1\to 2} \lesssim \mu^{\frac{d-1}{2}}$ from Lemma 2.3, $\|\mathcal{P}_\mu[\varphi^0]\|_{1\to\infty} \lesssim \mu^{-\frac{1}{2}}$, and $\|\mathcal{P}_\mu[\varphi^0]\|_{p_\circ\to q_\circ} \lesssim \mu^{-\frac{1}{2d+1}}$ at $(1/p_\circ, 1/q_\circ) = \mathfrak{B}$ from (3.3). Thus, interpolation gives

$$\|\mathcal{P}_\mu[\varphi^0]\|_{p\to q} \lesssim \mu^{d(\frac{1}{p}+\frac{1}{q})-\frac{2d+1}{2}} \tag{3.9}$$

for $(1/p, 1/q) \in [\mathfrak{B}, \mathfrak{C}]$.

Combining the estimates (3.6), (3.8), and (3.9) together with (2.8), we get the desired weak type (p, q) estimate for $\mathcal{P}_\mu$ when $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{C}]$.

4. Proof of Theorem 1.2: Sharpness

In this section we show that the estimate (1.5) is sharp and that (1.5) fails for $(1/p, 1/q) \in [\mathfrak{B}, \mathfrak{C}] \cup [\mathfrak{B}', \mathfrak{C}']$.

Proposition 4.1. *Let $d \geq 1$ and $1 \leq p \leq 2 \leq q \leq \infty$. For μ large enough, there is a constant C, independent of μ, such that*

$$\|\mathcal{P}_\mu\|_{p\to q} \geq C\mu^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{4.1}$$

$$\|\mathcal{P}_\mu\|_{p\to q} \geq C\mu^{d(\frac{1}{p}-\frac{1}{q})-1}, \tag{4.2}$$

$$\|\mathcal{P}_\mu\|_{p\to q} \geq C\mu^{\frac{2d-1}{2}-d(\frac{1}{p}+\frac{1}{q})}. \tag{4.3}$$

Proof of Theorem 1.2: Sharpness: By duality and (4.3) we obtain

$$\|\mathcal{P}_\mu\|_{p\to q} \geq C\mu^{d(\frac{1}{p}+\frac{1}{q})-\frac{2d+1}{2}}$$

for any $q \geq 2$. Combining this and the estimates in Proposition 4.1, we obtain

$$\|\mathcal{P}_\mu\|_{p\to q} \geq C\mu^{\varrho(p,q)}$$

for $1 \leq p \leq 2 \leq q \leq \infty$.

The failure of the strong type estimate (1.5) for p, q satisfying $(1/p, 1/q) \in [\mathfrak{B}, \mathfrak{C}] \cup [\mathfrak{B}', \mathfrak{C}']$ can be shown by using the L^p-L^q transplantation argument in [**9**, Lemma 3.5] (see the paragraph below Lemma 3.5 of [**9**]) because the twisted Laplacian L is also an elliptic operator on $\mathbb{R}^{2d}$. To do so, we define a projection operator $\mathcal{P}$ by

$$\mathcal{P} = \sum_{kn\leq\mu\leq(k+1)n} \mathcal{P}_\mu$$

for large k, $n > 0$, and set $P(z, z')$ as the kernel of $\mathcal{P}$. If we assume that $\|\mathcal{P}_\mu\|_{p\to q} \lesssim \mu^{d(\frac{1}{p}-\frac{1}{q})-1}$, by the triangle inequality, we have

$$\|\mathcal{P}\|_{p\to q} \lesssim k^{d(\frac{1}{p}-\frac{1}{q})-1} n^{d(\frac{1}{p}-\frac{1}{q})}.$$

This implies

$$n^d \Big| \iint P(z, z') f(n^{1/2}z') g(n^{1/2}z)\, dz\, dz' \Big| \lesssim k^{d(\frac{1}{p}-\frac{1}{q})-1} \|f\|_p \|g\|_{q'} \tag{4.4}$$

for $f, g \in C_c^\infty(\mathbb{R}^{2d})$. Let f and g be supported in a ball of radius r. If z' and z are in the support of $f(n^{1/2}\cdot)$ and $g(n^{1/2}\cdot)$, respectively, $z - z'$ is in a ball of radius $2rn^{-1/2}$, hence $|z - z'|$ is small enough if n is sufficiently large. Applying Hörmander's theorem [**7**, Theorem 5.1] (see also Theorem 3.6 of [**9**]), we see that

$$P(z, z') = (2\pi)^{-2d} \int_{kn \le |\xi|^2 \le (k+1)n} e^{i\psi(z,z',\xi)}\, d\xi + \mathcal{E}(z, z', k, n),$$

where $\psi(z, z', \xi) = \langle (x - x', y - y'), \xi \rangle + O(|z - z'|^2 |\xi|)$ with $z = x + iy, z' = x' + y'$, and $\mathcal{E}(z, z', k, n) = O(|kn|^{(2d-1)/2})$. Thus, by rescaling $(z, z') \to (n^{-1/2} z, n^{-1/2} z')$ we see that the estimate (4.4) implies

$$\Big| \iint \Big(\int_{k \le |\xi|^2 \le (k+1)} e^{i\psi(n^{-\frac{1}{2}}(z,z'), n^{\frac{1}{2}}\xi)}\, d\xi + O(k^{\frac{2d-1}{2}} n^{-\frac{1}{2}}) \Big) f(z') g(z)\, dz\, dz' \Big| \lesssim k^{d(\frac{1}{p} - \frac{1}{q}) - 1} \|f\|_p \|g\|_{q'}.$$

Letting n tend to ∞, this yields

$$\Big| \iint \int_{k \le |\xi|^2 \le (k+1)} e^{i\langle (x-x', y-y'), \xi \rangle}\, d\xi f(z') g(z)\, dz\, dz' \Big| \lesssim k^{d(\frac{1}{p} - \frac{1}{q}) - 1} \|f\|_p \|g\|_{q'}$$

for any $f, g \in C_c^\infty(\mathbb{R}^{2d})$, which is equivalent to

$$\Big\| \frac{1}{(2\pi)^{2d}} \int_{k \le |\xi|^2 \le (k+1)} e^{i\langle (x,y), \xi \rangle} \widehat{f}(\xi)\, d\xi \Big\|_q \lesssim k^{d(\frac{1}{p} - \frac{1}{q}) - 1} \|f\|_p$$

for any $f, g \in C_c^\infty(\mathbb{R}^{2d})$. After scaling and letting $k \to \infty$, we obtain the $2d$-dimensional restriction-extension estimate

$$\Big\| \int_{\mathbb{S}^{2d-1}} \widehat{f}(\xi) e^{2\pi i (x,y)\cdot\xi}\, d\sigma(\xi) \Big\|_q \lesssim \|f\|_p. \tag{4.5}$$

It was already known that (4.5) is true only if $(1/p, 1/q) \in \mathcal{R}_3$ (see [**2**], [**9**, Theorem 3.6]). □

Proof of the lower bounds (4.1) *and* (4.2)*:* We prove the estimates (4.1) and (4.2) by using a duality argument and the known sharpness result obtained by Koch and Ricci [**13**]. In fact, we will use the fact that there is a constant $C > 0$, independent of μ, such that for $2 \le q \le \infty$

$$\|\mathcal{P}_\mu\|_{q' \to q} \ge C \mu^{\varrho(q', q)}, \tag{4.6}$$

which follows from (1.3) and the TT^*-argument. Since we have $\varrho(p, q) = \max\big\{ -\frac{1}{2}\big(\frac{1}{p} - \frac{1}{q}\big), d\big(\frac{1}{p} - \frac{1}{q}\big) - 1, \frac{2d-1}{2} - d\big(\frac{1}{p} + \frac{1}{q}\big), d\big(\frac{1}{p} + \frac{1}{q}\big) - \frac{2d+1}{2} \big\}$, it is enough to show that (4.1) on $\big(\frac{1}{p}, \frac{1}{q}\big) \in \mathcal{R}_1$ and (4.2) on $\big(\frac{1}{p}, \frac{1}{q}\big) \in \mathcal{R}_3$.

We only show (4.1) since the same argument works for (4.2). We prove (4.1) by contradiction. Suppose (4.1) fails for some p, q with $p \neq q'$ and $(1/p, 1/q) \in \mathcal{R}_1$; then there are sequences c_k and μ_k such that

$$\|\mathcal{P}_{\mu_k}\|_{p\to q} = c_k \mu_k^{\varrho(p,q)},$$

$\mu_k \to \infty$, and $c_k \to 0$ as $k \to \infty$. Then, by duality we also have $\|\mathcal{P}_{\mu_k}\|_{q'\to p'} = c_k \mu_k^{\varrho(p,q)}$. By interpolation between these two estimates we get

$$\|\mathcal{P}_{\mu_k}\|_{r\to s} \le c_k \mu_k^{\varrho(r,s)}$$

for all r, s satisfying $(1/r, 1/s) \in [(1/p, 1/q), (1/q', 1/p')]$ with $1/r - 1/s = 1/p - 1/q$. In particular we get $\|\mathcal{P}_{\mu_k}\|_{r\to r'} \le c_k \mu_k^{\varrho(r,r')}$. This contradicts (4.6) because $c_k \to 0$ as $k \to \infty$. □

Proof of (4.3)*:* We make use of the formula (1.2), where the twisted kernel ς_k is given by the Laguerre function. In fact, let $\mathcal{L}_k^\alpha(t)$, $t \ge 0$, denote the normalized Laguerre function of type α given by

$$\mathcal{L}_k^\alpha(t) = \Big(\frac{k!}{\Gamma(k+\alpha+1)}\Big)^{1/2} t^{\alpha/2} e^{-t/2} L_k^\alpha(t).$$

We clearly have

$$\varsigma_k(z) = 2^{\frac{d-1}{2}} (\frac{k!}{(k+d-1)!})^{-1/2} |z|^{-(d-1)} \mathcal{L}_k^{d-1}(|z|^2/2).$$

There is a large body of literature concerning the asymptotic behavior of the Laguerre functions. We refer the reader to [**17, 6, 18**] and references therein. However, for our purpose we use the following relatively simple asymptotic formula.

Lemma 4.2 ([**15**, p. 422])**.** *Let* $\alpha \ge 0$ *and* $k \in \mathbb{N}$*. Then*

$$\mathcal{L}_k^\alpha(t) = \Big(\frac{2}{\pi}\Big)^{1/2} \frac{(-1)^k}{t^{1/4}(\nu - t)^{1/4}} \cos\Big(\frac{\nu(2\theta - \sin 2\theta) - \pi}{4}\Big) + O\Big(\frac{\nu^{1/4}}{(\nu - t)^{7/4}} + (\nu t)^{-3/4}\Big),$$

where $\nu = 4k + 2\alpha + 2$, $0 < t < \nu$, *and* $\theta = \cos^{-1}(t^{1/2}\nu^{-1/2})$.

Recalling $\mu = 2k + d$, from Lemma 4.2 we have

$$\varsigma_k(z) = 2^{\frac{d-1}{2}} \Big(\frac{2}{\pi}\frac{(k+d-1)!}{k!}\Big)^{1/2} \frac{(-1)^k}{|z|^{d-1}} \times \Big\{(|z|^2(\mu - 2^{-2}|z|^2))^{-1/4} \cos g(|z|) + O(\mu^{-3/2})\Big\} \tag{4.7}$$

for $\sqrt{\mu}/8 \le |z| \le \sqrt{\mu}/2$, where

$$g(s) = \frac{\mu}{2}(2\theta(s) - \sin 2\theta(s)) - \frac{\pi}{4}, \quad \theta(s) = \cos^{-1}\Big(\frac{s}{2\sqrt{\mu}}\Big).$$

Note that g is monotone decreasing and $\{g(t) : \sqrt{\mu}/8 \le t \le \sqrt{\mu}/3\}$ is an interval of length $\sim \mu$. So, there exist $\sqrt{\mu}/8 < t_1 < t_2 < \cdots < t_N < \sqrt{\mu}/3$, $N \sim \mu$, such that $|\cos g(t_j)| = 1$ for all j. Since $|g'(t)| \sim \sqrt{\mu}$, $t_{j+1} - t_j \sim 1/\sqrt{\mu}$ for all j. Also, $|\cos g(t)| \ge \cos(\pi/4) > 0$ whenever $|t - t_j| \le \pi/(8\sqrt{\mu})$.

To prove (4.3), we set $D_j := [t_j, t_j + \pi/(8\sqrt{\mu})]$, $1 \le j \le N$, and define f on $\mathbb{C}^d$ by

$$f(z) := \sum_{j=1}^{N} \chi_{D_j}(|z|)\varsigma_k(z).$$

It is easy to see that $|f(z)| \sim \mu^{-1/2}$, since $\frac{(k+d-1)!}{k!} \sim k^{d-1} \sim \mu^{d-1}$ from Stirling's formula. Hence we have

$$\int_{\mathbb{C}^d} |f(z)|^p \, dz \lesssim \int_{\sqrt{\mu}/8}^{\sqrt{\mu}/2} \mu^{-p/2} \, r^{2d-1} \, dr \lesssim \mu^{-p/2+d}.$$

We now observe $\mathcal{P}_\mu f$ near the origin. For $|z| \le \pi/(32\sqrt{\mu})$ and $w \in \mathbb{C}^d$ satisfying $|z - w| \in D_j$, we have $|w| \in [t_j - \pi/(8\sqrt{\mu}), t_j + 5\pi/(32\sqrt{\mu})]$ and $|\cos g(|w|)| \ge c > 0$ for some $c > 0$ independent of μ. This yields $\varsigma_k(z - w)\varsigma_k(w) > 0$ on $|z - w| \in D_j$ and $|z| \le \pi/(32\sqrt{\mu})$ if k is large enough. Also, $|\varsigma_k(w)| \sim \mu^{-2}$ for $|w| \in [t_j - \pi/(8\sqrt{\mu}), t_j + 5\pi/(32\sqrt{\mu})]$. Thus, if μ is large enough, for $|z| \le \pi/(32\sqrt{\mu})$ we obtain

$$\begin{aligned} |\mathcal{P}_\mu f(z)| &\ge (2\pi)^{-d} \sum_{j=1}^{N} \Big| \mathrm{Re}\Big(\int_{\mathbb{C}^d} \chi_{D_j}(|z - w|)\varsigma_k(z - w)\varsigma_k(w) e^{i\frac{1}{2}\operatorname{Im} z\cdot\overline{w}} \, dw \Big) \Big| \\ &\gtrsim \sum_{j=1}^{N} \mu^{-1} \int_{t_j + \frac{\pi}{32\sqrt{\mu}}}^{t_j + \frac{3\pi}{32\sqrt{\mu}}} r^{2d-1} \, dr \sim \mu^{d-1}, \end{aligned}$$

where the implicit constant is independent of μ. Therefore, for $1 \le p \le 2$ and μ large enough, we get

$$\begin{aligned}\|\mathcal{P}_\mu\|_{p\to q} &\ge \|\mathcal{P}_\mu f\|_q/\|f\|_p \ge \|\mathcal{P}_\mu f\|_{L^q(|z|\le\pi/(32\sqrt{\mu}))}/\|f\|_p \\ &\gtrsim \mu^{-d/q+d-1}\mu^{-d/p+1/2} \sim \mu^{\frac{2d-1}{2}-(\frac{d}{p}+\frac{d}{q})},\end{aligned}$$

which gives the lower bound (4.3). □

5. Resolvent estimate for the twisted Laplacian

In this section, we prove the uniform resolvent estimates for L in Theorem 1.3. Here we closely follow the argument for the Hermite resolvent estimate in [**10**], where the main ingredients were the uniform bound for the Hermite spectral projection, a kind of mixed norm estimate for the Hermite–Schrödinger propagator, and L^p-L^q boundedness of the fractional Hermite operator. For the twisted Laplacian L, the fractional integral operator L^{-s}, $s > 0$, is given by

$$L^{-s} = \sum_\mu \mu^{-s}\mathcal{P}_\mu = \frac{1}{\Gamma(s)}\int_0^\infty e^{-tL}t^{s-1}\,ds.$$

The L^p-L^q boundedness was already established by Nowak and Stempak [**16**].

Theorem 5.1 ([**16**]). *Let $s > 0$ and $1 \le p \le q \le \infty$. If $s > d$, L^{-s} is bounded from $L^p(\mathbb{C}^d)$ to $L^q(\mathbb{C}^d)$ for any $1 \le p \le q \le \infty$, and L^{-d} is bounded from $L^p(\mathbb{C}^d)$ to $L^q(\mathbb{C}^d)$ if and only if $(p,q) \ne (1,\infty)$. In addition, if $s<d$, then L^{-s} is bounded from $L^p(\mathbb{C}^d)$ to $L^q(\mathbb{C}^d)$ if and only if*

$$\frac{1}{p} - \frac{s}{d} \le \frac{1}{q} \quad \text{and} \quad \Big(\frac{1}{p},\frac{1}{q}\Big) \ne \Big(\frac{s}{d},0\Big),\quad \Big(1,\frac{d-s}{d}\Big).$$

We also need the mixed norm estimate for the Schrödinger propagator e^{-itL} in a certain range of p, q as follows.

Proposition 5.2. *Let $d \ge 2$ and Q be the closed quadrangle with vertices $(1/2,1/2)$, $\mathfrak{D}'$, $\mathfrak{F}'$, $((d+1)/2d,(d-1)/2d))$ from which the two points $\mathfrak{F}'$ and $((d+1)/2d,(d-1)/2d))$ are removed. If $(1/p,1/q) \in Q$, then*

$$\Big\|\int_{-\pi/2}^{\pi/2} |e^{-itL}f|\,dt\Big\|_q \lesssim \|f\|_p \tag{5.1}$$

and we also have restricted-weak type estimates if $(1/p,1/q) = \mathfrak{F}'$ or $((d+1)/2d,(d-1)/2d))$.

From (2.3) it follows that

$$\|e^{-itL}f\|_\infty \lesssim |\sin t|^{-d}\|f\|_1. \tag{5.2}$$

Since $d \geq 2$, combining this with (2.1), the standard argument [**11**] yields the endpoint Strichartz estimate

$$\|e^{-itL}f\|_{L_t^2([-\frac{\pi}{2},\frac{\pi}{2}];L_z^{2d/(d-1)}(\mathbb{C}^d))} \lesssim \|f\|_2. \tag{5.3}$$

Proof: From (2.1) and (5.3) it is clear that (5.1) holds for $(1/p, 1/q) = (1/2, 1/2)$, $(1/p, 1/q) = \mathfrak{D}'$. Thus, in view of interpolation it suffices to show the restricted-weak type estimate

$$\Big\| \int_{-\pi/2}^{\pi/2} |e^{-itL}f|\,dt \Big\|_{q,\infty} \lesssim \|f\|_{p,1} \tag{5.4}$$

with $(1/p, 1/q) = \mathfrak{F}'$, $((d+1)/2d, (d-1)/2d))$.

To show (5.4), we set $\psi^0 = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} - \psi$ so that

$$\sum_{j\geq 1} \psi_j(t) + \psi^0(t) = 1 \quad \text{for} \quad t \in [-\pi/2, \pi/2].$$

Then from (2.1), (5.3), and (5.2) the estimate

$$\Big\| \int |\psi^0(t)e^{-itL}f|\,dt \Big\|_q \lesssim \|f\|_p$$

holds with $(1/p, 1/q) = (1/2, 1/2), \mathfrak{D}', (1, 0)$. By interpolation we see that the above estimate holds for all p, q satisfying $(1/p, 1/q) \in Q$. Thus, to show (5.4) we only have to show that

$$\Big\| \int |\sum_{j\geq 1} \psi_j e^{-itL}f|\,dt \Big\|_{q,\infty} \lesssim \|f\|_{p,1} \tag{5.5}$$

with $(1/p, 1/q) = \mathfrak{F}'$, $((d+1)/2d, (d-1)/2d))$. We now claim that

$$\Big\| \int |\psi_j e^{-itL}f|\,dt \Big\|_q \lesssim 2^{(\frac{d}{p}-\frac{d}{q}-1)j}\|f\|_p \tag{5.6}$$

holds provided that $(1/p, 1/q)$ is contained in the closed triangle with vertices $(1/2, 1/2), \mathfrak{D}'$, and $(1, 0)$. Once we have this, (iii) of Lemma 2.2 gives the desired estimate (5.5) with $(1/p, 1/q) = \mathfrak{F}'$, $((d+1)/2d, (d-1)/2d))$. See Figure 1.

It remains to show (5.6). From (5.3) the estimate $\|\psi_j(t)e^{-itL}\|_{L_t^2 L_x^{\frac{2d}{d-1}}} \lesssim \|f\|_2$ follows. Using this estimate, by Hölder's and Minkowski's inequalities we obtain $\big\|\int |\psi_j e^{-itL}f|\,dt\big\|_{\frac{2d}{d-1}} \lesssim 2^{-\frac{1}{2}j}\|f\|_2$. Because of (2.3) we also have $\int \|\psi_j e^{-itL}f\|_\infty\,dt \lesssim 2^{(d-1)j}\|f\|_1$, and $\big\|\int|\psi_j e^{-itL}f|\,dt\big\|_2 \lesssim 2^{-j}\|f\|_2$ from (2.1). Interpolation among these estimates gives (5.6) for $(1/p, 1/q)$ in the closed triangle with vertices $(1/2, 1/2)$, $\mathfrak{D}'$, and $(1, 0)$. □

Once we have the estimates in Proposition 5.2, the desired resolvent estimates are established by following the argument used in [**10**]. For completeness, however, we give a brief proof of Theorem 1.3. We refer the reader to Section 3 of [**10**] for the details.

Proof of Theorem 1.3: The restricted-weak type estimates can be shown in a similar manner, so we only show the estimate (1.10). Since the adjoint operator of $(L-z)^{-1}$ is $(L-\bar{z})^{-1}$, which can be handled by the same argument, we may also assume $1/p \le 1/q'$ and furthermore $(1/p, 1/q) \in Q$. In fact, we show that if the estimate (5.1) holds and L^{-1} is bounded from L^p to L^q, then (1.10) holds.

For simplicity we only consider the case $z \in \mathbb{C}$ with $\Re z > d - 1/2$. The other cases can be handled by the same argument as we use to show the estimate for the term $\mathcal{E}$ below. Since $\operatorname{dist}(z, 2\mathbb{N}+d) \ge c_\circ > 0$, we write $z = 2n+d-2(a+ib)$ for some $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}$ satisfying $|a| < 1/2$ and $|(a,b)| \ge c_\circ/2$. Using a smooth symmetric function ζ supported in $(-1,1)$ and satisfying $\zeta(t) = 1$ on $(-1/2, 1/2)$, we decompose the resolvent operator $(L-z)^{-1}$ into two parts:

$$(L-z)^{-1} f = \mathcal{I} f + \mathcal{E} f,$$

where

$$\mathcal{I} f := \sum_{|k-n|<n} \frac{\zeta(\frac{k-n}{n})}{2(k-n+(a+ib))} \mathcal{P}_{2k+d} f, \tag{5.7}$$

$$\mathcal{E} f := \sum_{k} \frac{1-\zeta(\frac{k-n}{n})}{2(k-n+(a+ib))} \mathcal{P}_{2k+d} f. \tag{5.8}$$

From the choice of ζ, $\mathcal{I}$ is written as

$$\mathcal{I} f = \mathcal{I}_1 f + \mathcal{I}_2 f + \mathcal{I}_3 f,$$

where

$$\begin{aligned} \mathcal{I}_1 f &:= \frac{1}{2(a+ib)} \mathcal{P}_{2n+d} f, \\ \mathcal{I}_2 f &:= \sum_{k=1}^{n} \frac{(a+ib)\zeta(k/n)}{(k+a+ib)(-k+a+ib)} \mathcal{P}_{2(n-k)+d} f, \\ \mathcal{I}_3 f &:= \sum_{k=1}^{n} \frac{\zeta(k/n)}{2(k+a+ib)} (\mathcal{P}_{2(k+n)+d} f - \mathcal{P}_{2(n-k)+d} f). \end{aligned}$$

Then, for p, q satisfying $(1/p, 1/q) \in Q$, we obtain

$$\|\mathcal{I}_1\|_{p\to q}, \quad \|\mathcal{I}_2\|_{p\to q} \lesssim 1$$

uniformly in n and a, b satisfying $|(a,b)| \geq c_\circ/2$. Indeed, the estimate follows from the uniform bounds for $\mathcal{P}_\mu$, which are a direct consequence of Theorem 1.2 (or Proposition 5.2 with (2.2)). Using (2.2), we see that

$$\begin{aligned}\mathcal{I}_3 f &= \sum_{k=1}^{n} \frac{\zeta(k/n)}{2(k+a+ib)} \int_{-\pi/2}^{\pi/2} (e^{2itk} - e^{-2itk}) e^{it(2n+d)} e^{-itL} f\, dt \\ &= i \int_{-\pi/2}^{\pi/2} \sum_{k=1}^{n} \frac{\zeta(k/n)\sin(2kt)}{k+a+ib} e^{it(2n+d)} e^{-itL} f\, dt.\end{aligned}$$

Note that $\big|\sum_{k=1}^{n} \frac{\zeta(k/n)\sin(2kt)}{k+a+ib}\big| \leq C$ uniformly in n and a, b obeying $|(a,b)| \geq c_\circ/2$. Combining this with Proposition 5.2, we get $\|\mathcal{I}_3\|_{p\to q} \lesssim 1$ uniformly in n and a, b.

The term $\mathcal{E}$ is easier to deal with. Since $\mathcal{E}f = m_n(L) \circ L^{-1} f$ with

$$m_n(t) = t\Big(1 - \zeta\Big(\frac{t-2n-d}{2n}\Big)\Big)/(t-z), \quad z = 2n+d-2(a+ib),$$

and $\big|\frac{d^l}{dt^l} m_n(t)\big| \lesssim (1+t)^{-l}$ for $l = 0,1,2,\ldots,d+2$ whenever $t>0$, by applying the Marcinkiewicz multiplier theorem [**23**, Theorem 2.4.1] and Theorem 5.1, we obtain the desired result. □

Remark 1. When $d=1$, the uniform resolvent estimate (1.10) holds for any $1 < p \leq 2 \leq q < \infty$. Indeed, our proof of Theorem 1.3 works as long as we have the mixed norm estimate (5.1) for e^{-itL}. So, it suffices to show (5.1) for $1 < p \leq 2 \leq q < \infty$. Though the endpoint Strichartz estimate (5.3) fails with $d=1$, the estimate

$$\|e^{-itL} f\|_{L_t^r([-\frac{\pi}{2},\frac{\pi}{2}]; L_z^s(\mathbb{C}^d))} \lesssim \|f\|_2$$

holds true for any $r, s \geq 2$ satisfying $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$, $(r,s) \neq (2,\infty)$ (see [**11**]). Hence, this estimate combined with the argument in the proof of Proposition 5.2 yields (5.1) for $1 < p \leq 2 \leq q < \infty$.

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