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Semilinear formulation of a hyperbolic system of partial differential equations

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Abstract. In this paper, we solve the Cauchy problem for a hyperbolic system of first-order PDEs defined on a certain Banach space X. The system has a special semilinear structure because, on the one hand, the evolution law can be expressed as the sum of a linear unbounded operator and a nonlinear Lipschitz function but, on the other hand, the nonlinear perturbation takes values not in X but on a larger space Y which is related to X. In order to deal with this situation we use the theory of dual semigroups. Stability results around steady states are also given when the nonlinear perturbation is Fréchet differentiable. These results are based on two propositions: one relating the local dynamics of the nonlinear semiflow with the linearised semigroup around the equilibrium, and a second relating the dynamical properties of the linearised semigroup with the spectral values of its generator. The later is proven by showing that the Spectral Mapping Theorem always applies to the semigroups one obtains when the semiflow is linearised. Some epidemiological applications involving gut bacteria are commented

1. Introduction

The evolution law of many dynamical systems can be represented by a differential equation of the form $x'(t) = f(x(t))$ in which the dependent variable t refers to time. Roughly speaking, it is said that the system is well posed if every initial condition x_0 moves unambiguously and continuously into the future in accordance with the evolution law and if the trajectories described this way are continuous with respect to x_0. The well-posedness problem is usually referred as the Cauchy problem.

As it is well known, if the phase space X of the system is a Banach space, then the system is well-posed if the function $f : X \to X$ is Lipschitz. In addition, if f is differentiable, then the local dynamics around steady states can be studied in terms of the linearised system around them. Such results are really useful if X is finite dimensional, since in this case all linear functions from X into X are Lipschitz, which implies that all linear systems are well posed. However, if X has infinite dimension, there exist linear functions (or operators) that are not continuous from X into X and fail to be defined on the whole X, such as those appearing in partial differential equations. The main result characterizing the linear operators for which the initial

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value problem is well posed is the Hille–Yosida Theorem. Unfortunately, a general nonlinear theory is still incomplete. Some work has been devoted to solve the Cauchy problem for different classes of systems (see, for instance, the classical monographs of Pazy [22] and Henry [13] for semilinear semigroups and Miyadera [21] for dissipative systems). Similarly, different strategies exist to determine the asymptotic dynamics around stationary points of these systems (e.g. [5,15,20,26]). This range of techniques reflects the richness associated to differential equations in infinite dimension, as well as the fact that some generality has to be lost in order to obtain more accurate results.

This article is a contribution to the program summarised above. We focus on a class of dynamical systems (see (1.1) below) including the model treated in [2] to study the propagation of pathogenic bacteria inhabiting the guts of animals. Whereas in [2] the model is analysed in a rather formal way, the theoretical framework sustaining such analysis is developed here. This is done using some results about dual semigroups and their relations with evolution equations [6,7].

1.1. A class of first-order PDE systems

Let u and v be functions with domain $[0, 1] \times [0, \infty)$ taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let r be a $\mathbb{R}^k$ valued function with domain $[0, \infty)$. Let

$$\begin{aligned} g &: [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \\ f &: [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \\ h &: \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k \end{aligned}$$

be differentiable functions, and let $c : [0, 1] \to \mathbb{R}^m$ be a bounded function such that $c_i(x) \geq 1$ for all $x \in [0, 1]$ and $i \in \{1, , \ldots, m\}$, where we use an index i to denote the ith component of a vector valued function (such as u, v, r, g, f, h and c). Let Λ be a $m \times k$ real matrix and let $\operatorname{diag}(c(x))$ denote the $m \times m$ diagonal matrix whose entries are given by vector $c(x)$. Then consider the following system of first-order partial differential equations with initial condition:

$$\begin{cases} \partial_t u(x, t) = g(x, u(x, t), v(x, t)), \\ \partial_t v(x, t) = -\operatorname{diag}(c(x))\partial_x v(x, t) + f(x, u(x, t), v(x, t)) \\ r'(t) = h(v(1, t), r(t)) \\ u(x, 0) = u_0(x), \;\; v(x, 0) = v_0(x) \text{ and } r(0) = r_0, \end{cases} \tag{1.1}$$

with boundary condition

$$v(0, t) = \Lambda r(t). \tag{1.2}$$

The problem above is said to be well posed if, at least for small times t, it determines unambiguously the trajectory of the system and such trajectories present some continuity with respect to initial conditions. In order to prove that this is really the case the problem can be interpreted as a semilinear evolution problem defined on a certain Banach space X,

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u, v) \\ \frac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{diag}(c)\partial_x v + \mathcal{F}(u, v) \\ \frac{\mathrm{d}r}{\mathrm{d}t} = \mathcal{H}(v, r) \\ (v(0), u(0), r(0)) = (v_0, u_0, r_0) \in X \end{cases}, \tag{1.3}$$

and then try to proceed as in the case of ordinary differential equations. The first difficulty one encounters relates to the fact that the operator defining the right hand side of (1.3) is not smooth enough (i.e. Lipschitz) to apply the Picard iterative scheme, being the differential operator ∂_x the reason of that. To overcome this problem Ammon Pazy's book [22] develops a theory to treat the system composed only by the "difficult" (but linear) *principal part* (i.e. the differential operator or whatever that creates the loss in regularity) and after consider the whole system as a smooth perturbation of the system generated by just the principal part. The tool to link the perturbed system with the unperturbed one is the variation of constants formula, which makes possible to prove not only the well posedness of the problem but also the principle of linearised stability. Although this is a typical procedure to deal with semilinear partial differential equations, in our case some difficulties arise due to the nonlinearities $\mathcal{G}$ and $\mathcal{F}$ involved, which take the form of Nemytskij operators defined by g and f, namely $\mathcal{G}(u(\cdot, t), v(\cdot, t))(x) = g(x, u(x, t), v(x, t))$ and $\mathcal{F}(u(\cdot, t), v(\cdot, t))(x) = f(x, u(x, t), v(x, t))$. In fact, these difficulties were also highlighted in [18] for a hyperbolic system similar, but not reducible, to (1.3).

First of all, we have to decide which Banach space X we should use in order to study the system above. From a conceptual point of view, if u and v are densities on the interval [0, 1], it would be natural to consider spaces based on L^1 for these variables. However, the Nemytskij operators $\mathcal{G}$ and $\mathcal{F}$ defined on a space of integrable $\mathbb{R}$-valued functions on [0, 1] are very often not well defined and more importantly, they are Fréchet differentiable only if g and f are affine functions [17], which prevents us from using well defined but truncated versions of $\mathcal{G}$ and $\mathcal{F}$. This makes impossible to linearise the system around steady states in order to study their stability properties. Although this lack of differentiability does not invalidate the principle of linearised stability per se, it makes necessary to use ad hoc techniques to analyse the behaviour of the system around stationary points (see [10] for an example of that).

In order to avoid this lack of smoothness related to the Nemytskij operators on L^1, we may use spaces based on the sup norm. It is easy to prove that these operators inherit the smooth properties of their associated functions in spaces with the sup norm. In particular, if g is differentiable, then the Fréchet derivative of $\mathcal{G}$ at a point $(\bar{u}, \bar{v})$ is the operator

$$D\mathcal{G}(\bar{u}, \bar{v}) \begin{pmatrix} u \\ v \end{pmatrix} = D_2 g(\cdot, \bar{u}(\cdot), \bar{v}(\cdot))u(\cdot) + D_3 g(\cdot, \bar{u}(\cdot), \bar{v}(\cdot))v(\cdot), \tag{1.4}$$

and analogously for f. Specifically, the sup norm space X we are going to work with is

$$X = X_1 \times X_2 \tag{1.5}$$

with $X_1 = L^n_\infty := (L^\infty(0,1))^n$ and

$$X_2 = (C^m \times \mathbb{R}^k)_b := \left\{(v, r) \in C([0,1], \mathbb{R})^m \times \mathbb{R}^k \quad | \quad v(0) = \Lambda r\right\}.$$

Realise that the definition of $(C^m \times \mathbb{R}^k)_b$ comprises the boundary condition of the system. The reason why we choose X to have this form rather than a simpler one such as $L^n_\infty \times C([0,1], \mathbb{R})^m \times \mathbb{R}^k$ or $L^n_\infty \times L^m_\infty \times \mathbb{R}^k$ is that the trajectories would not be continuous on these spaces.

In order to follow the standard semilinear formulation of system (1.3) as it is described in [22], we consider the linear principal part and the nonlinear Lipschitz perturbation separately. Thus, the operator A defined as

$$A \begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{diag}(c(\cdot))v'(\cdot) \\ 0 \end{pmatrix} \tag{1.6}$$

with domain

$$D(A) = \{(u, v, r) \in X \mid (0, \mathrm{diag}(c(\cdot))v'(\cdot), 0) \in X\},$$

is the infinitesimal generator of a strongly continuous semigroup T. The prime symbol $'$ in the previous expressions refers to the generalised notion of derivative. Being the principal part of the system specified, the perturbation is the operator $\mathcal{P}$ that sends elements $(u, v, r) \in X$ into $\mathcal{P}(u, v, r) = (\mathcal{G}(u, v), \mathcal{F}(u, v), \mathcal{H}(v, r))$. At this point, however, we encounter an obstacle that prevents us from applying the standard semilinear formulation in a straightforward manner. It turns out that the range of the perturbation $\mathcal{P}$ is not within X but contained in the bigger space $Y = L^n_\infty \times L^m_\infty \times \mathbb{R}^k$. Since $\mathcal{P}$ is Lipschitz from X into Y, the problem is now not a lack of regularity but a lack of definition of the semigroup T on the space Y, in the sense that the variation of constants equation

$$\begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \\ r(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0(\cdot) \\ v_0(\cdot) \\ r_0 \end{pmatrix} + \int_0^t T(t-s)\mathcal{P} \begin{pmatrix} u(\cdot, s) \\ v(\cdot, s) \\ r(s) \end{pmatrix} \mathrm{d}s$$

is ambiguous. To overcome this difficulty the semigroup T should be extended into Y, and the natural way to do that is to define the operator A in Y instead of X. However, this procedure is not as easy as it could seem. In X some hypotheses are satisfied by A that guarantee the existence of a strongly continuous semigroup whereas in Y such hypotheses could not hold. In order to deal with semigroups that fail to be strongly continuous we apply the sun-dual framework developed in [6,7]. This theory extends the standard semilinear formulation by allowing perturbations that take values on certain Banach space related to the phase space in which the system is defined. It is worth pointing out that this theory has commonly been used to treat systems of delay differential equations [8,11], whereas here it is applied to a system of PDEs (similarly as in [18]).

This article is organised as follows. In Sect. 2 the theory of sun-dual semigroups is used to prove that the initial value problem (1.1) with boundary condition (1.2) is well posed (Theorem 1). In Sect. 3 a criterion to infer the (local) stability behaviour of a stationary solution of (1.1) is given (Theorem 4). This is done by showing that the linearised semigroup around any stationary solution of (1.1) is eventually norm continuous (Theorems 2 and 3), which implies that the behaviour of the linearised semigroup can be summarised in terms of the dominant eigenvalues and eigenvectors of its generator. In Section 4 we show that problem (1.1) and (1.2) can be used to model the growth of microorganisms along the intestines of animals, a topic with implications ranging from human health [23] to biodiversity [24]. The model is an extended version of the one model formulated in [1] and treated in [14,25]. The extension (first presented in [4] and later developed in [2]) includes environmental microbes, which makes possible to study reinfection events and transmission between hosts. In the same section we illustrate how the results of Sects. 2 and 3 are applied to this particular model. In Appendix A we recall the most important results (for our purposes) of the work done by Clement et al. [6,7], which can be used to study the existence and uniqueness of solutions to our problem as well as the stability properties of the steady states. Several definitions and results given in the appendix are used widely in Sects. 2 and 3.

2. Semilinear formulation of the problem

To deal with problem (1.3) we first give a formula for the semigroup T generated by A (see (1.6)) and then we show that T is indeed closed by $\odot*$-integration on Y (see Definition A2). This is done by checking that T has the structure of Example A1. Once this property of T has been verified, the well posedness of the problem will be derived as a consequence of Theorem A2. Finally, we show that the linearised semigroup S around any stationary point of (1.3) is eventually norm continuous, so that Theorem A7 gives a method to determine the stability of equilibria based only on the spectrum of the infinitesimal generator A_S of S.

2.1. Existence and uniqueness of solutions

The semigroup T on X generated by A can be obtained through the method of characteristics, and acts in the following way:

$$T(t)\begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} u(\cdot) \\ \tilde{v}(\cdot; t, v, r) \\ r \end{pmatrix} \tag{2.1}$$

where the ith component of $\tilde{v}(\cdot; t, v, r)$ is

$$\begin{aligned}(\tilde{v}(x; t, v, r))_i &= \tilde{v}_i(x; t, v_i, r) \\ &= \Lambda_i r \mathbb{1}_-(\varphi_i(-t, x)) + v_i(\varphi_i(-t, x))\mathbb{1}_+(\varphi_i(-t, x)),\end{aligned} \tag{2.2}$$

being Λ_i the ith row of Λ and $\varphi_i(t,x)$ the unique function satisfying $\partial_t\varphi_i(t,x) = c_i(\varphi_i(t,x))$ and $\varphi_i(0,x) = x$ (although function $c_i(x)$ is not necessarily Lipschitz, the uniqueness property can be deduced because $c_i(x) \geq 1$ for all x). Functions $\mathbb{1}_-$ and $\mathbb{1}_+$ stand for the indicator functions on $(-\infty, 0)$ and $[0,\infty)$ respectively. To give a meaning to $\varphi_i(-t,x)$ for any $-t < 0$ is enough to assume that the function c_i is prolonged by any positive constant (for instance 1) for negative values of its argument. Notice that $\varphi_i(\cdot,x)$ describes, as a function of t, the characteristic curve passing through the point x at time 0, so that $\varphi_i(-t,x)$ should be interpreted as the position that a point moving with velocity c had t units of times into the past. Therefore, a point that at time t is at position x, was at position $\varphi_i(-t,x)$ at time 0. In particular, if $\varphi_i(-t,x) < 0$ then we deduce that the point was outside the interval $[0,1]$ at time 0, hence the value of v_i at (t,x) is not given by the initial condition of v but by the boundary condition.

Remark 1. In order to keep track of the following results we recommend to consider the scalar case in which $n = 1$, $m = 1$ and $k = 1$. In this situation the subscript i can be dropped (since function v has only one component) and matrix Λ becomes a number. It may also help to consider function c to be constantly 1. Notice that in this case the function φ introduced above becomes $\varphi(t,x) = x - t$.

Clearly, T is a diagonal semigroup on $X = X_1 \times X_2 = L^n_\infty \times (C^m \times \mathbb{R}^k)_b$ (the detailed definition of X is given in (1.5)). Let T_1 and T_2 be the associated semigroups of T on X_1 and X_2 respectively. Since $Y = L^n_\infty \times L^m_\infty \times \mathbb{R}^k$, in order to show that T is closed by $\odot*$-integration on Y it is enough to verify that i) $L^m_\infty \times \mathbb{R}^k$ can be identified with a subspace of $X_2^{\odot_{T_2}*}$ and that ii) X_2 is sun-reflexive with respect to T_2 (as showed in Example A1). Notice that here Y has not exactly the same meaning as in Appendix A. There Y was a subspace of $X^{\odot*}$ whereas here Y is a representation of a subspace of $X^{\odot*}$. We proceed in this way because then we can consider X as a subspace of Y and avoid the use of the inclusion $j : X \to X^{\odot*}$ in the formulation of the results. Next we show that the two conditions mentioned above (i and ii) hold and, in addition, we specify how $T_2^{\odot*}$ is defined on $L^m_\infty \times \mathbb{R}^k$.

Proposition 1. *Let $X_2 = (C^m \times \mathbb{R}^k)_b$ and T_2 be as above. Then*

$$L^m_\infty \times \mathbb{R}^k \cong X_2^{\odot_{T_2}*} \quad \text{and} \quad X_2 \cong X_2^{\odot\odot_{T_2}}.$$

Moreover, $T_2^{\odot*}$ is the natural extension of T_2 into $L^m_\infty \times \mathbb{R}^k$, i.e.

$$T_2^{\odot*}(t)\begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} \tilde{v}(\cdot; t, v, r) \\ r \end{pmatrix}$$

with $\tilde{v}$ exactly given as in (2.2).

As a consequence, the semigroup T defined on X fulfills the hypothesis of Example A1 and hence is closed by $\odot*$-integration on Y.

Proof. The methodology of the proof is based on the one used in [11, Section II.5], and hence we only expose the main ideas (a more detailed version of the proof can be found in [3, Appendix B]). First the Riesz representation theorem is used to represent X_2^* as the space $\mathcal{M}_b^m \times \mathbb{R}^k$, where $\mathcal{M}_b$ is the set of real Borel measures μ satisfying $\mu(\{0\}) = 0$. Instead of $\mu(\{0\}) = 0$ other conditions could be imposed in order to identify Borel Measures (seen as functionals on $C([0, 1], \mathbb{R})^m \times \mathbb{R}^k$) that are equivalent on X_2 due to the boundary condition. The pairing between X_2^* and X_2 can be written as

$$\left\langle \begin{pmatrix} \mu \\ q \end{pmatrix}, \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle = \sum_{i=1}^{m} \int_0^1 v_i(s) \mathrm{d}\mu_i(s) + \langle q, r \rangle.$$

In order to work with functions instead of with measures, an isometric isomorphism between real Borel measures and normalised functions of bounded variation is used, so that X_2^* is represented as $\mathrm{NBV}_b^m \times \mathbb{R}^k$ with $\eta \in \mathrm{NBV}_b$ if η is a function of bounded variation, right continuous in $[0, 1]$ and satisfies $\eta(0) = 0$ (the norm in NBV_b is the total variation norm).

The next step consists in showing $X_2^{\odot T_2} \cong L^1(0, 1)^m \times \mathbb{R}^k$ where the norm in $L^1(0, 1)^m$ is weighted by function c, in the sense that

$$\|\nu\| = \sum_{i=1}^{m} \int_0^1 \left| \frac{\nu_i(s)}{c_i(s)} \right| \mathrm{d}s. \tag{2.3}$$

This is done taking into account the result $\overline{D(A_{T_2}^*)} = X_2^{\odot T_2}$ (which is a particular case of the equality $\overline{D(A_T^*)} = X^{\odot T}$ that holds for any strongly continuous semigroup T generated by A_T on a Banach space X, proved in [11, Proposition AII.3.8]). To apply this result first $D(A_{T_2}^*)$ is determined, which results in the pairs $(\eta, q) \in \mathrm{NBV}_b^m \times \mathbb{R}^k$ such that for all component of η there exists $\nu_i \in \mathrm{NBV}_b$ with $\nu_i(1) = 0$ satisfying

$$\eta_i(s) = \int_0^s \frac{\nu_i(\sigma)}{c_i(\sigma)} \mathrm{d}\sigma \qquad \forall s \in [0, 1].$$

Then the closure of $D(A_{T_2}^*)$ is shown to be isometrically isomorphic to

$$L^1(0, 1)^m \times \mathbb{R}^k$$

where the norm in $L^1(0, 1)^m$ is weighted by function c as stated, i.e. in the sense (2.3).

To represent $X_2^{\odot T_2} \subset X_2^*$ as $L^1(0, 1)^m \times \mathbb{R}^k$, we must define a pairing between $L^1(0, 1)^m \times \mathbb{R}^k$ and X_2. The natural pairing is the one defined by $\langle (\nu, q), (v, r) \rangle := \langle \phi((\nu, q)), (v, r) \rangle$ where ϕ is the isometric isomorphism between $L^1(0, 1)^m \times \mathbb{R}^k$ and $\overline{D(A_{T_2}^*)} \subset \mathrm{NBV}_b^m \times \mathbb{R}^k$, which results in

$$\left\langle \begin{pmatrix} \nu \\ q \end{pmatrix}, \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle = \sum_{i=1}^{m} \int_0^1 v_i(s) \frac{\nu_i(s)}{c_i(s)} \mathrm{d}s + r \cdot q.$$

Finally it is proven that $X_2^{\odot\odot T_2} \cong X_2$. Indeed, using $L^1(0,1)^m \times \mathbb{R}^k$ as a representation of $X_2^{\odot T_2}$ clearly implies $X_2^{\odot T_2 *} \cong L^\infty(0,1)^m \times \mathbb{R}^k$. In this case the norm of $L^\infty(0,1)^m$ is the standard one (i.e. is not affected by function c) if the pairing

$$\left\langle \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \nu \\ q \end{pmatrix} \right\rangle = \sum_{i=1}^m \int_0^1 \frac{v_i(s)}{c_i(s)} \nu_i(s) \mathrm{d}s + r \cdot q$$

is used, as we do. On the other hand, the semigroup $T_2^\odot$ on $(L^1(0,1))^m \times \mathbb{R}^k$ is specified by

$$T_2^\odot(t) \begin{pmatrix} \nu \\ q \end{pmatrix} = \begin{pmatrix} \tilde{\nu}(\cdot; t, \nu) \\ q + \sum_{i=1}^m \int_0^1 \frac{\nu_i(s)}{c_i(s)} \mathbb{1}_-(\varphi_i(-t,s)) \mathrm{d}s \ \Lambda_i \end{pmatrix}$$

where Λ_i must be interpreted as a vector of $\mathbb{R}^k$ and where the ith component of $\tilde{\nu}(\cdot; t, \nu)$ is

$$\tilde{\nu}_i(\cdot, t, \nu) = c_i(\cdot) \frac{\nu_i(\varphi_i(t,\cdot))}{c_i(\varphi_i(t,\cdot))} \partial_2 \varphi_i(t,\cdot) \mathbb{1}_+(\varphi_i(-t,1) - \cdot).$$

Similarly, $T_2^\odot$ is used to give an explicit formula for the semigroup $T_2^{\odot *}$ on $L_\infty^m \times \mathbb{R}^k$, which results in the natural extension of T_2 in $L_\infty^m \times \mathbb{R}^k$.

Then, the infinitesimal generator of $T_2^\odot$ on $L^1(0,1)^m \times \mathbb{R}^k$ is determined as

$$D(A_{T_2^\odot}) = \left\{ (\nu, q) \in L^1(0,1)^m \times \mathbb{R}^k \mid \nu \text{ is absolutely continuous and } \nu(1) = 0 \right\}$$

and $A_{T^\odot}(\nu, q) = (c\nu', \sum_{i=1}^m \nu_i(0)\, \Lambda_i)$. By saying that ν is absolutely continuous we mean that the condition is satisfied component-wise, i.e. for each component of ν. The adjoint of $A_{T_2^\odot}$ is consequently determined as

$$D(A^*_{T_2^\odot}) = \left\{ (v, r) \in (L^\infty(0,1))^m \times \mathbb{R}^k \mid v \text{ is Lipschitz and } v(0) = \Lambda r \right\} \quad (2.4)$$

and $A^*_{T_2^\odot}(v, r) = (-cv', 0)$. By using [11, Proposition AII.3.8] one more time, we obtain $X_2^{\odot\odot T_2}$ as the closure of $D(A^*_{T_2^\odot})$. When doing so we lose the Lipschitz condition on v but the continuity remains. Therefore,

$$X_2^{\odot\odot T_2} \cong \left\{ (v, r) \in C([0,1], \mathbb{R})^m \times \mathbb{R}^k \mid v(0) = \Lambda r \right\} = X_2,$$

as desired. □

Once shown that T is closed by $\odot *$-integration on Y, we can use Theorem A2 to conclude that one and only one semiflow Σ associated to (1.3) exists. This result is stated below in the form of a theorem.

Theorem 1. *Problem* (1.1) *is well posed on* $X = X_1 \times X_2$ *defined in* (1.5)*, i.e. there exists an open subset* Ω *of* $[0,\infty) \times X$ *(in the induced topology) and a unique function* Σ *from* Ω *to* X *satisfying the following:*

– *for all initial condition* $x = (u_0, v_0, r_0) \in X$ *there exists* $t_x \in (0, \infty]$ *such that* $[0, t_x) \times \{x\}$ *is the intersection of* Ω *with* $[0, \infty) \times \{x\}$*,*
– *for all* $x = (u_0, v_0, r_0) \in X$*, function* $\Sigma(\cdot; x)$ *from* $[0, t_x)$ *to* X *is a mild solution of* (1.1) *(in the sense of Definition* A3*). Thus,* Σ *satisfies the semigroup property, i.e.* $\Sigma(t+s, x) = \Sigma(t, \Sigma(s, x))$ *for positive* s *and* t *with* $t + s < t_x$*, whereas function* $\Sigma(t; \cdot)$ *is locally Lipschitz in* x*.*

2.2. Linearisation around steady states

Given a steady state $(\bar{u}, \bar{v}, \bar{r}) \in X$ of the semiflow Σ, Theorem A3 ensures that there exists a strongly continuous linear semigroup $S(t)$ on X such that $S(t) = D_2\Sigma(t; (\bar{u}, \bar{v}, \bar{r}))$. Moreover, using that $(v, r) \in D(A^*_{T_2^\odot})$ if and only if v is Lipschitz and $v(0) = \Lambda r$ (see (2.4)), it follows that the domain of the generator A_S of the semigroup $S(t)$ (see Theorem A1) can be written as

$$D(A_S) = \left\{ \begin{pmatrix} u \\ v \\ r \end{pmatrix} \in X \mid v \text{ is Lipschitz and } \begin{pmatrix} 0 \\ -c \cdot v' \\ 0 \end{pmatrix} + D\mathcal{H}(\bar{u}, \bar{v}, \bar{r}) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \in X \right\}$$

and then

$$A_S \begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -c \cdot v' \\ 0 \end{pmatrix} + D\mathcal{H}(\bar{u}, \bar{v}, \bar{r}) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \\ r \end{pmatrix} \in D(A_S). \tag{2.5}$$

Next, in order to study the asymptotic behaviour of Σ around $(\bar{u}, \bar{v}, \bar{r})$, we show that $S(t)$ is eventually norm continuous (see Theorem 3), so that Theorem A7 can be applied and a characterization of the local dynamics around $(\bar{u}, \bar{v}, \bar{r})$ can be given in terms of the eigenvalues of the operator A_S (see Theorem 4 at the end of the section). The proof is long and will occupy the rest of the section.

Let us start noticing that $D\mathcal{H}(\bar{u}, \bar{v}, \bar{r})$ has the form

$$D\mathcal{H}(\bar{u}, \bar{v}, \bar{r}) \begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} B_{11}u + B_{12}v \\ B_{21}u + B_{22}v \\ \tilde{K}(v, r) \end{pmatrix}.$$

where

$$\begin{aligned} B_{11} &\in L_\infty((0,1), \mathcal{M}_{n\times n}(\mathbb{R})), \\ B_{12} &\in L_\infty((0,1), \mathcal{M}_{n\times m}(\mathbb{R})), \\ B_{21} &\in L_\infty((0,1), \mathcal{M}_{m\times n}(\mathbb{R})), \\ B_{22} &\in L_\infty((0,1), \mathcal{M}_{m\times m}(\mathbb{R})), \end{aligned}$$

and $\tilde{K}$ is a bounded linear operator from X_2 into $\mathbb{R}^k$ (for instance, $B_{11}(x) = D_2 g(x, \bar{u}(x), \bar{v}(x))$). Thus, the generator A_S can be formally written as

$$A_S \begin{pmatrix} u \\ v \\ r \end{pmatrix} = A \begin{pmatrix} u \\ v \\ r \end{pmatrix} + B \begin{pmatrix} u \\ v \\ r \end{pmatrix} + K \begin{pmatrix} u \\ v \\ r \end{pmatrix},$$

where B and K are bounded operators from X into $X^{\odot_T *}$. Their explicit expressions are

$$B\begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} B_{11}u + B_{12}v \\ B_{21}u + B_{22}v \\ 0 \end{pmatrix} \quad \text{and} \quad K\begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{K}(v, r) \end{pmatrix}. \tag{2.6}$$

Notice that K is a compact operator because it takes values in a finite dimensional subspace of $X^{\odot *}$, namely $\{0\} \times \{0\} \times \mathbb{R}^k$. Thus, in order to show that S is eventually norm continuous it is enough to prove the eventually norm continuity of the simpler semigroup generated by $A + B$. This is so because compact perturbations of eventually norm continuous semigroups are also eventually norm continuous (see [12, Proposition III.1.14] and Theorem A8 in the appendix).

The proof of the eventual norm continuity of the semigroup S_B generated by $A + B$ is based on the series formula for S_B given in Theorem A1. From that theorem we know that the series

$$\sum_{k=0}^{\infty} S_k,$$

with $S_0 = T$ and $S_k = \int_0^{\cdot} T^{\odot *}(\cdot - s) B S_{k-1}(s) \mathrm{d}s$ for $n > 1$, converges uniformly (on compact time intervals) towards S_B. By the Uniform Convergence Theorem, we know that if function S_k is continuous in $[t_0, \infty)$ for each k, then so is S_B. Therefore, it is enough to prove that each term in the series defining S_B is continuous on $[1, \infty)$ in order to conclude that S_B is norm continuous from 1 onwards.

The specific value at which the functions defining the series become continuous is 1 due to the fact that $c_i(x) \geq 1$ for all i, which implies that $\varphi_i(-t, x) < 0$ for all $(t, x) \in (1, \infty) \times [0, 1]$. The proof consists in showing that the operator norm of $(S_k(t + h) - S_k(t))$ can be bounded as $\|(S_k(t + h) - S_k(t))\| < M_{t,k}|h|$ for all $k \in \mathbb{N}$, $t > 1$ and $|h|$ small enough, where $M_{t,k}$ is a constant that only depends on t and k. In order to do that the image of a point (u, v, r) by $S_k(t + h) - S_k(t)$ is expressed as the sum of several terms and each of them is properly bounded. Hence the methodology is technical, though the tricks are mostly elementary. Unfortunately, we failed in giving a proof by induction on the index k.

Next we are going to prove some intermediate steps that will lead to the result we desire. First, let us recall that the norms of X and $j(X)$ are equivalent (Proposition A.1), so that there exists $M > 0$ such that

$$\left\| j\begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_{X^{\odot *}} \leq \left\| \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_X \leq M \left\| j\begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_{X^{\odot *}}. \tag{2.7}$$

Then, for all $k \geq 1$ and $h \in (0, 1)$ observe that the definition of S_k implies

$$\begin{aligned}
&\left\| (S_k(t+h) - S_k(t)) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_X \\
&\leqslant M \left\| \int_0^{t+h} T^{\odot *}(t+h-s) B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s - \int_0^t T^{\odot *}(t-s) B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s \right\|_{X^{\odot *}} \qquad (2.8) \\
&\leqslant M \left\| \begin{pmatrix} \hat{u}(\cdot; t+h) - \hat{u}(\cdot; t) \\ \hat{v}(\cdot; t+h) - \hat{v}(\cdot; t) \\ 0 \end{pmatrix} \right\|_{X^{\odot *}} \\
&\leqslant M(\|\hat{u}(\cdot; t+h) - \hat{u}(\cdot; t)\|_{L^n_\infty} + \|\hat{v}(\cdot; t+h) - \hat{v}(\cdot; t)\|_{L^m_\infty})
\end{aligned}$$

where, defining the projection $\pi_u : L^n_\infty \times L^m_\infty \times \mathbb{R}^m \to L^n_\infty$ so that $\pi_u(u, v, r) = u$,

$$\hat{u}(\cdot; t) = \int_0^t \pi_u B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s, \qquad (2.9)$$

and, defining the projection $\pi_v : L^n_\infty \times L^m_\infty \times \mathbb{R}^m \to L^m_\infty$ so that $\pi_v(u, v, r) = v$,

$$\hat{v}(\cdot; t) = \int_0^t \pi_v T^{\odot *}(t-s) B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s. \qquad (2.10)$$

From inequality (2.8) it follows that in order to obtain a bound of the form $\|(S_k(t+h) - S_k(t))\| < M_t h$ one has to focus on the terms $\|\hat{u}(\cdot; t+h) - \hat{u}(\cdot; t)\|_{L^n_\infty}$ and $\|\hat{v}(\cdot; t+h) - \hat{v}(\cdot; t)\|_{L^m_\infty}$. This is what is done in the following lemmas.

Lemma 1. *For all $t > 0$ and $h \in (0, 1)$, there exists $\tilde{M}_t > 0$ such that the term $\|\hat{u}(\cdot; t+h) - \hat{u}(\cdot; t)\|_{L^n_\infty}$ defined in (2.8) can be bounded as*

$$\|\hat{u}(\cdot; t+h) - \hat{u}(\cdot; t)\|_{L^n_\infty} < \tilde{M}_t h \, \|(u, v, r)\|_X .$$

Proof. Since the operator norm of $S_{k-1}(s)$ is uniformly bounded within bounded intervals of time, using (2.9) one has

$$\begin{aligned}
&\|\hat{u}(\cdot; t+h) - \hat{u}(\cdot; t)\|_{L^n_\infty} \\
&\leqslant \left\| \pi_u \left(\int_0^{t+h} B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s - \int_0^t B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \mathrm{d}s \right) \right\|_{L^n_\infty} \\
&\leqslant \int_t^{t+h} \left\| B S_{k-1}(s) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_{X^{\odot *}} \mathrm{d}s \leqslant h \|B\| \sup_{s \in [0, t+1]} \|S_{k-1}(s)\| \left\| \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_X . \qquad (2.11)
\end{aligned}$$

□

Let us focus now on the terms $\|\hat{v}(\cdot;t+h)-\hat{v}(\cdot;t)\|_{L^m_\infty}$ of (2.8). Notice that B initially defined from X into $X^{\odot *}$ can be extended to a bounded operator from $X^{\odot *}$ into itself given by the same expression of B (see (2.6)). Thus, it is possible to enter the operators $T^{\odot *}(t-s_k)$ and B inside the integral that defines $S_k(s)$ in (2.10). That is, $\hat{v}(\cdot;t)$ can be written as

$$\hat{v}(\cdot;t)=\int_0^t \pi_v T^{\odot *}(t-s_k)B\int_0^{s_k} T^{\odot *}(s_k-s_{k-1})BS_{k-2}(s_{k-1})\begin{pmatrix}u\\ v\\ r\end{pmatrix}\mathrm{d}s_{k-1}\mathrm{d}s_k$$

$$=\int_0^t\int_0^{s_k} \pi_v T^{\odot *}(t-s_k)BT^{\odot *}(s_k-s_{k-1})BS_{k-2}(s_{k-1})\begin{pmatrix}u\\ v\\ r\end{pmatrix}\mathrm{d}s_{k-1}\mathrm{d}s_k,$$

and, inductively,

$$\hat{v}(\cdot;t)=\int_0^t\int_0^{s_k}\cdots\int_0^{s_2} \pi_v T^{\odot *}(t-s_k)BT^{\odot *}(s_k-s_{k-1})B\cdots T^{\odot *}(s_2-s_1)BT(s_1)\begin{pmatrix}u\\ v\\ r\end{pmatrix}\mathrm{d}s_1\cdots\mathrm{d}s_{k-1}\mathrm{d}s_k. \tag{2.12}$$

In order to simplify the above equation recall that $X^{\odot *}=L^n_\infty\times L^m_\infty\times\mathbb{R}^m$ and observe that the operators $T^{\odot *}(s_{l+1}-s_l)B : X^{\odot *}\to X^{\odot *}$ for $l\in\{1,\dots,k\}$ (setting $s_{k+1}=t$) can be synthesised as matrices of operators:

$$T^{\odot *}(s_{l+1}-s_l)B\sim\begin{pmatrix} B_{11} & B_{12} & 0\\ \tilde{T}(s_{l+1}-s_l)B_{21} & \tilde{T}(s_{l+1}-s_l)B_{22} & 0\\ 0 & 0 & 0\end{pmatrix},$$

where $\tilde{T}(s) : L^m_\infty\to L^m_\infty$ is defined as

$$(\tilde{T}(s)v)_i=v_i(\varphi_i(-s,\cdot))\mathbb{1}_+(\varphi_i(-s,\cdot))\qquad \forall i\in\{1,\dots,m\}. \tag{2.13}$$

The product of operators

$$\prod_{l=k}^{1}T^{\odot *}(s_{l+1}-s_l)B=T^{\odot *}(t-s_k)BT^{\odot *}(s_k-s_{k-1})B\cdots T^{\odot *}(s_2-s_1)B,$$

(with $s_{k+1}=t$) written as a matrix of operators, becomes:

$$J\sim\begin{pmatrix} J_{11} & J_{12} & 0\\ J_{21} & J_{22} & 0\\ 0 & 0 & 0\end{pmatrix}$$

where, for $(i_0, i_{k+1}) \in \{1,2\} \times \{1,2\}$,

$$J_{i_0 i_{k+1}} = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_k=1}^{2} J_{i_0 i_1 i_2 \cdots i_k i_{k+1}}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1)$$

with

$$\begin{aligned} &J_{i_0 i_1 i_2 \cdots i_k i_{k+1}}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) \\ &\quad = (T^{\odot *}(t - s_k)B)_{i_0 i_1} (T^{\odot *}(s_k - s_{k-1})B)_{i_1 i_2} \cdots (T^{\odot *}(s_2 - s_1)B)_{i_k i_{k+1}}. \end{aligned} \tag{2.14}$$

In particular, using $\tilde{v}(\cdot; s_1, v, r)$ defined in (2.2) the integrand in (2.12) can be expressed as

$$\pi_v \left(\prod_{l=k}^{1} T^{\odot *}(s_{l+1} - s_l) B \right) T(s_1) \begin{pmatrix} u \\ v \\ r \end{pmatrix} = J_{21} u + J_{22} \tilde{v}(\cdot; s_1, v, r)$$

so that (2.12) becomes the sum

$$\hat{v}(\cdot; t) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_k=1}^{2} I_{2\, i_1 i_2 \cdots i_k 1}(t, u) + \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_k=1}^{2} I_{2\, i_1 i_2 \cdots i_k 2}(t, v, r) \tag{2.15}$$

where

$$\begin{aligned} &I_{2\, i_1 i_2 \cdots i_k 1}(t, u) \\ &\quad = \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} J_{2\, i_1 i_2 \cdots i_k 1}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) u \, \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \mathrm{d}s_k \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &I_{2\, i_1 i_2 \cdots i_k 2}(t, v, r) \\ &\quad = \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} J_{2\, i_1 i_2 \cdots i_k 2}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) \tilde{v}(\cdot; s_1, v, r) \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \mathrm{d}s_k. \end{aligned} \tag{2.17}$$

Clearly, from (2.15) it follows

$$\begin{aligned} &\|\hat{v}(\cdot; t+h) - \hat{v}(\cdot; t)\|_{L_\infty^m} \\ &\quad \le \textstyle\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_k=1}^{2} \left\| I_{2\, i_1 i_2 \cdots i_k 1}(t+h, u) - I_{2\, i_1 i_2 \cdots i_k 1}(t, u) \right\|_{L_\infty^m} \\ &\qquad + \textstyle\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_k=1}^{2} \left\| I_{2\, i_1 i_2 \cdots i_k 2}(t+h, v, r) - I_{2\, i_1 i_2 \cdots i_k 2}(t, v, r) \right\|_{L_\infty^m}. \end{aligned} \tag{2.18}$$

In order to give a bound for $\|\hat{v}(\cdot; t+h) - \hat{v}(\cdot; t)\|_{L_\infty^m}$, we show that each summand on the right hand side in (2.18) can be bounded properly, i.e. by something of the form $h\tilde{M}_t \|(u, v, r)\|$. This is done by performing a change of variables to the "shifted integrals" (the ones evaluated at $t + h$) so that the new integrand coincides with the integrand of the "unshifted integrals" (the ones evaluated at t). The equality between integrands makes, on the one hand, that the difference between integrals vanishes over the domain of integration that is common to both integrals. On the other hand, the domains of integration that are specific to each integral have a Lebesgue measure proportional to h. These statements are developed in the following. To do so we treat separately the terms in the first summation from those in the second summation.

Lemma 2. *Let $I_{2\,i_1 i_2 \cdots i_k 1}(t, u)$ be defined as in (2.16). For all $t > 0$ and $h \in (0, 1)$, there exists $\tilde{M}_t > 0$ such that*

$$\left\| I_{2\,i_1 i_2 \cdots i_k 1}(t + h, u) - I_{2\,i_1 i_2 \cdots i_k 1}(t, u) \right\|_{L^m_\infty} < \tilde{M}_t h \, \|(u, v, r)\|_X .$$

Proof. Applying a translation τ_h to the integration variables so that

$$\tau_h(s_1, s_2, \ldots, s_k) = (s_1 + h, s_2 + h, \ldots, s_k + h)$$

the integral $I_{2\,i_1 i_2 \cdots i_k 1}(t + h, u)$ becomes

$$\int_{-h}^{t} \int_{-h}^{s_k} \cdots \int_{-h}^{s_2} J_{2\,i_1 i_2 \cdots i_k 1}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) u \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \mathrm{d}s_k .$$

Therefore, using $\|T^{\odot *}(s)\|_{X^{\odot *}} \leq 1$ for all $s \geq 0$, one has

$$\begin{aligned} &\left\| I_{2\,i_1 i_2 \cdots i_k 1}(t + h, u) - I_{2\,i_1 i_2 \cdots i_k 1}(t, u) \right\|_{L^m_\infty} \\ &= \Bigg\| \int_{-h}^{t} \int_{-h}^{s_k} \cdots \int_{-h}^{\min\{s_2, 0\}} \\ &\qquad J_{2\,i_1 i_2 \cdots i_k 1}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) u \int_{-h}^{t} \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \mathrm{d}s_k \Bigg\|_{L^m_\infty} \\ &\leq h(t + h)^{k-1} \|B\|^k \|(u, v, r)\| \leq h(t + 1)^{k-1} \|B\|^k \|(u, v, r)\| \end{aligned} \tag{2.19}$$

as desired. □

Let us consider now the terms in (2.18) of the form

$$\left\| I_{2\,i_1 i_2 \cdots i_k 2}(t + h, v, r) - I_{2\,i_1 i_2 \cdots i_k 2}(t, v, r) \right\|_{L^m_\infty} .$$

In this case it is not clear whether it is possible to make a change of variables that transforms the integrand of $I_{2\,i_1 i_2 \cdots i_k 2}(t+h, v, r)$ into the integrand of $I_{2\,i_1 i_2 \cdots i_k 2}(t, v, r)$ as we did in the proof of the previous lemma. The problem is that the integration variables do not only appear in the function that determines $J_{2\,i_1 i_2 \cdots i_k 2}$, but they also play a role in the term $\tilde{v}$. Apparently, finding a change of variables that transforms the integrand

$$J_{2\,i_1 i_2 \cdots i_k 2}(t + h - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1) \tilde{v}(\cdot; s_1, v, r)$$

into

$$J_{2\,i_1 i_2 \cdots i_k 2}(t-\sigma_k, \sigma_k-\sigma_{k-1}, \ldots, \sigma_2-\sigma_1)\tilde{v}(\cdot;\sigma_1, v, r)$$

amounts to solve the following system of $k+1$ equations and k unknows:

$$\begin{cases} t+h-s_k = t-\sigma_k \\ s_k - s_{k-1} = \sigma_k - \sigma_{k-1} \\ \vdots \\ s_2 - s_1 = \sigma_2 - \sigma_1 \\ s_1 = \sigma_1 \end{cases}, \tag{2.20}$$

which is impossible. However, the fact is that one of the equations in the system above is unnecessary to find the desired change of variables. To justify this let us distinguish the case $I_{2\,22\cdots 22}$ from the cases $I_{2\,i_1 i_2\cdots i_k 2}$ in which at least one index is 1.

Lemma 3. *Let* $I_{2\,i_1 i_2\cdots i_k 2}(t, v, r)$ *be defined as in* (2.17)*. Let* $i_l = 1$ *for some* $l \in \{1, \ldots, k\}$*. For all* $t > 0$ *and* $h \in (0, 1)$*, there exists* $\tilde{M}_t > 0$ *such that*

$$\left\| I_{2\,i_1 i_2\cdots i_k 2}(t+h, v, r) - I_{2\,i_1 i_2\cdots i_k 2}(t, v, r) \right\|_{L^m_\infty} < \tilde{M}_t h \left\|(u, v, r)\right\|_X .$$

Proof. Since $i_l = 1$, the term $(T^{\odot *}(s_k - s_{k-1})B)_{i_l i_{l+1}}$ appearing in (2.14) is

$$(T^{\odot *}(s_{k-l+1} - s_{k-l})B)_{i_l i_{l+1}} = (T^{\odot *}(s_{k-l+1} - s_{k-l})B)_{1 i_{l+1}} = B_{1 i_{l+1}},$$

so that $J_{2\,i_1 i_2\cdots i_k 2}$ is independent of the difference $s_{k-l+1} - s_{k-l}$. Thus, the desired change of variables can be obtained without imposing the equality

$$s_{k-l+1} - s_{k-l} = \sigma_{k-l+1} - \sigma_{k-l}.$$

This means that this equation can be removed from system (2.20) so that it becomes compatible. The solution of the reduced system is the desired change of variables, which is a translation τ_h given by

$$\tau_h(s_1, s_2, \ldots, s_k) = (s_1, \ldots, s_{k-l}, s_{k-l+1} + h \ldots, s_k + h).$$

By using this transformation on the "shifted integral" one has

$$\begin{aligned} & I_{2\,i_1 i_2\cdots i_k 2}(t+h, v, r) \\ & = \int_{-h}^{t} \int_{-h}^{s_k} \cdots \int_{-h}^{s_{k-l+2}} \int_{0}^{s_{k-l+1}+h} \int_{0}^{s_{k-l}} \cdots \int_{0}^{s_2} \\ & \quad J_{2\,i_1 i_2\cdots i_k 2}(t - s_k, \ldots, s_2 - s_1)\tilde{v}(\cdot; s_1, v, r)\mathrm{d}s_1 \cdots \mathrm{d}s_k, \end{aligned}$$

and, similarly as done in (2.19), one concludes

$$\begin{aligned} & \left\| I_{2\,i_1 i_2\cdots i_k 2}(t+h, v, r) - I_{2\,i_1 i_2\cdots i_k 2}(t, v, r) \right\|_{L^m_\infty} \\ & \leq h(t+1)^{k-1} \|B\|^k \|(u, v, r)\|. \end{aligned} \tag{2.21}$$

□

The previous proof cannot be applied to give an analogous bound for the terms of the form

$$\|I_{222\cdots 22}(t+h,v,r)-I_{222\cdots 22}(t,v,r)\|_{L^m_\infty}.$$

The problem is that, in this case, the term $J_{222\cdots 22}$ depends on all the differences $s_{l+1}-s_l$ with $l\in\{1,\dots k\}$, so that all the first k rows of system (2.20) have to be imposed. The bound, however, can be proven by showing that the last equation of (2.20) is not needed to give a suitable change of variables.

Lemma 4. *Let $d_i=s_{i+1}-s_i$ for $i\in\{1,\dots,k\}$ and $s_{k+1}=t$. If $t>1$, the functions*

$$\mathbb{1}_+(\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot))$$

and

$$\mathbb{1}_-(\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot))$$

seen as elements of $L^\infty(0,1)$ are the constant functions 0 and 1 respectively.

Proof. Notice that for all $x\in[0,1]$ and $t>1$ one has

$$\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)<0.$$

Indeed, using that $\partial_t\varphi_j(t,x)\geq 1$ for all $j\in\{1,\dots,m\}$, which implies $\varphi_j(-d,x)\leq\varphi_j(0,x)-d=x-d$ for all $d>0$, we deduce

$$\begin{aligned}
&\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)(x)\\
&\quad\leq\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)(x)-s_1\\
&\quad\leq\varphi_{l_3}(-d_2,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)(x)-d_1-s_1\leq\cdots\leq\\
&\quad\leq\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)(x)-\textstyle\sum_{i=1}^{k-2}d_i-s_1\\
&\quad\leq\varphi_i(-d_k,x)-\textstyle\sum_{i=1}^{k-1}d_i-s_1\leq x-\sum_{i=1}^{k}d_i-s_1\\
&\quad=x-\textstyle\sum_{i=1}^{k}(s_{i+1}-s_i)-s_1=x-t<0
\end{aligned}$$

since $t>1$ and $x\in[0,1]$. □

Lemma 5. *Let $J_{222\cdots 22}(t-s_k,s_k-s_{k-1},\dots,s_2-s_1)$ be defined as in (2.14). For all $t>1$ the following holds*

$$\begin{aligned}
&J_{222\cdots 22}(t-s_k,s_k-s_{k-1},\dots,s_2-s_1)\tilde{v}(\cdot;s_1,v,r)\\
&\quad=J_{222\cdots 22}(t-s_k,s_k-s_{k-1},\dots,s_2-s_1)\Lambda r.
\end{aligned}\tag{2.22}$$

Proof. Recall the definition of $\tilde{T}$ given in (2.13). Denoting $d_l=s_{l+1}-s_l$, the ith component of $J_{222\cdots 22}(d_k,\dots,d_1)v$ is

$$\begin{aligned}
(J_{222\cdots 22}(d_k,\dots,d_1)v)_i&=(\tilde{T}(d_k)B_{22}\tilde{T}(d_{k-1})B_{22}\cdots\tilde{T}(d_1)B_{22}v)_i\\
&=\sum_{l_k=1}^{m}\cdots\sum_{l_1=1}^{m}\tilde{T}_i(d_k)B_{22,il_k}\tilde{T}_{l_k}(d_{k-1})B_{22,l_kl_{k-1}}\cdots\tilde{T}_{l_2}(d_1)(B_{22,l_2l_1}v_{l_1})\\
&=\sum_{l_k=1}^{m}\cdots\sum_{l_1=1}^{m}(\tilde{T}_i(d_k)B_{22,il_k}\tilde{T}_{l_k}(d_{k-1})B_{22,l_kl_{k-1}}\cdots\tilde{T}_{l_2}(d_1)B_{22,l_2l_1})\\
&\qquad(\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)v_{l_1}).
\end{aligned}\tag{2.23}$$

In the last equality we have used that, for all $i \in \{1, \dots, m\}$ and for all $f, g \in L^\infty(0, 1)$, the operator $\tilde{T}_i(d)$ satisfies

$$\begin{aligned}\tilde{T}_i(d)(fg) &= f(\varphi_i(-d,\cdot))g(\varphi_i(-d,\cdot))\mathbb{1}_+(\varphi_i(-d,\cdot))\\ &= (f(\varphi_i(-d,\cdot))\mathbb{1}_+(\varphi_i(-d,\cdot)))(g(\varphi_i(-d,\cdot))\mathbb{1}_+(\varphi_i(-d,\cdot)))\\ &= (\tilde{T}_i(d)f)(\tilde{T}_i(d)g), \end{aligned} \tag{2.24}$$

so that for all triad $b, f, g \in L^\infty(0, 1)$ (for instance $b = B_{22,l_3l_2}$, $f = B_{22,l_2l_1}$ and $g = v_{l_1}$) one has

$$b\tilde{T}_i(d)(fg) = (b\tilde{T}_i(d)f)(\tilde{T}_i(d)g).$$

In particular, the ith component of $J_{222\cdots22}(d_k, \dots, d_1)\tilde{v}(\cdot; s_1, v, r)$ is

$$(J_{222\cdots22}(d_k, \dots, d_1)\tilde{v}(\cdot; s_1, v, r))_i = \mathrm{SUM}_v + \mathrm{SUM}_r$$

with

$$\begin{aligned}\mathrm{SUM}_v = \textstyle\sum_{l_k=1}^m \cdots \sum_{l_1=1}^m &(\tilde{T}_i(d_k)B_{22,il_k}\tilde{T}_{l_k}(d_{k-1})B_{22,l_kl_{k-1}}\cdots\tilde{T}_{l_2}(d_1)B_{22,l_2l_1})\\ &(\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)(v_{l_1}(\varphi_{l_1}(-s_1,\cdot))\mathbb{1}_+(\varphi_{l_1}(-s_1,\cdot))))\end{aligned}$$

and

$$\begin{aligned}\mathrm{SUM}_r = \textstyle\sum_{l_k=1}^m \cdots \sum_{l_1=1}^m &(\tilde{T}_i(d_k)B_{22,il_k}\tilde{T}_{l_k}(d_{k-1})B_{22,l_kl_{k-1}}\cdots\tilde{T}_{l_2}(d_1)B_{22,l_2l_1})\\ &(\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)(\Lambda_{l_1}r\mathbb{1}_-(\varphi_{l_1}(-s_1,\cdot)))).\end{aligned}$$

Since the operators $\tilde{T}_i(d)$ satisfy

$$\tilde{T}_i(d)(fg) = f(\varphi_i(-d,\cdot))g(\varphi_i(-d,\cdot))\mathbb{1}_+(\varphi_i(-d,\cdot)) = (\tilde{T}_i(d)f)g(\varphi_i(-d,\cdot)),$$

for all $f \in L^\infty(0, 1)$ and $g \in L^\infty(\mathbb{R})$ (and agreeing that the product of a function f in $L^\infty(0, 1)$ with a function g in $L^\infty(\mathbb{R})$ is the product of f with the projection of g in $L^\infty(0, 1)$), the factors in each summand of SUM_v that depend on v, i.e. the terms of the form

$$\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)(v_{l_1}(\varphi_{l_1}(-s_1,\cdot))\mathbb{1}_+(\varphi_{l_1}(-s_1,\cdot))),$$

can be written as the product

$$\begin{aligned}&(\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)v_{l_1}(\varphi_{l_1}(-s_1,\cdot)))\\ &\qquad\mathbb{1}_+(\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)),\end{aligned}$$

whereas the factors in each summand of SUM_r that depend on r can be written as

$$\begin{aligned}&(\tilde{T}_i(d_k)\tilde{T}_{l_k}(d_{k-1})\cdots\tilde{T}_{l_2}(d_1)\Lambda_{l_1}r)\\ &\qquad\mathbb{1}_-(\varphi_{l_1}(-s_1,\cdot)\circ\varphi_{l_2}(-d_1,\cdot)\circ\cdots\circ\varphi_{l_k}(-d_{k-1},\cdot)\circ\varphi_i(-d_k,\cdot)).\end{aligned}$$

Now, by Lemma (4) we conclude that SUM_v is zero whereas SUM_r equals

$$\sum_{l_k=1}^m \cdots \sum_{l_1=1}^m (\tilde{T}_i(d_k) B_{22,il_k} \tilde{T}_{l_k}(d_{k-1}) B_{22,l_k l_{k-1}} \cdots \tilde{T}_{l_2}(d_1) B_{22,l_2 l_1}) \\ (\tilde{T}_i(d_k) \tilde{T}_{l_k}(d_{k-1}) \cdots \tilde{T}_{l_2}(d_1) \Lambda_{l_1} r),$$

which clearly coincides with the ith component of $J_{222\cdots 22}(d_k, \ldots, d_1)\Lambda r$ if one uses the expression (2.23). That is, for all $i \in \{1, \ldots, m\}$ one has

$$(J_{222\cdots 22}(d_k, \ldots, d_1)\tilde{v}(\cdot; s_1, v, r))_i = (J_{222\cdots 22}(d_k, \ldots, d_1)\Lambda r)_i \ ,$$

which implies the equality stated in (2.22). □

Lemma 6. *Let $I_{222\cdots 22}(t, v, r)$ be defined as in (2.17). For all $t > 1$ and $h \in (0, 1)$, there exists $\tilde{M}_t > 0$ such that*

$$\|I_{222\cdots 22}(t + h, v, r) - I_{222\cdots 22}(t, v, r)\|_{L^m_\infty} < \tilde{M}_t h \, \|(u, v, r)\|_X \, .$$

Proof. Lemma 5 implies that, if $t > 1$, function $\tilde{v}$ in the integrand of $I_{222\cdots 22}(t, v, r)$ can be replaced by the constant Λr. Then, the solution of system (2.20) with the last equation removed gives a change of variables, namely $\tau_h(s_1, s_2, \ldots, s_k) = (s_1 + h, s_2 + h, \ldots, s_k + h)$, such that the integral $I_{222\cdots 22}(t + h, v, r)$ transforms into

$$\int_{-h}^{t} \int_{-h}^{s_k} \cdots \int_{-h}^{s_2} J_{222\cdots 22}(t - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1)\Lambda r \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \mathrm{d}s_k.$$

Finally, similarly as it is done in (2.19) and taking into account $\Lambda r = v(0)$, one concludes

$$\begin{aligned} &\|I_{222\cdots 22}(t + h, v, r) - I_{222\cdots 22}(t, v, r)\|_{L^m_\infty} \\ &\quad \le h(t + 1)^{k-1} \|B\|^k \|(u, v, r)\|. \end{aligned} \tag{2.25}$$

□

Lemma 7. *For all $t > 1$ and $h \in (0, 1)$, there exists $\tilde{M}_t > 0$ such that the term $\|\hat{v}(\cdot; t + h) - \hat{v}(\cdot; t)\|_{L^m_\infty}$ defined in (2.8) can be bounded as*

$$\|\hat{v}(\cdot; t + h) - \hat{v}(\cdot; t)\|_{L^m_\infty} \le \tilde{M}_t h \, \|(u, v, r)\|_X \, .$$

Proof. By using the bounds (2.19), (2.21) and (2.25) in inequality (2.18) it follows

$$\|\hat{v}(\cdot; t + h) - \hat{v}(\cdot; t)\|_{L^m_\infty} \le 2^{k+1} h(t + 1)^{k-1} \|B\|^k \left\| \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\|_X . \tag{2.26}$$

□

Theorem 2. *Let T and $T^{\odot *}$ be defined by (2.1) on X and $X^{\odot_T *}$ respectively. Let B be defined by (2.6). The functions*

$$S_0(t) = T(t),$$
$$S_k(t) = j^{-1} \int_0^t T^{\odot *}(t-s) B S_{k-1}(s) ds, \qquad \forall k \in \mathbb{N}$$

from $[0, \infty)$ to $\mathcal{L}(X)$ (the Banach space of the bounded linear operators on X with the operator norm) are continuous on $[1, \infty)$.

Proof. The case $k = 0$ follows immediately from the definition of $T(\cdot)$ in (2.1). Indeed, let $t > 1$ and $h > 1 - t$. Then

$$\left\| (T(t+h) - T(t)) \begin{pmatrix} u \\ v \\ r \end{pmatrix} \right\| = \left\| \begin{pmatrix} u - u \\ r - r \\ r - r \end{pmatrix} \right\| = 0,$$

that is $\|T(t+h) - T(t)\| = 0$ for all $h > 1 - t$, and in particular the limit as h tends to zero is also zero.

The general case $k > 0$ is a consequences of the previous lemmas. If $h \in (0, 1)$ and $t > 0$, then Lemmas 1 and 7 applied to inequality (2.8) yield

$$\|(S_k(t+h) - S_k(t))\| \leq h M_t \tag{2.27}$$

with

$$M_t = M(\|B\| \sup_{s \in [0, t+1]} \|S_{k-1}(s)\| + 2^{k+1}(t+1)^{k-1}\|B\|^k).$$

If $t > 1$ and $h \in (\max\{1 - t, -1\}, 0)$, then inequality (2.27) can be applied as

$$\|(S_k(t+h) - S_k(t))\| = \|(S_k(t+h+|h|) - S_k(t+h))\| \leq k_{t+h}|h| \leq M_t|h|,$$

since, on the one hand, $t + h$ is still bigger than 1 and, on the other hand, the constant M_t is increasing as a function of t.

By combining the results for positive and negative h, one finally concludes that for $t > 1$ and $h \in (\max\{1 - t, -1\}, 1)$ there exists a constant M_t (which depends on t) such that

$$\|S_k(t+h) - S_k(t)\|_X \leq |h| M_t.$$

□

As a corollary of Theorem 2 and the uniform convergence of the series defining S_B, it follows that

Theorem 3. *The semigroup S_B generated by $A + B$ (in the sense of Theorem A1) is eventually norm continuous. Moreover, since K is a compact operator from X into $X^{\odot *}$, the semigroup S generated by $A + B + K$ (in the sense of Theorem A1, using either $A + (B + K)$ or $(A + B) + K$) is eventually norm continuous (as an application of Theorem A8).*

Finally, the eventual norm continuity of the linearised semigroup S makes possible to apply Theorem A7, so that the following result on linearisation specific to problem (1.1) can be stated:

Theorem 4. *Let $(\bar{u}, \bar{v}, \bar{r}) \in X$ be a steady state of the semiflow Σ associated to (1.1) and let A_S be the operator defined in (2.5). Then,*

(i) *$(\bar{u}, \bar{v}, \bar{r})$ is locally asymptotically stable if $s(A_S) < 0$,*

(ii) *$(\bar{u}, \bar{v}, \bar{r})$ is unstable if there exists $\omega > 0$ such that the spectrum of A_S within the region $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \omega\}$ is non-empty and is composed only by a finite number of eigenvalues with finite algebraic multiplicity.*

3. Applications and discussion

The system presented in Sect. 2 was originally inspired by an epidemiological problem involving pathogens spreading through the oral-fecal route [2,4] (see also [1,14]). Specifically, in [2] we analysed a biological system involving n hosts (animals) and m species of microorganisms living in their intestines, either as free particles in the lumen or attached to the epithelial wall. By calling $u_{h,s}(x,t)$ and $v_{h,s}(x,t)$ the densities of attached and luminal microbes of type s in the host h respectively, and $r_s(t)$ the density of microbes s in the soil, all of them at time t, the dynamical equations are

$$\begin{cases} \partial_t u_{h,s}(t,x) = g_{h,s}(x, u_h(t,x), v_h(t,x)), \\ \partial_t v_{h,s}(t,x) = -\partial_x(c_h(x)v_{h,s}(t,x)) + f_{h,s}(x, u_h(t,x), v_h(t,x)), \\ \frac{\mathrm{d}r_s(t)}{\mathrm{d}t} = m_s(r(t)) + \sum_{h\in H} c_h(l_h)v_{h,s}(t,l_h) - \sum_{h\in H} \lambda_{h,s}r_s(t), \end{cases} \tag{3.1}$$

with $h \in \{1,\ldots,n\} =: H$ and $s \in \{1,\ldots,m\} =: S$. In the above equations the notation $u_h = (u_{h,1},\ldots,u_{h,m})$, $v_h = (v_{h,1},\ldots,v_{h,m})$ and $r = (r_1,\ldots,r_m)$ is used. The parameter l_h is the intestine length of host h (thus, the spatial domain of both u_h and v_h is $[0,l_h]$) and $c_h(x)$ stands for the velocity of its intestinal flow (notice that it is implicitly assumed that advection dominates diffusion). The functions $g_{h,s}$ and $f_{h,s}$ take into account the ecological processes happening locally at the position x of the intestine. Besides replication and mortality of bacteria, these functions may also reflect migration between epithelium and lumen, competition interactions or whatever we are interested in. Similarly, the function m_s describes the ecology in the external media of the population of type s. Finally, we assume that microbes enter the intestine in a rate which is proportional to their amount in the soil. Thus, $\lambda_{h,s}$ represents a kind of ingestion rate of particles of type s by host h. Consequently, a boundary condition for $v_{h,s}$ must be incorporated relating such reinfection term, which is

$$c_h(0)v_{h,s}(0,t) = \lambda_{h,s}r_s \qquad \forall (h,s) \in H \times S. \tag{3.2}$$

The diversity in gut lengths across hosts makes that, a priori, system (3.1) has not the form of system (1.1). Fortunately, we can perform a change in the spatial variables to rewrite (3.1) properly in the form of a system as (1.1). Such a change is possible because microbes within a host do not affect what happens in a different host. Indeed, by defining $\tilde{v}_h(x,t) := v_h(l_h x, t)$ and $\tilde{u}_h(x,t) := u_h(l_h x, t)$, we have, on the one hand, that the spatial domain of $\tilde{u}_{h,s}$ and $\tilde{v}_{h,s}$ is the interval [0, 1] for all $(h,s) \in H \times S$ and, on the other hand, that system (3.1) transforms into

$$\begin{cases} \partial_t \tilde{u}_{h,s}(t,x) = g_{h,s}(l_h x, \tilde{u}_h(t,x), \tilde{v}_h(t,x)), \\ \partial_t \tilde{v}_{h,s}(t,x) = -\partial_x(c_h(l_h x)\tilde{v}_{h,s}(t,x)) + f_{h,s}(l_h x, \tilde{u}_h(t,x), \tilde{v}_h(t,x)), \\ \frac{\mathrm{d}r_s(t)}{\mathrm{d}t} = m_s(r(t)) + \sum_{h\in H} c_h(l_h)\tilde{v}_{h,s}(t,1) - \sum_{h\in H} \lambda_{h,s} r_s(t). \end{cases} \tag{3.3}$$

with the boundary condition

$$\tilde{v}_{h,s}(0,t) = \frac{\lambda_{h,s}}{c_h(0)} r_s(t) \qquad \forall (h,s) \in H \times S.$$

In particular, by considering vectors $\tilde{u}$ and $\tilde{v}$ to be indexed by one number instead of two, for example by writing $\tilde{u}_{m(h-1)+s}$ instead of $\tilde{u}_{h,s}$ (analogously for $\tilde{v}$), then the initial value problem associated to (3.3) has the form of problem (1.1). Thus, in order to apply Theorems 1 and 4 to system (3.3) the phase space we have to work on is the Banach space

$$X = X_1 \times X_2$$

with $X_1 = L^\infty(0,1)^{n\times m}$ and

$$X_2 = \left\{ (\tilde{v}, r) \in C([0,1], \mathbb{R})^{n\times m} \times \mathbb{R}^m \mid \tilde{v}_{m(h-1)+s}(0) = \frac{\lambda_{h,s}}{c_h(0)} r_s \quad \forall (h,s) \in H \times S \right\}.$$

In [2] problem (3.1) was analysed formally more than theoretically. In the present paper we have proved that Theorems 1 and 4 apply to problem (3.3) (which is equivalent to system (3.1)), so that, as a corollary, we can state that the procedures followed in [2] are well supported by the theory of dual semigroups. In addition, Theorem 4 also implies that the conjecture used in [2] about the asymptotic dynamics of the system around steady states is true.

To illustrate the theory developed in this paper let us consider the simplest case of the ones treated in [2]: problem (3.1) and (3.2) with a single host, a single bacterial strain and assuming that the intestinal flow velocity is constant at all points (i.e. that $c(x)$ is constant). In this case, and using $g(u,v,x) = u\gamma_1(u) - \alpha u + \delta v$, $f(u,v,x) = v\gamma_2(v) + \alpha u - \delta v$ and $m(r) = -\mu r$ the equations reduce to

$$\begin{cases} \partial_t u(t,x) = u(t,x)\gamma_1(u(t,x)) - \alpha u(t,x) + \delta v(t,x), \\ \partial_t v(t,x) = -c\partial_x(v(t,x)) + v(t,x)\gamma_2(v(t,x)) + \alpha u(t,x) - \delta v(t,x), \\ \frac{\mathrm{d}r(t)}{\mathrm{d}t} = -\mu r(t) + cv(t,l) - \lambda r(t), \\ cv(t,0) = \lambda r(t). \end{cases} \tag{3.4}$$

Here α and δ represent attachment and detachment rates (to and from the intestinal epithelium), μ is a mortality rate of the bacteria found in the environment (outside the host) and γ_1 and γ_2 are per capita reproduction rates in the epithelium and in the lumen of the intestine. In [2] it is shown that if γ_1 and γ_2 are decreasing and negative for large arguments, then (3.4) has at most one positive stationary solution. Such a positive equilibrium exists if and only if

- $\gamma_1(0) \geq \delta$, or
- $\gamma_1(0) < \delta$ and

$$\frac{\lambda}{\lambda+\mu} e^{\frac{l}{c}\left(\gamma_2(0)-\alpha+\frac{\alpha\delta}{\delta-\gamma_1(0)}\right)} > 1.$$

The local behaviour of this equilibria, denoted by $(\bar{u}(\cdot), \bar{v}(\cdot), \bar{r})$, can be determined applying Theorem 4. To do this first we specify the generator A_S of the linearised semigroup around this equilibria (see (2.5) and also (1.4)), which is

$$A_S \begin{pmatrix} u \\ v \\ r \end{pmatrix} = \begin{pmatrix} a_1(\cdot)u(\cdot) + \alpha v(\cdot) \\ -c \cdot v'(\cdot) + a_2(\cdot)v(\cdot) + \delta u(\cdot) \\ cv(l) - \lambda r - \mu r \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \\ r \end{pmatrix} \in D(A_S),$$

where

$$\begin{aligned} a_1(\cdot) &= \gamma_1(\bar{u}(\cdot)) - \delta + \gamma_1'(\bar{u}(\cdot))\bar{u}(\cdot), \\ a_2(\cdot) &= \gamma_2(\bar{v}(\cdot)) - \alpha + \gamma_2'(\bar{v}(\cdot))\bar{v}(\cdot). \end{aligned}.$$

Then the spectral values of A_S are determined by studying which operators $A_S - \eta \mathrm{Id}$ defined on $D(A_S)$ fail to have a continuum inverse. This leads to the characteristic equation whose solutions are the eigenvalues of A_S and which can be used to give the spectral bound of A_S (see Lemma 3.5 in [2] for more details). For this particular example it turns out that the positive stationary solution, when does exist, is always locally asymptotically stable.

Beyond the model treated in [2], problem (1.3) can also be applied to determine if coexistence scenarios exist in the chemostat model of [1,14] and, in general, to any system of living beings inhabiting networks where advection is much more intense than diffusion. In the particular case of gut microogranisms, however, the coexistence of different bacteria has implications not only on the microbial ecology, but also on the host development. Indeed, physiological diversity between microbes may create distinguished biofilm patterns along the intestine which, in turn, could play a role during tissue differentiation. A "pathogenic" pattern would result in intestines more vulnerable to infections or in microbiomes unable to supply essential nutrients. In this sense, improving our understanding about stable gastro-intestinal ecosystems will be helpful for therapists dedicated to reshape the microbiome of patients with dysfunctional microbial patterns.

In addition to the biological implications of the model proposed above, the present work has motivated some mathematical questions. Recall that in Sect. 3 we proved that a certain semigroup was eventually norm continuous. This property implied the Spectral Mapping Theorem, which essentially gives a one to one relation between the spectrum of the semigroup evaluated at t and the spectrum of its generator. The proof, however, relied on the particular structure of the system we considered and, from our point of view, this was not optimal. By studying alternative ways to treat the problem, we realised that the kind of semigroup we were analysing was obtained by perturbing an eventually continuous semigroup by a bounded operator (in fact, the bounded operator was atypical in the sense that it was defined from the base space X to $X^{\odot_T *}$, with T being the unperturbed semigroup, but let us set aside this particularity now). A natural question at this point is whether the perturbation of an eventually continuous semigroup by a bounded perturbation is also eventually norm continuous. As Nagel and Engel show in their book, this is not true in general (see [12, Example III.1.15], in which the unperturbed semigroup is even nilpotent), although if the bounded perturbation is also compact, then the perturbed semigroup does inherit the eventual continuity of the unperturbed semigroup (see Theorem A8). Since in our case the compactness of the perturbation was not something we could assume, deducing the eventual continuity of the perturbed semigroup by means of general results seemed unfeasible. However, our final goal was not to show that the perturbed semigroup was eventually continuous, we wanted to prove the Spectral Mapping Theorem for the perturbed semigroup, which is something weaker. In this sense, we would need a general result as the following.

Conjecture 1. *Let X be a Banach space. Let T be an eventually norm continuous semigroup on X generated by A. Let B be a bounded operator on X. Then the Spectral Mapping Theorem holds for the semigroup S generated by $A + B$, i.e. $\sigma(S(t)) \setminus \{0\} = e^{t\,\sigma(A+B)}$ holds for all $t \geq 0$.*

Similarly, a weaker version of the above conjecture that, if proved to be true, would also imply Theorem 4 as a corollary is the following.

Conjecture 2. *Let X be a Banach space. Let T be an eventually norm continuous semigroup on X generated by A. Let B be a bounded operator on X. Then the growth bound of the semigroup S generated by $A + B$ is equal to the spectral bound of the operator $A + B$, i.e.*

$$\omega_0(S) = s(A + B).$$

Our attempts to prove or disprove any of these two conjectures were unfruitful, but since we have not found any reference to these questions in the specialised literature [12,26], we present it here for those with more expertise.

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Appendix

A Sun-dual formalism and evolution equations

In this appendix an introduction of the work done by Clement et al. [6,7] on sun-dual semigroups is given. Although similar brief summaries exist written by the same authors [8], we include our own in order to present a more comprehensible work and because slight variations are introduced to deal with non-sun-reflexive spaces for which the variation of constants formula is well defined (see [9] for another reference where this semilinear formulation is undertaken without the sun-reflexivity hypothesis). Thus some of the theorems stated below are not exactly the same as the ones found in [6,7], even though the arguments to prove them can be applied essentially in the same way.

A.1 Linear theory

Let X be a Banach space, and denote by X^* its dual space. An element $x^* \in X^*$ is, by definition, a linear continuous operator from X to $\mathbb{R}$. We denote the image of an element $x \in X$ by x^* with the bracket $\langle x^*, x\rangle$.

Given a closed operator C on X, its adjoint operator C^* is a linear operator from X^* to X^* with domain

$$D(C^*) = \{x^* \in X^* \mid \exists \zeta^* \in X^* \text{ such that } \langle x^*, Cx\rangle = \langle \zeta^*, x\rangle \ \forall x \in D(C) \subset X\}.$$

It turns out that if $x^* \in D(C^*)$ then only one $\zeta^* \in X^*$ exists satisfying $\langle x^*, Cx\rangle = \langle \zeta^*, x\rangle$ for all $x \in D(C)$, so that the image of x^* by C^* is defined unequivocally as $C^*x^* = \zeta^*$. In particular if C is a bounded operator, then C^* is also a bounded operator and satisfies $\langle C^*x^*, x\rangle = \langle x^*, Cx\rangle$ for all $x^* \in X^*$ and $x \in X$.

Given a strongly continuous semigroup T on X, the sun-dual space of X relative to T is a subspace of the dual space X^* defined by:

$$X^{\odot_T} := \left\{x^* \in X^* \mid \|T^*(t)x^* - x^*\| \to 0 \text{ as } t \downarrow 0\right\},$$

where $T^*(t)$ is the adjoint of the operator $T(t)$. We denote by $T^{\odot}(t)$ the restriction of $T^*(t)$ to $X^{\odot_T}$, so that $T^{\odot}$ is, by construction, a strongly continuous semigroup on $X^{\odot_T}$ whose infinitesimal generator is denoted as $A_{T^{\odot}}$. Repeating this procedure on the pair $(T^{\odot}, X^{\odot_T})$ we define the double-sun-dual of X relative to T as:

$$X^{\odot\odot_T} := (X^{\odot_T})^{\odot_{T^{\odot}}}.$$

The *canonical injection* $j : X \hookrightarrow X^{\odot_T *} := (X^{\odot_T})^*$ is determined by the pairing

$$\langle j(x), x^{\odot} \rangle = \langle x^{\odot}, x \rangle \qquad \forall x^{\odot} \in X^{\odot_T}.$$

In [6] it is shown that

$$\|j(x)\|_{X^{\odot_T *}} \leq \|x\|_X \leq M \|j(x)\|_{X^{\odot_T *}}, \tag{A.1}$$

where M is a constant which depends on T. Among other things, this implies that

$$j(X) \subset X^{\odot\odot_T} \tag{A.2}$$

since it is easily checked that $T^{\odot *}(t) j(x) = jT(t)x$ and then

$$\|T^{\odot *}(t) j(x) - j(x)\|_{X^{\odot_T *}} \leq \|T(t)x - x\|_X \longrightarrow 0 \quad \text{as} \quad t \downarrow 0.$$

The canonical injection makes possible to introduce two important concepts for the development of the theory.

Definition A1. X is said to be sun-reflexive relative to a strongly continuous semigroup T on X if $j(X) = X^{\odot\odot_T}$.

Definition A2. Let T be a strongly continuous semigroup on X. Let Y be a subspace of $X^{\odot_T *}$. We say that T is closed by $\odot*$-integration on Y if, for all $f \in C([0, \infty), Y)$ and for all $t \geq 0$,

$$\int_0^t T^{\odot *}(t - s) f(s) \mathrm{d}s \in j(X). \tag{A.3}$$

The integral in (A.3) must be understood as an element of $X^{\odot_T *}$, and specifically (due to Bochner integral properties) as the functional satisfying

$$\left\langle \int_0^t T^{\odot *}(t - s) f(s) \mathrm{d}s, x^{\odot} \right\rangle = \int_0^t \langle T^{\odot *}(t - s) f(s), x^{\odot} \rangle \mathrm{d}s.$$

Notice that a semigroup T is always closed by $\odot*$-integration on $j(X)$ since

$$\int_0^t T^{\odot *}(t - s) f(s) \mathrm{d}s = j \int_0^t T(t - s) j^{-1} f(s) \mathrm{d}s.$$

A well-known result of the theory states that the kind of integrals given by (A.3) take values not in the whole space $X^{\odot_T *}$ but in the subset $X^{\odot\odot_T}$.

Proposition A1 ([6, Theorem 3.2]). *Let T be a strongly continuous semigroup on X and $f \in C([0,\infty), X^{\odot_T *})$. Then*

$$\int_0^t T^{\odot *}(t-s) f(s) \in X^{\odot \odot_T} \qquad \forall t \geq 0.$$

Taking into account this proposition together with inclusion (A.2) it follows:

Corollary A1. *If X is sun-reflexive relative to T, then T is closed by $\odot*$-integration on $X^{\odot_T *}$.*

As already commented, most propositions in the series of papers [6,7] as well as in the book [11] assume that a given Banach space X is sun-reflexive relative to a semigroup T. However, it is possible to prove similar results by means of analogous arguments assuming the closedness of T by $\odot*$-integration on a subspace Y of $X^{\odot_T *}$. Let us give the reformulated statements we need for the present thesis.

Theorem A1 ([6, Theorem 4.2]). *Let T be a strongly continuous semigroup generated by A and closed by $\odot*$-integration on $Y \subset X^{\odot_T *}$. Let B be a bounded linear operator from X into Y. Then the equation*

$$S(t)x = T(t)x + j^{-1} \int_0^t T^{\odot *}(t-s) B S(s) x ds \tag{A.4}$$

uniquely defines a strongly continuous semigroup S on X. The partial sums of

$$\sum_{k=0}^{\infty} S_k,$$

with $S_0 = T$ and

$$S_{k+1}(t) = j^{-1} \int_0^t T^{\odot *}(t-s) B S_k(s) ds \qquad \forall t \geq 0,$$

converge towards S uniformly on compact intervals, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,\tau]} \|S(t) - \sum_{k=0}^{n} S_k(t)\| = 0 \qquad \forall \tau > 0.$$

The generator of S is A_S with domain

$$D(A_S) = \{x \in X \mid j(x) \in D(A^*_{T\odot}) \ \ \text{and} \ \ A^*_{T\odot} j(x) + Bx \in j(X)\}$$

and images $A_S x = j^{-1}(A^*_{T\odot} j(x) + Bx)$.

The semigroup implicitly defined by equation (A.4) makes natural to ask for its sun-dual spaces, that is for $X^{\odot_S}$ and $X^{\odot\odot_S}$. The following result shows that these spaces are determined just by the generator A, so they do not depend on the perturbation B.

Proposition A2 ([6, Lemma 4.3]). *Let T, A, B and S be as in Theorem A1. Then $X^{\odot_S} = X^{\odot_T}$ and $X^{\odot\odot_S} = X^{\odot\odot_T}$.*

Not only the sun-dual spaces are invariant with respect bounded perturbations from X to Y. The property of being closed by $\odot*$-integration on Y is also satisfied by the perturbed semigroup S. Notice that this makes sense since Proposition A2 ensures that Y is a subspace of $X^{\odot_S *}$. This is stated formally in the following proposition. The same proposition relates the evolution family obtained when T is perturbed by $B + f$ with the evolution family obtained when S is perturbed by f, where f is a continuous function of time from $[0, \infty)$ to Y.

Proposition A3 ([7, Proposition 2.5]). *Let T, A, B and S be as in Theorem A1. Then S is closed by $\odot*$-integration on Y. Moreover, for every $x \in X$ and every function $f \in C([0, \infty), Y)$, $u(t; x)$ defined as*

$$u(t; x) = S(t)x + j^{-1} \int_0^t S^{\odot*}(t - \tau) f(\tau) d\tau$$

is the only solution of

$$u(t; x) = T(t)x + j^{-1} \int_0^t T^{\odot*}(t - \tau)(Bu(\tau; x) + f(\tau)) d\tau.$$

Corollary A2. *Let T, A, B and S be as in Theorem A1. Let B_1 and B_2 be bounded operators from X into $Y \subset X^{\odot_T *}$ such that $B = B_1 + B_2$, and let S_1 be the semigroup obtained when T is perturbed by B_1. Then S is equal to the semigroup obtained when S_1 is perturbed by B_2 (in the sense of Theorem A1).*

The above reformulation of the perturbation theory for dual semigroups is useful when, on the one hand, the unperturbed semigroup T is defined on a non-sun-reflexive space X, but, on the other hand, T is closed by $\odot*$-integration on some subspace Y bigger than $j(X)$ (so that the standard semilinear formulation is not enough to solve the problem). At this point it is mandatory to show that semigroups satisfying these properties do exist.

Example A1. Let X_1 and X_2 be Banach spaces. Consider $X = X_1 \times X_2$ with the usual norm, and let $T = \mathrm{diag}(T_1, T_2)$ be a strongly continuous semigroup with a "diagonal" structure, i.e. $T(t)(x_1, x_2) = (T_1(t)x_1, T_2(t)x_2)$ for all $t \geq 0$. Let X_2 be sun-reflexive with respect T_2. For $i \in \{1, 2\}$ let j_i be the canonical inclusion from X_i into $X_i^{\odot_{T_i} *}$ and define $Y = j_1(X_1) \times X_2^{\odot_{T_2} *}$. Then T is closed by $\odot*$-integration on Y.

Proof. Take $f \in C([0, \infty), Y)$ arbitrary, and define f_1 and f_2 the component functions of f in $j_1(X_1)$ and $X_2^{\odot_{T_2} *}$ respectively. On the one hand, since j_1 is a linear bounded operator and $j_1 T_1(t) = T_1^{\odot*}(t) j_1$, then

$$\int_0^t T_1^{\odot*}(t - s) f_1(s) \mathrm{d}s = j_1 \left(\int_0^t T_1(t - s) j_1^{-1} f_1(s) \mathrm{d}s \right) \in j_1(X_1).$$

On the other hand, since f_2 is norm continuous from $[0,\infty)$ to $X_2^{\odot T_2 *}$, Proposition A1 implies

$$\int_0^t T_2^{\odot *}(t-s) f_2(s)\mathrm{d}s \in X_2^{\odot\odot T_2},$$

and this is enough because $X_2^{\odot\odot T_2} = j_2(X_2)$ due to the sun-reflexivity condition. □

Notice that the subspace Y in the previous example is in general bigger than $j(X)$ because $X_2^{\odot T_2 *}$ is in general bigger than $X_2^{\odot\odot T_2}$. Notice also that X could be non-sun-reflexive if, for example, $T_1(t) = \mathrm{Id}$ for all $t \geq 0$ and X_1 were non-reflexive. Arguably one could say that the example above is very degenerate due to the diagonal structure of the semigroup T. However, Proposition A3 allows us to take any perturbation of T by a bounded linear perturbation from X to Y, which give a collection of less trivial examples.

A.2. Semi-linear theory

Consider the initial value problem

$$\begin{cases} \frac{dv(t)}{dt} = A_T v(t) + j^{-1}\left(\mathcal{H}(v(t))\right) \\ v(0) = x \in X \end{cases}, \tag{A.5}$$

where A_T is the generator of a strongly continuous semigroup T which is closed by $\odot *$-integration on Y (see Definition A2) and $\mathcal{H} : X \to Y \subset X^{\odot_T *}$ is a Lipschitz function.

Since $\mathcal{H}$ takes values in $Y \subset X^{\odot_T *}$, system (A.5) is a non-standard initial value problem which even fails to be well defined because in general $j(X) \subsetneq Y$ (notice that a classical solution $v(\cdot)$ of (A.5) must satisfy $\mathcal{H}(v(t)) \in j(X)$ for all $t \geq 0$). However, such a problem admits a generalised version of the variation of constants equation, which takes the form

$$v(t) = T(t)x + j^{-1}\left(\int_0^t T^{\odot *}(t-s)\mathcal{H}(v(s))\mathrm{d}s\right), \tag{A.6}$$

thanks to the fact that T is closed by $\odot *$-integration on Y.

The functions $v : [0,\infty) \to X$ satisfying this integral equation are their solutions. A classical solution of (A.5) (if any) is a solution of (A.6), which motivates the notion of mild solution:

Definition A3. A function $v : [0,\infty) \to X$ is a mild solution of (A.5) if it is continuous and it satisfies the integral equation (A.6).

Two fundamental properties of mild solutions are given in the following theorem.

Theorem A2 ([7, Theorem 3.1]). *For every $x \in X$ there exists a unique mild solution $v(\cdot; x)$ of (A.5). Moreover, v satisfies the semigroup property $v(t+s; x) = v(t; v(s; x))$ and $v(t;\cdot)$ is Lipschitz. We refer to $v(\cdot;\cdot)$ as the semiflow of (A.5).*

When $\mathcal{H}$ is Fréchet differentiable, a result on the differentiability with respect to initial conditions around a steady state can be given. Here, by steady state we mean an element $\bar{x} \in X$ such that

$$\bar{x} = T(t)\bar{x} + j^{-1}\left(\int_0^t T^{\odot *}(t-s)\mathcal{H}(\bar{x})\mathrm{d}s\right) \qquad \forall t \geq 0.$$

As it occurs in the standard semilinear formulation, the steady states satisfying the above integral conditions coincide with the equilibrium points of system (A.5), i.e., the points $\bar{x} \in X$ such that

$$\bar{x} \in D(A), \quad \mathcal{H}(\bar{x}) \in j(X) \quad \text{and} \quad A\bar{x} + j^{-1}(\mathcal{H}(\bar{x})) = 0.$$

Theorem A3. ([11, Proposition VII.5.6]) *Let $\bar{x}$ be a steady state of (A.6). Assume $\mathcal{H}$ is Fréchet differentiable in $\bar{x}$ and define $B := \mathcal{H}'(\bar{x}) \in \mathcal{B}(X, Y)$. Then the semiflow $v(\cdot;\cdot)$ given by (A.6) is uniformly Fréchet differentiable at $\bar{x}$, i.e. for all $t_1 > 0$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that if $||x - \bar{x}|| < \delta$ and $t \in [0, t_1]$ then*

$$||v(t;x) - \bar{x} - D_x v(t;\bar{x})(x - \bar{x})|| < \varepsilon ||x - \bar{x}||.$$

The family of linear operators S defined as $S(t) := D_x v(t;\bar{x}) : X \longrightarrow X$ for all $t \geq 0$, is given implicitly by

$$S(t)x = T(t)x + j^{-1}\left(\int_0^t T^{\odot *}(t-s)BS(t)x\mathrm{d}s\right) \qquad \forall t \geq 0, \tag{A.7}$$

and explicitly by the series

$$S = \sum_{n=0}^{\infty} S_n \tag{A.8}$$

with $S_0 = T$ and

$$S_n(t) = j^{-1}\left(\int_0^t T^{\odot *}(t-s)BS_{n-1}(s)\mathrm{d}s\right) \qquad \forall t \geq 0.$$

Moreover, the partial sums in (A.8) converge uniformly on compact intervals.

Notice that $S(t)$ is well defined because, since T is closed by $\odot *$-integration on Y and B is continuous and takes values in Y, Theorem A1 ensures the existence of a unique strongly continuous semigroup S which is solution of (A.7). The generator of S is also given by Theorem A1.

From the previous theorem it follows that, for all fixed $t > 0$, $v(t;x)$ can be approximated by $\bar{x} + S(t)(x - \bar{x})$ for those x close enough to $\bar{x}$. It seems a good strategy to infer the behavior of $v(\cdot;x)$ close to $\bar{x}$ by means of the stability properties of S. However, we must justify carefully the validity of this procedure because, a priori, the asymptotic behavior of S could not determine the stability of the equilibrium $\bar{x}$.

To our knowledge, it is yet an open question if it could exist nonlinear semiflows with an asymptotically stable equilibrium for which the linearised system around it is unstable. Fortunately, some results in the literature can be applied to our system in order to discard this pathological possibility. Before exposing them let us recall a couple of concepts.

Definition A4. The growth bound of a strongly continuous linear semigroup T on a Banach space X is defined as

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t} \quad \forall t \geq 0\}.$$

Definition A5. A strongly continuous linear semigroup T is said to be exponentially stable if its growth bound is strictly negative. Similarly, T is said to be exponentially unstable if its growth bound is strictly positive.

If S is exponentially stable (i.e. there exists $M \geq 1$ and $\omega > 0$ such that $\|S(t)\| \leq Me^{-\omega t}$), it can be shown that $\bar{x}$ is a locally asymptotically stable equilibrium (the continuity of $v(\cdot; x)$ makes possible to apply essentially the same argument used in the theory of ODEs [19]).

Theorem A4 ([7, Theorem 4.2]). *Let $\bar{x}$, v and S as in Theorem A3. If $\omega_0(S) < 0$ then $\bar{x}$ is locally asymptotically stable in the Lyapunov sense. More precisely, there exist $\omega > 0$ and $\delta > 0$ such that if $\|x - \bar{x}\| < \delta$,*

$$e^{\omega t}(v(t; x) - \bar{x}) \to 0 \quad \text{as} \quad t \to \infty.$$

In order to give an instability result some additional hypotheses other than $\omega_0(S) > 0$ have to be assumed.

Theorem A5 ([7, Theorem 4.3]). *Let $\bar{x}$, v and S as in Theorem A3, and denote A_S the generator of S. Assume $\omega_0(S) > 0$ and that X admits a decomposition*

$$X = X_1 \oplus X_2 \tag{A.9}$$

into S-invariant subspaces with X_1 finite-dimensional. For $i \in \{1, 2\}$ let S_i be the restriction of S to X_i and let A_{S_i} be the corresponding generators. If

$$\omega_0(S_2) < \min\{\text{Re}\lambda \mid \lambda \in \text{Spectrum}(A_{S_1})\}$$

then $\bar{x}$ is unstable, i.e. there exist $M > 0$ and a sequence $\{x_n, t_n\}_{n \geq 1} \subset X \times \mathbb{R}$ satisfying $x_n \to \bar{x}$ and $t_n \to \infty$ such that $\|(v(t_n, x_n) - \bar{x}\| \geq M$.

The above results are useful provided we have a method to study the dynamics of S. In general this cannot be done in a straightforward manner because S is given either implicitly or as a series. To overcome this problem the generator A_S can be used, since its expression is usually simpler than S and some relations are known between the growth bound of a semigroup T and the spectral bound of its infinitesimal generator A_T, denoted by $s(A_T)$. In general one can only say that $s(A_T) \leq \omega_0(T)$, since counterexamples exist in which the strict inequality holds (see [12, Chapter 5] and [26] for a review on asymptotics of semigroups). However, if T is *eventually norm continuous*, the relation $s(A_T) = \omega_0(T)$ as well as a mapping linking the spectrum of $T(t)$ with the spectrum of A_T does hold.

Theorem A6 ([12, Theorem IV.3.10]). *Let T be an eventually norm continuous semigroup on a Banach space X with generator A_T. Then T satisfies the Spectral Mapping Theorem: for all $t \geq 0$, the spectrum of $T(t)$ and the spectrum of A_T, denoted by $\sigma(T(t))$ and $\sigma(A_T)$ respectively, satisfy the following set relation*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A_T)}.$$

In particular the equality $\omega_0(T) = s(A_T)$ holds.

Taking this into account, Theorems A4 and A5 can be modified in terms that they only involve information about the spectrum of A_S.

Theorem A7. *Let $\bar{x}$, v and S as in Theorem A3, and denote A_S the generator of S. If S is eventually norm continuous then*

(i) *$\bar{x}$ is locally asymptotically stable if $s(A_S) < 0$,*
(ii) *$\bar{x}$ is unstable if there exists $\omega > 0$ such that the spectrum of A_S within the region $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \omega\}$ is non-empty and is composed only by a finite number of eigenvalues with finite algebraic multiplicity.*

Proof. Statement (i) follows directly from Theorem A4 and the relation $s(A_S) = \omega_0(S)$ due to the eventually norm continuity of S. To prove statement (ii) we use the decomposition theorem [16, Theorem 6.17] by taking a Jordan curve enclosing the eigenvalues to the right of ω. For full details see [3, Theorem 1.2.21]. □

As in the standard theory of perturbation semigroups, the property of being eventually norm continuous is preserved under compact perturbations. Specifically:

Theorem A8. *Let T be an eventually norm continuous semigroup generated by the linear operator A and closed by $\odot*$-integration on $Y \subset X^{\odot_T *}$. Let K be a compact operator from X into Y. Then the semigroup S generated by $A + K$ (in the sense of Theorem A1) is eventually norm continuous.*

Proof. The same arguments as in [12, Proposition III.1.14], where this result is stated in the standard case (i.e. when the perturbation K is defined from X to X) can be applied in the sun-dual framework. For full details see [3, Theorem 1.2.22]. □

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