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Extending Modelling Activity Diagrams as a tool to characterise mathematical modelling processes

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Abstract: In this paper, we present a qualitative study in which we analyse the video-recordings of four groups of students solving Fermi Problems. Previous studies show that Secondary School students solve this type of problems using complex problem solving processes and developing mathematical models. In order to analyse the students' problem solving processes, so-called Modelling Activity Diagrams were used. The results of the present study demonstrate that solving Fermi problems is a complex matter, and that some of the theoretical tools used in the field of Mathematical Education fail to adequately reflect this level of complexity. In addition, Modelling Activity Diagrams are presented as a more detailed analysis tool to characterise student choices and actions, as well as to make the structure of the Fermi problem addressed more visible.

Keywords: Mathematical modelling, Modelling Activity Diagrams, Fermi problems, Secondary School.

1. Introduction

Since the early work of Pollak (1979), there has been a growing interest in introducing activities that promote contextualised work and mathematical modelling in the field of Mathematical Education at different educational levels (Vorhölter, Kaiser, & Borromeo Ferri, 2014). In this paper we present a qualitative study focusing on identifying and developing tools to aid researchers in the analysis of mathematical modelling processes students engage in. Existing literature features several theoretical developments that attempt to describe students' modelling processes. Most of these theoretical perspectives are based on the division of the students' work into two domains by separating the real world from the mathematical realm. Particularly, it is mostly

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accepted that students go through different stages in a cyclical manner when engaged in modelling as exemplified for example by the so-called modelling cycle of Blum and Leiss (2007). In fact, the concept of a modelling cycle is featured in the Common Core State Standards (CCSSI, 2010) to describe the mathematical activity developed within modelling processes. It has been also used as defining modelling processes in articles related to educational practices, such as done by Anhalt and Cortez (2015), and also as a heuristic tool for task design (Czocher, 2017).

However, there are opposing voices on the prevalence of the modelling cycle in the literature. Several authors claim that, though useful in the design of tasks or for explaining modelling to teachers and teaching staff, the modelling cycle does not show most of the actual work done by the students (Cai et al., 2014). In fact, some studies (e.g. Ärlebäck, 2009; Czocher, 2016; Aymerich & Albarracín, 2016) highlight the difficulties in the qualitative identification of the stage of the modelling process corresponding to each episode of the students' work. In order to overcome these difficulties, we propose a construct that extends Modelling Activity Diagrams (MAD) to characterise the performance of the students in a mathematical modelling process as an alternative to the modelling cycle. For this purpose, we used video-data of students aged 15 to 16 engaged in solving Fermi problems (c.f. Ärlebäck, 2009), and we extended the information provided by the original MADs by including a representation of the internal structure of the sub-problems the students engaged in to solve the main problem.

Thus, the aim of the present study is to explore the possibilities of Modelling Activity Diagrams, extended to in a more detailed way represent the complexity of modelling processes and to determine whether this extended tool actually provide further information on the structures underlying the processes the students engaged in.

2. Mathematical modelling as a process: from the Modelling Cycle to Modelling Activity Diagrams

Mathematical modelling is a problem solving process of contextualised problems that involves elaborating a mathematical model to describe the phenomenon studied. Lesh and Harel (2003) define mathematical models as conceptual systems that describe other systems. These authors understand that models are made up of a) a set of concepts to describe or explain the mathematical objects relevant to the phenomenon studied, as well as the relations, actions, patterns, and regularities that are attributed to the problem solving situation; and b) the procedures used to create useful constructions, manipulations, or predictions for achieving clearly recognized goals. The system that comprises the model can be expressed in multiple ways, such as in terms of algebraic relations, written symbols, sketches or diagrams, natural language or metaphors based on experience.

The way students elaborate mathematical models to describe phenomena or to carry out predictions is an object of discussion and there are currently different theoretical positions and perspectives one can take (Borromeo Ferri, 2006). In general, current literature accepts the cyclical nature of modelling processes. Even though there are several ways of determining the structure of a modelling process, it is understood that it can be divided into different stages and that the students go through these sequentially, moving on to the following stage once they consider the work of the current stage has been concluded satisfactorily. When working in the classroom with students, the starting point is a problem or situation (in the real world) and from a simplification and abstraction process, a mathematical model is generated (within the mathematical realm) that facilitates the use of mathematical knowledge and techniques to reach a solution. When the students attempt to validate the result obtained, they decide whether it is good enough solution to the problem (back in the real-life domain), and if not they start the modelling process again, aiming to revise and reconstruct the mathematical model generated (by reformulating it or developing it further by incorporating or modifying certain elements), in order to obtain a new solution. In this way, the process is repeated through different iterations, as many as the students consider necessary to obtain a satisfactory model and result.

This view on modelling processes is not shared by all researchers and has received criticism based on being an oversimplification of the processes described and of the students' decisions (Barbosa, 2009; Villa-Ochoa, Bustamante & Berrio, 2010). In addition, there is also no clear consensus on describing the nature of the stages of the so-called modelling cycle (Perrenet & Zwaneveld, 2012), which makes it difficult to establish a collectively shared basis and understanding for the used analysis tools in this field.

To provide examples of the many views on mathematical modelling in the literature, we in the following briefly dicuss some of the theoretical perspectives developed to describe and understand the processes involved in modelling. The first example is that of Pollak (1979), in which a platonic differentiation between the mathematical domain and the rest of the world is established (Figure 1). This separation inevitably leads to the mathematisation process of a phenomenon and to the interpretation of mathematical models elaborated in the mathematical realm within their real context as validation.

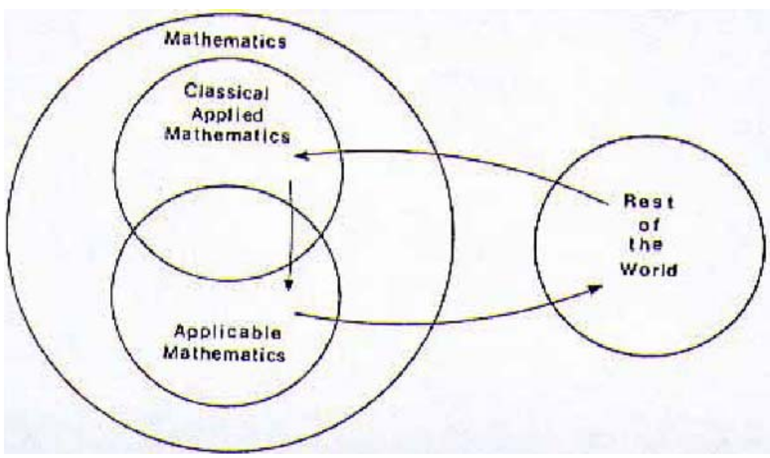

Figure 1 – Pollak's (1979) modelling cycle (Pollak, 1970, p. 233)

Berry and Davies (1996) expanded the view introduced by Pollak by focusing on different stages of the modelling process, such as the formulation of a mathematical model or realising the need to revise and reformulate the model, as well as write up their solution in a report. This conceptualisation of the modelling cycle inherits the aforementioned division between the domains of the real world and the mathematical realm. Berry and Davies (1996) describe the modelling process as cyclic, understanding that the modellers/solvers go through the different stages in their work while they improve the models and results obtained.

In recent years, other theoretical interpretations have been introduced to represent modelling processes with the intention of collecting processes that were not present in previous formulations (Perrenet & Zwaneveld, 2012). An example along this line is the work of Geiger (2011), that includes the use of technology as an active element of the different stages of the process, even though it maintains a view on modelling as a cyclical process. One of the most recognised perspective of the mathematical modelling processes in the field of Mathematical Education is the conceptualisation and representation of Blum and Leiss (2007), shown in Figure 2.

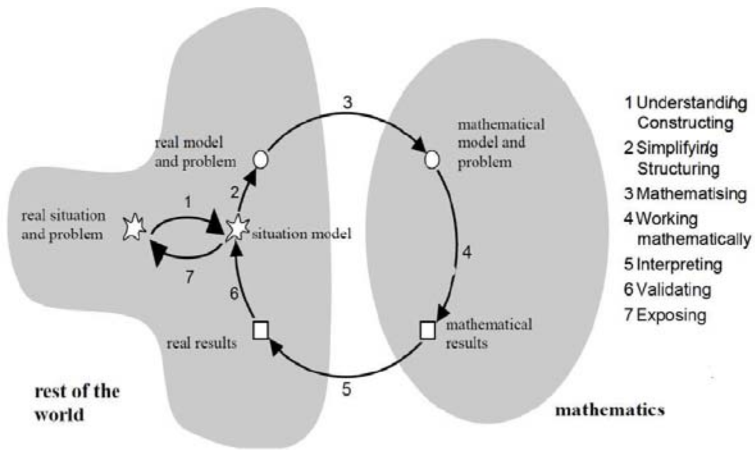

Figure 2 – The modelling cycle of Blum and Leiss (2007, p. 1626)

This diagram shows the differentiation between the mathematical and real-life domains, distinguishing between the models based in real life and their abstraction constructed in the mathematical realm. It also maintains a cyclical view of the process. This representation finalises the work started by Blum (2002), in which the following modelling stages are detailed: 1) simplification of a real-life problem to a real-life model; 2) mathematisation of the real model, thus moving on to the mathematical model; 3) search for a solution based on the mathematical model (working mathematically); 4) interpretation of the solution to the mathematical model; and 5) validation of the mathematical solution, interpreting it in the context of the real-life problem. The conceptualisation of Blum and Leiss (2007) has been widely accepted to describe the modelling process but has shown to be difficult to use as a tool for analysis when it comes to describe and explain the students' work. Some researchers have directed their efforts towards elaborating indicators to identify the stages described by Blum and Leiss (2007) in the students' real-life activities, as in the work of Borromeo Ferri (2007), or that of Sol, Giménez and Rosich (2011). The existence of this type of study is an example of the difficulties that researchers come across when using a concrete theoretical framework in their research.

Other researchers have adopted a different approach, and argue that existing modelling cycle frameworks in some respects provide a too simplified picture of modelling, since the path the students follow throughout a modelling process is non-linear. In addition, they also argue that it is hard to assign the students' activities when engaged in modelling to either belonging to the real world or to the mathematical domain, suggesting that the distinction between the two domains is not always possible, and that the boundary between them not is clear cut (Ärlebäck, 2009). Some positions found in existing literature oppose the aforementioned meaning given to the word *reality*. Blum and Borromeo Ferri (2009), based on the perspective of Pollak (1979), understand reality as everything that is outside mathematics, its concepts and procedures. Thus, they define reality as *the rest of the world*, which includes nature, society, everyday life and the rest of scientific disciplines. According to Barbosa (2009), the notion of a reality that is external to mathematics holds a view on mathematical models as a rough portrait of reality. Barbosa also introduce a different perspective in the sense that they consider mathematics to be part of reality and suggest that mathematical models are not necessarily skewed or partial by nature.

When faced with the need to describe the modelling processes implemented by students, Borromeo Ferri (2007) proposes so-called individual modelling routes with the aim of broadening the modelling cycle's reach. These routes are based on arrow diagrams that show the students' work within the theoretical diagram of the modelling cycle, evidencing that the cycle fails to show the decisions the students make during the modelling process. In a previous study, Ärlebäck (2009) includes Borromeo Ferri's approach, and presents Modelling Activity Diagrams (MAD) as a bidimensional graph that depicts the type of modelling activities the students engage in when solving modelling problems. These activities proposed by Ärlebäck (2009) and used to characterise the modelling processes in terms of MADs can be found in Table 1.

Table 1 – Activities that make up MADs

R: Reading	Reading the statement of the task and understanding it
M: Modelling	Simplifying and structuring the task mathematically
E: Estimating	Making quantitative estimates
C: Calculating	Performing mathematical calculations, such as arithmetic calculations, working with equations, drawing sketches or diagrams
V: Validating	Interpreting, verifying and validating the results, calculations and the model itself in its real-life context
W: Writing	Summarising the findings and results in a report, drafting the solving process as well as the solution

Figure 3 shows a MAD that reveals the complexity of the mathematical process developed by the students, leading to their modelling work being qualified as idiosyncratic, in relation to both the students' characteristics as protagonists of the mathematical activity and the nature of the problem that prompted the activity in the first place.

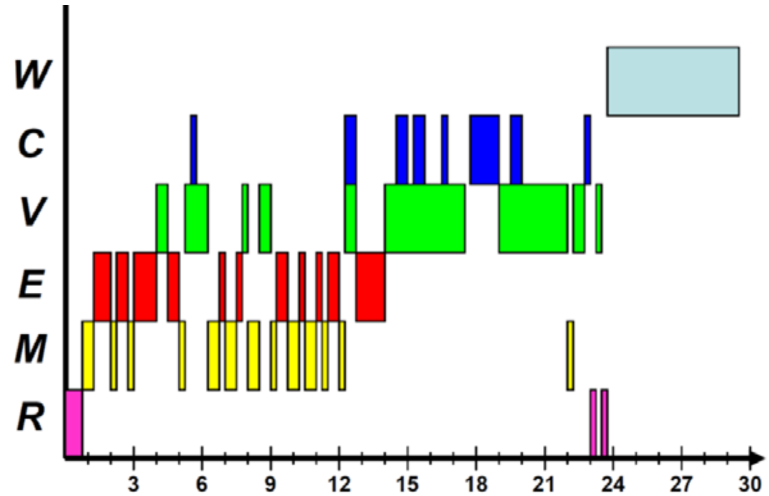

Figure 3 – Modelling Activity Diagram for the resolution of the problem of estimating the time needed to walk up the Empire State Building (Ärlebäck, 2009, p. 345)

3. Fermi Problems

The use of so-called Fermi Problems in our work allows us to propose simple situations that require mathematical analysis promoting the construction of models. Following Ärlebäck (2009), we consider Fermi problems to be open, non-standard problems that require students to make assumptions about the problem situation and estimates of certain quantities before they engage in, often, a series of simple calculations. Based on his previous investigations, Ärlebäck (2011) argues that working with Fermi problems may be useful for the purpose of introducing modelling in the classroom, since they are accessible to students and do not necessarily depend on any kind of previous mathematical knowledge. Examples of Fermi problems used in different investigations are: the estimation of the number of cars on a stretch of road (Peter-Koop, 2009), the estimation of the time needed to walk up the stairs of the Empire State Building in New York (Ärlebäck, 2009) or the estimation of the number of people participating in a demonstration (Albarracín & Gorgorió, 2014). More examples of Fermi problems and their resolutions can be found in Weinstein and Adam (2009) and Weinstein (2012).

Some research studies conducted to date on the use of Fermi problems in the teaching and learning of mathematics have been shown to facilitate the introduction of mathematical modelling in Primary and Secondary School classrooms. Peter-Koop (2009) used Fermi problems with students in upper Primary School (ages 10 to 12) to analyse their solving strategies. Based on her investigations, Peter-Koop concluded that a) students solve problems in a myriad of ways, b) students develop new mathematical knowledge in arriving at their solutions, and c) the problem solving processes the students displayed indeed were multi-cyclic in nature and relatable to the modelling cycle. Ärlebäck (2009), introducing MADs as an analytic tool, analysed the modelling activities of a group of Secondary School students engaged in solving a Fermi problem. With this work, he highlighted the importance of social relations and extra-mathematical knowledge in the problem solving situation. Albarracín and Gorgorió (2014) analysed the solving strategies used by Secondary School students on several Fermi problems and observed a large number of possible strategies and ways of constructing the models used to arriving at their answers. Ferrando, Albarracín, Gallart, García-Raffi and Gorgorió (2017) characterised the output produced by Secondary School students from a sequence of Fermi problem by identifying the procedures and concepts that shaped their models, highlighting elements that differ between the outputs of students with previous experience in modelling activities and that of students without any previous experience. Finally, we wish to acknowledge the work of Czocher (2016), who used Fermi problems with university students in the field of Engineering and analysed their performance in terms of MADs. Czocher confirms that the MAD diagrams reveal their mathematical thinking about variables and constraints related to the problem contexts. Czocher adds an interesting dimension to the interpretation of the MAD when she writes that when a task has become routine for an individual, the modelling route displayed by that individual for solving that task resemble the idealized working process of the modelling cycle (e.g. Blum & Leiss, 2007).

Efthimiou and Llewellyn (2007) characterised Fermi problems by how these were formulated since they always appear to be open questions offering little or no specific information to the solver to direct them in the solving process. The key aspect to Fermi problems from the perspective of mathematical modelling is the need to conduct a detailed analysis of the situation presented in the statement of the problem, with the objective of decomposing the original problem into simpler problems – addressed as sub-problems in the present work – to reach the solution of the original question by means of reasonable estimates and reasoning. Such reasoned conjectures can be replaced by classroom activities to further the extraction of information and its critical interpretation (Sriraman, & Knott, 2009) or problem solving processes linked to reality (Ärlebäck, 2009). Analyses of the sub-problem types that can be found in Fermi problems have not been covered in the existing literature, though several studies have shown that these depend on the students' approaches when tackling the problem (Peter-Koop, 2009; Albarracín & Gorgorió, 2014). In an attempt to establish a systematic approach to the solving processes of Fermi problems to promote the development of analytical skills in university Economics and Business students, Anderson and Sherman (2010) proposed a type of representation that structures the solving process by explicitly differentiating the sub-problems the students had to deal with. In the case of estimating the number of hotdogs consumed at major league baseball (MLB) games each season in the US, Anderson and Sherman (2010) presented the structure shown in Figure 4. Together with their students, they differentiated between the values that needed to be estimated – such as the number of hotdogs consumed by an attendant and match – from unchanging values that can be looked up beforehand – such as the number of matches in a season.

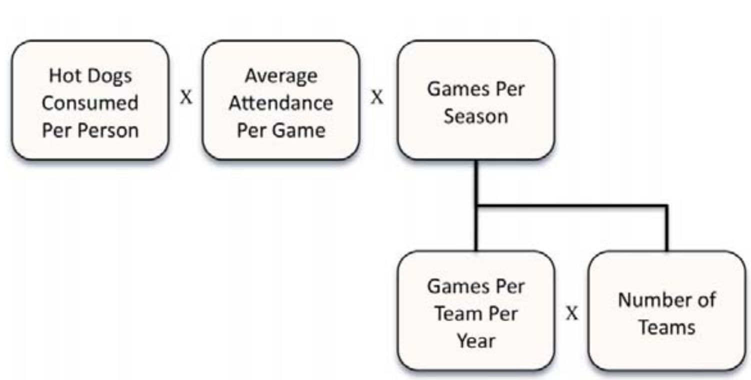

Figure 4 – Structure of a Fermi problem by Anderson and Sherman (2010, p. 37)

4. Methodology

The aim of our research presented in this paper is to develop an extension of the Modelling Activity Diagrams for modelling processes and to investigate the potential and possibilities of this extended framework. For this purpose, we explicitly included a representation of the sub-problems elaborated by the students to achieve their objective within the activity (e.g. solving a Fermi problem). The data, collected at a Secondary School of a municipality located in the metropolitan area of Barcelona, consisted of the video-recordings of the problem solving processes of four groups engaged in solving a Fermi problem. Each group consisted of three regular students of 4th grade of Secondary School (ages 15 to 16), which in the following will be referred to as groups A, B, C and D respectively. The composition of the groups was based on the answers the students had given to a previous questionnaire in which the students were requested to detail their action plan to solve two Fermi problems – following the same process disclosed and discussed in Albarracín and Gorgorió (2014). The groups were constructed by choosing students that both suggested different strategies and provided highly detailed answers, with the intention of minimising the amount of data to be collected. The problem chosen for the students to work on deals with the estimation of the number of objects needed to fill a large volume and is presented to the students in a specific context. The problem statement provided to the students was the following:

> Water is a scarce resource and it is necessary to be aware of the use we make of it. We have organised a debate to address the amount of water used for different purposes, and, in order to provide concrete data, we need to answer the following question: *How many bathtubs can we fill with the water of a public swimming pool?*
>
> Calculate the requested amount approximately and describe the steps followed to reach the solution.

The video-recording was transcribed in terms of the utterances the students made. We consider each single un-interrupted verbal contribution of a student as one utterance. We analysed the data by creating a MAD for each of the recordings obtained, following the coding process detailed in next section. In each of the four cases, the resulting MADs are reinterpreted by additionally identifying the sub-problems used by the students to solve the main problem. The analysis developed by Albarracín and Gorgorió (2014) allows us to anticipate the models used by the students when working on problems such as that of the students worked on in this study (and given above). Notably, the generally most common model for this type of problem and this particular age-group of students, and the only one observed in this study, is the *iteration of a unit*, in which the students determine the volume of a (larger) base unit and compare it to that of its containing (smaller) object, decomposing the problem into three different sub-problems – the calculation of each volume and a final comparison – following the diagram shown in Figure 5.

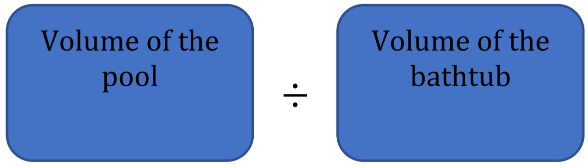

Figure 5 – Structure of the Fermi problem used in this study

With this second level of analysis based on identifying the modelling activities present in each sub-problem, we intend to describe and investigate the relationship between the MAD and the underlying structure of the modelling process induced by the sub-problems.

5. Analysis process

In this section, we illustrate the analysis process of the video-recordings and the way different MAD modelling activities are coded, based on the data collected. Our intention is to highlight the decisions made by the researchers when determining the categories – the type of activities – in which the students were identified to work in, and to provide examples from the data collected.

R-Reading:

In general, students read the problem statement together, with one of the members reading the problem out loud. If this is the case, this activity it is easy to identify for researchers in the recording. Sometimes the students decided to re-read the statement, usually after a discussion that required some kind of reformulation resulting in the group agreeing to read the problem statement again. For example, the following is an extract from the video-recording of group D's work when deciding to read the problem out loud:

Duna:	*Okay, let's read it out loud. Read it.*
Dídac:	*No, you read it.*
Domingo:	*Well, I'll read it. 2013 is the international year on water cooperation (…) How many bathtubs full of water would we need to fill a swimming pool?*

M-Developing the model:

This category collects all instances regarding the creation of the mathematical model, both when approaching the problem more generally by discussing aspects related to a potential real-life model – as referred to in Blum and Leiss (2007) – as well as when elaborating the actual mathematical model.

As an example of an instance in which students created real-life models, we present the following excerpt of a conversation, in which the students of group D attempt to decide which concepts will lay the foundation for their model. In this case, they discussed different standards to compare volumes.

Dídac:	*Look, the easiest way is to imagine a bathtub and imagine you fill it. A bathtub, another bathtub, one bathtub next to another, and one on top of the other, and so on until you manage to fill the whole swimming pool. (Makes gestures as if placing bathtubs one next to the other and stacking them)*
Duna:	*Of course.*
Domènec:	*(Addressing Dídac) What do you suggest doing?*

Duna: *Let's see, let's imagine something more like this, more dynamic, no calculations. I mean, more normal.*

Dídac: *More normal? Let's see, when you say more normal, what do you mean? Without using mathematics?*

Duna: How many balls... How many balls with a volume of I don't know how much… (makes gestures as if placing large footballs in a bathtub and pushing them to the other side of the pool)

Dídac: *Sure, we don't know the calculations yet, but we will be calculating volumes, right?*

As an example of an instance when students constructed their mathematical model, we present the following extract of when the students in group C modelled the shape of a bathtub and approximated it to a cuboid, disregarding its real shape. At this specific moment of the recording, they were addressing the width of the bathtub:

Claudia: *Mine (the bathtub) is like this, and goes inwards. I mean it's not completely straight on the sides. (Makes gestures showing a wedge shape, indicating that the base is smaller and shapes are rounded)*

Carlos: *I know, but that doesn't matter because it's not going to make much difference. (Regarding the fact that they will consider the walls of the bathtub to be parallel to each other)*

E-Estimating:

To pinpoint the moments when the students made estimates, we paid attention to details in the conversations related to utterances involving discussing what values to assign to certain quantities needed to complete a calculation. In this example, group C is estimating the depth of a swimming pool by comparing it to its height.

Carlos: *How much is that? 50 by 25 metres.*

Cesc: *By two of depth.*

Carlos: *Two of depth? only?*

Cesc: *More or less.*

Carlos: *Two and a half. (Raising his arm to compare with his own height and in reference to his real-life experience in swimming pools)*

Cesc: *Well, two and a half.*

C-Calculating:

The times when the students are performing their calculations are easy to detect in the video-recording, since often one of the members of the group either explicitly explains the mental calculations being performed or otherwise writes them down on paper. The following dialogue shows and instance when a mental calculation is being made.

> *Arnau:* *OK, 12 times 10… 120, and times 4…*
>
> *Ada:* *OK, times 4, equals 480.*

V-Validating:

An essential aspect of modelling processes is the validation of results obtained. This takes place just after finding partial or final results – since the students can decide to validate a result for a sub-problem. Validations can end in the group coming to an agreement, as in the following example for group B.

> *Bárbara:* *Is that reasonable?*
>
> *Blanca:* *What?*
>
> *Bárbara:* *Is that reasonable, 54 bathtubs?*
>
> *Berta:* *Yes, I think so.*
>
> *Blanca:* *Yes.*

We also identified some validation processes which required the group members to recall and reconsider certain knowledge and facts about the context of the problem situation in order to resolve tensions in the group, in order to resolve opposing viewpoints and discrepancies before obtaining consensus and results. In the following example, group C calculated the capacity of a bathtub from its estimated dimensions, they obtained 450 litres and decided this number is too large.

> *Cesc:* *It can't be 450 litres.*
>
> *Carlos:* *Well, with these measurements it's approximate.*
>
> *Cesc:* *It's just that I don't know if they (the measurements) are acceptable.*
>
> *Carlos:* *We imagined a bathtub. Well now what's missing is that there are lots of types of bathtubs, for the rich, for the poor… We chose a standard one.*
>
> *Cesc:* *Well, not quite standard. That's not the one we chose. Mine is surely smaller.*
>
> *Carlos:* *It's just that 450 is a lot.*

W-Writing:

The times the students wrote their report are easy to identify in the video-recordings, though they are not always verbalised. The following is however an example of an instance where writing is evidenced during group A's teamwork:

> *Aina:* *Well, in order to avoid writing the whole thing, we can just write what we got at the end.*
>
> *Arnau:* *Yes.*
>
> *Ada:* *Go on, dictate it and I'll write.*

6. Results: Modelling Activity Diagrams obtained

In the following we present the MADs obtained from the video-recordings of the four groups of students, as a result of the qualitative analysis developed.

6.1. Group A: Ada, Aina and Arnau

The students in group A worked from the assumption of a swimming pool being 25 metres in length, arriving at an estimated result that 480 bathtubs would fit inside the swimming pool. The participation in the discussion was not equally distributed, and the problem was mainly modelled and discussed by Ada and Arnau, while Aina drafted the report and participated in the validation process. The total number of utterances for each of the members of the team was the following: Ada, 33; Aina, 26; and Arnau, 37. The corresponding MAD of the groups work is shown below (Figure 6):

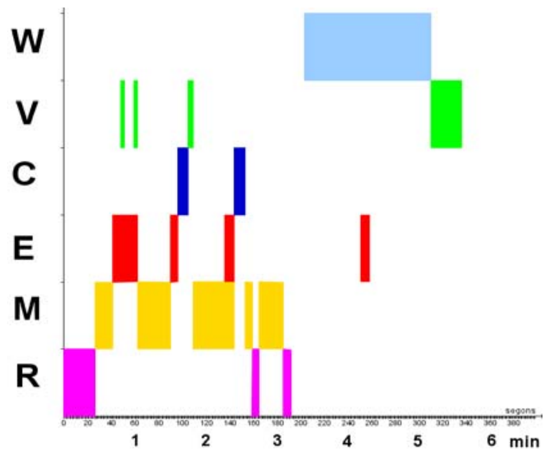

Figure 6 – MAD that describes group A's modelling process

In figure 6 we can see the switching between the activities the students engaged in, denoting the complexity of how groups A's modelling process unfolded and by first inspection conveying the picture of a somewhat disorganised process. However, if we analyse the sub-problem that the students engaged in and how the coded activities are distributed in relation to these a clear structure is revealed, as seen in Figure 7. This graph shows the activities the students engaged in to solve each of the sub-problem they tackled. The students started with working on deciding on their swimming pool of choice and estimating its volume (sub-problem one), and then worked on estimating the volume of a bathtub (sub-problem two). The third sub-problem consists of calculating the number of bathtubs per swimming pool, before finally re-organising the information obtained in order to present a solution and write the report.

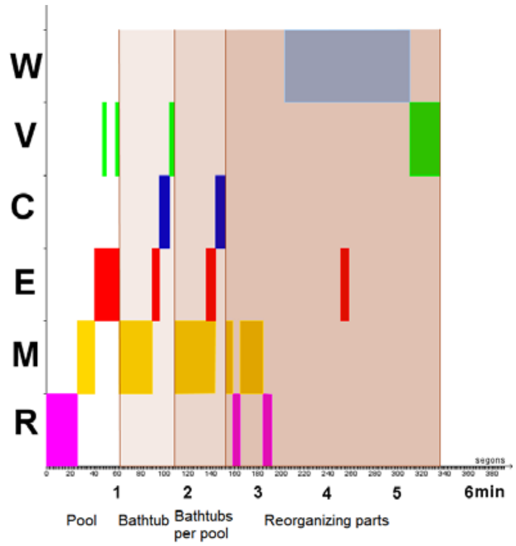

Figure 7 – MAD with separation by sub-problems for group A's modelling process

6.2. Group B: Blanca, Bárbara and Berta

Group B chose a small leisure swimming pool and the result of their work was an estimate that the pool contained of 54 bathtubs. The three team members all participated actively throughout the whole activity. The number of utterances was the following: Blanca, 55; Bárbara, 63; and Berta, 64. The activities the students engaged in in terms of a MAD is shown in Figure 8.

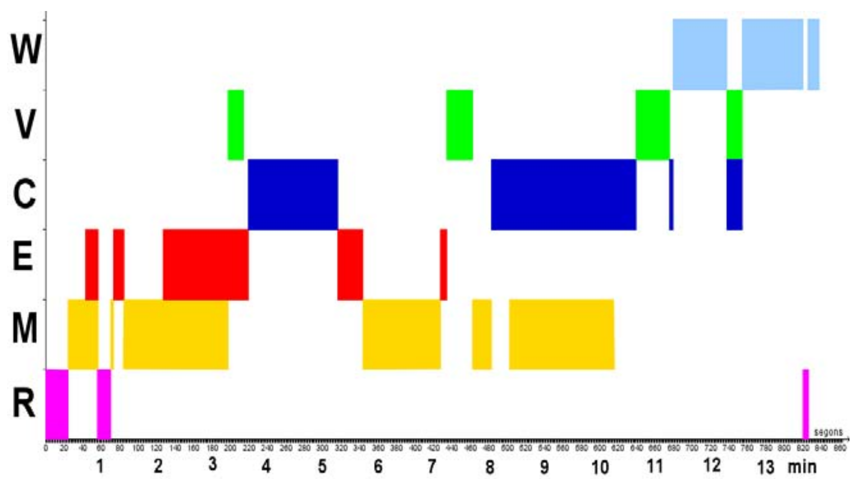

Figure 8 – MAD describing group B's modelling process

Group B way of working was similar to the path taken by group A, beginning with estimating the volume of the pool and the turning to estimating the volume of the bathtub, as shown in Figure 9. However, group B worked with different units of measure (cubic metres for the volume of the pool and litres for the bathtub) compared with group A (they used cubic metres in both problems). As a consequence, this gave raise to an intermediate episode (time between 5:12 and 7:39) in which they calculated and reconsidered the problem, resulting in the deciding to acknowledge and make use of the work they had done but to convert all their estimates to litres. Finally, the students divided the volume of the swimming pool by the volume of the bathtub. The analysis of the students work presented below in Figure 9 shows that also the students in group B proceeded by completing the different sub-problems, but that they additionally had to coordinate the results obtained with respect to units to give structure to the final solution.

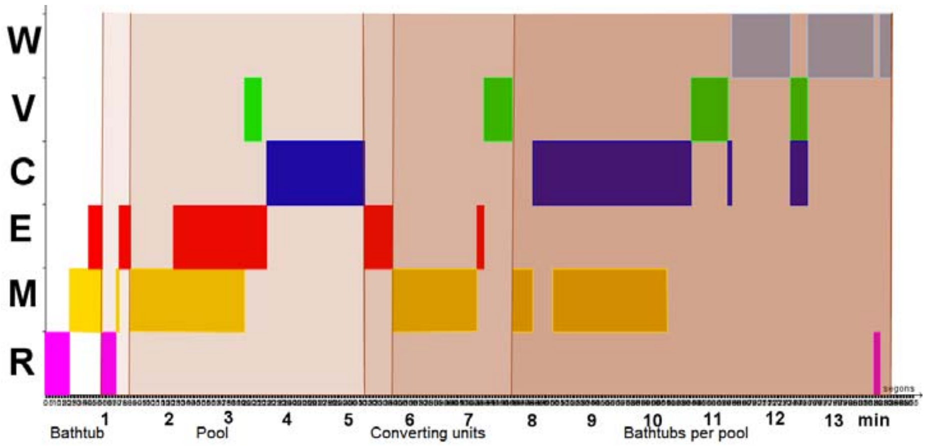

Figure 9 – MAD with separation into sub-problems for group B's modelling process

The graph above (e.g. Figure 9) shows how the students engaged in two different activities overlapped each other at the same time on two occasions. This can be seen in Figure 9 from 2:12 to 3:20 and from 8:19 to 10:10. In the particular case of the first of the two instances this is due to the fact that Blanca detached herself from the teamwork to clarify her ideas, attempting to solve the situation separately, and during this time the group engaged in two different parallel problem solving processes.

6.3. Group C: Carlos, Cesc and Claudia

The students in group C choose to consider an Olympic swimming pool and arrived at an estimate that it contains 6,944 bathtubs. Carlos and Cesc immediately started discussing the problem and how to solve it. Claudia however was somewhat hesitant in the beginning, stating that without checking the internet she would not be able to solve the problem. The total number of utterances was: Carlos, 59; Cesc, 45; and Claudia, 45. The MAD representing the work of group C is shown in Figure 10.

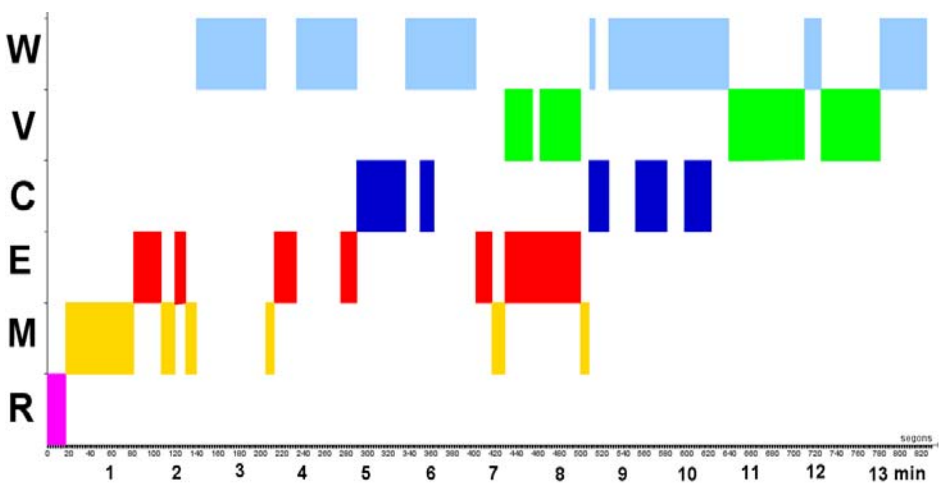

Figure 10 – MAD that describes group C's modelling process

The analysis by sub-problems of group C's MAD (Figure 11) shows that the students started the solving the problem by deciding the particulars of the pool and the bathtub they would work with. The activities group C engaged in to solve the sub-problems concerning the estimation of the volume of the pool and of the bathtub follow a non-linear pattern, which reveals that the students faced up to difficulties determining the volume of the bathtub by using their own bodies as a reference for estimating bathtub's length, wide and deep. Also in the case of group C, the students engaged in two different and overlapping activities during the estimation of the volume of the bathtub. What happened was that the students temporarily worked more individually using their own bodies as a reference to simultaneously estimate and validate their results.

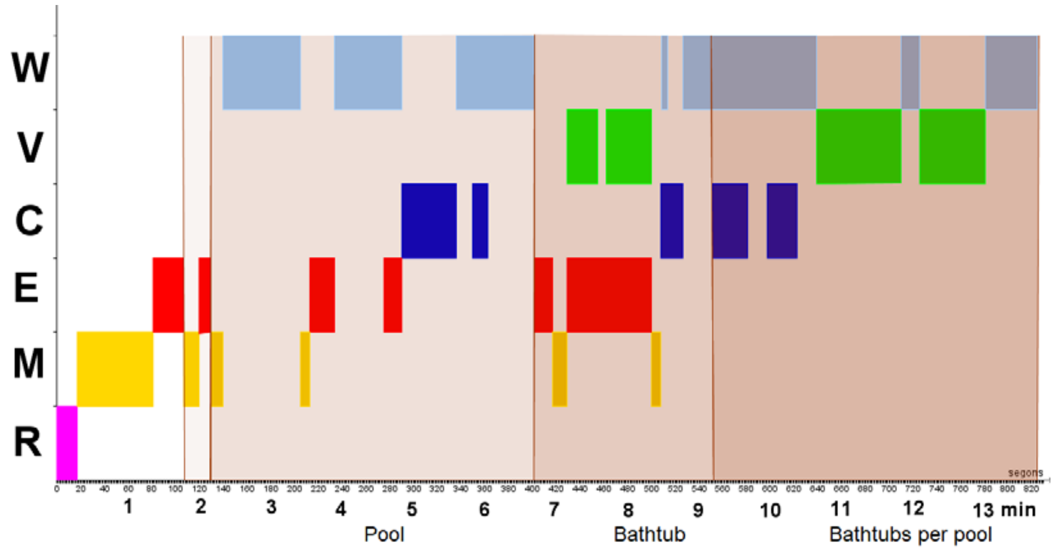

Figure 11 – MAD with a separation into sub-problems of group C's modelling process

6.4. Group D: Diego, Duna and Domingo

Group D, like group C, used an Olympic swimming pool for their calculations and estimated you could manage to fit 10,416 bathtubs in this particular pool. The participation was quite evenly distributed between the three group members. The total number of utterances was: Dídac, 90; Duna, 101; and Domingo, 85. The actitivies the students in group D engaged in in terms of a MAD when solving the problem is given below (Figure 12):

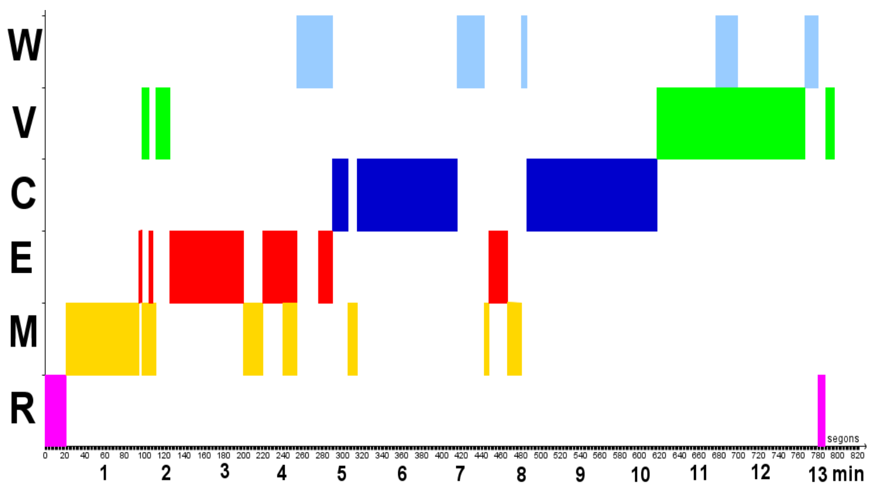

Figure 12 – MAD that describes group D's modelling process

Group D obtained the volume of the bathtub by initially attempting to come up with an answer by simply throwing out a number that felt right to them. This phase started at the beginning of the activity and lasted for 3 minute and 20 seconds. However, after having realised the flaws in this kind of naïve estimation process, they began in revising their answer and in calculating the volume of the bathtub with more precision. This transition includes a validation process in which the students doubt the possibility of obtaining an adequate value without performing any calculations. In this way the students between minute 3:20 and 7:30 calculated the volume of the bathtub by estimating the lengths of a parallelepiped. Next, they proceeded to engage in a similar process to approximate the volume of the swimming pool and ended the activity by estimating the number of bathtubs that would fit in a swimming pool with a division.

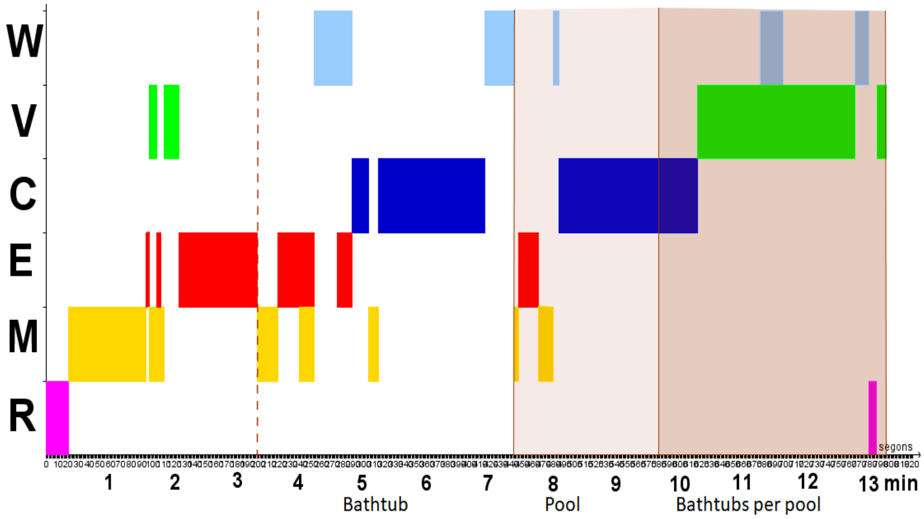

Figure 13 – MAD with a separation by sub-problems of group D's modelling process

7. Discussion and conclusions

In the present study, we analysed four groups of students aged 15 to 16 when engaged in solving the Fermi problem *how many bathtubs could we fill with the water of a public swimming pool?* We analysed the modelling processes the students engaged in by identifying the different modelling activities in terms of the MAD framework proposal in Ärlebäck (2009), and in addition extended this framework by adding a layer of interpretation by including a sub-division of the episodes in which the students solved different sub-problems in order to solve the Fermi problem.

From a methodological point of view, we reaffirm the idea that characterizing the students' different activities they engage in when solving a modelling problem is more straight forward and clearer in terms of MAD activities compared to that in terms of stages of the modelling cycle. However, we must emphasize that analysing the tasks carried out by a group is a complex process since we only can rely on the observable elements the collected data provide. Sometimes the students in the group split their focuses and interests when working toward the solution, resulting in the students not working together. Due to their inherent structure the MAD diagrams make it possible to show this fact explicitly by mapping out different bars for the different activities for the time period in question. In this way, we consider MAD diagrams to be both a richer and more robust tool from a methodological point of view when it comes to make the connection between the activities the students actually engage in during the problem solving process and the codes aiming to capture these activities explicit and visable.

Another aspect to note is that the results shows that even in the case of students engaged in solving a Fermi problem that is accessible to students, the modelling process involved in arriving to a solution is complex. This complexity is manifested in the large number of the instances students engage in different actions needed to characterize the problem solving process, but also in the significant differences between the problem solving processes of the different groups. This study allows us to affirm that although this complexity exists, the extended MAD diagrams allow us to reveal the internal structure of the proposed Fermi problem, thus more clearly showing the work pattern of the students. Group A's initial MAD (Figure 6) shows a complex problem solving process that is, however, clearly possible to divide into the solving of several sub-problems, displayed using the extended MAD (Figure 7). For group A the problem given does not pose great difficulties, neither from a decision-making point of view regarding the modelling process, nor in the workings of the mathematical tools used, i.e. resulting in an extended MAD of the students problem solving process that in a trustworthy way reflects the activities the students engaged in. In part the modelling behaviour displayed by the students in groups A is concurrent with the findings of Czocher (2016), that when students experience the modelling problem at hand as routine, the problem solving process is more or less straight forward.

In the specific case of group A's performance, we conducted a parallel analysis using the modelling cycle of Blum and Leiss (2007), following the instructions of Sol, Giménez and Rosich (2011), as we did in Aymerich and Albarracín (2016), to identify the different stages in the modelling process. The result of this analysis of group A's performance can be seen in Figure 12, which shows a complex problem solving process involving many on the stages in the modelling cycle. One can note however, that non of the students modelling routes are closed, and that the students seem to be struggling in the initial phase of solving the problem.

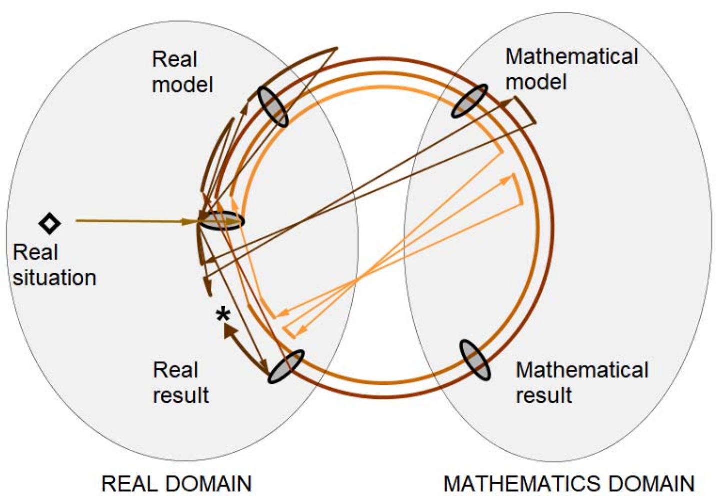

Figure 14 – Analysis of group A's modelling cycle

Coming back to the point made by Czocher (2016), that student's modelling processes of routine problems tends to better line up with the stages in the modelling cycle, we conclude that this by no mean obvious in terms of the analysis of students' modelling routes as depictured in Figure 14. Rather, the impression conveyed by Figure 14 is that the problem solving process the students engaged in is an ill-structured and complex one. Here, the resulting diagram from the extended MAD (e.g. Figure 13) provide a more accurate and nuanced representation of the students engagement with the problem.

The comparison between the two approaches to analyse the data leads us to conclude that the extended MAD do facilitate an analytical narrative that is closer to the actual activities the students engage in. It does this in the sense that the extended MAD provides more localised information that allows for a clearer and more detailed interpretation of the students' actions and decisions.

An example of this extra explanatory information the extend MAD makes it easier to identify, is the activities the students in group B engage in when they detected the inconsistencies in their calculated values due to the units chosen when considering the volumes of the pool and bathtub respectively. The extended MAD also facilitates the qualitative identification of types of activities in which the students make decisions that involve settling on the characteristics of the problem that are not given, as well as in the discussions on measurements of the pool or the bathtub of group C (Figure 11). The lack of specificity in the formulation of the given problem is part of the nature of Fermi problems and promotes discussion between students, as well as connecting mathematical and extra-mathematical knowledge.

Pelesko in Cai et al. (2014) stated that the modelling cycles unintentionally hide much of the real work of mathematical modelling. The analytical approach developed in this study shows that extended MAD diagrams can be a helpful tool to visualize some of these hidden elements, and that previously seemingly messy and haphazard problem solving processes can be understood as highly structured. In fact, we are hopeful that extended MAD diagrams can be adjusted and readily applied to study also mathematical modelling processes beyond mere Fermi problems. Given that the MADs were designed to specifically study these types of problems, one of the activities included in the original analysis is estimation, but we suggest that this activity is broaden, or complemented with other types of activites, so that activites such as for example *making measurement*, *statistical data collection*, *experimentation*, *simulation* are included in the next version of the extended MAD.

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