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Irregular periodic functions: when Algebra met Analysis

J. Ángel Cid i F. Adrián F. Tojo

1 A brief glimpse into the periodic world

Periodicity is a deceptively simple property to state that has important ramifications. We say that $T \in \mathbb{R}$ is a *period* of a function $f\colon \mathbb{R} \to \mathbb{R}$ whenever

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R},$$

and we say that f is *periodic*, or T*–periodic*, if it has a nontrivial period $T \neq 0$ (of course, we may assume $T > 0$ without loss of generality).

Periodic functions appear naturally in Mathematical Analysis at the time of introducing trigonometric functions such as the sine or the cosine, and continue to play a crucial role in more advanced topics.

Arguably, one of the most basic and fruitful models in Physics is provided by the harmonic oscillator, where a particle of mass m is subject to a restoring force that is proportional to the distance of the particle to the equilibrium point, $x = 0$ (think for instance of a particle joined to a spring). Newton's second law of motion provides the differential equation satisfied by $x(t)$, the displacement of the particle with respect to the equilibrium, namely,

$$m\,\ddot{x}(t) = -k\,x(t), \;\; k > 0. \tag{1}$$

All of the solutions of (1) are periodic and explicitly given by the formula

$$x(t) = a\cos\left(\sqrt{\frac{k}{m}}\,t\right) + b\sin\left(\sqrt{\frac{k}{m}}\,t\right) \quad \text{for } a, b \in \mathbb{R}.$$

As equation (1) also models air vibrations, and thus the human perception of sound, [3], a pure musical tone can be mathematically represented by a periodic wave of the form

$$a\cos(\omega\, t) + b\sin(\omega\, t) \text{ for } a, b \in \mathbb{R} \text{ and } \omega > 0,$$

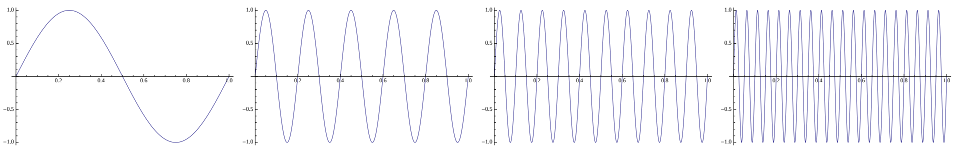

Figure 1: Successive sine waves with frequencies 1 Hz, 5 Hz, 10 Hz and 20 Hz, respectively.

with period $\frac{2\pi}{\omega}$ and hence frequency $\frac{\omega}{2\pi}$ cicles per second (Herz). For instance the middle A (LA in Romance languages, 440Hz) corresponds to $\omega = 880\pi$.

In general, a plucked string will vibrate as a superposition of waves given by integral multiples of its natural frequency

$$\sum_{n=1}^{\infty} a_n \cos(n\,\omega\,t) + b_n \sin(n\,\omega\,t) \text{ for } a_n, b_n \in \mathbb{R},$$

and the decomposition of a signal into its frequencial components was just the original motivation of Fourier Analysis.

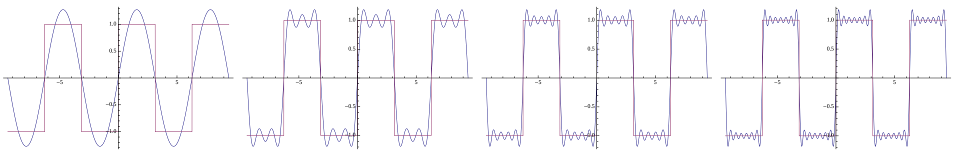

Figure 2: Periodic Fourier approximations to a square wave.

We can also say that periodicity is a central topic in the study of Differential Equations, [14]: some simple models with periodic solutions are well understood, such as the free pendulum equation, the Lotka-Volterra predator-prey system or the gravity interaction between two masses. However, others, such as the n-body problem or Hilbert's 16th problem, have a very intricate behavior and are a current topic of research.

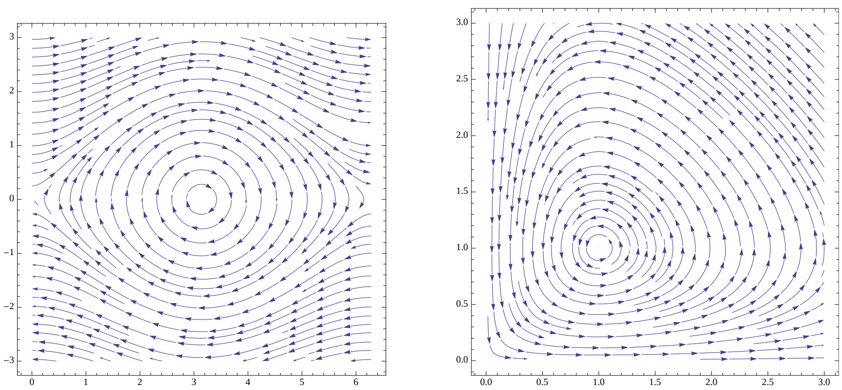

Figure 3: Some periodic orbits for the free pendulum (left) and the Lotka–Volterra predator–prey systems (right).

2 The structure of the period set

Despite its analytical foundations, periodicity is also of an eminently algebraic nature: clearly, the set of all the periods of f, G_f, is an additive subgroup of $\mathbb{R}$ and, reciprocally, for any additive subgroup $G \neq \{0\}$ of $\mathbb{R}$, there exists a periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $G = G_f$. Indeed, it suffices to consider

$$f(x) = \chi_G(x) := \begin{cases} 1, & \text{if } x \in G, \\ 0, & \text{if } x \in \mathbb{R} \setminus G. \end{cases}$$

However, for the usual periodic functions $f\colon \mathbb{R} \to \mathbb{R}$ considered in Calculus courses, which are also continuous, the structure of G_f is much simpler: either f is constant, and therefore $G_f = \mathbb{R}$, or $G_f = T\mathbb{Z}$ for some $T > 0$. In fact, the following important properties for periodic continuous functions are well-known (for the convenience of the reader, we will provide a complete proof of the first two properties in the Appendix):

1. If $f\colon \mathbb{R} \to \mathbb{R}$ is periodic, nonconstant and continuous at a point, then f has a minimal positive period T and $G_f = T\mathbb{Z}$.
2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be nonconstant, continuous and T and S–periodic functions, respectively. Then $f + g$ is periodic if and only if $T/S \in \mathbb{Q}$.
3. If $f\colon \mathbb{R} \to \mathbb{R}$ is measurable and T and S–periodic with $T/S \in \mathbb{R} \setminus \mathbb{Q}$, then it is constant almost everywhere [12, Theorem 2.2].

For instance, Property 2 ensures that the function

$$h(x) = \sin(x) + \sin(\sqrt{2}\, x), \ x \in \mathbb{R},$$

is not periodic (it is in fact an *almost-periodic* function [4]). Notice also that Property 1 prevents the coexistence of incommensurable periods (that is, those of which the quotient is irrational) for a continuous and nonconstant function, a fact that still holds true under the weaker assumptions in Property 3.

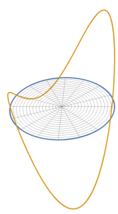

Figure 4: The periodic curve $(\sin(t), \cos(t), \sin(t) + \sin(2t))$.

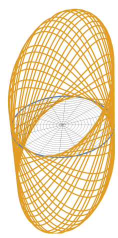

Figure 5: The almost–periodic curve $(\sin(t), \cos(t), \sin(t) + \sin(\sqrt{2}\, t))$.

The goal of this note is to show that, in the absence of some sort of regularity, this type of results can fail quite spectacularly, in particular, that the graph of a periodic function can be dense in $\mathbb{R}^2$, something that the function χ_G defined before does not satisfy.

3 The Cauchy functional equation

Let us consider now the Cauchy functional equation

$$f(x+y) = f(x) + f(y). \tag{2}$$

A solution $f\colon \mathbb{R} \to \mathbb{R}$ of (2) is called an *additive* function. Cauchy showed that any continuous additive function should be of the form $f(x) = c\,x$, with $c = f(1)$. In fact, Darboux pointed out that continuity just at one point, or even local boundedness, is enough to get the same conclusion. Firstly, we are going to establish a basic result.

Definition 1. $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-*linear* if it is additive and $f(r\,x) = r\,f(x)$ for all $x \in \mathbb{R}$ and all $r \in \mathbb{Q}$.

Lemma 2. *$f\colon \mathbb{R} \to \mathbb{R}$ is a solution of (2) if and only if f is $\mathbb{Q}$-linear.*

Proof. If f is $\mathbb{Q}$-linear then, in particular, is a solution of (2). So, let us prove the reciprocal implication: assuming that $f\colon \mathbb{R} \to \mathbb{R}$ is additive it is enough to prove that $f(rx) = rf(x)$ for all $x \in \mathbb{R}$ and all $r \in \mathbb{Q}$. Indeed:

1. $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$ for all $x \in \mathbb{R}$. Now, by induction it follows that $f(n\,x) = n\,f(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

2. From 1 it follows that $m\,f(x) = f(m\,x) = f\left(n\frac{m\,x}{n}\right) = n\,f\left(\frac{m\,x}{n}\right)$. Therefore, $f\left(\frac{m}{n}x\right) = \frac{m}{n}f(x)$ for all $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$.

3. Since $f(0) = f(2 \cdot 0) = 2f(0)$, we have that $f(0) = 0$.

4. Furthermore, $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$, so $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, that is, f is an odd function.

Finally, as a consequence of 2, 3 and 4 we obtain the desired result. □

Now, from Lemma 2 is clear that any additive function satisfies that $f(r) = rf(1)$ for all $r \in \mathbb{Q}$. Therefore Cauchy's assertion is clear: continuity of f, together with the density of rational numbers in the real line, imply that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Also, the first remark of Darboux follows easily because the continuity of an additive function f at some $x_0 \in \mathbb{R}$ implies its continuity at any point $x \in \mathbb{R}$:

$$\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-x_0+x_0+h) = f(x)-f(x_0)+\lim_{h\to 0} f(x_0+h) = f(x).$$

Subsequently, many authors, among them Banach and Sierpinski, proved that measurable additive functions are also of the form $f(x) = c\,x$, [16]. So, additive functions that are not linear should be quite pathological: discontinuous everywhere –actually not locally bounded at any point– and non measurable. Notice that this "wild" behaviour of discontinuous solutions is also shared by the solutions of other more general functional equations, [1, 2].

The existence of discontinuous additive functions was settled by Hamel in [8], where he introduced the concept of a Hamel basis in order to define $\mathbb{Q}$-linear functions not of the form $f(x) = c\,x$. His construction, depending on the Axiom of Choice, also showed the density in $\mathbb{R}^2$ of its graph. Surprinsingly, the graph of an additive function should be either connected or totally disconnected but there exist discontinuous additive functions with connected graph, [10, 15].

In the following section we will present irregular periodic functions as discontinuous solutions of (2).

4 Periodic irregular functions

In the sequel we show how to obtain functions with dense graph and having any given proper $\mathbb{Q}$-vector subspace of $\mathbb{R}$ as set of periods. Note that such a function having an arbitrarily small positive period is sometimes called *microperiodic*, [6]. The key ingredients in the proof are the existence of a Hamel basis for any vector space and the possibility of uniquely constructing a linear function by prescribing arbitrary coefficients to the elements of a basis. These facts (maybe in the finite dimensional setting) are well–known for undergraduate students after a first Linear Algebra course. So, in our opinion, it is very illustrative they learn how these concepts come into play to produce some rather 'pathological' functions.

Still, it is worth noticing that infinite dimensional (topological) vector spaces are much more troublesome than finite dimensional ones. Results that we take for granted in the finite dimensional case, such as the continuity of linear maps, the compactness of the unit ball or the mere existence of a basis have to be thought carefully. We will not deepen in the details, but we will provide a definition of (Hamel) basis and the statement regarding its existence.

Definition 3. Let V be a vector space over a field $\mathbb{K}$ and $v_1, \dots, v_n \in V$. We say $v_1, \dots, v_n$ are *linearly independent* if, given $a_k \in \mathbb{K}$, $k = 1, \dots, n$, such that $\sum_{k=1}^{n} a_k v_k = 0$, then we have that $a_k = 0$ for every $k = 1, \dots, n$. A set $A \subset V$ is a set of *linearly independent vectors* if any $v_1, \dots, v_n \in A$, $n \in \mathbb{N}$, are linearly independent.

$B \subset V$ is called a *Hamel basis* (or simply a *basis*) of V if

1. B is a set of linearly independent vectors.

2. Given $v \in V$, there exist $n \in \mathbb{N}$, $v_1, \dots, v_n \in B$ and $a_1, \dots, a_k \in \mathbb{K}$ such that $v = \sum_{k=1}^{n} a_k v_k$.

With the help of the Axiom of Choice we can prove that any vector space has a Hamel basis —see [9, Theorem 3.19, Exercise 3.26]—. In fact, we have the following result.

Lemma 4. *If V is a vector space and A a set of linearly independent vectors, there exists a basis B of V such that $A \subset B$.*

Now we are ready to prove the result.

Theorem 5. *Let W be a proper $\mathbb{Q}$–vector subspace of $\mathbb{R}$. Then there exists a periodic function $f\colon \mathbb{R} \to \mathbb{R}$ with $G_f = W$ and such that $\operatorname{graph}(f)$ is dense in $\mathbb{R}^2$.*

Proof. If we restrict our attention to additive functions $f\colon \mathbb{R} \to \mathbb{R}$ it is clear that for every $x \in \mathbb{R}$

$$f(x+T) - f(x) = f(x) + f(T) - f(x) = f(T),$$

so T is a period for f if and only if $f(T) = 0$, that is, $G_f = \ker(f)$.

Hence, it is enough to construct a $\mathbb{Q}$–linear function $f\colon \mathbb{R} \to \mathbb{R}$ with the prescribed kernel W. In order to do that, consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and a basis B_1 of W. Since the vectors of B_1 are $\mathbb{Q}$–linearly independent, we can extend B_1 to a (infinite) Hamel basis B of $\mathbb{R}$ over $\mathbb{Q}$. Now we define $f(b) = 0$ for $b \in B_1$ and $f(b) = b$ for $b \in B \backslash B_1$. We can extend f to $\mathbb{R}$ uniquely by $\mathbb{Q}$–linearity. Then, f is $\mathbb{Q}$–linear by construction and, clearly, $W \subset \ker(f)$. On the other hand, if $w \in \ker(f)$, since

$$w = \sum_{b \in B_1} \alpha_b b + \sum_{b' \in B \backslash B_1} \beta_{b'} b'$$

where only a finite amount of the $\alpha_b, \beta_{b'} \in \mathbb{Q}$ are nonzero, we have that

$$0 = f(w) = \sum_{b \in B_1} \alpha_b f(b) + \sum_{b' \in B \backslash B_1} \beta_{b'} f(b') = \sum_{b' \in B \backslash B_1} \beta_{b'} b'.$$

So, $w \in W$ and therefore $\ker(f) = W$.

Finally, the density of $\operatorname{graph}(f) \subset \mathbb{R}^2$ follows from the elegant argument given originally by Hamel in [8, p. 462]: it is not possible to have $f(x) = c\,x$ for some $c \in \mathbb{R}$ and all $x \in \mathbb{R}$, for, otherwise, either $W = \ker(f) = \{0\}$ or $W = \ker(f) = \mathbb{R}$, which is contrary to the assumptions. Then it follows that there must exist $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ such that $\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}$. Thus, $v_1 = (x_1, f(x_1))$ and $v_2 = (x_2, f(x_2))$ are independent vectors (over $\mathbb{R}$) of $\mathbb{R}^2$ (as they have different slope) and, consequently, the set $\{p\,v_1+q\,v_2 \;:\; p,q\in\mathbb{Q}\}$, is dense in $\mathbb{R}^2$. To see this, take any point $v\in\mathbb{R}^2$ and $\varepsilon\in\mathbb{R}^+$. Since v_1 and v_2 are $\mathbb{R}$–linearly independent, $v=\alpha\,v_1+\beta\,v_2$ for some $\alpha,\beta\in\mathbb{R}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ we can take $p,q\in\mathbb{Q}$ such that $|\alpha-p|<\frac{\varepsilon}{2\|v_1\|}$ and $|\beta-q|<\frac{\varepsilon}{2\|v_2\|}$. Thus,

$$|v-(p\,v_1+q\,v_2)\|=\|(\alpha-p)v_1+(\beta-q)v_2\|\le|\alpha-p|\|v_1\|+|\beta-q|\|v_2\|<\varepsilon.$$

Finally, from the $\mathbb{Q}$-linearity of f we have that for each $p,q\in\mathbb{Q}$

$$\begin{aligned}p\,v_1+q\,v_2&=(p\,x_1+q\,x_2,p\,f(x_1)+q\,f(x_2))\\&=(p\,x_1+q\,x_2,f(p\,x_1+q\,x_2))\in\operatorname{graph}(f),\end{aligned}$$

which implies that graph(f) is dense in $\mathbb{R}^2$. □

Theorem 5 allows us to show that, if we were to not ask for f and g to be continuous (or measurable), the thesis of Properties 1, 2 and 3 in Section 2 do not necessarily hold. We illustrate this fact in Example 6, where we use Theorem 5 to build quite an irregular function.

Example 6. Consider the set $W=\{p\,x_1+q\,x_2 \;:\; p,q\in\mathbb{Q}\}$ where $x_1,x_2>0$ are two positive real numbers such that $x_2/x_1\notin\mathbb{Q}$. It is clear that $\dim_{\mathbb{Q}}(W)=2$. We now consider a $\mathbb{Q}$–linear function f such that $\ker(f)=W$ as in Theorem 5, with $B_1=\{x_1,x_2\}$ and B any Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ that extends B_1. Thus, for $p,q\in\mathbb{Q}$, we define

$$f(p\,x_1+q\,x_2+y)=y \text{ for any } y \text{ that is a } \mathbb{Q}\text{–linear combination of } B\backslash B_1.$$

Now, for $T=x_1$ and $S=x_2$ it follows that f is both T and S–periodic with $T/S\in\mathbb{R}\setminus\mathbb{Q}$. So, by Corollary 11 in the Appendix, f is not continuous at any point.

Furthermore, taking $g=f$, it follows that $f+g$ is periodic with $G_{f+g}=W$, so the Property 2 in Section 2 can not be extended to discontinuous functions.

Let us make this example more explicit: take $W=\{p+q\,\pi \;:\; p,q\in\mathbb{Q}\}$ and $B_1=\{1,\pi\}$. Now we can consider a basis B of $\mathbb{R}$ over $\mathbb{Q}$ that contains all the elements of the form $\{\pi^n\}_{n\in\mathbb{Z}}$ (note that this is a set of linearly independent vectors over $\mathbb{Q}$ since π is a transcendent number, [13, Theorem 9.11].

With this we have enough information to plot the function f given by

$$f(p+q\,\pi+y)=y \text{ for any } p,q\in\mathbb{Q} \text{ and } y \text{ a } \mathbb{Q}\text{–linear combination of } B\backslash\{1,\pi\},$$

over a dense subset of $\mathbb{R}$ as Figure 6 shows.

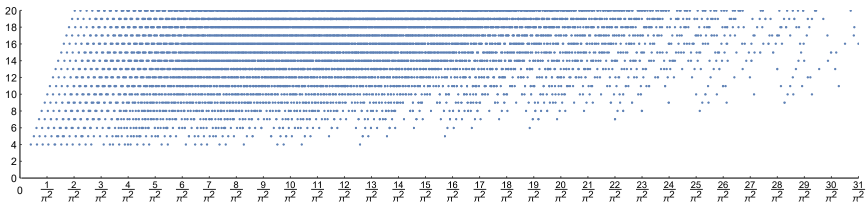

Figure 6: Representation of some (almost 10000) points in the graph of f. The points represented are of the form $\alpha_1\pi^{-1} + \cdots + \alpha_n\pi^{-n}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$.

5 Appendix: some properties of periodic continuous functions

For the sake of completeness we provide here proofs of the Properties 1 and 2 in Section 2. The proof of Property 3 falls outside the elementary scope of this paper. For more results on periodic functions the reader may refer to [7, 11, 12].

5.1 Background results

We will need the following notation in order to talk about periods. Given $a, b \in \mathbb{R}$ and $X, Y \subset \mathbb{R}$ we define:

$$\begin{aligned} a\,X :=& \{a\,k \;:\; k \in X\}, \\ a\,X + b\,Y :=& \{a\,j + b\,k \;:\; j \in X,\; k \in Y\}, \\ XY :=& \{\,j\,k \;:\; j \in X,\; k \in Y\}. \end{aligned}$$

It is clear that:

1. $a(bX) = (ab)X$.
2. $a(bX + cY) = (ab)X + (ac)Y$.
3. $a\mathbb{Z} = b\mathbb{Z}$ iff $b = \pm a$.
4. In general, $a\mathbb{Z} + b\mathbb{Z} \neq (a + b)\mathbb{Z}$.

Now, we start by recalling Bézout's identity.

Lemma 7 (Bézout's identity [5, Theorem 2.5])**.** *Let $a, b \in \mathbb{Z}\backslash\{0\}$. Then there exist $x, y \in \mathbb{Z}$ such that*

$$a\,x + b\,y = \mathrm{mcd}(a, b), \tag{3}$$

satisfying

$$|x| \le \left|\frac{b}{\mathrm{mcd}(a,b)}\right|, \quad |y| \le \left|\frac{a}{\mathrm{mcd}(a,b)}\right|.$$

We have the following interesting consequences.

Corollary 8.

1. *For any* $a, b \in \mathbb{Z}\backslash\{0\}$ *we have that*

$$a\mathbb{Z} + b\mathbb{Z} = \operatorname{mcd}(a, b)\mathbb{Z}.$$

2. *Given any two irreducible fractions* $p/q, \widetilde{p}/\widetilde{q} \in \mathbb{Q}\backslash\{0\}$ *we have that*

$$\frac{p}{q}\mathbb{Z} + \frac{\widetilde{p}}{\widetilde{q}}\mathbb{Z} = \frac{\operatorname{mcd}(p\widetilde{q}, \widetilde{p}q)}{q\widetilde{q}}\mathbb{Z}.$$

3. *For every* $r, s \in \mathbb{R}\backslash\{0\}$ *such that* $r/s = p/q \in \mathbb{Q}$ *where* p/q *is an irreducible fraction we have that*

$$r\mathbb{Z} + s\mathbb{Z} = \frac{s}{q}\mathbb{Z}.$$

Proof.

1. Let $x, y \in \mathbb{Z}$ be given by Bézout's identity for a and b, that is, $a\,x + b\,y = \operatorname{mcd}(a, b)$. Then, if $k \in \mathbb{Z}$,

$$\operatorname{mcd}(a, b)\, k = (a\,x + b\,y)k = a(x\,k) + b(y\,k) \in a\mathbb{Z} + b\mathbb{Z},$$

and, therefore $\operatorname{mcd}(a, b)\mathbb{Z} \subset a\mathbb{Z} + b\mathbb{Z}$. On the other hand, if $j, k \in \mathbb{Z}$,

$$\begin{aligned} aj + bk &= \operatorname{mcd}(a, b)\frac{a}{\operatorname{mcd}(a, b)}j + \operatorname{mcd}(a, b)\frac{b}{\operatorname{mcd}(a, b)}k \\ &= \operatorname{mcd}(a, b)\left(\frac{a}{\operatorname{mcd}(a, b)}j + \frac{b}{\operatorname{mcd}(a, b)}k\right) \in \operatorname{mcd}(a, b)\mathbb{Z}. \end{aligned}$$

Thus, $a\mathbb{Z} + b\mathbb{Z} \subset \operatorname{mcd}(a, b)\mathbb{Z}$.

2. We want to solve the equation $x\mathbb{Z} = \frac{p}{q}\mathbb{Z} + \frac{\widetilde{p}}{\widetilde{q}}\mathbb{Z}$. Then, multiplying by $q\,\widetilde{q}$ and using 1,

$$q\,\widetilde{q}\,x\mathbb{Z} = p\,\widetilde{q}\mathbb{Z} + \widetilde{p}\,q\mathbb{Z} = \operatorname{mcd}(p\,\widetilde{q}, \widetilde{p}\,q)\mathbb{Z}.$$

Therefore, $q\,\widetilde{q}\,x = \pm\operatorname{mcd}(p\,\widetilde{q}, \widetilde{p}\,q)$ and we can take

$$x = \frac{\operatorname{mcd}(p\,\widetilde{q}, \widetilde{p}\,q)}{q\,\widetilde{q}}.$$

3. We want to solve the equation $x\mathbb{Z} = r\mathbb{Z} + s\mathbb{Z}$. Then, dividing by s and using 2,

$$\frac{x}{s}\mathbb{Z} = \frac{r}{s}\mathbb{Z} + \mathbb{Z} = \frac{p}{q}\mathbb{Z} + \mathbb{Z} = \frac{\mathrm{mcd}(p,q)}{q}\mathbb{Z} = \frac{1}{q}\mathbb{Z}.$$

Thus, $\frac{x}{s} = \pm\frac{1}{q}$, so we can take $x = \frac{s}{q}$.

□

Next we present a very useful result when the ratio r/s is irrational.

Lemma 9. *Let $r, s \in \mathbb{R}\backslash\{0\}$, $r/s \in \mathbb{R}\backslash\mathbb{Q}$. Then $r\mathbb{Z} + s\mathbb{Z}$ is dense in $\mathbb{R}$.*

Proof. Let $X = r\mathbb{Z} + s\mathbb{Z}$. We will first check that there are numbers in X as close to zero as we want. Let $t := r/s$ and let us fix $\varepsilon \in \mathbb{R}^+$. Let $n \in \mathbb{N}$ be such that $n > |s|/\varepsilon$. Consider the ordered set of $n+1$ real numbers

$$Y := (kt - \lfloor kt \rfloor)_{k=1}^{n+1} \subset [0,1]\backslash\mathbb{Q}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Let us divide now the interval $[0,1)$ in n subintervals of equal length $1/n$ of the form

$$I_k = \left[\frac{k}{n}, \frac{k+1}{n}\right), \quad k = 0, \dots, n-1.$$

Observe that $\bigcup_{k=0}^{n-1} I_k = [0,1)$. Since the numbers in Y are irrational each one of them is in the interior of one of the intervals I_k. Since there are $n+1$ elements in Y and n intervals, by the pigeonhole principle, there is at least one interval I_k with two elements of Y, which implies that there exist $j, k \in \{1, \dots, n+1\}$ such that

$$0 < k\,t - \lfloor k\,t \rfloor - (j\,t - \lfloor j\,t \rfloor) < \frac{1}{n}.$$

Let $x = k - j, y = \lfloor j\,t \rfloor - \lfloor k\,t \rfloor \in \mathbb{Z}$. Then, $r\,x + s\,y \in X$ and, since $r/s \notin \mathbb{Q}$,

$$\begin{aligned} 0 < |r\,x + s\,y| &= |s(t\,x + y)| = |s|\,|t(k-j) + \lfloor j\,t \rfloor - \lfloor k\,t \rfloor| \\ &= |s|\,|k\,t - \lfloor k\,t \rfloor - (j\,t - \lfloor j\,t \rfloor)| < \frac{|s|}{n} < \varepsilon. \end{aligned}$$

Now, since $0 \in X$ and $X = (-1)X$ it is enough to prove that X is dense in $(0, +\infty)$. Then, fix $b, \varepsilon \in \mathbb{R}^+$ and take $a \in X$ such that $a \in (0, \varepsilon)$ (again, as $X = (-1)X$, we can assume that $a > 0$). By the Archimedean property and the well ordering principle, there exists $n \in \mathbb{N}$ such that

$$(n-1)\,a < b \leq n\,a.$$

Then, $0 \leq n\,a - b < a < \varepsilon$ and since $na \in X$, the proof is finished. □

5.2 Main properties

Proposition 10. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a T and S–periodic function.*

1. *If $j,k\in\mathbb{Z}$ are such that $jT+kS>0$, then f is $jT+kS$–periodic.*
2. *If $T/S=p/q$ where $p,q\in\mathbb{N}$ and p/q is irreducible, then f is S/q–periodic.*
3. *If $T/S\in\mathbb{R}\backslash\mathbb{Q}$, then, for every $\delta\in\mathbb{R}^+$, there exists $\tilde{\delta}\in(0,\delta)$ such that f is $\tilde{\delta}$–periodic.*
4. *If for every $\delta\in\mathbb{R}^+$ there exists $\tilde{\delta}\in(0,\delta)$ such that f is $\tilde{\delta}$–periodic and f is continuous at some $x\in\mathbb{R}$, then f is constant.*

Proof.

1. Since f is T and S–periodic, we have that $f(x)=f(x+jT+kS)$ for every $x\in\mathbb{R}$, so f is $jT+kS$–periodic.

2. By Corollary 8, point 3, there exist $j,k\in\mathbb{Z}$ such that $jT+kS=S/q>0$. Therefore, by 1, we have that f is S/q periodic.

3. Let $\delta\in\mathbb{R}^+$ be fixed. By Lemma 9, $T\,\mathbb{Z}+S\,\mathbb{Z}$ is dense in $\mathbb{R}$, so there exist $j,k\in\mathbb{Z}$ such that $0<\tilde{\delta}:=jT+kS<\delta$. From 1 it follows that f is $\tilde{\delta}$–periodic.

4. Let $y\in\mathbb{R}$ and $\varepsilon\in\mathbb{R}^+$. Since f is continuous at x, there exists $\delta\in\mathbb{R}^+$ such that

$$|f(x)-f(z)|<\varepsilon$$

for every $z\in(x-\delta,x+\delta)$. Let $\widetilde{\delta}<\delta$ a period of f. Then, as a consequence of the Archimedean property and the well ordering principle, there exists $n\in\mathbb{Z}$ such that

$$0\le n\,\widetilde{\delta}-(x-y)<\widetilde{\delta}<\delta.$$

Hence, $y+n\widetilde{\delta}\in(x-\delta,x+\delta)$ and

$$|f(y)-f(x)|=\left|f(y+n\,\widetilde{\delta})-f(x)\right|<\varepsilon.$$

Since ε was fixed arbitrarily, we deduce that $f(y)=f(x)$, and since y was fixed arbitrarily, we conclude that f is constant.

□

Points 3 and 4 of Proposition 10 can be summarized into the following corollary.

Corollary 11. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a T and S–periodic function that is continuous at a point. If $T/S\in\mathbb{R}\backslash\mathbb{Q}$, then f is constant.*

Now we are ready to establish the structure of the period set for a continuous function.

Theorem 12. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be periodic, continuous at a point and not constant. Let $\mathcal{T}$ be the set of positive periods of f. Then $\mathcal{T}$ has a minimum T called* minimum period *and* $\mathcal{T} = \{nT : n \in \mathbb{N}\}$.

Proof. Assume there does not exist a minimum period. Since $\mathcal{T}$ is bounded from below by 0, there exists $\inf \mathcal{T} \in \mathbb{R}$ and a strictly decreasing sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $T_n \to \inf \mathcal{T}$. Clearly,

$$0 < T_n - T_{n+1} \to 0,$$

and since f is T_n and T_{n+1}–periodic, so it is $(T_n - T_{n+1})$–periodic as well. Hence there exist periods as small as we want, with together with the continuity of f at a point, imply by Proposition 10 point 4, that f is constant, contradicting the hypothesis. So, $\mathcal{T}$ has a minimum $T > 0$.

Clearly $\{nT : n \in \mathbb{N}\} \subset \mathcal{T}$. Suppose now there exists $\widetilde{T} \in \mathcal{T}$ such that $\widetilde{T} \neq nT$ for all $n \in \mathbb{N}$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$(n_0 - 1)T < \widetilde{T} < n_0 T,$$

wich implies that $0 < n_0T - \widetilde{T} < T$, but this is a contradiction since $n_0T - \widetilde{T} \in \mathcal{T}$ and $T = \min \mathcal{T}$. □

Finally, we can characterize when the sum of two continuous and non constant periodic functions is also periodic. Notice the difference with the discontinuous case presented in Example 6.

Theorem 13. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be T and S–periodic, respectively.*

1. *If $T/S = p/q$ where $p, q \in \mathbb{N}$, then $f + g$ is qT–periodic.*
2. *If $T/S \in \mathbb{R}\backslash\mathbb{Q}$, f and g are continuous and nonconstant, then $f + g$ is not periodic.*

Proof. 1. If $T/S = p/q$, then $p\,S = q\,T$. Hence, for every $x \in \mathbb{R}$,

$$\begin{aligned}(f+g)(x) =& f(x) + g(x) = f(x + q\,T) + g(x + p\,S)\\ =& f(x + q\,T) + g(x + q\,T) = (f + g)(x + qT).\end{aligned}$$

2. Let us assume that $h = f + g$ is H–periodic. Then, for every $x \in \mathbb{R}$,

$$f(x + H) + g(x + H) = f(x) + g(x),$$

so $l(x) := g(x) - g(x + H) = f(x + H) - f(x)$. Therefore, l is both T and S–periodic and, since it is continuous as well and $T/S \in \mathbb{R}\backslash\mathbb{Q}$, we have, by Corollary 11, that $l(x) = k$ is constant. Then, for every $x \in \mathbb{R}$, $f(x+H) = f(x) + k$ and $g(x+H) = g(x) - k$.

If $k = 0$, we have that $f(x+H) = f(x)$ and $g(x+H) = g(x)$, so f and g are H–periodic. Since f and g are not constant, $T/H, S/H \in \mathbb{Q}$, but then $T/S = (T/H)/(S/H) \in \mathbb{Q}$, which is a contradiction.

If $k \neq 0$, we have by induction that $f(nH) = f(0) + n\,k$ for all $n \in \mathbb{N}$ and then

$$\lim_{n\to\infty} f(nH) = \lim_{n\to\infty} (f(0) + n\,k) = \operatorname{sgn}(k) \cdot \infty,$$

but this is impossible since f is continuous and periodic on $\mathbb{R}$, so it is a bounded function.

□

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J. Ángel Cid
Departamento de Matemáticas
Universidade de Vigo
CITMAga, Santiago de Compostela
angelcid@uvigo.es

F. Adrián F. Tojo
Departamento de Estatística,
Análise Matemática e Optimización
Universidade de Santiago de Compostela
CITMAga
fernandoadrian.fernandez@usc.es

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