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Sobolev Embeddings for Fractional Hajłasz-Sobolev Spaces in the Setting of Rearrangement Invariant Spaces

Joaquim Martín[1] · **Walter A. Ortiz**[1]

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Abstract
We obtain symmetrization inequalities in the context of Fractional Hajłasz-Sobolev spaces in the setting of rearrangement invariant spaces and prove that for a large class of measures our symmetrization inequalities are equivalent to the lower bound of the measure.

Keywords Sobolev inequality · Fractional Hajłasz-Sobolev spaces · Metric measure spaces

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1 Introduction

Let us consider a metric measure space (Ω, d, μ) where μ is a Borel measure on (Ω, d) such that $0 < \mu(B) < \infty$, for every ball B in Ω. We will always assume $\mu(\Omega) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in \Omega$. Let X be a rearrangement invariant (r.i.) space on Ω (see Section 2.2.1 below). In this paper, we introduce the fractional Hajłasz-Sobolev spaces $M^{s,X}(\Omega)$ for $s > 0$, and we will focus on understanding the relation between Sobolev embeddings theorems for spaces $M^{s,X}(\Omega)$ and the growth properties of the measure μ.

Let $s > 0$ and let X be a r.i. space on Ω. We say that $f \in M^{s,X}(\Omega)$, if $f \in X$, and there exits a non-negative measurable function $g \in X$ such that

$$|f(x) - f(y)| \leq d(x, y)^s (g(x) + g(y)) \quad \mu - a.e. \ x, y \in \Omega. \tag{1}$$

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✉ Joaquim Martín
jmartin@mat.uab.cat

Walter A. Ortiz
waortiz@mat.uab.cat

[1] Department of Mathematics, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

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A function g satisfying (1) will be called a $s-$gradient of f. We denote by $D^s(f)$ the collection of all $s-$gradients of f. The **homogeneous Hajłasz-Sobolev space** $\dot{M}^{s,X}(\Omega)$ consists of all functions $f \in X$ for which

$$\|f\|_{\dot{M}^{s,X}(\Omega)} = \inf_{g \in D^s(u)} \|g\|_X$$

is finite. The **Hajłasz-Sobolev space** $M^{s,X}(\Omega)$ is $\dot{M}^{s,X}(S) \cap X$ equipped with the norm

$$\|f\|_{M^{s,X}(\Omega)} = \|f\|_X + \|f\|_{\dot{M}^{s,X}(\Omega)}.$$

When $X = L^p(\Omega)$, $1 \leq p \leq \infty$, we shall write $M^{s,p}(\Omega)$ instead of $M^{s,X}(\Omega)$.

Remark 1 In the context of metric spaces, the spaces $M^{1,p}(\Omega)$ were first introduced by Hajłasz (see [11] and [12]). They play an important role in the area of analysis called analysis on metric spaces and a lot of papers have focused on this subject (see for example [3, 13–16], and the references quoted therein). When the measure μ is doubling[1], spaces $M^{1,X}(\Omega)$ have been considered in some particular cases, for example, Hajłasz-Lorentz-Sobolev spaces $M^{1,L^{p,q}}(\Omega)$ (see [19]) and Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,L^{\Phi}}(\Omega)$, where L^{Φ} is an Orlicz space (see [34]). Also in the doubling case, fractional spaces $M^{s,p}(\Omega)$ were introduced and studied in [35] (see also [17] and [18]).

For $p > 1$, $M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ (see [11]), whereas for $p = 1$, $M^{1,1}(\mathbb{R}^n)$ coincides with the Hardy.Sobolev space $H^{1,1}(\mathbb{R}^n)$ (see [24, Thm 1]) and if $0 < s < 1$, then $M^{s,p}(\mathbb{R}^n) = B^s_{p,\infty}(\mathbb{R}^n)$ (see [35]). Notice that in $\mathbb{R}^n$, if $s > 1$, then $M^{s,p}(\mathbb{R}^n)$ is trivial (contains only constant functions). However, if Ω is a fractal, then $M^{s,p}(\Omega)$ for $s > 1$ may be non-trivial (see [17]).

It is well known that the lower bound for the growth of the measure

$$\mu(B(x,r)) \geq b r^{\alpha}, \tag{2}$$

implies Sobolev embedding theorems for Hajłasz-Sobolev spaces $M^{1,p}$(see [11] and [12]).

The converse problem, i.e. when the embedding

$$M^{1,p}(X) \subset L^q(X), \ q > p \tag{3}$$

implies a lower bound for the growth of the measure, has been considered by several authors (see [10, 18, 20–23] and the references quoted therein). In the recent paper [2], R. Alvarado, P. Górka and P. Hajłasz show that in fact if (3) holds with $q = \alpha p/(\alpha - p)$, then lower bound for the growth (2) holds.

The purpose of this paper is to obtain an analogous result for $M^{s,X}$ spaces. This will be done by obtaining pointwise estimates between the special difference $f^{**}(t) - f^*(t)$ (called the oscillation[2] of f) and the function g (see Theorem 1 below), i.e. we will see that for a wide range of measures, condition (2) implies

$$f^{**}(t) - f^*(t) \leq C t^{s/\alpha} g^{**}(t), \tag{4}$$

for every $f \in M^{s,L^1+L^\infty}$ and $g \in D^s(f)$. Moreover, if $0 < s \leq 1$, then (4) implies (2).

[1] μ is said to be doubling provided there exists a constant $C > 0$ such that

$$\mu(2B) \leq C\mu(B) \text{ for all balls } B \subset \Omega.$$

[2] Here f^* is the decreasing rearrangement of f, $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, for all $t > 0$, (see Section 2.2).

Symmetrization inequalities imply Sobolev inequalities in the setting of rearrangement invariant spaces. Indeed, from (4) we obtain: for any r.i. space X with upper Boyd[3] index $\bar{\alpha}_X < 1$, we have

$$\left\| t^{-s/\alpha}(f^{**}(t) - f^{*}(t)) \right\|_X \leq c \, \|g\|_X ,$$

where $c = c(s, \alpha, X)$.

Notice that we avoid one common drawback of the usual approaches to Sobolev inequalities which require the choice of specific norms before one starts the analysis. Instead, we work with pointwise symmetrization inequalities which are *universal* and it is the inequalities themselves that select the *correct* spaces.

For example, in the particular case of $X = L^p$ (see Corollary 1 below) we obtain that if $1 > s/\alpha > \frac{1}{p}$, then[4]

$$\left\| t^{-s/\alpha}(f^{**}(t) - f^{*}(t)) \right\|_{L^p} \simeq \left\| t^{-s/\alpha} f^{**}(t) \right\|_{L^p} = \|f\|_{L^{p^*,p}}$$

where $p_s^* = \frac{\alpha p}{\alpha - sp}$, i.e.

$$\left(\int_0^\infty \left(t^{\frac{1}{p} - \frac{1}{\alpha}} f^{**}(t) \right)^p \frac{dt}{t} \right)^{1/p} \leq C \left(\int_\Omega g^p \right)^{1/p} .$$

On the other hand, since $p < p_s^*$ we have that

$$L^{\frac{\alpha p}{\alpha - sp}, p} \subset L^{\frac{\alpha p}{\alpha - sp}}$$

in particular, if $s = 1$, then we get

$$\left(\int_\Omega |f|^{p_1^*} d\mu \right)^{1/p_1^*} \leq \left(\int_0^\infty \left(t^{\frac{1}{p} - \frac{1}{\alpha}} f^{**}(t) \right)^p \frac{dt}{t} \right)^{1/p} \leq C \left(\int_\Omega g^p d\mu \right)^{1/p} .$$

Remark 2 The technique to obtain Sobolev oscillation type inequalities has been developed by M. Milman and J. Martín (see [27, 28] and [29]) and provide a considerable simplification in the theory of embeddings of Sobolev spaces based on rearrangement invariant spaces.

The paper is organized as follows. In Section 2, we introduce the notation and the standard assumptions used in the paper, in Section 3, we will obtain oscillation type inequalities for spaces $M^{s,X}$, we will see that they are equivalent to the lower bound for the growth of the measure and will obtain Sobolev type embedding of $M^{s,X}$ into a rearrangement invariant spaces. Finally, in the Appendix A we will give some properties of the measures we will be working with.

2 Preliminaries

In this section we establish some further notation and background information and we provide more details about metrics spaces and r.i. spaces that we will working with.

[3]The restriction on the Boyd indices is only required to guarantee that the inequality $\|g^{**}\|_X \leq c_X \|g\|_X$, holds for all $g \in X$.

[4]As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\, f$, while the symbol $f \preceq g$ means that $f \leq c\, g$, and $f \succeq g$ means that $f \geq c\, g$.

2.1 Metric Spaces

Let (Ω, d) be a metric space. As usual a ball B in Ω with a center x and radius $r > 0$ is a set $B = B(x, r) := \{y \in \Omega; d(x, y) < r\}$. Throughout the paper by a metric measure space we mean a triple (Ω, d, μ), where μ is a Borel measure on (Ω, d) such $0 < \mu(B) < \infty$, for every ball B in Ω, we also assume that $\mu(\Omega) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in \Omega$.

We will say that a measure μ is $\alpha-$lower bounded if there are $b, \alpha > 0$ such that

$$\mu(B(x, r)) \geq br^{\alpha}, \tag{5}$$

for all $x \in \Omega$ and $r > 0$.

For simplicity we assume in what follows that $\mu(B(x, r)) \geq r^{\alpha}$.

In what follows we will, furthermore, assume that the measure μ is continuous, i.e. μ satisfies that the map $r \to \mu(B(x, r))$ is continuous[5] or that μ is doubling, i.e. there exists a constant C_D such that, for all $x \in \Omega$ and for all $r > 0$, we have that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r)).$$

Notice that in both cases there is a constant $c = c_{\mu} \geq 1$ such that given $t > 0$, for all $x \in \Omega$, there is a positive number $r(x)$ such that

$$t \leq \mu(B(x, r(x)) \leq ct.$$

In the doubling case, given $x \in \Omega$, consider $r_0(x) = \sup\{r : \mu(B(x, r)) < t\}$ and take r such that $r < r_0(x) < 2r$, then

$$t \leq \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) \leq C_D t.$$

In what follows we call these measures $c-$almost continuous[6].

2.2 Background on Rearrangement Invariant Spaces

For measurable functions $f : \Omega \to \mathbb{R}$, the distribution function of f is given by

$$\mu_f(t) = \mu\{x \in \Omega : |f(x)| > t\} \quad (t > 0).$$

The **decreasing rearrangement** f_{μ}^{*} of f is the right-continuous non-increasing function from $[0, \infty)$ into $[0, \infty]$ which is equimeasurable with f. Namely,

$$f_{\mu}^{*}(s) = \inf\{t \geq 0 : \mu_f(t) \leq s\}.$$

We will write in what follows f^* instead of f_{μ}^{*}.

In particular, for $t > 0$ we get (see [6, Prop. 1.7. Chapter 2])

$$\mu(f^*(t)) = \mu\left(\{x \in \Omega : |f(x)| > f^*(t)\}\right) \leq t. \tag{6}$$

It is easy to see that for any measurable set $E \subset \Omega$

$$\int_E |f(x)|\, d\mu \leq \int_0^{\mu(E)} f^*(s) ds. \tag{7}$$

Since f^* is decreasing, the function f^{**}, defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \tag{8}$$

[5] In the Appendix A we describe measures with this property.

[6] An example of an $\alpha-$lower bounded measure that does not satisfy this condition is given in the Appendix A

is also decreasing and, moreover,

$$f^* \leq f^{**}.$$

Remark 3 An elementary computation shows that

$$\frac{\partial}{\partial t} f^{**}(t) = -\frac{f^{**}(t) - f^*(t)}{t}$$

and that the function $t \to t\,(f^{**}(t) - f^*(t))$ is increasing. Moreover, it is well known and easy to see

$$t\left(f^{**}(t) - f^*(t)\right) = \int_{\{x \in \Omega : |f(x)| > f^*(t)\}} \left(|f(x)| - f^*(t)\right) dt. \tag{9}$$

2.2.1 Rearrangement Invariant Spaces

We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces and refer the reader to [6, 25], for a complete treatment. We say that a Banach function space $X = X(\Omega)$ on (Ω, d, μ) is rearrangement-invariant (r.i.) space, if $g \in X$ implies that all $\mu-$ measurable functions f with the same decreasing rearrangement function with respect to the measure μ, i.e. such that $f^* = g^*$, also belong to X, and, moreover, $\|f\|_X = \|g\|_X$.

For any r.i. space $X(\Omega)$ we have

$$L^\infty(\Omega) \cap L^1(\Omega) \subset X(\Omega) \subset L^1(\Omega) + L^\infty(\Omega),$$

with continuous embedding.

A r.i. space $X(\Omega)$ can be represented by an r.i. space on the interval $(0, \mu(\Omega))$, with Lebesgue measure, $\bar{X} = \bar{X}(0, \mu(\Omega))$, such that

$$\|f\|_X = \|f^*\|_{\bar{X}},$$

for every $f \in X$. A characterization of the norm $\|\cdot\|_{\bar{X}}$ is available (see [6, Theorem 4.10 and subsequent remarks]). Typical examples of r.i. spaces are the L^p-spaces, Lorentz spaces and Orlicz spaces.

The associated space $X'(\Omega)$ of $X(\Omega)$ is the r.i. space of all measurable functions h for which the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{g \neq 0} \frac{\int_\Omega |g(x)h(x)|\, d\mu}{\|g\|_{X(\Omega)}} \tag{10}$$

is finite. Note that by the definition (10), the generalized Hölder inequality

$$\int_\Omega |g(x)h(x)|\, d\mu \leq \|g\|_{X(\Omega)}\, \|h\|_{X'(\Omega)} \tag{11}$$

holds.

Let $X(\Omega)$ be an r.i. space. Then, the function $\phi_X : [0, \infty) \to [0, \infty)$ given by

$$\phi_X(s) = \left\|\chi_{[0,s)}\right\|_{\bar{X}},$$

is called the fundamental function of $X(\Omega)$. The fundamental function ϕ_X of any r.i. space $X(\Omega)$ is quasiconcave, in the sense that it is non-decreasing on $[0, \infty)$, $\phi_X(s) = 0$ and $\phi_X(s)/s$ is non-increasing on $(0, \infty)$. Moreover, one has that (see [6, Theroem 5.2. Chapter 2.])

$$\phi_{X'}(s)\phi_X(s) = s, \quad (s > 0). \tag{12}$$

Classically conditions on r.i. spaces are given in terms of the Hardy defined by

$$P^{(q)}f(t)=\left(\frac{1}{t}\int_0^t |f(x)|^q\right)^{1/q},\quad Q_\lambda^{(q)}f(t)=\left(\frac{1}{t^\lambda}\int_t^\infty |f(x)|^q\,\frac{dx}{x^{1-\lambda}}\right)^{1/q},\quad t>0,$$

here $0<q<\infty,\ 0\le\lambda<1$.

The boundedness of these operators on r.i. spaces can be described in terms of the so called **Boyd indices**[7] defined by

$$\bar{\alpha}_X=\inf_{s>1}\frac{\ln h_X(s)}{\ln s}\quad\text{and}\quad \underline{\alpha}_X=\sup_{s<1}\frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the compression/dilation operator E_s on $\bar{X}$, defined for $s>0$, by $E_s f(t)=f^*(\frac{t}{s})$. For example if $X=L^p$ with $p>1$, then $\bar{\alpha}_X=\underline{\alpha}_X=\frac{1}{p}$. It is well known that (see [26], and [33])

$$\overline{\alpha}_X<\frac{1}{q}\Leftrightarrow P^{(q)}\text{ is bounded on }X, \tag{13}$$

$$\underline{\alpha}_X>\frac{\lambda}{q}\Leftrightarrow Q_\lambda^{(q)}\text{ is bounded on }X. \tag{14}$$

3 Symmetrization Inequalities and Embeddings for Fractional Hajłasz-Sobolev Spaces

The method of proof of the following theorem follows the ideas of [30, Theorem 2] (see also [31]).

Theorem 1 *Let (Ω,d,μ) be a metric measure space such that μ is $c-$almost continuous and $\alpha-$lower bounded. Let $s>0$, $f\in M^{s,L^1+L^\infty}$ and $g\in D^s(f)$. Let $0<p\le 1$. Then, for all $t>0$, we have*

$$\left(\left(|f|^p\right)^{**}(t)-\left(|f|^p\right)^{*}(t)\right)^{1/p}\le Ct^{s/\alpha}\left(\left(g^p\right)^{**}(t)\right)^{1/p}, \tag{15}$$

where $C=C(c,p)$ is a constant that just depends on c and p.

Proof Take $t>0$ if $\left(|f|^p\right)^{**}(t)-\left(|f|^p\right)^{*}(t)=0$, then (15) is obvious, otherwise let

$$A=\left\{x\in\Omega:|f(x)|^p>\left(|f|^p\right)^*(t)\right\},$$

notice that by (9) the set A is not empty. Given $x\in A$, since μ is $c-$almost continuous, there is a radius $r(x)$ such that

$$2t\le\mu(B(x,r(x)))\le 2ct.$$

Let $r=\min((2t)^{1/\alpha},r(x))$, and for every $x\in A$, set

$$A_x=\left\{y\in B(x,r):|f(y)|^p\le\left(|f|^p\right)^*(t)\right\}.$$

From

$$B(x,r)=(B(x,r)\cap A)\cup A_x$$

[7] Introduced by D.W. Boyd in [7].

we see that

$$2t \leq \mu(B(x,r)) \leq \mu(A) + \mu(A_x) \leq t + \mu(A_x), \quad \text{(by (6))}$$

whence

$$t \leq \mu(A_x). \tag{16}$$

Then

$$\begin{aligned} I &= \int_A \left(|f(x)|^p - \left(|f|^p\right)^*(t)\right) d\mu(x) \\ &\leq \int_A \left(|f(x)|^p - \frac{1}{\mu(A_x)} \int_{A_x} |f(y)|^p \, d\mu(y)\right) d\mu(x) \\ &= \int_A \left(\frac{1}{\mu(A_x)} \int_{A_x} \left(|f(x)|^p - |f(y)|^p\right) d\mu(y)\right) d\mu(x) \\ &\leq \frac{1}{t} \int_A \int_{A_x} \left||f(x)|^p - |f(y)|^p\right| d\mu(y) d\mu(x) \ \text{(by (16))} \\ &\leq \frac{1}{t} \int_A \int_{B(x,r)} \left||f(x)|^p - |f(y)|^p\right| d\mu(y) d\mu(x) \ (\text{since} A_x \subset B(x,r)) \\ &= J. \end{aligned}$$

Taking into account that

$$\left||x|^p - |y|^p\right| \leq |x-y|^p,$$

we get

$$\begin{aligned} J &\leq \frac{1}{t} \int_A \int_{B(x,r)} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \\ &\leq \frac{1}{t} \int_A \int_{B(x,r)} d(x,y)^{sp} \left(g(x) + g(y)\right)^p d\mu(y) d\mu(x). \\ &\leq \frac{1}{t} \int_A \int_{B(x,r)} r^{sp} \left(g(x)^p + g(y)^p\right) d\mu(y) d\mu(x) \\ &\leq \frac{(2t)^{sp/\alpha}}{t} \int_A \int_{B(x,r)} \left(g(x)^p + g(y)^p\right) d\mu(y) d\mu(x) \\ &\leq \frac{(2t)^{sp/\alpha}}{t} \left(\int_A \int_{B(x,r)} g(x)^p d\mu(y) d\mu(x) + \int_A \int_{B(x,r)} g(y)^p d\mu(y) d\mu(x)\right) \\ &\leq \frac{(2t)^{sp/\alpha}}{t} \left(\int_A g(x)^p \mu\left(B(x,r)\right) d\mu(x) + \int_A \left(\int_0^{\mu(B(x,r))} g^*(z)^p dz\right) d\mu(x)\right) \ \text{(by (7))} \\ &\leq \frac{(2t)^{sp/\alpha}}{t} \left(2ct \int_0^t g^*(z)^p dz + \int_A \left(\int_0^{2ct} g^*(z)^p dz\right) d\mu(x)\right) \\ &\leq \frac{(2t)^{sp/\alpha}}{t} \left(2ct \int_0^t g^*(z)^p dz + t \int_0^{2ct} g^*(z)^p dz\right) \ \text{(by (8))} \\ &\leq (2c)^2 2^{sp/\alpha} t^{sp/\alpha} \int_0^t g^*(z)^p dz. \end{aligned}$$

Finally, the formula (see Remark 3)

$$t\left(\left(|f|^p\right)^{**}(t) - \left(|f|^p\right)^*(t)\right) = \int_A \left(|f(x)|^p - \left(|f|^p\right)^*(t)\right) d\mu(x)$$

yields

$$\left(\left(|f|^p\right)^{**}(t)-\left(|f|^p\right)^{*}(t)\right)^{1/p}\leq Ct^{s/\alpha}\left(\left(g^p\right)^{**}(t)\right)^{1/p},$$

as we wished to show. □

Theorem 2 *Let (Ω, d, μ) be a metric measure space such that μ is $c-$almost continuous. Then the following statements are equivalent*

(i) *μ is $\alpha-$lower bounded.*

(ii) *If $0<s\leq 1$, then for every $f\in M^{s,L^1+L^\infty}$ and $g\in D^s(f)$, we have that*

$$f^{**}(t)-f^{*}(t)\leq Ct^{s/\alpha}g^{**}(t). \tag{17}$$

Proof If (i) holds, then by Theorem 1 we get (ii). Assume that (17) holds. Fix $x_0\in\Omega$ and $r>0$. Define the function f_{r,x_0} by

$$f_{r,x_0}(x)=\begin{cases}(r-d(x_0,x))^s & \text{if } d(x_0,x)\leq r,\\ 0 & \text{if } d(x_0,x)>r\end{cases}$$

It is easy see that $g_{r,x_0}(x)=\chi_{B(x_0,r)}$, satisfies that

$$\left|f_{r,x_0}(x)-f_{r,x_0}(y)\right|\leq d(x,y)^s\left(g_{r,x_0}(x)+g_{r,x_0}(y)\right).$$

An elementary computation shows that

$$\mu_{f_{r,x_0}}(\lambda)=\begin{cases}\mu\left(B(x_0,r-\lambda^{1/s})\right) & \text{if } 0<\lambda\leq r^s,\\ 0 & \text{if } \lambda>r^s.\end{cases} \tag{18}$$

By hypothesis,

$$\left(f_{r,x_0}\right)^{**}(t)-\left(f_{r,x_0}\right)^{*}(t)\leq Ct^{s/\alpha}\left(g_{r,x_0}\right)^{**}(t)$$

Thus,

$$\begin{aligned}&\left(f_{r,x_0}\right)^{**}(0)-\left(f_{r,x_0}\right)^{**}(2\mu(B(x_0,r)))\\ &=\int_0^{2\mu(B(x_0,r))}\left(\left(f_{r,x_0}\right)^{**}(t)-\left(f_{r,x_0}\right)^{*}(t)\right)\frac{dt}{t}\\ &\leq C\int_0^{2\mu(B(x_0,r))}t^{s/\alpha-1}\left(g_{r,x_0}\right)^{**}(t)dt.\end{aligned} \tag{19}$$

But

$$\left(g_{r,x_0}\right)^{**}(t)=\frac{1}{t}\int_0^t\chi_{[0,\mu(B(x_0,r)))}(s)ds=\min\left(1,\frac{\mu(B(x_0,r))}{t}\right),$$

thus

$$\begin{aligned}I&=\int_0^{2\mu(B(x_0,r))}t^{s/\alpha-1}\left(g_{r,x_0}\right)^{**}(t)dt\\ &=\int_0^{\mu(B(x_0,r))}t^{s/\alpha-1}dt+\mu(B(x_0,r))\int_{\mu(B(x_0,r))}^{2\mu(B(x_0,r))}t^{s/\alpha-2}dt\\ &\preceq\mu(B(x_0,r))^{s/\alpha}.\end{aligned} \tag{20}$$

Combining (19) and (20) we get

$$\left(f_{r,x_0}\right)^{**}(0)-\left(f_{r,x_0}\right)^{**}(2\mu(B(x_0,r)))\preceq\mu(B(x_0,r))^{s/\alpha}. \tag{21}$$

On the other hand

$$\left(f_{r,x_0}\right)^{**}(0)=\left\|f_{r,x_0}\right\|_{L^\infty}=r^s \tag{22}$$

so from (18) we get that $\left(f_{r,x_0}\right)^*(t) = 0$ if $t > \mu(B(x_0, r))$, therefore

$$\begin{aligned}\left(f_{r,x_0}\right)^{**}(2\mu(B(x_0,r))) &= \frac{1}{2\mu(B(x_0,r))}\int_0^{2\mu(B(x_0,r))}\left(f_{r,x_0}\right)^*(t)dt \\ &= \frac{1}{2\mu(B(x_0,r))}\int_0^{\mu(B(x_0,r))}\left(f_{r,x_0}\right)^*(t)dt \\ &= \frac{1}{2\mu(B(x_0,r))}\left\|f_{r,x_0}\right\|_{L^1} \\ &\leq \frac{1}{2\mu(B(x_0,r))}\left(r^s\mu(B(x_0,r)\right) = \frac{r^s}{2}. \end{aligned} \tag{23}$$

Thus

$$\begin{aligned}\frac{r^s}{2} &= r^s - \frac{r^s}{2} \leq \left(f_{r,x_0}\right)^{**}(0) - \left(f_{r,x_0}\right)^{**}(2\mu(B(x_0,r))) \quad \text{(by (22) and (21))} \\ &\preceq \mu(B(x_0,r))^{s/\alpha} \quad \text{(by (21))}\end{aligned}$$

which implies that μ is $\alpha-$lower bounded up to constants. □

Remark 4 In Kalis' 2007 PhD thesis at FAU (see also [20]) was proved that Sobolev embeddings estimates for Hömander vector fields imply a lower bound for the growth of the measure.

Theorem 3 provides us the following Sobolev embedding result for Fractional Hajłasz-Sobolev spaces.

Theorem 3 *Let (Ω, d, μ) be a metric measure space such that μ is $c-$almost continuous and $\alpha-$lower bounded. Let X be a r.i. space. Let $f \in M^{s,X}$ and $g \in D^s(f)$.*

(i) *If $s/\alpha < 1$,*

(a) *If $\underline{\alpha}_X > s/\alpha$, then*

$$\left\|t^{-s/\alpha}f^{**}(t)\right\|_{\bar{X}} \preceq \|g\|_X .$$

(b) *If $\bar{\alpha}_X < s/\alpha$, then*

$$\|f\|_{L^\infty} \preceq \|g\|_X + \|f\|_{L^1+L^\infty} .$$

(ii) *If $s/\alpha = 1$, then*

$$\sup_{t>0}\phi_X(t)\frac{(f^{**}(t) - f^*(t))}{t} \preceq \|g\|_X .$$

(iii) *If $s/\alpha > 1$, then*

$$\|f\|_{L^\infty} \preceq \|g\|_X + \|f\|_{L^1+L^\infty} .$$

Proof Case (i) Assume $s/\alpha < 1$, then:

(a) Condition $\underline{\alpha}_X > s/\alpha$, implies $f^{**}(\infty) = 0$, so

$$t^{-s/\alpha}f^{**}(t) = t^{-s/\alpha}\int_t^\infty z^{\frac{s}{\alpha}}\frac{(f^{**}(z) - f^*(z))}{z^{\frac{s}{\alpha}}}\frac{dz}{z} = Q^{(1)}_{\frac{s}{\alpha}}\left[\frac{(f^{**}(\cdot) - f^*(\cdot))}{(\cdot)^{\frac{s}{\alpha}}}\right](t).$$

Hence

$$\begin{aligned}\left\| t^{-s/\alpha} f^{**}(t)\right\|_{\bar{X}} &= \left\| Q^{(1)}_{\frac{s}{\alpha}}\left[\frac{(f^{**}(\cdot)-f^{*}(\cdot))}{(\cdot)^{\frac{s}{\alpha}}}\right](t)\right\|_{\bar{X}} \\ &\leq C\left\|\left(\frac{f^{**}(t)-f^{*}(t)}{t^{s/\alpha}}\right)\right\|_{\bar{X}} \ (\text{since } \underline{\alpha}_X > s/\alpha). \\ &\leq C\left\| t^{-s/\alpha} f^{**}(t)\right\|_{\bar{X}}.\end{aligned}$$

Therefore, if $\bar{\alpha}_X < 1$, we have that

$$\begin{aligned}\left\| t^{-s/\alpha} f^{**}(t)\right\|_{\bar{X}} &\simeq \left\|\left(\tfrac{f^{**}(t)-f^{*}(t)}{t^{s/\alpha}}\right)\right\|_{\bar{X}} \\ &\preceq \| g^{**}(t)\|_{\bar{X}} \\ &\preceq \|g\|_X .\end{aligned}$$

In case that $\bar{\alpha}_X = 1$, let $0 < p < 1$, and consider the function $|f|^p$. By (15) we have that

$$\left(t^{-sp/\alpha}\left(|f|^p\right)^{**}(t) - \left(|f|^p\right)^{*}(t)\right)^{1/p} \preceq \left(\left(g^p\right)^{**}(t)\right)^{1/p}. \tag{24}$$

The formula

$$\begin{aligned}\left(|f|^p\right)^{**}(t) &= \int_t^\infty \left(\left(|f|^p\right)^{**}(z) - \left(|f|^p\right)^{*}(z)\right)\frac{dz}{z} \\ &= \int_t^\infty z^{\frac{sp}{\alpha}}\left(\left[z^{-\frac{sp}{\alpha}}\left(\left(|f|^p\right)^{**}(z) - \left(|f|^p\right)^{*}(z)\right)\right]^{1/p}\right)^p \frac{dz}{z},\end{aligned}$$

yields

$$\begin{aligned}t^{-\frac{s}{\alpha}}\left(\left(|f|^p\right)^{**}(t)\right)^{1/p} &= \left(t^{-\frac{sp}{\alpha}}\int_t^\infty z^{\frac{sp}{\alpha}}\left(z^{-\frac{s}{p\alpha}}\left[\left(\left(|f|^p\right)^{**}(z)-\left(|f|^p\right)^{*}(z)\right)\right]^{1/p}\right)^p\frac{dz}{z}\right)^{1/p} \\ &= Q^{(p)}_{\frac{sp}{\alpha}}\left((\cdot)^{\frac{-s}{p\alpha}}\left[\left(\left(|f|^p\right)^{**}(\cdot) - \left(|f|^p\right)^{*}(\cdot)\right)\right]^{1/p}\right)(t),\end{aligned}$$

Since $\underline{\alpha}_X > \frac{s}{\alpha}$, the operator $Q^{(p)}_{\frac{s}{\alpha}}$ is bounded on X (by (14)), thus

$$\begin{aligned}\left\| t^{-\frac{s}{\alpha}} f^{**}\right\|_{\bar{X}} &\simeq \left\| t^{-\frac{s}{\alpha}}\left(\left(|f|^p\right)^{**}(t)\right)^{1/p}\right\|_{\bar{X}} \\ &= \left\| Q^{(p)}_{\frac{s}{p\alpha}}\left((\cdot)^{\frac{-s}{p\alpha}}\left[\left(\left(|f|^p\right)^{**}(\cdot) - \left(|f|^p\right)^{*}(\cdot)\right)\right]^{1/p}\right)(t)\right\|_{\bar{X}} \\ &\preceq \left\|\left(z^{-\frac{s}{p\alpha}}\left[\left(\left(|f|^p\right)^{**}(z) - \left(|f|^p\right)^{*}(z)\right)\right]^{1/p}\right)\right\|_{\bar{X}} \\ &\preceq \left\|\left(\left(g^p\right)^{**}(t)\right)^{1/p}\right\|_{\bar{X}} \ (\text{by } (24)) \\ &\preceq \|g\|_X \ (\text{by } (13)).\end{aligned}$$

(b) If $\bar{\alpha}_X < s/\alpha$, by Theorem 2.3 of [8], we have that

$$\begin{aligned}\|f\|_{L^\infty} &\leq \left\|\frac{f^{**}(t)-f^{*}(t)}{t^{s/\alpha}}\right\|_{\bar{X}} + \|f\|_{L^1+L^\infty} \\ &\preceq \left\| g^{**}(t)\right\|_{\bar{X}} \preceq \|g\|_X .\end{aligned}$$

Case (ii) If $s/\alpha = 1$, then

$$\frac{f^{**}(t)-f^{*}(t)}{t} \leq Cg^{**}(t) = \frac{1}{t}\int_0^t g^{*}(s)ds \leq \|g\|_X \frac{\phi_{X'}(t)}{t} \ (\text{by } (11)).$$

Consequently,

$$\phi_X(t)\frac{(f^{**}(t)-f^{*}(t))}{t}\preceq \|g\|_X \text{ (by (12)).}$$

Case (3) If $s/\alpha>1$, then

$$\frac{f^{**}(t)-f^{*}(t)}{t^{s/\alpha}}\preceq \frac{1}{t}\int_0^t g^*(s)ds\leq \|g\|_X\,\frac{\phi_{X'}(t)}{t}.$$

On the other hand

$$\begin{aligned}f^{**}(0)-f^{**}(1) &= \int_0^1 \left(f^{**}(z)-f^{*}(z)\right)\frac{dz}{z}=\int_0^1 z^{s/\alpha-1}\left(\frac{f^{**}(z)-f^{*}(z)}{z^{s/\alpha}}\right)dz\\ &\preceq \|g\|_X\int_0^1 z^{s/\alpha-1}\frac{\phi_{X'}(z)}{z}dz\\ &\leq \|g\|_X\,\phi_{X'}(1)\int_0^1 z^{s/\alpha-2}dz\\ &\preceq \|g\|_X\,.\end{aligned}$$

Finally, using that $\|f\|_{L^\infty}=f^{**}(0)$, we obtain

$$\|f\|_{L^\infty}\leq \|g\|_X+\|f\|_{L^1+L^\infty}\,.$$

As we wished to show. □

Remark 5 In the classical setting, embedding theorems for $W^{1,p}(\mathbb{R}^n)$ have different behavior when $p<n,\ p=n$,or $p>n$. The point is that

$$|B(x,r)|\simeq r^n. \tag{25}$$

In the metric-measure context the counterpart condition (25) is provided by the lower bound for the growth of the measure (5). Notice also the different character on the embeddings for $p<n,\ p=n$, or $p>n$ in the r.i. context is done by the role of the Boyd index.

Corollary 1 *Under conditions of Theorem 3, in the particular case of $X=L^p$, $(0<p<\infty)$, we obtain*

(i) *If $s/\alpha<1$,*

  (a) *If $s/\alpha>\frac{1}{p}$, then*

$$\|f\|_{L^{p^*,p}}\preceq \|g\|_{L^p}\,.$$

  where $p^*=\frac{\alpha p}{\alpha-sp}$.

  (b) *If $\frac{1}{p}=s/\alpha$, then*

$$\left(\int_0^1\left(\frac{f^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^p\frac{dt}{t}\right)^{1/p}\preceq \|g\|_{L^p}+\|f\|_{L^1+L^\infty}\,.$$

  (c) *If $\frac{1}{p}<s/\alpha$, then*

$$\|f\|_{L^\infty}\leq \|g\|_{L^p}+\|f\|_{L^1+L^\infty}\,.$$

(ii) *If $s/\alpha=1$, then*

$$\sup_{t>0} t^{1/p}\frac{(f^{**}(t)-f^{*}(t))}{t}\preceq \|g\|_{L^p}\,.$$

In particular, if $L^p = L^1$, *we obtain* $\|f\|_{L^\infty_w} \preceq \|f\|_{M^{s,L^p}}$, *where* L^∞_w *is the weak* L^∞*−space*[8]

(iii) *If* $s/\alpha > 1$, *then*

$$\|f\|_{L^\infty} \leq \|g\|_{L^p} + \|f\|_{L^1+L^\infty}.$$

Proof Except $(i.b)$, the remaining statements are a particular case of Theorem 3. To see $(i.b)$, by Theorem 1, we have that

$$t^{-1/p}\left(f^{**}(t) - f^*(t)\right) \leq Cg^{**}(t).$$

Hence

$$\left(\int_0^\infty \left(t^{-1/p}\left(f^{**}(t) - f^*(t)\right)\right)^p dt\right)^{1/p} \preceq \|g\|_{L^p},$$

and by [4, Lemma 5.4] we have that

$$\left(\int_0^1 \left(\frac{f^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^p \frac{dt}{t}\right)^{1/p} \preceq \left(\int_0^\infty \left(t^{-1/p}\left(f^{**}(t) - f^*(t)\right)\right)^p dt\right)^{1/p} + \|f\|_{L^1+L^\infty}.$$

□

Appendix A

A.1 Description of Measures μ Satisfying that the Map $r \to \mu(B(x,r))$ is Continuous

Let (Ω, d) be a metric measure space, we say that a measure μ is metrically continuous with respect to metric d if all $x \in \Omega$ and all $r > 0$ it holds that

$$\lim_{d(x,y)\to 0} \mu(B(x,r)\Delta B(y,r)) = 0$$

where $A\Delta B$ stands for a symmetric difference of sets $A, B \subset \Omega$ and is defined as follows: $A\Delta B := (A\backslash B) \cup (B\backslash A)$.

The following lemma collects some basic facts about continuity of a measure with respect to the metric (see [9] and [1] for the proof).

Lemma 1 *Let* (Ω, d, μ) *be a metric space with a Borel regular measure* μ. *Then the following hold:*

(i) *If* μ *is continuous with respect to the metric* d, *then the map* $x \to \mu(B(x,r))$ *is continuous in* d.

(ii) *If for every* $x \in \Omega$ *and every* $r > 0$ *it holds that* $\mu(\partial B(x,r)) = 0$, *then* μ *is continuous with respect to the metric* d.

(iii) *If for every* $x \in \Omega$ *the function* $r \to \mu(B(x,r))$ *is continuous, then* μ *is continuous with respect to the metric* d.

It is easy to see that if we take $\mathbb{R}^n$ with Lebesgue measure (or with an absolutely continuous measure respect to the Lebesgue measure) with the Euclidean distance, then

[8] Bennett, DeVore and Sharpley introduced the space weak L^∞ defined as $\|f\|_{L^\infty_w} = \sup_{t>0}(f^{**}(t) - f^*(t))$ in [5] where studied its relationship with functions of bounded mean oscillation

this measure is metrically continuous. In fact we have more (see [9, Proposition 2.1]) if (Ω, d, μ) and (Ω, d, ν) are metric measure spaces then if $\mu \prec\prec \nu$ and ν is metrically continuous, then μ is metrically continuous too.

An important example is the following (see [32]).

Lemma 2 *Let μ be a nonnegative Radon measure on $\mathbb{R}^n$. Assume that for any point $p \in \mathbb{R}^n$, $\mu(\{p\}) = 0$, then we choose the coordinate axes in such a way that $\mu(\partial Q) = 0$ for all cubes Q with sides parallel to the axes. In particular the function $\ell \to \mu(Q(x, \ell))$ where ℓ denotes the length of the edge and x is the center of Q, is continuous.*

A.2 An Example of an $\alpha-$lower Bounded Measure which is not $c-$almost Continuous

Consider $\mathbb{R}^2$ with the distance $d_\infty(x, y) = \max\{|x|, |y|\}$ and the measure $\mu =$ Lebesgue measure in the plane + length measure in vertical axis + length measure in vertical straight line passing through $(1, 0)$. It is easy to see that

$$\frac{r^2}{4} \leq \mu(B(x, r))$$

however, is not $c-$almost continuous.

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Declarations

Conflict of Interests The authors declare that they have no conflict of interest.

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