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# Set-Valued Kurzweil–Henstock–Pettis Integral

L. DI PIAZZA[1] and K. MUSIAŁ[2]
[1]*Department of Mathematics, University of Palermo, via Archirafi 34, 90123 Palermo, Italy.*
*e-mail: dipiazza@math.unipa.it*
[2]*Institute of Mathematics, Wrocław University, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.*
*e-mail: musial@math.uni.wroc.pl*

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**Abstract.** It is shown that the obvious generalization of the Pettis integral of a multifunction obtained by replacing the Lebesgue integrability of the support functions by the Kurzweil–Henstock integrability, produces an integral which can be described – in case of multifunctions with (weakly) compact convex values – in terms of the Pettis set-valued integral.

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## 1. Introduction

There is abundant literature dealing with Bochner and Pettis integration of Banach space-valued multifunctions (see El Amri and Hess [5] for further references) of several types. In particular, quite recently Ziat [17, 18] and El Amri and Hess [5] presented a nice characterization of Pettis integrable multifunctions having as their values convex weakly compact subsets of a Banach space.

The definitions of such integrals involve in some way the Lebesgue integrability of the support functions. The theory of integration introduced by Lebesgue at the beginning of the twentieth century is a powerful tool which, perhaps because of its abstract character, does not have the intuitive appeal of the Riemann integral. Besides, as Lebesgue himself observed in his thesis [15], his integral does not integrate all unbounded derivatives and so it does not provide a solution for the problem of primitives, i.e. for the problem of recovering a function from its derivative. Moreover the Lebesgue theory does not cover nonabsolutely convergent integrals. In 1957 Kurzweil [14] and, independently, in 1963 Henstock [9], by a simple modification of Riemann's method, gave a new definition of integral, which is more general than that of Lebesgue. The Kurzweil–Henstock integral retains the intuitive appeal of the Riemann definition, and, at the same time, has the power of Lebesgue's one. Moreover it integrates all derivatives. In the last thirty years the theory of nonabsolute integrals has gone on considerably and the researches in this field are still active and far to be complete (for a survey on the subject, we refer to [1]).

This is the motivation to consider, also in case of multifunctions, the Kurzweil–Henstock integral for real valued functions. In this paper, in particular, we study the obvious generalization of the Pettis integral of a multifunction obtained by replacing the Lebesgue integrability of the support functions by the Kurzweil–Henstock integrability (we call such an integral Kurzweil–Henstock–Pettis). In Theorem 1 we prove a surprising and unexpected characterization of the new integral in terms of the Pettis integral: the Kurzweil–Henstock–Pettis integral is in some way a translation of the Pettis integral.

In case of measurable multifunctions with convex weakly compact values, the Pettis integrability of the selections is a necessary and sufficient condition for the Pettis integrability of the multifunction. We show that a similar characterization holds true also in case of the Kurzweil–Henstock–Pettis integrability of multifunctions.

## 2. Basic Facts

Let $[0, 1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure. $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $[0, 1]$ and if $E \in \mathcal{L}$, then $|E|$ denotes its Lebesgue measure. A *partition $P$ in* $[a, b] \subset [0, 1]$ is a collection $\{(I_1, t_1), \dots, (I_p, t_p)\}$, where $I_1, \dots, I_p$ are nonoverlapping subintervals of $[a, b]$ and $t_i$ is a point of $I_i$, $i = 1, \dots, p$. If $\bigcup_{i=1}^{p} I_i = [a, b]$, we say that $P$ is a *partition of* $[a, b]$. A *gauge* on $[a, b]$ is a positive function on $[a, b]$. For a given gauge $\delta$ on $[a, b]$, we say that a partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is *$\delta$-fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \dots, p$.

DEFINITION 1 ([3, 6]). Let $X$ be any Banach space. A function $f\colon [0, 1] \to X$ is said to be *Henstock integrable* on $[a, b] \subset [0, 1]$ if there exists $w \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge $\delta$ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{p} f(t_i)|I_i| - w \right\| < \varepsilon,$$

for each $\delta$-fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[a, b]$. We set $w =: (H) \int_a^b f \, \mathrm{d}t$.

We denote the set of all Henstock integrable functions on $[0, 1]$, taking their values in $X$, by $\mathcal{H}([0, 1], X)$. In case when $X$ is the real line, $f$ is called *Kurzweil–Henstock integrable*, or simply *KH-integrable* and the space of all KH-integrable functions is denoted by $\mathcal{KH}[0, 1]$.

It is useful to recall some basic results in the theory of real valued KH-integrable functions. The proofs can be found, for example, in [8].

PROPOSITION 1. *Let $f\colon [0, 1] \to \mathbf{R}$ be a function.*

(a) *If $f$ is Lebesgue integrable on* $[0, 1]$*, then it is also KH-integrable.*
(b) *If $f$ is KH-integrable on* $[0, 1]$*, then $f$ is measurable.*
(c) *If $f$ is KH-integrable on* $[0, 1]$*, then $f$ is KH-integrable on every subinterval of* $[0, 1]$.
(d) *$f$ is Lebesgue integrable on* $[0, 1]$ *if and only if both $f$ and $|f|$ are KH-integrable.*
(e) *If $f = F'$ is a derivative, then $f$ is KH-integrable and* $(KH)\int_0^s f(t)\,\mathrm{d}t = F(s) - F(0)$*, for each* $s \in [0, 1]$.

Throughout this paper $X$ is a separable Banach space with dual $X^*$. The closed unit ball of $X^*$ is denoted by $\mathcal{B}(X^*)$. $c(X)$ denotes the collection of all nonempty closed convex subsets of $X$. $cwk(X)$ (resp. $cwk(X^{**})$) denotes the family of all nonempty convex weakly compact subsets of $X$ (resp. of the bidual $X^{**}$ of $X$), $ck(X)$ (resp. $ck(X^{**})$) the family of all nonempty convex compact subsets of $X$ (resp. of $X^{**}$) and $cb(X)$ (resp. $cb(X^{**})$) the family of all nonempty closed bounded convex subsets of $X$ (resp. of $X^{**}$).

For every $C \in c(X)$ the *support function of* $C$ is denoted by $s(\cdot, C)$ and defined on $X^*$ by $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$. If $C = \emptyset$, $s(\cdot, C)$ is identically $-\infty$. Otherwise $s(\cdot, C)$ does not take the value $-\infty$.

Any map $\Gamma\colon [0, 1] \to c(X)$ is called a *multifunction*.

A multifunction $\Gamma$ is said to be *measurable* if for each open subset $O$ of $X$, the set $\{t \in [0, 1] : \Gamma(t) \cap O \neq \emptyset\}$ is a measurable set. $\Gamma$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. It is known that in case of $cwk(X)$-valued multifunctions the scalar measurability yields the measurability (cf. [10], Proposition 2.39). The reverse implication always holds true (cf. [10], Proposition 2.3.2). $\Gamma\colon [0, 1] \to c(X)$ is said to be *graph measurable* if the set $\{(t, x) \in [0, 1] \times X : x \in \Gamma(t)\}$ is a member of the product $\sigma$-algebra generated by $\mathcal{L}$ and the Borel subsets of $X$ in the norm topology. In case of considering of a complete probability space and a Banach space the graph measurability of a $c(X)$-valued multifunction coincides with its measurability (cf. [10], Theorem 2.1.35).

$\Gamma$ is said to be *scalarly integrable* (resp. *scalarly Kurzweil–Henstock integrable*) if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is *integrable* (resp. *Kurzweil–Henstock integrable*). A function $f\colon [0, 1] \to X$ is called a *selection of* $\Gamma$ if, for every $t \in [0, 1]$, one has $f(t) \in \Gamma(t)$. A selection $f$ is said to be *measurable* if the function $f$ is strongly measurable (i.e. $f$ is a limit of an almost everywhere convergent sequence of measurable simple functions).

DEFINITION 2. A measurable multifunction $\Gamma\colon [0, 1] \to cb(X)$ is *Dunford* (*respectively Kurzweil–Henstock–Dunford*) *integrable* or simply D-*integrable* (resp. KHD-*integrable*), if it is scalarly integrable (resp. scalarly Kurzweil–Henstock integrable) and for each nonempty set $A \in \mathcal{L}$ (resp. subinterval $[a, b] \subseteq [0, 1]$), there exists a nonempty set $W_A \in cb(X^{**})$ (resp. $W_{[a,b]} \in cb(X^{**})$) such that

$$s(x^*, W_A) = (L)\int_A s(x^*, \Gamma(t))\,\mathrm{d}t, \tag{1}$$

$$\left(\text{resp. } s(x^*, W_{[a,b]}) = (KH)\int_a^b s(x^*, \Gamma(t))\,\mathrm{d}t\right) \tag{2}$$

for all $x^* \in X^*$ ($L$ stands for the Lebesgue integral).

If $W_A \in \mathcal{C}$ (resp. $W_{[a,b]} \in \mathcal{C}$), for each $A \in \mathcal{L}$ (resp. $[a,b] \subseteq [0,1]$), and $\mathcal{C}$ is a subspace of $cb(X)$, then $\Gamma$ is said to be *Pettis integrable*, or simply *P-integrable* (resp. *Kurzweil–Henstock–Pettis integrable* or simply *KHP-integrable*) in $\mathcal{C}$. We call $W_A$ (resp. $W_{[a,b]}$) the *Pettis* (resp. *Kurzweil–Henstock–Pettis*) *integral of* $\Gamma$ over $A$ (resp. $[a,b]$) and we set $W_A =: (P)\int_A \Gamma(t)\,\mathrm{d}t$ (resp. $W_{[a,b]} =: (KHP)\int_a^b \Gamma(t)\,\mathrm{d}t$).

We note that when a multifunction is a function $f\colon [0,1] \to X$, then the sets $W_A$ and $W_{[a,b]}$ are reduced to vectors in $X$, the equalities (1) and (2) turn into

$$\langle x^*, W_A\rangle = (L)\int_A x^* f(t)\,\mathrm{d}t,$$

$$\left(\text{resp. } \langle x^*, W_{[a,b]}\rangle = (KH)\int_a^b x^* f(t)\,\mathrm{d}t\right)$$

and we say in that case that the function $f$ is *Pettis* (resp. *Kurzweil–Henstock–Pettis*) *integrable*. It is perhaps worth to recall in this place that Gamez and Mendoza [7] proved that a function $f\colon [0,1] \to X$ is KHD-integrable if and only if $f$ is scalarly KH-integrable.

An extensive study of Banach valued Pettis integral can be found in [16].

Given a multifunction $\Gamma\colon [0,1] \to cb(X)$ by the symbols $\mathcal{S}_{KHP}(\Gamma)$ and $\mathcal{S}_P(\Gamma)$ we denote the families of all measurable selections of $\Gamma$ that are respectively Kurzweil–Henstock–Pettis integrable and Pettis-integrable. It is a consequence of [13] that if $X$ is separable, then for each measurable multifunction $\Gamma\colon [0,1] \to cb(X)$ the family of measurable selections of $\Gamma$ is not empty.

DEFINITION 3. A measurable multifunction $\Gamma\colon [0,1] \to cwk(X)$ is said to be *Aumann–Kurzweil–Henstock–Pettis integrable* if $\mathcal{S}_{KHP}(\Gamma) \neq \emptyset$. Then we define

$$(AKHP)\int_0^1 \Gamma(t)\,\mathrm{d}t := \left\{(KHP)\int_0^1 f(t)\,\mathrm{d}t : f \in \mathcal{S}_{KHP}(\Gamma)\right\}.$$

It is clear that each Henstock integrable function is also KHP-integrable. The reverse implication is not so obvious. In [7] there is an example of a KHP-integrable function $f\colon [0,1] \to c_0$ (the authors say there on the Denjoy–Pettis integral) which is not Pettis integrable. We are going to show that same function is not Henstock integrable. It will follow from this that the collection of KHP-integrable functions is larger than that of Henstock integrable ones.

EXAMPLE 1. Consider a sequence of intervals $A_n = [a_n, b_n] \subseteq [0, 1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $f\colon [0, 1] \to c_0$ by

$$f(t) = \left(\frac{1}{2|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{2|A_{2n}|}\chi_{A_{2n}}(t)\right)_{n=1}^{\infty}.$$

The function $f$ is Dunford integrable, $(D)\int_{[0,1]} f = 0$ and $(D)\int_J f$ belongs to $c_0$ for each subinterval $J \subset [0, 1]$ (see [7]). Consequently, $f$ is KHP-integrable in $[0, 1]$ and $(KHP)\int_{[0,1]} f = 0$.

Let us consider now any gauge $\delta$ on $[0, 1]$. Set $I^0 = [c, 1]$, where $c > 1 - \delta(1)$ and $b_{2n_0-1} < c < a_{2n_0}$, for a suitable natural number $n_0$. Then, for $i = 1, \dots, 2n_0 - 1$, let $P_i = \{(I_s^i, y_s^i) : s = 1, \dots, p_i\}$ be a $\delta$-fine partition of $[a_i, b_i]$ and let $\hat{P}_i = \{(J_s^i, z_s^i) : s = 1, \dots, q_i\}$ be a $\delta$-fine partition of $[b_i, a_{i+1}]$, for $i = 1, \dots, 2n_0 - 2$, and of $[b_{2n_0-1}, c]$ for $i = 2n_0 - 1$. Moreover assume that if $z_s^i = b_i$ or, respectively, $z_s^i = a_{i+1}$for some indices $i$ and $s$ then the corresponding interval $J_s^i$ satisfies the additional condition $|J_s^i| < \frac{1}{4(2n_0-1)}|A_i|$ or, respectively, $|J_s^i| < \frac{1}{4(2n_0-1)}|A_{i+1}|$. By the construction the family $P = \{(I^0, 1)\} \cup (\bigcup_i P_i) \cup (\bigcup_i \hat{P}_i)\}$ is a $\delta$-fine partition of $[0, 1]$. We have

$$\begin{aligned}
&\left\|\sum_{(I,t)\in P} f(t)|I|\right\| \\
&= \left\| f(1)|I^0| + \sum_{i=1}^{2n_0-1}\sum_{s=1}^{p_i} f(y_s^i)|I_s^i| + \sum_{\{i:z_s^i=b_i\}} f(z_s^i)|J_s^i| + \right. \\
&\qquad \left. + \sum_{\{i:z_s^i=a_{i+1}\}} f(z_s^i)|J_s^i| \right\| \\
&\geqslant \left\|\sum_{i=1}^{2n_0-1}\sum_{s=1}^{p_i} f(y_s^i)|I_s^i|\right\| - \left\|\sum_{\{i:z_s^i=b_i\}} f(z_s^i)|J_s^i|\right\| - \left\|\sum_{\{i:z_s^i=a_{i+1}\}} f(z_s^i)|J_s^i|\right\| \\
&\geqslant \frac{1}{2} - \sum_{\{i:z_s^i=b_i\}} \frac{|J_s^i|}{2|A_i|} - \sum_{\{i:z_s^i=a_{i+1}\}} \frac{|J_s^i|}{2|A_{i+1}|} \\
&\geqslant \frac{1}{2} - (2n_0-1)\frac{1}{8(2n_0-1)} - (2n_0-1)\frac{1}{8(2n_0-1)} = \frac{1}{4}.
\end{aligned}$$

So taking into account that $(D)\int_{[0,1]} f = 0$, the inequalities show that $f$ cannot be $H$-integrable. □

It seems to be a good place to put here a remark concerning the problem of primitives for Banach space valued functions.

PROPOSITION 2. *If* $f\colon [0, 1] \to X$ *is weakly differentiable, then its weak derivative* $f'$ *is KHP-integrable and* $(KHP)\int_0^s f'(t)\,\mathrm{d}t = f(s) - f(0)$ *for each* $s \in [0, 1]$.

*Proof.* The existence of the weak derivative at a point $t$ means that there is a point $f'(t) \in X$ such that

$$\lim_{\Delta t \to 0} \frac{x^* f(t + \Delta t) - x^* f(t)}{\Delta t} = x^* f'(t)$$

for each $x^* \in X^*$. This in particular means that each $x^* f$ is differentiable and so by (e) of Proposition 1, $x^* f(s) - x^* f(0) = (KH) \int_0^s (x^* f)'(t)\,\mathrm{d}t$ for every $s \in [0, 1]$. But by the assumption, we have $(x^* f)' = x^* f'$ what yields

$$x^* f(s) - x^* f(0) = (KH) \int_0^s x^* f'(t)\,\mathrm{d}t$$

and means exactly that $f(s) - f(0) = (KHP) \int_0^s f'(t)\,\mathrm{d}t$. □

In the above proof one may assume the weak continuity of $f$ everywhere and the weak differentiability of $f$ nearly everywhere, i.e. except for a countable set (cf. [8]).

## 3. A Characterization of KHP-Integrable Multifunctions

We begin with an easy fact (it is true in a more general case of $c(X)$ instead of $cb(X)$, but we do not want to enter into details concerning the definition of the Pettis integral in such a case, see [5]).

LEMMA 1. *Let $G$: $[0, 1] \to cb(X)$ be Pettis integrable in $cb(X)$. If the null function is a selection of $G$, then for every measurable sets $A$ and $B$ such that $A \subseteq B$ we have $W_A \subseteq W_B$.*

*Proof.* Suppose that $x_0 \in W_A \setminus W_B$. Then, due to the Hahn–Banach theorem, there is $x_0^*$ such that $x_0^*(x_0) > \sup_{x \in W_B} x_0^*(x)$. Consequently,

$$\begin{aligned}(L) \int_A s(x_0^*, G(t))\,\mathrm{d}t &= s(x_0^*, W_A) \geqslant x_0^*(x_0) > \sup_{x \in W_B} x_0^*(x) \\ &= s(x_0^*, W_B) = (L) \int_B s(x_0^*, G(t))\,\mathrm{d}t\end{aligned}$$

what contradicts the nonnegativity of the support functions of $G$. Thus $W_A \subseteq W_B$. □

LEMMA 2. *If $\Gamma$: $[0, 1] \to cwk(X)$ is KHP-integrable, then each measurable selection of $\Gamma$ is KHP-integrable.*

*Proof.* If $f$ is a measurable selection of $\Gamma$, then for each $x^* \in X^*$ and $t \in [0, 1]$ we have the inequality

$$-s(-x^*, \Gamma(t)) \leqslant x^* f(t) \leqslant s(x^*, \Gamma(t)). \tag{3}$$

So, if $f$ is measurable, then the Kurzweil–Henstock integrability of the function $x^*f$ follows immediately by the Kurzweil–Henstock integrability of $s(x^*, \Gamma(t))$, for each $x^* \in X^*$. Indeed, we get from (3) the inequalities,

$$0 \leqslant x^*f(t) + s(-x^*, \Gamma(t)) \leqslant s(x^*, \Gamma(t)) + s(-x^*, \Gamma(t)).$$

The function $s(x^*, \Gamma(\cdot)) + s(-x^*, \Gamma(\cdot))$ is nonnegative and KH-integrable, hence it is Lebesgue integrable (see (d) of Proposition 1). Consequently also $x^*f(\cdot) + s(-x^*, \Gamma(\cdot))$ is Lebesgue integrable. Finally $x^*f(t) = [x^*f(t) + s(-x^*, \Gamma(t))] - s(-x^*, \Gamma(t))$ and so $x^*f \in \mathcal{KH}[0, 1]$. Hence for each $[a, b] \subseteq [0, 1]$ we have

$$-s(-x^*, W_{[a,b]}) \leqslant (KH) \int_a^b x^*f(t)\,\mathrm{d}t \leqslant s(x^*, W_{[a,b]}).$$

Since $W_{[a,b]}$ is convex weakly compact, the function $x^* \to s(x^*, W_{[a,b]})$ is $\tau(X^*, X)$-continuous, where $\tau(X^*, X)$ is the Mackey topology of $X^*$. Consequently the functional $x^* \to (KH) \int_a^b x^*f(t)\,\mathrm{d}t$ is also $\tau(X^*, X)$-continuous.

It follows that there is $x_{[a,b]} \in X$ such that

$$(KH) \int_a^b x^*f(t)\,\mathrm{d}t = \langle x^*, x_{[a,b]} \rangle. \qquad \square$$

LEMMA 3. *If all measurable selections of a measurable multifunction $\Gamma$: $[0, 1] \to cwk(X)$ are KHP-integrable, then for every $x^* \in X^*$ and every $[a, b] \subseteq [0, 1]$, we have*

$$s\left(x^*, (AKHP) \int_a^b \Gamma(t)\,\mathrm{d}t\right) = (KH) \int_a^b s(x^*, \Gamma(t))\,\mathrm{d}t.$$

*Proof.* It is enough to prove the assertion for the unit interval. If $f \in \mathcal{S}_{KHP}(\Gamma)$, then $x^*f(t) \leqslant s(x^*, \Gamma(t))$ and so we get immediately the inequality

$$s\left(x^*, (AKHP) \int_0^1 \Gamma(t)\,\mathrm{d}t\right) \leqslant (KH) \int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t.$$

To prove the reverse inequality let us fix $x^*$ and $\varepsilon > 0$. Then set

$$\Gamma_\varepsilon(t) := \Gamma(t) \cap \{x \in X : x^*(x) \geqslant s(x^*, \Gamma(t)) - \varepsilon\}.$$

One can easily see that $\Gamma_\varepsilon$ is graph measurable and so it is measurable (cf. [10], Theorem 2.1.35). If $f$ is a measurable selection of $\Gamma_\varepsilon$, then clearly $x^*f(t) \geqslant s(x^*, \Gamma(t)) - \varepsilon$ and further

$$(KH) \int_0^1 x^*f(t)\,\mathrm{d}t \geqslant (KH) \int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t - \varepsilon.$$

This completes the proof. $\square$

LEMMA 4. *If all measurable selections of a scalarly KH-integrable multifunction* $\Gamma$: $[0,1] \to cwk(X)$ *are KHP-integrable, then* $(AKHP) \int_0^1 \Gamma(t)\,\mathrm{d}t$ *is a convex weakly compact set.*

*Proof.* Notice first that the scalar KH-integrability of $\Gamma$ yields the scalar measurability of $\Gamma$ and hence its measurability (because we consider $cwk(X)$-valued multifunctions).

Let us fix a measurable selection $f$ of $\Gamma$ and let $G(t) := \Gamma(t) - f(t)$. By the assumption $f$ is KHP-integrable and so $G$ is also AKHP-integrable. Let $I_{[0,1]} := (AKHP) \int_0^1 G(t)\,\mathrm{d}t$ and let $D$ be a countable weak*-dense subset of $B(X^*)$. It is enough to prove the convexity and weak compactness of $I_{[0,1]}$. As the convexity is obvious we will try to prove the weak compactness of $I_{[0,1]}$. To do it take a sequence of points $x_n \in I_{[0,1]}$. Then there exist $g_n \in \mathcal{S}_{KHP}(G)$ with

$$x_n = (KHP) \int_0^1 g_n(t)\,\mathrm{d}t.$$

We have for each $n \in \mathbf{N}$, each $t \in [0,1]$ and each $x^*$ the inequalities

$$-s(-x^*, G(t)) \leqslant x^* g_n(t) \leqslant s(x^*, G(t)) \tag{4}$$

and the support functions $s(x^*, G(t))$ are Lebesgue integrable (because they are nonnegative and KH-integrable). Consequently also each $x^* g_n$ is Lebesgue integrable and

$$(L) \int_0^1 |x^* g_n(t)|\,\mathrm{d}t \leqslant (L) \int_0^1 s(x^*, G(t))\,\mathrm{d}t + (L) \int_0^1 s(-x^*, G(t))\,\mathrm{d}t.$$

Due to the countability of $D$ and $L_1$-boundedness of each sequence $\langle x^* g_n \rangle$ we can apply Bukhvalov–Lozanovskij's theorem [2] to find $h_n \in \mathrm{conv}\{g_n, g_{n+1}, \ldots\}$ such that for each $x^* \in D$ the sequence $\langle x^* h_n \rangle$ is a.e. convergent to a measurable function $h_{x^*}$. (We could apply also Komlos' theorem [11] instead, but the proof of the result of Bukhvalov–Lozanovskij is much more elementary.)

As for each $t$ and $n$ we have $h_n(t) \in G(t)$ and $G(t)$ is weakly compact there is a weak cluster point $h(t) \in G(t)$ of $\langle h_n(t) \rangle$. It follows that there is a set $N$ of Lebesgue measure zero such that for each $x^* \in D$ and each $t \notin N$ we have

$$x^* h(t) = \lim_n x^* h_n(t) = h_{x^*}(t).$$

But as $D$ separates points of $X$ and $\langle h_n(t) \rangle$ is weakly relatively compact it follows that for each $t \notin N$ the sequence $\langle h_n(t) \rangle$ is weakly convergent to $h(t)$. The weak measurability of $h$ is immediate and as $X$ is separable the Pettis theorem yields its strong measurability. Moreover, as a consequence of the KHP integrability of all measurable selections of $\Gamma$ and of the KHP-integrability of $f$, we have $h \in \mathcal{S}_{KHP}(G)$.

Taking into account (4) and the Lebesgue dominated convergence theorem we get for each $x^* \in X^*$ the relation

$$\lim_n \left\langle x^*, (KHP) \int_0^1 h_n(t)\,\mathrm{d}t \right\rangle = \lim_n (L) \int_0^1 x^* h_n(t)\,\mathrm{d}t$$
$$= (L) \int_0^1 x^* h(t)\,\mathrm{d}t$$
$$= \left\langle x^*, (KHP) \int_0^1 h(t)\,\mathrm{d}t \right\rangle. \tag{5}$$

Put now $y_n = (KHP) \int_0^1 h_n(t)\,\mathrm{d}t$. Then $y_n \in I_{[0,1]}$, $y_n \in \mathrm{conv}\{x_n, x_{n+1}, \ldots\}$ and the sequence $\langle y_n \rangle$ is weakly convergent to $x = (KHP) \int_0^1 h(t)\,\mathrm{d}t$. Thus, given an arbitrary sequence of elements $x_n \in I_{[0,1]}$ there is a convex combination of points $y_n \in \mathrm{conv}\{x_n, x_{n+1}, \ldots\}$ and $x \in I_{[0,1]}$ such that $y_n \to x$ weakly. Consequently, the set $I_{[0,1]}$ is weakly compact (cf. [12], §24). □

THEOREM 1. *Let $\Gamma$: $[0, 1] \to cwk(X)$ be a scalarly Kurzweil–Henstock integrable multifunction. Then the following conditions are equivalent:*

(i) *$\Gamma$ is KHP-integrable in $cwk(X)$;*
(ii) *$\mathcal{S}_{KHP}(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_{KHP}(\Gamma)$ there exists a multifunction $G$: $[0, 1] \to cwk(X)$ such that $\Gamma(t) = G(t) + f(t)$ and $G$ is Pettis integrable in $cwk(X)$;*
(iii) *there exists $f \in \mathcal{S}_{KHP}(\Gamma)$ and a multifunction $G$: $[0, 1] \to cwk(X)$ such that $\Gamma(t) = G(t) + f(t)$ and $G$ is Pettis integrable in $cwk(X)$;*
(iv) *$\mathcal{S}_{KHP}(\Gamma) \neq \emptyset$ and for every $f, h \in \mathcal{S}_{KHP}(\Gamma)$, $h - f$ is Pettis integrable;*
(v) *for all $[a, b] \subseteq [0, 1]$, $(AKHP) \int_a^b \Gamma(t)\,\mathrm{d}t$ belongs to $cwk(X)$ and*

$$s\left(x^*, (AKHP) \int_a^b \Gamma(t)\,\mathrm{d}t\right) = (KH) \int_a^b s(x^*, \Gamma(t))\,\mathrm{d}t$$

*for all $x^* \in X^*$;*
(vi) *each measurable selection of $\Gamma$ is KHP-integrable.*

*Proof.* (i) $\Rightarrow$ (ii) Let $f \in \mathcal{S}_{KHP}(\Gamma)$ be quite arbitrary (existing by Lemma 2). Define $G$: $[0, 1] \to cwk(X)$ by setting $G(t) := \Gamma(t) - f(t)$. Then $s(x^*, G(t)) \geqslant 0$ for all $x^*$ and $t \in [0, 1]$. Moreover,

$$s(x^*, \Gamma(t)) = s(x^*, G(t)) + x^* f(t), \tag{6}$$

and so KH-integrability of $s(x^*, \Gamma(\cdot))$ and of $x^* f$ yields the Lebesgue integrability of $s(x^*, G(\cdot))$. Thus, we have for each $x^* \in X^*$

$$(L) \int_0^1 s(x^*, G(t))\,\mathrm{d}t = (KH) \int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t - (KH) \int_0^1 x^* f(t)\,\mathrm{d}t$$

$$= s(x^*, W_{[0,1]}) - (KH) \int_0^1 x^* f(t)\,\mathrm{d}t$$
$$= s\left(x^*, W_{[0,1]} - (KHP) \int_0^1 f(t)\,\mathrm{d}t\right).$$

And since $W_{[0,1]}$ belongs to $cwk(X)$, also the set $W_{[0,1]} - (KHP) \int_0^1 f(t)\,\mathrm{d}t$ belongs to $cwk(X)$.

Since the zero function is a Pettis integrable selection of $G$ and $s(x^*, G(\cdot))$ is Lebesgue integrable, it follows from Theorem 3.7 of [5] that for every $E \in \mathcal{L}$ there is a closed convex set $C_E \subset X$ such that

$$(L) \int_E s(x^*, G(t))\,\mathrm{d}t = s(x^*, C_E).$$

But as the set $W_{[0,1]} - (KHP) \int_0^1 f(t)\,\mathrm{d}t$ is weakly compact, we may apply Lemma 1 to get Pettis integrability of $G$ in $cwk(X)$.

(ii) $\Rightarrow$ (iv) Let $f \in \mathcal{S}_{KHP}(\Gamma)$ and set $G(t) := \Gamma(t) - f(t)$. Now if $h$ is a measurable selection of $\Gamma$ then $g = h - f$ is a measurable selection of $G$ and we have the inequality

$$-s(-x^*, G(t)) \leqslant x^* g(t) \leqslant s(x^*, G(t)). \tag{7}$$

Hence, for each $E \in \mathcal{L}$ there is $W_E \in cwk(X)$ such that

$$-s(-x^*, W_E) = -(L) \int_E s(-x^*, G(t))\,\mathrm{d}t$$
$$\leqslant (L) \int_E x^* g(t)\,\mathrm{d}t$$
$$\leqslant (L) \int_E s(x^*, G(t))\,\mathrm{d}t = s(x^*, W_E).$$

Since $W_E$ is weakly compact its support function is $\tau(X^*, X)$-continuous. It follows that $x^* \to (L) \int_E x^* g(t)\,\mathrm{d}t$ is $\tau(X^*, X)$-continuous. This implies the Pettis integrability of $g$.

(iv) $\Rightarrow$ (ii) Take an $f \in \mathcal{S}_{KHP}(\Gamma)$. Then, by the assumption, each measurable selection $g$ of $G = \Gamma - f$ is Pettis integrable and so, by Theorem 5.4 of [5], $G$ is Pettis integrable in $cwk(X)$.

(ii) $\Rightarrow$ (v) Let $f \in \mathcal{S}_{KHP}(\Gamma)$ be such that the multifunction $G(t) = \Gamma(t) - f(t)$ is Pettis integrable in $cwk(X)$. According to Theorem 5.4 of [5] we have $(P) \int_0^1 G(t)\,\mathrm{d}t = \{(P) \int_0^1 f(t)\,\mathrm{d}t : f \in \mathcal{S}_P(G)\}$. Since $(P) \int_0^1 G(t)\,\mathrm{d}t$ is a convex weakly compact set and

$$(AKHP) \int_0^1 \Gamma(t)\,\mathrm{d}t = (P) \int_0^1 G(t)\,\mathrm{d}t + (KHP) \int_0^1 f(t)\,\mathrm{d}t,$$

also the set $(AKHP) \int_0^1 \Gamma(t)\,\mathrm{d}t$ is convex weakly compact.

Now we prove the second part of the assertion. Its proof is similar to the proof of Lemma 3.

We take $[a, b] = [0, 1]$ for simplicity and fix $x^* \in X^*$. Since $s(x^*, \Gamma(t)) = s(x^*, G(t)) + x^* f(t)$, by the hypotheses the support function $s(x^*, \Gamma(\cdot))$ is KH-integrable. Then, taking into account that $s(x^*, G(t)) \geqslant 0$

$$(KH)\int_0^1 x^* f \,\mathrm{d}t \leqslant (KH)\int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t$$

for each $f \in \mathcal{S}_{KHP}(\Gamma)$. Hence,

$$s\left(x^*, (AKHP)\int_0^1 \Gamma(t)\,\mathrm{d}t\right) \leqslant (KH)\int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t.$$

Take now an arbitrary $\varepsilon > 0$ and define a new multifunction $H\colon [0, 1] \to cwk(X)$ by setting for each $t \in [0, 1]$

$$H(t) := \Gamma(t) \cap \{x \in X : x^*(x) \geqslant s(x^*, \Gamma(t)) - \varepsilon\}.$$

If $h$ is a measurable selection of $H$, $s(x^*, \Gamma(t)) - \varepsilon \leqslant x^* h(t) \leqslant s(x^*, \Gamma(t))$ and so

$$(KH)\int_0^1 x^* h \,\mathrm{d}t \geqslant (KH)\int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t - \varepsilon.$$

Since $h$ is also a selection of $\Gamma$, the assertion follows.

(v) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious.

By Lemma 1 (i) implies (vi) and (vi) $\Rightarrow$ (i) follows from Lemmata 3 and 4. This completes the proof. □

*Remark 1.* Theorem 1 remains true if $cwk(X)$ is replaced by $ck(X)$. The proof requires only obvious changes.

COROLLARY 1. *If $\Gamma$, $G$ and $f$ are as in Theorem* 1*, then the (KHP)-integral of $\Gamma$ is a translation of the Pettis integral of $G$:*

$$(KHP)\int_a^b \Gamma(t)\,\mathrm{d}t = (P)\int_a^b G(t)\,\mathrm{d}t + (KHP)\int_a^b f(t)\,\mathrm{d}t$$

*for all* $[a, b] \subseteq [0, 1]$.

Analysis of the proof of Theorem 1 gives also the following result:

THEOREM 2. *Let $\Gamma\colon [0, 1] \to cb(X)$ be a measurable and scalarly KH-integrable multifunction. If $\mathcal{S}_{KHP}(\Gamma) \neq \emptyset$, then the following conditions are equivalent:*

(j) *$\Gamma$ is KHP-integrable in $cb(X)$;*

(jj) *for every* $f \in \mathcal{S}_{KHP}(\Gamma)$ *the multifunction* $G\colon [0,1] \to cb(X)$, *given by* $G(t) = \Gamma(t) - f(t)$, *is Pettis integrable in* $cb(X)$;

(jjj) *there exists* $f \in \mathcal{S}_{KHP}(\Gamma)$ *such that the multifunction* $G\colon [0,1] \to cb(X)$, *given by* $G(t) = \Gamma(t) - f(t)$, *is Pettis integrable in* $cb(X)$.

*We have then*

$$(KHP)\int_a^b \Gamma(t)\,\mathrm{d}t = (P)\int_a^b G(t)\,\mathrm{d}t + (KHP)\int_a^b f(t)\,\mathrm{d}t$$

*for all* $[a,b] \subseteq [0,1]$.

Without the assumption $\mathcal{S}_{KHP}(\Gamma) \neq \emptyset$ we get only the following:

THEOREM 3. *Let* $\Gamma\colon [0,1] \to cb(X)$ *be a measurable and scalarly KH-integrable multifunction. If* $\Gamma$ *is KHP-integrable in* $cb(X)$, *then for every measurable selection* $f$ *of* $\Gamma$ *the multifunction* $G\colon [0,1] \to cb(X)$ *given by* $G(t) = \Gamma(t) - f(t)$ *is Pettis integrable in* $cb(X)$. *We have then*

$$(KH)\int_a^b s(x^*, \Gamma(t))\,\mathrm{d}t = (L)\int_a^b s(x^*, G(t))\,\mathrm{d}t + (KH)\int_a^b x^* f(t)\,\mathrm{d}t$$

*for all* $[a,b] \subseteq [0,1]$ *and all* $x^* \in X^*$.

*Proof.* Let $f$ be a measurable selection of $\Gamma$ and let $G$ be defined in the way described above. As a measurable selection of a scalarly KH-integrable multifunction is scalarly KH-integrable, we see that $\Gamma(t) = G(t) + f(t)$, where $f$ is scalarly KH-integrable. Since $G$ has at least one Bochner integrable selection (the null function), according to Corollary 3.8 of [5], for every $E \in \mathcal{L}$ there is $D_E \in c(X)$ with

$$s(x^*, D_E) = (L)\int_E s(x^*, G(t))\,\mathrm{d}t$$

for all $x^* \in X^*$. Hence, we have for every $x^* \in X^*$

$$s(x^*, D_{[0,1]}) + (KH)\int_0^1 x^* f(t)\,\mathrm{d}t = (KH)\int_0^1 s(x^*, \Gamma(t))\,\mathrm{d}t \neq \pm\infty.$$

It follows that $s(x^*, D_{[0,1]}) \neq \pm\infty$ for all $x^* \in X^*$. The Banach–Steinhaus theorem yields now $D_{[0,1]} \in cb(X)$. And so applying Lemma 1 we get that also every $D_E$ is bounded, proving the Pettis integrability of $G$ in $cb(X)$. □

The Theorems 1 and 2 show that under suitable assumptions a measurable and scalarly (KH)-integrable multifunction is a sum of a Pettis integrable multifunction and a selection. In case of a $cwk(X)$-valued multifunction all selections of the Pettis integrable component are Pettis integrable.

We finish with an example showing that a decomposition theorem for a Pettis integrable in $cwk(X)$ multifunction $G$ in the form $G(t) = H(t) + f(t)$, where $f \in \mathcal{S}_P(G)$ and all measurable selections of $H$ are Bochner integrable, is in general false. In particular $H$ cannot be taken integrably bounded.

EXAMPLE 2. Let $f\colon [0,1] \to X$ be a Pettis but not Bochner integrable function (it is well known that if $X$ is infinite-dimensional, then there are such functions). We define a multifunction $G\colon [0,1] \to ck(X)$ by setting $G(t) = \mathrm{conv}\{0, f(t)\}$. Since for every $x^* \in X^*$ we have $s(x^*, G(t)) = [x^* f(t)]^+$, the multifunction $G$ is scalarly integrable and measurable. Moreover all measurable selections of $G$ are Pettis integrable and so $G$ itself is Pettis integrable in $ck(X)$ (see [5]). Suppose there is $h \in \mathcal{S}_P(G)$ which is not Bochner integrable and $H(t) := G(t) - h(t)$ is such that all its measurable selections are Bochner integrable. We have $H(t) = \mathrm{conv}\{0, f(t)\} - h(t) = \mathrm{conv}\{-h(t), f(t) - h(t)\}$. But the function $-h$ is a non-Bochner integrable selection of $H$.

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