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A remark on differentiable functions with partial derivatives in L^p

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Abstract

We consider a definition of p,δ-variation for real functions of several variables which gives information on the differentiability almost everywhere and the absolute integrability of its partial derivatives on a measurable set. This definition of p,δ-variation extends the definition of n-variation of Malý and the definition of p-variation of Bongiorno. We conclude with a result of change of variables based on coarea formula.

1. Introduction

Let Ω be an open set of $\mathcal{R}^n$ and let E be a subset of Ω. We denote by $\mathcal{L}^n(\cdot)$ (respectively by $\mathcal{L}_e^n(\cdot)$) the measure (respectively the outer measure) of Lebesgue in $\mathcal{R}^n$. For every $x \in \mathcal{R}^n$ and for every real number $r \geqslant 0$ we set $B(x,r) = \{y \in \mathcal{R}^n\colon \|y-x\| \leqslant r\}$. A function $\delta : E \to [0,+\infty]$ is a *gage* on E if $\mathcal{L}_e^n(\{x \in E\colon \delta(x) = 0\}) = 0$. We denote with $\Delta(E)$ the family of all the gages on E. For every open set $G \subset \Omega$ the function $\delta_G : G \to [0,+\infty]$ associating to each $x \in G$ its distance from the boundary of G is a gage on G. For every $\eta > 0$ the function $\delta_\eta : \Omega \to [0,+\infty]$ with $\delta_\eta = \min\{\eta, \delta_\Omega\}$ is a gage on Ω.

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A partition P in Ω is a countable disjoint family $\{B(x_i, r_i)\}$ with $B(x_i, r_i) \subset \Omega$ for all i. If for all i, $x_i \in E$, the partition $P = \{B(x_i, r_i)\}$ is called *tagged* by E. For every gage δ on E, a partition P in Ω is called *δ-fine* if $r_i < \delta(x_i)$ as soon as $\delta(x_i) > 0$ and $r_i = 0$ otherwise. We denote with $\mathcal{P}(E, \delta)$ the family of all partitions δ-fine in Ω that are tagged by E.

Let $f : \Omega \to \mathcal{R}$ and let $B(x, r) \subset \Omega$. We denote with $\omega(f, B(x, r))$ the oscillation of the function f in $B(x, r)$, that is the diameter of the image $f(B(x, r))$. For every positive number p, we set

$$f_p\big(B(x, r)\big) = \omega^p\big(f, B(x, r)\big)r^{n-p}$$

and for every $P = \{B(x_i, r_i)\} \in \mathcal{P}(E, \delta)$,

$$f_p(P) = \sum_i f_p\big(B(x_i, r_i)\big).$$

For every $\delta \in \Delta(E)$, we associate to the function f the extended real number

$$V_p(f, E, \delta) = \sup\big\{f_p(P):\ P \in \mathcal{P}(E, \delta)\big\}.$$

$V_p(f, E, \delta)$ is the *p, δ-variation* of the function f over E. If there exists $\delta \in \Delta(E)$ such that $V_p(f, E, \delta) < +\infty$, we say that f is of *bounded p, δ-variation* on E.

In [4] Malý proved that the functions with bounded n-variation, that is, the functions with *bounded n, δ_Ω-variation* on Ω, are differentiable almost everywhere in Ω and have gradient in $L^n(\Omega)$. In [1] D. Bongiorno proved that the functions with bounded p-variation, that is, the functions with *bounded p, δ_η-variation* in Ω with $1 \leqslant p \leqslant n$, are differentiable almost everywhere in Ω. In this paper, we show that the functions with bounded p, δ-variation, in a measurable subset E of Ω, are differentiable almost everywhere in E and have partial derivatives that belong to $L^p(E)$. The variation introduced in this paper is weaker in comparison to those considered in [1,4]. We conclude with a result of change of variables, based on coarea formula, for the functions that have bounded p, δ-variation.

2. Properties of the functions with bounded p, δ-variation

In this section, Ω will denote an open set of $\mathcal{R}^n$, E a measurable subset of Ω and p a positive real number. To $f : \Omega \to \mathcal{R}$ we associate the function $\mathrm{lip}(f, \cdot) : \Omega \to [0, +\infty]$ defined by

$$\mathrm{lip}(f, x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{\|y - x\|}.$$

We have the following result.

Theorem 1. *If the function $f : \Omega \to \mathcal{R}$ has bounded p, δ-variation in E, then it is differentiable almost everywhere in E.*

Proof. Let $\delta \in \Delta(E)$ be such that $V_p(f, E, \delta) < +\infty$. We set

$$E_\infty = \big\{x \in E:\ \mathrm{lip}(f, x) = +\infty\big\} \tag{1}$$

and

$$E_0 = \{x \in E\colon\ \delta(x) > 0\}. \tag{2}$$

By Stepanoff's Theorem [3, Theorem 3.1.9], it is enough to prove that

$$\mathcal{L}_e^n(E_\infty) = 0.$$

For every positive integer k, we consider the set

$$E_k = \{x \in E_0\colon\ \forall\sigma > 0\ \exists y \in \Omega \text{ with } \|y - x\| \leqslant \sigma \text{ and } |f(y) - f(x)| > k\|y - x\|\}.$$

If we set $B = \mathcal{L}^n(B(0,1))$, we will show that

$$\mathcal{L}_e^n(E_k) \leqslant Bk^{-p} V_p(f, E, \delta).$$

To every point $x \in E_k$, we associate the family $\mathcal{B}(x)$ of the closed balls $B(x, r) \subset \Omega$ with $0 < r < \delta(x)$ such that $\omega(f, B(x, r)) > kr$. The family $\bigcup_{x \in E_k} \mathcal{B}(x)$ forms a Vitali cover for E_k. Hence there exists a countable disjoint subfamily $\{B(x_i, r_i)\}$ with

$$x_i \in E_k, \qquad r_i < \delta(x_i), \qquad B(x_i, r_i) \subset \Omega \quad \text{and} \quad \omega(f, B(x_i, r_i)) > kr_i,$$

for all i, such that

$$\mathcal{L}_e^n\Big(E_k \Big\backslash \bigcup_i B(x_i, r_i)\Big) = 0.$$

We have

$$\begin{aligned}\mathcal{L}_e^n(E_k) &\leqslant \mathcal{L}_e^n\Big(E_k \Big\backslash \bigcup_i B(x_i, r_i)\Big) + \mathcal{L}_e^n\Big(\bigcup_i B(x_i, r_i)\Big) = \mathcal{L}^n\Big(\bigcup_i B(x_i, r_i)\Big)\\ &= \sum_i \mathcal{L}^n(B(x_i, r_i)) = B\sum_i r_i^n \leqslant Bk^{-p} \sum_i \omega^p(f, B(x_i, r_i)) r_i^{n-p}.\end{aligned}$$

Consequently,

$$\mathcal{L}_e^n(E_k) \leqslant Bk^{-p}\, V_p(f, E, \delta).$$

From

$$E_0 \cap E_\infty \subset \bigcap_{k=1}^{+\infty} E_k,$$

we get that

$$\mathcal{L}_e^n(E_\infty) \leqslant Bk^{-p}\, V_p(f, E, \delta)$$

for all k and for $k \to +\infty$ we obtain $\mathcal{L}_e^n(E_\infty) = 0$. □

Theorems 2 and 3 give results regarding the link between the integrability of the function $\operatorname{lip}(f, \cdot)$ and the p, δ-variation of f.

Theorem 2. *If the function* $f:\Omega\to\mathcal{R}$ *is such that* $V_p(f,E,\delta)<+\infty$, *with* $\delta\in\Delta(E)$, *then*

$$\int_E \operatorname{lip}^p(f,\cdot)\,dx \leqslant C V_p(f,E,\delta), \quad \textit{where } C\in\mathcal{R}_+.$$

Proof. For all $x\in E$, we assume that $0\leqslant\delta(x)<1$ and we consider the function $h:\Omega\to[0,+\infty[$ defined by $h(x)=\operatorname{lip}^p(f,x)$ if $x\in E_0\setminus E_\infty$ and $h(x)=0$ otherwise, where E_∞ and E_0 are as in (1) and (2). Let $g:\Omega\to[0,+\infty[$ be an upper semicontinuous function with $g\leqslant h$. Proceeding as in [4, Theorem 3.3], we deduce that

$$\int_E h\,dx=\int_\Omega h\,dx=\sup\left\{\int_\Omega g\,dx:\ g \text{ is u.s.c., } 0\leqslant g\leqslant h\right\}\leqslant C V_p(f,E,\delta).$$

We obtain the conclusion observing that

$$\int_E \operatorname{lip}^p(f,\cdot)\,dx=\int_E h\,dx. \qquad \square$$

Theorem 3. *Let* E *be a measurable subset of* Ω *with* $\mathcal{L}^n(E)<+\infty$. *If the function* $f:\Omega\to\mathcal{R}$ *is such that* $\int_E \operatorname{lip}^p(f,\cdot)\,dx<+\infty$, *then there exists* $\delta\in\Delta(E)$ *such that*

$$V_p(f,E,\delta)\leqslant C\int_E \operatorname{lip}^p(f,\cdot)\,dx, \quad \textit{with } C\in\mathcal{R}_+.$$

Proof. From $\int_E \operatorname{lip}^p(f,\cdot)\,dx<+\infty$, we deduce that $\mathcal{L}^n(E_\infty)=0$. Let $h:\Omega\to[0,+\infty[$ be the function defined by $h(x)=\operatorname{lip}^p(f,x)$ if $x\in E\setminus E_{\,\infty}$ and $h(x)=0$ otherwise.

Let $G\subset\Omega$ be an open set such that $E\subset G$ and $\mathcal{L}^n(G)<+\infty$. For a fixed $\epsilon>0$, we consider the function $\delta_\epsilon\in\Delta(E)$ defined as follows: $\delta_\epsilon(x)=0$ in every point of E where the derivative of $\int_{B(x,r)} h\,dx$ does not coincide with $\operatorname{lip}^p(f,\cdot)$. Moreover, for any other $x\in E$, we choose $\delta_\epsilon(x)<\delta_G(x)$ so that for every ball $B(x,r)\subset G$ with $r<\delta_\epsilon(x)$,

$$\left|\int_{B(x,r)} h\,dx-\operatorname{lip}^p(f,\cdot)\mathcal{L}^n\big(B(x,r)\big)\right|<\epsilon\mathcal{L}^n\big(B(x,r)\big) \tag{3}$$

and

$$\omega\big(f,B(x,r)\big)\leqslant 3\operatorname{lip}(f,x)r \quad \text{if } \operatorname{lip}(f,x)>0 \tag{4}$$

or

$$\omega\big(f,B(x,r)\big)\leqslant \epsilon^{1/p}r \quad \text{if } \operatorname{lip}(f,x)=0 \tag{5}$$

holds.

For every partition $P=\{B(x_i,r_i)\}\in\mathcal{P}(E,\delta_\epsilon)$ we consider sets $I_1=\{i:\ \operatorname{lip}(f,x_i)>0\}$ and $I_2=\{i:\ \operatorname{lip}(f,x_i)=0\}$. Using (3)–(5) we get

$$\sum_i \omega^p\big(f, B(x_i, r_i)\big) r_i^{n-p}$$
$$\leqslant \sum_{i\in I_1} 3^p \operatorname{lip}^p(f, x_i) r_i^n + \sum_{i\in I_2} \epsilon r_i^n$$
$$= 3^p B^{-1} \sum_{i\in I_1} \operatorname{lip}^p(f, x_i)\mathcal{L}^n\big(B(x_i, r_i)\big) + \epsilon B^{-1} \sum_{i\in I_2} \mathcal{L}^n\big(B(x_i, r_i)\big)$$
$$< 3^p B^{-1}\left(\mathcal{L}^n(G)\epsilon + \sum_i \int\limits_{E\cap B(x_i,r_i)} \operatorname{lip}^p(f,\cdot)\,dx\right).$$

Since ϵ is arbitrary and $\mathcal{L}^n(G) < +\infty$, it follows that

$$V_p(f, E, \delta_\epsilon) \leqslant C \int\limits_E \operatorname{lip}^p(f,\cdot)\,dx,$$

with $C \in \mathcal{R}_+$, and the proof is completed. □

The following result concerning the differentiability and the integrability of partial derivatives of the function with bounded p, δ-variation.

Theorem 4. *Let $f : \Omega \to \mathcal{R}$, let $E \subset \Omega$ be a set with finite measure and let p be a positive real number. Then the following conditions are equivalent*:

(i) *the function f is differentiable almost everywhere in E with partial derivatives belonging to $L^p(E)$*;
(ii) *f is a function with bounded p, δ-variation in E.*

Proof. (i) ⇒ (ii). By [2, Theorem 3] there exist an increasing sequence (E_k) of measurable subsets of E and an increasing sequence (M_k) of positive numbers, with

$$\mathcal{L}^n\Big(E\setminus\bigcup E_k\Big) = 0 \quad \text{and} \quad M_1^p\mathcal{L}^n(E_1) + \sum_{k=1}^{+\infty} M_{k+1}^p \mathcal{L}^n(E_{k+1}\setminus E_k) < +\infty$$

such that, for every $x \in E_k$, $\operatorname{lip}(f, x) < M_k$. It follows that

$$\int\limits_E \operatorname{lip}^p(f,\cdot)\,dx = \int\limits_{E_1} \operatorname{lip}^p(f,\cdot)\,dx + \sum_{k=1}^{+\infty} \int\limits_{E_{k+1}\setminus E_k} \operatorname{lip}^p(f,\cdot)\,dx$$
$$\leqslant M_1^p\mathcal{L}^n(E_1) + \sum_{k=1}^{+\infty} M_{k+1}^p \mathcal{L}^n(E_{k+1}\setminus E_k).$$

The proof of (ii) is obtained using Theorem 3.

(ii) $\Rightarrow$ (i). Theorem 1 assures that f is differentiable almost everywhere in E and consequently f has partial derivatives f_{x_i} $(i = 1, 2, \ldots, n)$ almost everywhere in E. Being $|f_{x_i}| \leqslant \mathrm{lip}(f, \cdot)$ $(i = 1, 2, \ldots, n)$ we have

$$\int_E |f_{x_i}|^p \, dx \leqslant \int_E \mathrm{lip}^p(f, \cdot) \, dx$$

and Theorem 2 gives that $\int_E |f_{x_i}|^p \, dx < +\infty$. □

3. Change of variables via coarea formula

In this section using the technique of Malý a result of change of variables is obtained via coarea formula for the functions having bounded p, δ-variation.

Theorem 5. *Let $f : \Omega \to \mathcal{R}^m$ with $m < n$. If the function f has bounded p, δ-variation in Ω with $m < p$ and $\delta(x) > 0$ for every $x \in \Omega$, then*

$$\int_{\mathcal{R}^m} \mathcal{H}^{n-m}\big(E \cap f^{-1}(y)\big) \, dy = 0 \tag{6}$$

as soon as $\mathcal{L}^n(E) = 0$ and $E \subset \Omega$.

Proof. For fixed $\eta > 0$, let G be an open set with $E \subset G$ and $\mathcal{L}^n(G) < \eta$. For every $x \in E$ we consider a closed ball $B(x, r(x)) \subset G$ and $0 < 2r(x) < \delta_{\eta,G}(x)$, where $\delta_{\eta,G}(x) = \min\{\eta, \delta(x), \delta_G(x)\}$ for every $x \in E$. Besicovitch's Theorem assures that there exist N sets $A_1, \ldots, A_N \subset E$, with N depending only on n, such that

$$E \subset \bigcup_{i=1}^{N} \bigcup_{x \in A_i} B\big(x, r(x)\big)$$

and for every $i = 1, \ldots, N$, the family $\{B(x, r(x)){:}\ x \in A_i\}$ is disjoint. Then for every $y \in \mathcal{R}^m$ we have

$$\mathcal{H}_\eta^{n-m}\big(E \cap f^{-1}(y)\big) \leqslant C \sum_{i=1}^{N} \sum_{x \in A_i} \big\{r^{n-m}(x){:}\ x \in A_i,\ y \in f\big(B\big(x, r(x)\big)\big)\big\},$$

where C is a constant that can vary from member to member in what follows. Consequently

$$\int_{\mathcal{R}^m} \mathcal{H}_\eta^{n-m}\big(E \cap f^{-1}(y)\big) \, dy$$

$$\leqslant C \sum_{i=1}^{N} \sum_{x \in A_i} r^{n-m}(x) \mathcal{L}^m\big(f\big(B\big(x, r(x)\big)\big)\big)$$

$$\leqslant C \sum_{i=1}^{N} \sum_{x \in A_i} r^{n-m}(x) \omega^m\big(f, B\big(x, r(x)\big)\big)$$

$$\leqslant C\sum_{i=1}^{N}\biggl(\sum_{x\in A_i} r^n(x)\biggr)^{(p-m)/p}\biggl(\sum_{x\in A_i} r^{n-p}(x)\omega^p\bigl(f, B\bigl(x, r(x)\bigr)\bigr)\biggr)^{m/p}$$
$$\leqslant C\eta^{(p-m)/p}\bigl(V_p(f,\Omega,\delta_{\eta,G})\bigr)^{m/p}.$$

As $\eta \to 0$, we obtain (6). □

Let $f:\Omega \to \mathcal{R}^m$ with $m < n$. We denote by $f'(x)$ the Jacobi matrix of all the partial derivatives of f at x and by $J_m f(x)$ the row matrix having as elements the minors of order m of $f'(x)$.

Theorem 6. *Let $f:\Omega \to \mathcal{R}^m$, with $m < n$, be a function with bounded p,δ-variation in Ω, with $m < p$ and $\delta(x) > 0$ for all $x \in \Omega$. For every measurable function u on a measurable set $E \subset \Omega$ such that $u\|J_m f\| \in L^1(E)$, we have that*

$$\int_E u(x)\bigl\|J_m f(x)\bigr\|\,dx = \int_{\mathcal{R}^m}\biggl(\int_{E\cap f^{-1}(y)} u(x)\,d\mathcal{H}^{n-m}\biggr)dy. \tag{7}$$

Proof. Since Theorem 1 holds for functions with values in $\mathcal{R}^m$ the function f is differentiable almost everywhere in Ω. Therefore, a succession (f_j) of Lipschitz functions of $\mathcal{R}^n$ to $\mathcal{R}^m$ exists such that

$$\mathcal{L}^n\biggl(\Omega\Bigm\backslash\bigcup_j\bigl\{x\colon\ f_j(x) = f(x) \text{ and } f_j'(x) = f'(x)\bigr\}\biggr) = 0.$$

Since (7) holds for Lipschitz functions [3, Theorem 3.2.12] it is enough to examine the case $\mathcal{L}^n(E) = 0$ when the function u is the characteristic function of the set E. Under such hypotheses, we obtain (7) using Theorem 5. □

Remark. Let $E \subset \Omega$ and $\delta \in \Delta(E)$. We say that a function $f:\Omega \to \mathcal{R}$ is p,δ-*absolutely continuous* in E if for every $\varepsilon > 0$ there exists $\bar\eta > 0$ such that

$$\sum_i \omega^p\bigl(f, B(x_i, r_i)\bigr)r_i^{n-p} < \varepsilon,$$

for each $\{B(x_i, r_i)\} \in \mathcal{P}(E,\delta)$ with $\sum_i \mathcal{L}^n(B(x_i, r_i)) < \bar\eta$.

We observe that Theorem 5 holds also if f is p,δ-absolutely continuous in Ω and $p \geqslant m$. In fact, we fix $\varepsilon > 0$ and choose $\bar\eta > 0$ as in the definition of p,δ-absolutely continuous function. Proceeding as in the proof of Theorem 5, for every $\eta \leqslant \bar\eta$, we deduce that

$$\int_{\mathcal{R}^m} \mathcal{H}_\eta^{n-m}\bigl(E \cap f^{-1}(y)\bigr)\,dy \leqslant C\eta^{(p-m)/p}\varepsilon^{m/p}$$

and we obtain that (6) holds if $p \geqslant m$.

Since Theorem 1 holds for p,δ-absolutely continuous functions, we deduce that Theorem 6 is also valid if f is p,δ-absolutely continuous and $p \geqslant m$.

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