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Property (w) and perturbations ☆

Pietro Aiena [a,*], Maria Teresa Biondi [b]

[a] *Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy*

[b] *Departamento de Matemáticas, Facultad de Ciencias, Universidad UCLA de Barquisimeto, Venezuela*

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Abstract

A bounded linear operator $T \in L(X)$ defined on a Banach space X satisfies property (w), a variant of Weyl's theorem, if the complement in the approximate point spectrum $\sigma_a(T)$ of the Weyl essential approximate spectrum $\sigma_{wa}(T)$ coincides with the set of all isolated points of the spectrum which are eigenvalues of finite multiplicity. In this note, we study the stability of property (w), for a bounded operator T acting on a Banach space, under perturbations by finite rank operators, by nilpotent operator and quasi-nilpotent operators commuting with T.

Keywords: Localized SVEP; Weyl's theorems; Browder's theorems; Property (w)

1. Definitions and basic results

Throughout this paper, X will denote an infinite-dimensional complex Banach space, $L(X)$ the algebra of all bounded linear operators on X. For an operator $T \in L(X)$ we shall denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $T(X)$. We recall that an operator $T \in L(X)$ is called *upper semi-Fredholm* if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is called *lower semi-Fredholm* if $\beta(T) < \infty$. Let $\Phi_+(X)$ and $\Phi_-(X)$ denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$,

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* Corresponding author.
E-mail addresses: paiena@unipa.it (P. Aiena), mtbiondi@hotmail.com (M.T. Biondi).

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while the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Define

$$W_+(X) := \{T \in \Phi_+(X): \operatorname{ind} T \leqslant 0\}$$

and

$$W_-(X) := \{T \in \Phi_-(X): \operatorname{ind} T \geqslant 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X): \operatorname{ind} T = 0\}.$$

The classes of operators defined above generate the following spectra. The *Weyl spectrum* is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C}: \lambda I - T \notin W(X)\},$$

while the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{\mathrm{wa}}(T) := \{\lambda \in \mathbb{C}: \lambda I - T \notin W_+(X)\}.$$

The *approximate point spectrum* is canonically defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C}: \lambda I - T \text{ is not bounded below}\}.$$

Note that $\sigma_{\mathrm{wa}}(T)$ is the intersection of all approximate point spectra $\sigma_{\mathrm{a}}(T + K)$ of compact perturbations K of T, see for instance [1, Theorem 3.65]. Write $\operatorname{iso} K$ for the set of all isolated points of $K \subseteq \mathbb{C}$. It is known that if $K \in L(X)$ is a finite-rank operator commuting with T, then

$$\lambda \in \operatorname{acc} \sigma_{\mathrm{a}}(T) \quad \Leftrightarrow \quad \lambda \in \operatorname{acc} \sigma_{\mathrm{a}}(T + K), \tag{1}$$

for a proof see Theorem 3.2 of [9].

The classes $W_+(X)$, $W_-(X)$ and $W(X)$ are stable under some perturbations. In fact we have:

Theorem 1.1. *Let $T \in L(X)$ and $K \in L(X)$ be a compact operator. Then*

(i) $T \in W_+(X) \Leftrightarrow T + K \in W_+(X)$.
(ii) $T \in W_-(X) \Leftrightarrow T + K \in W_-(X)$.
(iii) $T \in W(X) \Leftrightarrow T + K \in W(X)$.

Proof. The implication (i) is a consequence of the well-known fact if $T \in \Phi_+(X)$, then $T + K \in \Phi_+(X)$ with $\operatorname{ind}(T + K) = \operatorname{ind}(T)$. The same happens for $T \in \Phi_-(X)$ or $T \in \Phi(X)$. □

For an operator T the *ascent* is defined as $p := p(T) = \inf\{n \in \mathbf{N}: \ker T^n = \ker T^{n+1}\}$, while the *descent* is defined as let $q := q(T) = \inf\{n \in \mathbf{N}: T^n(X) = T^{n+1}(X)\}$, the infimum over the empty set is taken ∞. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$ (see [15, Proposition 38.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Proposition 50.2 of Heuser [15].

The class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X): p(T) < \infty\},$$

while the class of all *lower semi-Browder operators* is defined

$$B_-(X) := \{T \in \Phi_-(X): q(T) < \infty\}.$$

The class of all *Browder operators* is defined

$$B(X) := B_+(X) \cap B_-(X) = \{T \in \Phi(X):\ p(T) = q(T) < \infty\}.$$

We have

$$B(X) \subseteq W(X), \qquad B_+(X) \subseteq W_+(X), \qquad B_-(X) \subseteq W_-(X),$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin B_+(X)\}.$$

Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. The proof of the following result may be found in Rakočević [23]:

Theorem 1.2. *Let $T \in L(X)$ and K be a Riesz operator commuting with T. Then*

(i) $T \in B_+(X) \Leftrightarrow T + K \in B_+(X)$.
(ii) $T \in B_-(X) \Leftrightarrow T + K \in B_-(X)$.
(iii) $T \in B(X) \Leftrightarrow T + K \in B(X)$.

The single-valued extension property was introduced by Dunford [10,11] and has an important role in local spectral theory, see the recent monograph by Laursen and Neumann [16]. In this article we shall consider the following local version of this property, which has been studied in [3,6,12], see also the recent monograph by Aiena [1].

Definition 1.3. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

An operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$ and from the identity theorem for analytic function it easily follows that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum $\sigma(T)$.

Note that

$$p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{2}$$

and dually

$$q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda, \tag{3}$$

see [1, Theorem 3.8]. Furthermore, from definition of SVEP we have

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \tag{4}$$

An important subspace in local spectral theory is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \left\{x \in X:\ \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\right\}.$$

We also have [3]

$$H_0(\lambda I - T) \text{ closed} \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \tag{5}$$

Remark 1.4. It should be noted that the implications (2)–(5) are equivalences if we assume that $\lambda I - T \in \Phi_\pm(X)$, see [1, Chapter 3].

2. Property (ω) and perturbations

For a bounded operator $T \in L(X)$, define $p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T)$. If $\lambda \in p_{00}^a(T)$, then $p(\lambda I - T) < \infty$, and, since $\lambda I - T$ is upper semi-Fredholm from Remark 1.4 it then follows that $\lambda \in \operatorname{iso}\sigma_a(T)$, so

$$p_{00}^a(T) \subseteq \pi_{00}^a(T) := \left\{\lambda \in \operatorname{iso}\sigma_a(T):\ 0 < \alpha(\lambda I - T) < \infty\right\}.$$

Define

$$\pi_{00}(T) := \left\{\lambda \in \operatorname{iso}\sigma(T):\ 0 < \alpha(\lambda I - T) < \infty\right\}.$$

Following Harte and W.Y. Lee [14] we shall say that T satisfies *Browder's theorem* if

$$\sigma_w(T) = \sigma_b(T),$$

or equivalently $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$, where $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$. Evidently, $p_{00}(T) \subseteq p_{00}^a(T)$ for every $T \in L(X)$.

A bounded operator $T \in L(X)$ is said to satisfy *a-Browder's theorem* if

$$\sigma_{wa}(T) = \sigma_{ub}(T),$$

or equivalently $\sigma_a(T) \setminus \sigma_{wa}(T) = p_{00}^a(T)$. If T is a finite-rank operator commuting with T, from Theorems 1.1 and 1.2 it then easily follows the following equivalence:

$$T \text{ satisfies } a\text{-Browder's theorem} \quad \Leftrightarrow \quad T + K \text{ satisfies } a\text{-Browder's theorem}.$$

Following Coburn [8], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \tag{6}$$

Weyl's theorem entails Browder's theorem. In fact, we have:

Theorem 2.1. *(See [2].) $T \in L(X)$ satisfies Weyl's theorem precisely when T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T)$.*

The following two variants of Weyl's theorem has been introduced by Rakočević [21,22].

Definition 2.2. A bounded operator $T \in L(X)$ is said to satisfy property (w) if

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{00}(T),$$

while $T \in L(X)$ is said to satisfy a-Weyl's theorem if

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{00}^a(T).$$

Property (w) has been also studied in recent paper [5]. As observed in [21] and [5], we have:

$$\text{either } a\text{-Weyl's theorem or property } (w) \text{ for } T \quad \Rightarrow \quad \text{Weyl's theorem holds for } T,$$

and examples of operators satisfying Weyl's theorem but not property (w) may be found in [5]. Property (w) is fulfilled by a relevant number of Hilbert space operators, see [5], and this property for T is equivalent to Weyl's theorem for T or to a-Weyl's theorem whenever T^* satisfies SVEP [5, Theorem 2.16]. For instance, property (w) is satisfied by generalized scalar operator, or if the Hilbert adjoint T' has property $H(p)$ [5, Corollary 2.20]. Note that

$$\text{property } (w) \text{ for } T \quad \Rightarrow \quad a\text{-Browder's theorem holds for } T,$$

and precisely we have:

Theorem 2.3. *(See [5].) If $T \in L(X)$, the following statements are equivalent*:

(i) *T satisfies property (w)*;
(ii) *a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}(T)$.*

It should be noted that property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [5] for examples.

Lemma 2.4. *Suppose that $T \in L(X)$ satisfies property (w) and K is a finite rank operator commuting with T such that $\sigma_a(T+K) = \sigma_a(T)$. Then $p_{00}^a(T+K) \subseteq \pi_{00}(T+K)$.*

Proof. Let $\lambda \in p_{00}^a(T+K)$ be arbitrary given. Then $\lambda \in \operatorname{iso}\sigma_a(T+K)$ and $\lambda I - (T+K) \in B_+(X)$, so $\alpha(\lambda I - (T+K)) < \infty$. Since $\lambda I - (T+K)$ has closed range, the condition $\lambda \in \sigma_a(T+K)$ entails that $0 < \alpha(\lambda I - (T+K))$. Therefore, in order to show that $\lambda \in \pi_{00}(T+K)$, we need only to prove that λ is an isolated point of $\sigma(T+K)$.

We know that $\lambda \in \operatorname{iso}\sigma_a(T)$. Also, by Theorem 1.2 we also have $\lambda I - (T+K) - K = \lambda I - T \in B_+(X)$ so that $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = p_{00}^a(T)$.

Now, by assumption T satisfies property (w) so, by Theorem 2.3, $p_{00}^a(T) = \pi_{00}(T)$. Moreover, T satisfies Weyl's theorem and hence, by Theorem 2.1,

$$\pi_{00}(T) = p_{00}(T) = \sigma(T) \setminus \sigma_b(T).$$

Therefore, $\lambda I - T$ is Browder and hence also $\lambda I - (T+K)$ is Browder, so

$$0 < p\big(\lambda I - (T+K)\big) = q\big(\lambda I - (T+K)\big) < \infty$$

and hence λ is a pole of the resolvent of $T+K$. Consequently, λ an isolated point of $\sigma(T+K)$, as desired. □

Define

$$\pi_{0f}(T) := \big\{\lambda \in \operatorname{iso}\sigma(T)\colon \alpha(\lambda I - T) < \infty\big\}.$$

Obviously, $\pi_{00}(T) \subseteq \pi_{0f}(T)$.

Lemma 2.5. *If $T, K \in L(X)$, K is a Riesz operator commuting with T, then $\pi_{00}(T+R) \cap \sigma_a(T) \subseteq \operatorname{iso}\sigma(T)$.*

Proof. Clearly,

$$\pi_{00}(T+R)\cap\sigma_{\mathrm{a}}(T)\subseteq\pi_{0f}(T+R)\cap\sigma(T),$$

and by Lemma 2.3 of [20] the last set is contained in $\operatorname{iso}\sigma(T)$. □

A bounded operator $T\in L(X)$ is said to be *a-isoloid* if every isolated point of $\sigma_{\mathrm{a}}(T)$ is an eigenvalue of T.

Theorem 2.6. *Suppose that $T\in L(X)$ is a-isoloid and K is a finite rank operator commuting with T such that $\sigma_{\mathrm{a}}(T+K)=\sigma_{\mathrm{a}}(T)$. If T satisfies property (w), then $T+K$ satisfies property (w).*

Proof. Suppose that T satisfies property (w). Then, by Theorem 2.3, T satisfies a-Browder's theorem, and hence also $T+K$ satisfies a-Browder's theorem.

By Theorem 2.3, in order to show that $T+K$ satisfies property (w) it suffices only to prove the equality $p_{00}^a(T+K)=\pi_{00}(T+K)$. The inclusion $p_{00}^a(T+K)\subseteq\pi_{00}(T+K)$ follows from Lemma 2.4, so we need only to show the opposite inclusion $\pi_{00}(T+K)\subseteq p_{00}^a(T+K)$.

We first show the inclusion

$$\pi_{00}(T+K)\subseteq p_{00}(T). \tag{7}$$

Let $\lambda\in\pi_{00}(T+K)$. By assumption $\lambda\in\operatorname{iso}\sigma(T+K)$ and $\alpha(\lambda I-(T+K))>0$ so $\lambda\in\operatorname{iso}\sigma_{\mathrm{a}}(T+K)$, and hence $\lambda\in\operatorname{iso}\sigma_{\mathrm{a}}(T)$. By Lemma 2.5 we then conclude that λ is an isolated point of $\sigma(T)$. Furthermore, since T is a-isoloid, we have also $0<\alpha(\lambda I-T)$.

We show now that $\alpha(\lambda I-T)<\infty$. To see this, note first that the restriction $\lambda I-(T+K)\mid M$ of $\lambda I-(T+K)$ on $M:=\ker(\lambda I-T)$ coincides with the restriction of K on M, so $(\lambda I-(T+K))\mid M$ has both finite-dimensional kernel and finite-dimensional range. From this it then follows that $M=\ker(\lambda I-T)$ is finite-dimensional, and consequently $\alpha(\lambda I-T)<\infty$.

Therefore the inclusion $\pi_{00}(T+K)\subseteq\pi_{00}(T)$ is proved. Now, since property (w) entails that T satisfies Weyl's theorem, by Theorem 2.1, we then have $\pi_{00}(T+K)\subseteq p_{00}(T)$ and hence the inclusion (7) is established. Consequently, if $\lambda\in\pi_{00}(T+K)$, then $\lambda I-T$ is Browder. By Theorem 1.2 it then follows that $\lambda I-(T+K))$ is also Browder, hence

$$\lambda\in\sigma(T+K)\setminus\sigma_{\mathrm{b}}(T+K)=p_{00}(T+K)\subseteq p_{00}^a(T+K),$$

as desired. □

In the sequel we shall consider nilpotent perturbations of operators satisfying property (w). It easy to check that if N is a nilpotent operator commuting with T, then $\sigma(T)=\sigma(T+N)$ and $\sigma_{\mathrm{a}}(T)=\sigma_{\mathrm{a}}(T+N)$.

Lemma 2.7. *Suppose that $T\in L(X)$ satisfies property (w). If N is a nilpotent operator that commutes with T, then $p_{00}^a(T+N)\subseteq\pi_{00}(T+N)$.*

Proof. Suppose that $\lambda\in p_{00}^a(T+N)$. Then

$$\lambda\in\sigma_{\mathrm{a}}(T+N)\setminus\sigma_{\mathrm{ub}}(T+N)=\sigma_{\mathrm{a}}(T)\setminus\sigma_{\mathrm{ub}}(T)=p_{00}^a(T).$$

Since T satisfies property (w) we then have, by Theorem 2.3, $p_{00}^a(T)=\pi_{00}(T)$. Hence λ is an isolated point of $\sigma(T)=\sigma(T^*)$ and therefore both T and T^* have SVEP at λ. Since $\lambda I-T\in$

$B_+(X)$ it then follows that $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Furthermore, since $\lambda \in \pi_{00}(T)$ we also have $\alpha(\lambda I - T) < \infty$ and Theorem 3.4 of [1] entails that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, thus $\lambda I - T$ is Browder and hence also $\lambda I - (T + N)$ is Browder, by Theorem 1.2. Hence λ is an isolated point of $\sigma(T + N)$ and $\alpha(\lambda I - (T + N)) < \infty$.

On the other hand, $\lambda I - (T + N)$ has closed range and since $\lambda \in \sigma_a(T + N)$ it then follows that $0 < \alpha(\lambda I - (T + N))$. Thus $\lambda \in \pi_{00}(T + N)$. □

Theorem 2.8. *Suppose that $T \in L(X)$ is a-isoloid and suppose that N is a nilpotent operator that commutes with T. If T satisfies property (w), then $T + N$ satisfies property (w).*

Proof. Observe first that $\sigma_{wa}(T + N) = \sigma_{wa}(T)$, see Theorem 2.13 of [9], and by Theorem 1.2 we also have $\sigma_{ub}(T + N) = \sigma_{ub}(T)$. Since a-Browder's theorem holds for T, by Theorem 2.3, it then follows that $\sigma_{ub}(T + N) = \sigma_{wa}(T + N)$, i.e. $T + N$ satisfies a-Browder's theorem. By Theorem 2.3 and Lemma 2.7 we have only to prove the inclusion

$$\pi_{00}(T + N) \subseteq p_{00}^a(T + N). \tag{8}$$

Let $\lambda \in \pi_{00}(T + N)$ be arbitrary given. There is no harm if we assume $\lambda = 0$. Clearly, $0 \in \operatorname{iso}\sigma(T + N) = \operatorname{iso}\sigma(T)$. Let $p \in \mathbb{N}$ be such that $N^p = 0$. If $x \in \ker(T + N)$, then

$$T^p x = (-1)^p T^p x = 0,$$

thus $\ker(T + N) \subseteq \ker T^p$. Since by assumption $0 < \alpha(T + N)$ it then follows that $\alpha(T^p) > 0$ and this obviously implies that $0 < \alpha(T)$. By assumption we also have $\alpha(T + N) < \infty$ and this implies that $\alpha(T + N)^p < \infty$, see Remark 2.6 of [4]. It is easily seen that if $x \in \ker T$, then

$$(T + N)^p x = N^p x = 0,$$

so $\ker T \subseteq \ker(T + N)^p$ and hence $\alpha(T) < \infty$. Therefore, $0 \in \pi_{00}(T)$ and consequently $\pi_{00}(T + N) \subseteq \pi_{00}(T)$. Now, since T satisfies Weyl's theorem we have

$$\pi_{00}(T) = p_{00}(T) \subseteq p_{00}^a(T).$$

The inclusion (8) will be then proved if we show that $p_{00}^a(T) = p_{00}^a(T + N)$. But this is immediate, since $\sigma_a(T) = \sigma_a(T + N)$ and $\sigma_{ub}(T) = \sigma_{ub}(T + N)$, so the proof is complete. □

Example 2.9. The following example shows that both Theorem 2.8 and Theorem 2.6 fail if we do not assume that the nilpotent operator N, and the finite rank operator K do not commute with T.

Let $X := \ell^2(\mathbb{N})$ and T and N be defined by

$$T(x_1, x_2, \dots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \dots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and

$$N(x_1, x_2, \dots) := \left(0, -\frac{x_1}{2}, 0, 0, \dots\right), \quad (x_n) \in \ell^2(\mathbb{N}).$$

Clearly, N is a nilpotent finite rank operator, T is a quasi-nilpotent operator satisfying Weyl's theorem. Since T is decomposable, then T satisfies property (w), see Corollary 2.10 of [5]. On the other hand, it is easily seen that $0 \in \pi_{00}(T + N)$ and $0 \notin \sigma(T + N) \setminus \sigma_w(T + N)$, so that $T + N$ does not satisfies Weyl's theorem, and hence does not satisfies property (w). Note that $\sigma_a(T + K) = \sigma_a(T)$.

Example 2.10. The following example shows that and Theorem 2.6 fails if we do not assume that T is a-isoloid. Let $S:\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})$ be an injective quasi-nilpotent operator, and let $U:\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})$ be defined:

$$U(x_1,x_2,\dots):=(-x_1,0,0,\dots)\quad\text{with } (x_n)\in\ell^2(\mathbb{N}).$$

Define on $X:=\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$ the operators T and K by

$$T:=\begin{pmatrix} I & 0\\ 0 & S\end{pmatrix},\qquad K:=\begin{pmatrix} U & 0\\ 0 & 0\end{pmatrix}.$$

Clearly, K is a finite-rank operator and $KT=TK$. It is easy to check that

$$\sigma(T)=\sigma_{\mathrm{w}}(T)=\sigma_{\mathrm{a}}(T)=\{0,1\}.$$

Since $\alpha(T)=0$, then T is not a-isoloid. Now, both T and T^* have SVEP, since $\sigma(T)=\sigma(T^*)$ is finite. Moreover, $\pi_{00}(T)=\sigma(T)\setminus\sigma_{\mathrm{w}}(T)=\emptyset$, so T satisfies Weyl's theorem, and hence by Theorem 2.16 of [5], T satisfies property (w).

On the other hand,

$$\sigma(T+K)=\sigma_{\mathrm{w}}(T+K)=\{0,1\},$$

and $\pi_{00}(T+K)=\{0\}$, so that Weyl's theorem does not hold for $T+K$ and this implies that property (w) does not hold for $T+K$. Note that $\sigma_{\mathrm{a}}(T+K)=\sigma_{\mathrm{a}}(T)$.

Generally, property (w) is not transmitted from T to a quasi-nilpotent perturbation $T+Q$. For instance, take $T=0$, and $Q\in L(\ell^2(\mathbb{N}))$ defined by

$$Q(x_1,x_2,\dots)=\left(\frac{x_2}{2},\frac{x_3}{3},\dots\right)\quad\text{for all } (x_n)\in\ell^2(\mathbb{N}).$$

Then Q is quasi-nilpotent and

$$\{0\}=\pi_{00}(Q)\neq\sigma_{\mathrm{a}}(Q)\setminus\sigma_{\mathrm{aw}}(Q)=\emptyset.$$

Hence T satisfies property (w) but $T+Q=Q$ fails this property.

We want now to show that property (w) is preserved under injective quasi-nilpotent perturbations. We need first some preliminary results.

Lemma 2.11. *Let $T\in L(X)$ be such that $\alpha(T)<\infty$. Suppose that there exists an injective quasi-nilpotent operator $Q\in L(X)$ such that $TQ=QT$. Then $\alpha(T)=0$.*

Proof. Suppose that there exists $0\neq x\in\ker T$. By the commutative assumption we have $TQ^n=Q^nTx=0$ for all $n=0,1,\dots$, so that the sequence $(Q^nx)_{n=0,1,\dots}$ is contained in $\ker T$. We claim that $(Q^nx)_{n=0,1,\dots}$ is a sequence of linearly independent vectors. In fact, suppose that

$$c_0x+c_1Qx+\cdots+c_nQ^nx=0.$$

Write $p(\lambda):=c_0+c_1\lambda+\cdots+c_n\lambda^n$. If $\lambda_1,\dots,\lambda_n$ are the zeros of $p(\lambda)$, then

$$p(\lambda)=c_n\prod_{i=1}^{n}(\lambda_i-\lambda)$$

and

$$0=p(Q)x=c_n\prod_{i=1}^{n}(\lambda_iI-Q)x.$$

Since Q is injective and quasi-nilpotent, then all $(\lambda_i I - Q)$ are injective and hence the condition $p(Q)x = 0$ entails that $c_n = 0$. An inductive argument then shows that $c_{n-1} = \cdots = c_0 = 0$, as claimed. Now, $Q^n x \in \ker T$ for all $n = 0, 1, \ldots$, and this contradicts our assumption $\alpha(T) < \infty$. □

Theorem 2.12. *Suppose that $T \in L(X)$ and Q be a quasi-nilpotent operator commuting with T. The following statements hold*:

(i) $\sigma_{\mathrm{aw}}(T) = \sigma_{\mathrm{aw}}(T + Q)$.
(ii) *If Q is injective, then $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + Q)$.*

Proof. (i) It is well known that if $T \in \Phi_+(X)$ and K is a Riesz operator commuting with T, then $T + \lambda K \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}$, see [24]. Suppose that $\lambda \notin \sigma_{\mathrm{aw}}(T)$. There is no harm if we suppose that $\lambda = 0$. Then $T \in W_+(X)$ and hence $T + \mu Q \in \Phi_+(X)$ for all $\mu \in \mathbb{C}$. Clearly, T and $T + Q$ belong to the same component of the open set $\Phi_+(X)$, so $\operatorname{ind} T = \operatorname{ind}(T + Q) \leqslant 0$, and hence $0 \notin \sigma_{\mathrm{aw}}(T + Q)$. This shows $\sigma_{\mathrm{aw}}(T + Q) \subseteq \sigma_{\mathrm{aw}}(T)$. By symmetry then

$$\sigma_{\mathrm{aw}}(T) = \sigma_{\mathrm{aw}}(T + Q - Q) \subseteq \sigma_{\mathrm{aw}}(T + Q),$$

so the equality $\sigma_{\mathrm{aw}}(T) = \sigma_{\mathrm{aw}}(T + Q)$ is proved.

(ii) Suppose that $\lambda \notin \sigma_{\mathrm{a}}(T)$. Then $\lambda I - T$ is bounded below and hence $\lambda I - T \in W_+(X)$. By part (i) then $\lambda I - (T + Q) \in W_+(X)$, and since Q commutes with $\lambda I - (T + Q)$ from Lemma 2.11 we deduce that $\alpha(\lambda I - (T + Q)) = 0$. Since $\lambda I - (T + Q)$ has closed range we then conclude that $\lambda \notin \sigma_{\mathrm{a}}(T + Q)$. This shows that $\sigma_{\mathrm{a}}(T + Q) \subseteq \sigma_{\mathrm{a}}(T)$. A symmetric argument may be used for obtaining the opposite inclusion, so also the equality $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + Q)$ is proved. □

Theorem 2.13. *Suppose that $T \in L(X)$ and Q an injective quasi-nilpotent operator commuting with T. If T satisfies property (w), then also $T + Q$ satisfies property (w).*

Proof. Since T satisfies property (w) from Theorem 2.12 we have

$$\sigma_{\mathrm{a}}(T + Q) \setminus \sigma_{\mathrm{aw}}(T + Q) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{aw}}(T) = \pi_{00}(T). \tag{9}$$

To show property (w) for $T + Q$ it suffices to prove that

$$\pi_{00}(T) = \pi_{00}(T + Q) = \emptyset.$$

Suppose that $\pi_{00}(T) \neq \emptyset$ and let $\lambda \in \pi_{00}(T)$. From (9) we know that $\lambda I - T \in W_+(X)$, and hence by Lemma 2.11 it then follows that $\alpha(\lambda I - T) = 0$, a contradiction.

To show that $\pi_{00}(T + Q) = \emptyset$ suppose that $\lambda \in \pi_{00}(T + Q)$. Then $0 < \alpha(\lambda I - (T + Q)) < \infty$, so there exists $x \neq 0$ such that $(\lambda I - (T + Q))x = 0$. Since Q commutes with $\lambda I - (T + Q)$ the same argument of the proof of Lemma 2.11 shows that $\alpha(\lambda I - (T + Q) = \infty$, a contradiction. □

Theorems 2.6 and 2.8 apply to several classes of operators. A bounded operator $T \in L(X)$ is said to have *property* $H(p)$ if for all $\lambda \in \mathbb{C}$ there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p.$$

In [5] it has been observed that if $T^* \in L(X)$ has property $H(p)$, then T satisfies property (w). Moreover, T is a-isoloid. Indeed, from the implication (5), we see that property $H(p)$ for T^*

entails that T^* has SVEP and hence, by [1, Corollary 2.45], $\sigma_a(T) = \sigma(T) = \sigma(T^*)$. Since every isolated point of $\sigma(T)$ is a pole of the resolvent [2, Lemma 3.3], and hence an eigenvalue of T it then follows that T is a-isoloid. By Theorems 2.6 and 2.8 it then follows that property (w) holds for $T + K$, where K is nilpotent or a finite rank operator commuting with T. It should be noted that property $H(p)$ holds for a relevant number of operators. In [19, Example 3] Oudghiri observed that every *generalized scalar operator* and every *subscalar operator* T (i.e. T is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces) has property $H(p)$, see [16] for definitions and properties. Consequently, property $H(p)$ is satisfied by p-hyponormal operators and *log*-hyponormal operators [17, Corollary 2], w-hyponormal operators [18], M-hyponormal operators [16, Proposition 2.4.9], and totally paranormal operators [7]. Also totally $*$-paranormal operators have property $H(1)$ [13].

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