Fluctuation and Noise Letters
Vol. 5, No. 2 (2005) L305–L311

World Scientific
www.worldscientific.com

NONMONOTONIC PATTERN FORMATION IN THREE SPECIES LOTKA–VOLTERRA SYSTEM WITH COLORED NOISE

A. FIASCONARO*, D. VALENTI and B. SPAGNOLO
Dipartimento di Fisica e Tecnologie Relative and INFM, Group of Interdisciplinary Physics†
Università di Palermo, Viale delle Scienze pad. 18, I–90128 Palermo, Italy
*afiasconaro@gip.dft.unipa.it

Received 14 January 2005
Revised 19 March 2005
Accepted 10 April 2005
Communicated by Werner Ebeling and Bernardo Spagnolo

A coupled map lattice of generalized Lotka–Volterra equations in the presence of colored multiplicative noise is used to analyze the spatiotemporal evolution of three interacting species: one predator and two preys symmetrically competing each other. The correlation of the species concentration over the grid as a function of time and of the noise intensity is investigated. The presence of noise induces pattern formation, whose dimensions show a nonmonotonic behavior as a function of the noise intensity. The colored noise induces a greater dimension of the patterns with respect to the white noise case and a shift of the maximum of its area towards higher values of the noise intensity.

Keywords: Statistical mechanics; population dynamics; noise induced effects; Lotka–Volterra equations.

1. Introduction

The addition of noise in mathematical models of population dynamics can be useful to describe the observed phenomenology in a realistic and relatively simple form. This noise contribution can give rise to non trivial effects, modifying sometimes in an unexpected way the deterministic dynamics. Examples of noise induced phenomena are stochastic resonance, noise delayed extinction, temporal oscillations and noise-induced pattern formation [1–5]. Biological complex systems can be modelled as open systems in which interactions between the components are nonlinear and a noisy interaction with the environment is present [6]. Recently it has been found that nonlinear interaction and the presence of multiplicative noise can give rise to pattern formation in population dynamics of spatially extended systems [7–9]. The real noise sources are correlated and their effects on spatially extended systems have

†Electronic address: http://gip.dft.unipa.it

been investigated in Refs. [3] (see cited references there) and [4]. In this paper we study the spatio-temporal evolution of an ecosystem of three interacting species: two competing preys and one predator, in the presence of a colored multiplicative noise. We find a nonmonotonic behavior of the average size of the patterns as a function of the noise intensity. The effects induced by the colored noise, in comparison with the white noise case [9], are: (i) pattern formation with a greater dimension of the average area, (ii) a shift of the maximum of the area of the patterns towards higher values of the multiplicative noise intensity.

2. The Model

To describe the dynamics of our spatially distributed system, we use a coupled map lattice (CML) [9, 10] with a multiplicative noise

$$
\begin{aligned}
x_{i,j}^{n+1} &= \mu x_{i,j}^{n}(1 - x_{i,j}^{n} - \beta^{n} y_{i,j}^{n} - \alpha z_{i,j}^{n}) + x_{i,j}^{n} X_{i,j}^{n} + D\sum_{p}(x_{p}^{n} - x_{i,j}^{n}), \\
y_{i,j}^{n+1} &= \mu y_{i,j}^{n}(1 - y_{i,j}^{n} - \beta^{n} x_{i,j}^{n} - \alpha z_{i,j}^{n}) + y_{i,j}^{n} Y_{i,j}^{n} + D\sum_{p}(y_{p}^{n} - y_{i,j}^{n}), \\
z_{i,j}^{n+1} &= \mu_z z_{i,j}^{n}[-1 + \gamma(x_{i,j}^{n} + y_{i,j}^{n})] + z_{i,j}^{n} Z_{i,j}^{n} + D\textstyle\sum_{p}(z_{p}^{n} - z_{i,j}^{n}),
\end{aligned}
\tag{1}
$$

where $x_{i,j}^n$, $y_{i,j}^n$ and $z_{i,j}^n$ are respectively the densities of preys x, y and the predator z in the site (i, j) at the time step n. Here α and γ are the interaction parameters between preys and predator, D is the diffusion coefficient, μ and μ_z are scale factors. $\sum_p$ indicates the sum over the four nearest neighbors in the map lattice. $X(t), Y(t), Z(t)$ are Ornstein–Uhlenbeck processes with the statistical properties

$$
\langle \chi(t) \rangle = 0, \quad \langle\langle \chi(t)\chi(t+\tau) \rangle\rangle = \frac{q}{2\tau_c} e^{-\tau/\tau_c}, \tag{2}
$$

and

$$
\langle X_{i,j}^n Y_{i,j}^m \rangle = \langle X_{i,j}^n Z_{i,j}^m \rangle = \langle Y_{i,j}^n Z_{i,j}^m \rangle = 0 \quad \forall\, n, m, i, j \tag{3}
$$

where τ_c is the correlation time of the process, q is the noise intensity, and $\chi(t)$ represents the three continuous stochastic variables $(X(t), Y(t), Z(t))$, taken at time step n. The boundary conditions are such that no interaction is present out of lattice. Because of the environment temperature, the interaction parameter $\beta(t)$ between the two preys can be modelled as a periodical function of time

$$
\beta(t) = 1 + \epsilon + \eta cos(\omega t). \tag{4}
$$

Here $\eta = 0.2$, $\omega = \pi 10^{-3}$ and $\epsilon = -0.1$. The interaction parameter $\beta(t)$ oscillates around the critical value $\beta_c = 1$ in such a way that the dynamical regime of Lotka–Volterra model for two competing species changes from coexistence of the two preys ($\beta < 1$) to exclusion of one of them ($\beta > 1$). The parameters used in our simulations are the same of [9], in order to compare the results with the white noise case. Specifically they are: $\mu = 2$; $\alpha = 0.03$; $\mu_z = 0.02$, $\gamma = 205$ and $D = 0.1$. The noise intensity q varies between 10^{-11} and 10^{-2}. With this choice of parameters the intraspecies competition among the two prey populations is stronger compared to the interspecies interaction preys–predator ($\beta \gg \alpha$), and both prey populations can therefore stably coexist in the presence of the predator [11]. To evaluate the species correlation over the grid we consider the correlation coefficient r^n between a couple of them at the step n as

$$r^n = \frac{\sum_{i,j}^{N}(w_{i,j}^n - \bar{w}^n)(k_{i,j}^n - \bar{k}^n)}{\left[\sum_{i,j}^{N}(w_{i,j}^n - \bar{w}^n)^2 \sum_{i,j}^{N}(k_{i,j}^n - \bar{k}^n)^2\right]^{1/2}}, \tag{5}$$

where N is the number of sites in the grid (100x100), the symbols w^n, k^n represent one of the three species concentration x, y, z, and $\bar{w}^n, \bar{k}^n$ are the mean values of the same quantities in all the lattice at the time step n. From the definition (5) it follows that $-1 \leq r^n \leq 1$.

3. Colored Noise Effects

We quantify our analysis by considering the maximum patterns, defined as the ensemble of adjoining sites in the lattice for which the density of the species belongs to the interval $[3/4\ max, max]$, where max is the absolute maximum of density in the specific grid. The various quantities, such as pattern area and correlation parameter, have been averaged over 50 realizations, obtaining the mean values below reported. We evaluated for each spatial distribution, in a temporal step and for a given noise intensity value, the following quantities referring to the maximum pattern (MP): mean area of the various MPs found in the lattice and correlation r between two preys, and between preys and predator.

From the deterministic analysis we observe: (i) for $\epsilon < 0$ ($\beta < 1$) a coexistence regime of the two preys, characterized in the lattice by a strong correlation between them and the predator lightly anti-correlated with the two preys; (ii) for $\epsilon > 0$ ($\beta > 1$) wide exclusion zones in the lattice, characterized by a strong anti-correlation between preys. Because of the periodic variation of the interaction parameter $\beta(t)$, an interesting activation phenomenon for $\epsilon < 0$ takes place: the two preys, after an initial transient, remain strongly correlated for all the time, in spite of the fact that the parameter $\beta(t)$ takes values greater than 1 during the periodical evolution. We focus on this dynamical regime to analyze the effect of the noise. We found that the noise acts as a trigger of the oscillating behavior of the species correlation r giving rise to periodical alternation of coexistence and exclusion regime. Even a very small amount of noise is able to destroy the coexistence regime periodically in time. This gives rise to a periodical time behavior of the correlation parameter r, with the same periodicity of the interaction parameter $\beta(t)$ (see Eq. (4)), which turns out almost independent of the noise intensity and of the correlation time τ_c (see Fig. 1(a)). This periodicity reflects the periodical time behavior of the mean area of the patterns. A nonmonotonic behavior of the pattern area as a function of time is observed for all values of noise intensity investigated. This behavior becomes periodically in time for lower values of noise intensity, when higher values of correlation time τ_c are considered. In Figs. 1(b–d) we show the time evolution of the mean area of the maximum patterns, for $q = 10^{-4}$ and for three values of correlation time, namely $\tau_c = 1, 10, 100$. The periodicity of the nonmonotonic behavior of the area of MPs is clearly observed.

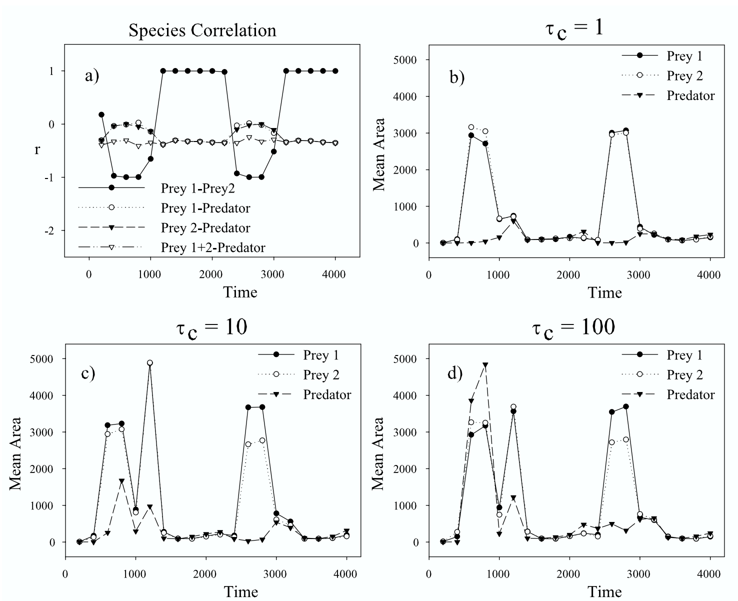

Fig. 1. (a) The correlation coefficient between preys and predator as a function of time; (b–d) Mean area of the maximum patterns of the species as a function of time, for three values of correlation time $\tau_c = 1, 10, 100$ and for $q = 10^{-4}$. The correlation plot (a) is quite the same for all the τ_c investigated.

To analyze the noise induced pattern formation we focus on the correlation regime between preys $r_{12} = 1$, where pattern formation appears. In fact when the preys are highly anticorrelated with species correlation parameter $r_{12} = -1$, a big clusterization of preys is observed, with large patches of preys enlarging to all the available space of the lattice. This scenario, observed also in the white noise case [9], is confirmed by the analysis of the time series of the species. These large patches appear, in the anticorrelation regime corresponding to the exclusion regime of the two preys, with smooth contours and low intensity of species density for lower noise intensities and higher correlation time values.

The study of the area of the pattern formation as a function of noise intensity with colored noise shows two main effects: (1) the increase of the pattern dimension and (2) a shift of the maximum toward higher values of the noise intensity. As expected, for low values of the correlation time we observe the same results than in the white noise case. These effects are well visible in Fig. 2 where the three curves show the nonmonotonic behavior of the area of the maximum pattern as a function of noise intensity. The interaction step here considered is 1400, which correspond to the biggest pattern area found in our calculations. The first curve ($\tau_c = 1$) is quite the same found in the white noise case. The value of maximum in the third curve ($\tau_c = 100$) is not so different from the previous one ($\tau_c = 10$), because its value is approaching the maximum possible value of 10,000 into the used grid.

The pattern formation is visible in Fig. 3, where we report three patterns of the two preys and the predator for the following values of noise intensity:

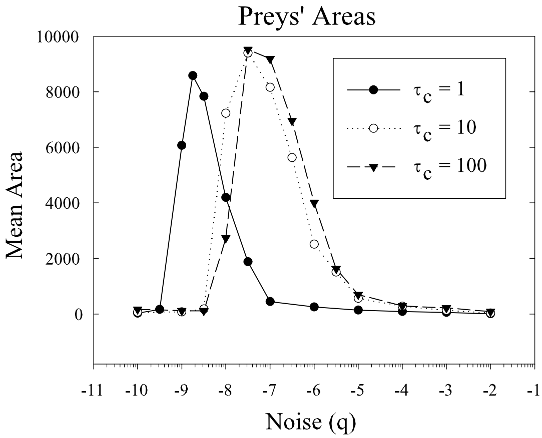

Fig. 2. Semi-Log plot of the mean area of the maximum patterns for all species as a function of noise intensity, at iteration step 1400 for the three correlation time here reported. See the text for the values of the other parameters.

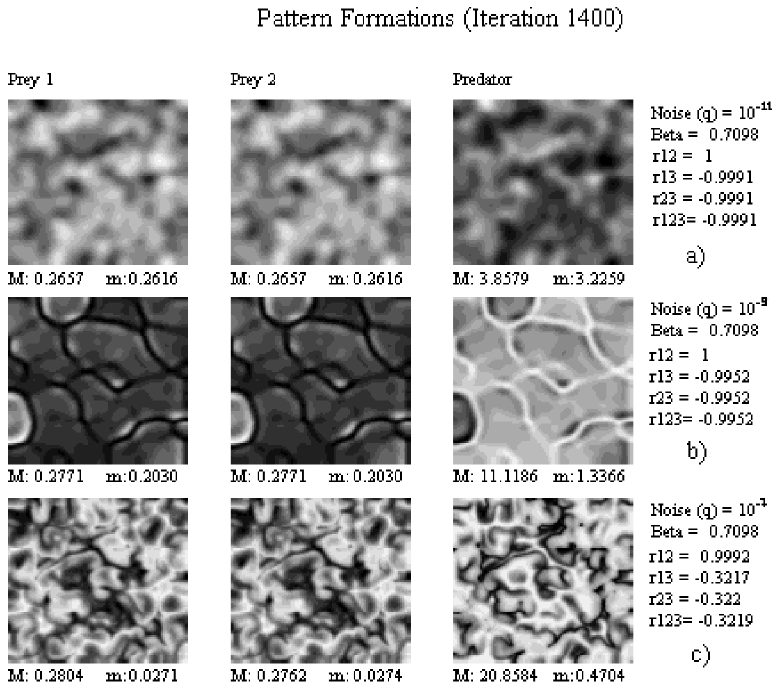

Fig. 3. Pattern formation for preys and predator with homogeneous initial distribution, at time iteration 1400 for $\tau_c = 1$ and noise intensity: $q = 10^{-11}, 10^{-9}, 10^{-7}$. $r_{12}, r_{13}, r_{23}, r_{123}$ represent respectively prey-prey, prey1-predator, prey2-predator and total preys-predator correlation. See the text for the values of the other parameters.

$q = 10^{-11}, 10^{-9}, 10^{-7}$ and $\tau_c = 1$. The initial spatial distribution is homogeneous and equal for all species, that is $x_{ij}^{init} = y_{ij}^{init} = z_{ij}^{init} = 0.25$ for all sites (i, j). We see that a spatial structure emerges with increasing noise intensity. At very low noise intensity ($q = 10^{-11}$), the spatial distribution appears almost homogeneous without strong pattern formation (see Fig. 3(a)). We considered here only structured pattern, avoiding big clusterization of density visible in the case of anticorrelated preys. At intermediate noise intensity ($q = 10^{-9}$) spatial patterns appear. As we can see the structure disappears by increasing the noise intensity (see Fig. 3(c)). Consistently with Fig. 2, we find that for higher correlation time τ_c the qualitative shape of the patterns shown in Fig. 3 are repeated, but with a shift of the maximum area (darkest patterns) toward higher values of the noise intensity.

4. Conclusions

The noise-induced pattern formation in a coupled map lattice of three interacting species, described by generalized Lotka–Volterra equations in the presence of multiplicative colored noise, has been investigated. We find nonmonotonic behavior of the mean area of the maximum patterns as a function of noise intensity for all the correlation time investigated. For increasing values of the correlation time τ_c we observe an increase of the area of the pattern and a shift of the maximum value towards higher values of the noise intensity. The nonmonotonic behavior is also found for the area of the patterns as a function of the evolution time.

Acknowledgments

This work was supported by INTAS Grant 01–0450, by INFM and MIUR.

References

[1] D. Valenti, A. Fiasconaro and B. Spagnolo, *Stochastic resonance and noise delayed extinction in a model of two competing species*, *Physica A* **331** (2004) 477–486.

[2] A. A. Zaikin and L. Schimansky-Geier, *Spatial patterns induced by additive noise, Phys. Rev. E* **58** (1998) 4355–4360.

[3] J. García-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems*, Springer-Verlag, New York (1999).

[4] J. García-Ojalvo and J. M. Sancho, *Colored noise in spatially extended systems*, *Phys. Rev. E* **49** (1994) 2769–2778.

[5] M. Ibanes, J. M. Sancho, J. Buceta and K. Lindenberg, *Noise-driven mechanism for pattern formation, Phys. Rev. E* **67** (2003) 021113–1–8.

[6] S. Ciuchi, F. de Pasquale and B. Spagnolo, *Self regulation mechanism of an ecosystem in a non Gaussian fluctuation regime, Phys. Rev. E* **54** (1996) 706–716.

[7] B. Spagnolo, M. Cirone, A. La Barbera and F. de Pasquale, *Noise induced effects in population dynamics, J. Phys.: Cond. Matter* **14** (2002) 2247–2255.

[8] B. Spagnolo, A. Fiasconaro and D. Valenti, *Noise induced phenomena in Lotka–Volterra systems, Fluctuation and Noise Letters* **3**(2) (2003) L177–L185.

[9] A. Fiasconaro, B. Spagnolo and D. Valenti, *Nonmonotonic behavior of spatiotemporal pattern formation in a noisy Lotka–Volterra system, Acta Physica Polonica B* **35**(4) (2004) 1491–1500.

[10] Special issue CML models, ed. K. Kaneko [Chaos **2** (1992) 279–460].

[11] A. D. Bazykin, *Nonlinear dynamics of interacting populations*, World Scientific, Singapore (1998).

[12] J. García Lafuente, A. García, S. Mazzola, L. Quintanilla, J. Delgado, A. Cuttitta and B. Patti, *Hydrographic phenomena influencing early life stages of the Sicilian Channel anchovy, Fishery Oceanography* **11**(1) (2002) 31–44.